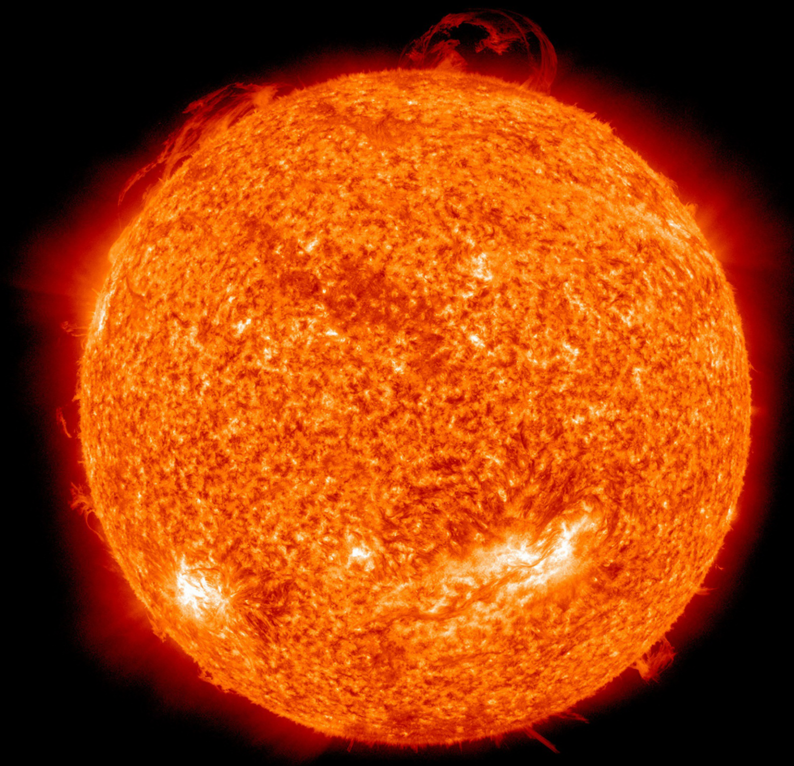
INSIDE STARS

The 3rd revised edition

L. Borissova and D. Rabounski



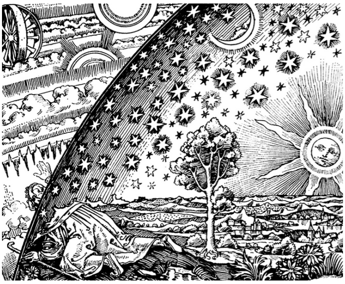
Inside Stars

A theory of the internal constitution of stars
and sources of stellar energy
based on the General Theory of Relativity

by Larissa Borissova and Dmitri Rabounski

The 3rd revised edition

First published in 2013
The 2nd edition published 2014



New Scientific Frontiers
London, 2023

Summary: — This book introduces a mathematical theory of the internal constitution of stars and sources of stellar energy, created using the mathematical methods of General Relativity. This is an alternative to the traditional theory of gaseous stars, which was introduced in the 1920s based on classical mechanics and thermodynamics. On the contrary, the consideration of a star and its field in the framework of General Relativity, which is presented in this book, comes to a model of liquid stars. Such a star is homogeneous inside, with a tiny core (about a few kilometers in radius) in the centre. The core is separated from the main substance of the star by a collapse surface with a radius corresponding to the star's mass. Despite the fact that almost all the mass of the star is outside the core (the core is not a black hole), the gravitational force tends to infinity on the surface of the core due to the space breaking in the star's internal field. Such a superstrong gravitational force is sufficient to transfer the necessary kinetic energy to the light atomic nuclei of stellar substance in order to start the process of thermonuclear fusion. The energy produced by thermonuclear fusion is the energy with which stars glow: each star's tiny core is its glowing "inner sun", and the stellar energy produced in it is then transferred to the star's physical surface by thermal conduction. A new classification of stars according to the space breaking in their fields has been introduced: ordinary stars (ranging from dwarfs to supergiants), Wolf-Rayet stars, neutron stars (and pulsars), and also black holes are considered. The introduced liquid model of stars is consistent with new observational evidence for the state of condensed matter inside stars; in particular, that the Sun is composed of high-temperature liquid metallic hydrogen.

This book was typeset using the LaTeX typesetting system.

New Scientific Frontiers is a publisher registered with Nielsen Book Services Ltd., Woking, Surrey, UK.

ISBN: 978-1-7392930-6-2

Published in the United Kingdom.

Contents

Preface

A scientist often encounters established ideas that were once the subject of debate, sometimes controversy. Often, we use those ideas with no knowledge of their historical development, nor of the assumptions on which they are based. We rarely stop to ponder the validity of an established idea. This is not surprising as this is how we have been building our edifice of physical theories, by standing on the shoulders of giants, to paraphrase Isaac Newton.

Yet established ideas and theories need to be challenged and revisited when new data or new theories that contradict or shed new light on them, become available. We need not be afraid of new information that risk overturning accepted ideas. After all, this is how new paradigms arise and how progress is achieved.

The question of whether stars are gaseous or liquid is one debate that most scientists are oblivious to. Yet this was a subject of vigorous debate in the late 19th and early 20th centuries, with well-known physicists lining up behind both sides of the question. Larissa Borissova and Dmitri Rabounski provide a summary of the history of this debate and a personal perspective on how they were pulled into it.

Recent evidence for liquid stars, in particular the extensive research performed by Pierre-Marie Robitaille who has proposed the liquid metallic hydrogen model of the Sun*, leads us to revisit this question. Interestingly enough, stellar plasmas are modelled using Magnetohydrodynamics, i.e. magnetic fluid dynamics, a combination of Maxwell's equations of electromagnetism and the Navier-Stokes equations of fluid mechanics†. Magnetohydrodynamics is also used to model liquid met-

*Robitaille P.-M. A high temperature liquid plasma model of the Sun. *Progress in Physics*, 2007, vol. 3, no. 1, 70–81.

†Tajima T. and Shibata K. Plasma Astrophysics. Perseus Publishing, Cambridge, 2002; Kulsrud R. M. Plasma Physics for Astrophysics. Princeton University Press, Princeton, 2005.

als. This is an indication that the theory of liquid stars is highly plausible as an explanation of solar and stellar astrophysical data.

My personal interest in this research area stems from the astrophysical research I performed on stellar atmospheres of Wolf-Rayet stars at the University of Ottawa's Department of Physics for my thesis on "Laser Action in C IV, N V and O VI Plasmas Cooled by Adiabatic Expansion". Wolf-Rayet stars exhibit mass loss and an expanding stellar atmosphere. This results in population inversion of certain atomic transitions due to the rapid cooling of the expanding plasma and the recombination of the free electrons into higher excited ionic states and laser action in the corresponding emission lines. This physical mechanism has been proposed as the explanation for the prominent spectral lines observed in the spectra of Wolf-Rayet stars.

In this book, Larissa Borissova and Dmitri Rabounski provide a general relativistic theory of the internal constitution of liquid stars, a model that was lacking till now. This they accomplish by using a mathematical formalism first introduced by Abraham L. Zelmanov for calculating physically observable quantities in a four-dimensional pseudo-Riemannian space, known as the "theory of chronometric invariants". This mathematical formalism allows to calculate physically observable (chronometrically invariant) tensors of any rank, based on operators of projection onto the time line and the spatial section of the observer. The basic idea is that physically observable quantities obtained by an observer should be the result of a projection of four-dimensional quantities onto the time line and onto the spatial section (local three-dimensional space) of the observer.

This analysis allows them to propose a classification of stars based on three main types: ordinary stars (ranging from white dwarfs to supergiants), of which Wolf-Rayet stars are a subtype, neutron stars and pulsars and collapsars (i.e. black holes). Their theory also provides a model of the internal constitution of our solar system. It provides an explanation for the presence of the asteroid belt, the general structure of the planets inside and outside that orbit and the net emission of energy by the planet Jupiter.

The ultimate test of any theory of stellar structure is the stellar mass-luminosity relation which is the main empirical relation of observational astrophysics. Using their theory, the authors can calculate the pressure inside stars as a function of radius, including the central pressure. As pointed out by the authors, the temperature of an incompressible liquid star does not depend on the pressure, but only on the source of stellar energy (as opposed to a gas, in particular as given by the equation of state of an ideal gas). The authors compare the calculated energy release by the proposed mechanism of thermonuclear fusion of the light atomic nuclei in the Hilbert core (the “inner sun”) of stars with the empirical mass-luminosity relation of observational astrophysics, to determine the density of the liquid stellar substance in the Hilbert core.

Pulsars and neutron stars are found to be stars whose physical radius is close to the radius of their Hilbert core. They are modelled by introducing an electromagnetic field in the theory due to their rotation and gravitation. Electromagnetic radiation is found to be emitted only from the poles of those stars, along the axis of rotation of the stars.

This book represents a solid contribution to our understanding of stellar structure from a general relativistic perspective. It provides a general relativistic underpinning to the theory of liquid stars. It raises new ideas on the constitution of stars and planetary systems and proposes a new approach to stellar structure and evolution which is bound to help us better understand stellar astrophysics.

Ottawa, September 2, 2013

Pierre Millette

Astrophysics research on stellar atmospheres, Department of Physics, University of Ottawa

Foreword

Three decades ago, in 1983, I began to study the history of the theory of gaseous stars. I was inspired to do this by Prof. Kyril Stanyukovich (1916–1989), an outstanding scientist in the field of gas dynamics and General Relativity, with whom Larisa and I were on friendly terms for long time. Stanyukovich told me that soon after Hans Bethe proposed thermonuclear fusion as a source of stellar energy, in 1939 astrophysicists began trying to adapt the gas model of stars to thermonuclear fusion. In many cases their assumptions were so artificial in relation to gas dynamics itself that only the absence of another theory could justify their models. Stanyukovich also talked about many of the obvious evidence for gas dynamics that would inherently contradict the gas model of stars.

Then I read the primary papers on the theory of gaseous stars published in the early 20th century. I found that the "core" of this theory, consisting of the equations of mechanical and thermal equilibrium inside stars, does not depend on whether the stars are made of gas or something else. Only then, introducing into these equations the equation of state of an ideal gas, the theory gives the so-called gaseous stars and all the variety of the gas models.

Then we got carried away with other research studies, mainly on General Relativity, so that astrophysics fell behind our attention by almost 25 years.

In the summer of 2007, Prof. Pierre-Marie Robitalle visited us for the first time. Working in the Ohio State University, Pierre spent many years doing deep experimental research in the fields of thermal physics and nuclear magnetic resonance (which produces microwave radiation). He drew our attention to new astrophysical evidence for the liquid Sun and stars, which appeared only in the last decade. When Pierre-Marie was walking with me in the afternoon in a nearby park, he pointed to the disc of the Sun in the sky and said: "Look, it is a liquid ball." But that summer did not allow us to create a detailed mathematical theory of liquid stars.

A few months after this event, in the same 2007, Larisa and I undertook to translate into English two classic papers on General Relativity written in 1916 by Karl Schwarzschild. In one of the works mentioned above, he introduced the space metric of a sphere filled with an incompressible liquid, which then brought him great posthumous fame. We knew that the Schwarzschild metric of a liquid sphere could not be used as a model for liquid stars. This is due to the specific limitation contained in the metric. However, immediately after reading the initial derivation of the metric, published in his 1916 paper, we found that the limitation was introduced artificially by him in order to make the gravitational field of the liquid sphere free from a breaking (discontinuity). If it were possible to deduce a real metric of a liquid sphere, free from any artificial limitations pre-imposed on the geometry of the space, we could create a mathematical theory of liquid stars.

The way forward was finally found: we knew what to do next. To check, Larisa immediately deduced the true metric of a liquid sphere, then calculated some consequences for the liquid Sun. She found that when the Sun is represented as a liquid sphere, its gravitational field has a space breaking corresponding to the maximum concentration of substance in the asteroid belt; thus the space breaking in the Sun's gravitational field prevents substance from forming as a planet in that orbit. So, we made sure that we are on the right path. (David Jones, Editor of the *New Scientist*, wrote in 1981: "As is known, all major scientific discoveries had been made in the course of working on other problems or as a result of random observations.")

That is the story in a nutshell. In the spring of 2013, we completed the mathematical theory of liquid stars. This theory provides three basic liquid models according to General Relativity, which together cover all known types of stellar objects ranging from supergiants to neutron stars and black holes. This book presents the main elements of this theory, with the exception of the details of the stellar energy mechanism (this is left outside the scope of a book devoted mainly to the internal constitution of stars).

Puschino, August 6, 2013 Dmitri Rabounski

Acknowledgements

Larissa and I would like to thank Pierre-Marie Robitaille (Ohio, USA) for many days of scientific discussion and encouragement during his two long term visits to us across the ocean and later. We are most grateful to him. His contribution is truly invaluable.

We are also very grateful to Pierre A. Millette (Ottawa, Canada), who, being an expert in both stellar astrophysics and General Relativity, carefully read the draft of this book and made helpful suggestions.

We also would like to express our sincerest thanks to Indranu Suhendro and his wife, Susanne Billharz, USA. Our common discussion of this book and their editing assistance has made the book much better than it was in the first draft.

We are also grateful to Patrick Marquet (Calais, France). Our discussions with him on General Relativity always give us a lot of new things, including an impetus for new scientific research.

Our special thanks go to Anatole V. Belyakov (Tverin Kariela, Russia), who undertook the translation of this book into Russian.

In the 3rd English edition, we have completely revised the entire text of the book and made many necessary corrections.

Puschino, February 15, 2023 Dmitri Rabounski

Chapter 1 Problem Statement

1.1 A new theory of the internal constitution of stars

In this book, we introduce a new mathematical theory of the internal constitution of stars and sources of stellar energy. The theory is based on the joint consideration of a star and its field according to General Relativity.

This is an alternative to the traditional theory of stars introduced in the 1920s by Arthur Eddington [1] and others in the framework of classical mechanics and thermodynamics.

As is known, the conventional theory has led to the *model of gaseous stars*: stars are considered as gaseous spheres, consisting mainly of hydrogen and a very inhomogeneous interior, so that the hydrogen of the extremely hot and dense central region is used as fuel for the stellar energy generation process. It is assumed, following Hans Bethe [2], that this exothermic process is a thermonuclear fusion producing helium from hydrogen. The other two variants of the gas model differ in detail from Eddington's theory. Edward Milne [3] had proposed that there are two (or more) different states of substance inside a star. Nikolai Kozyrev [4] had come up with a peculiar picture of low density and temperature inside stars and a non-nuclear source of stellar energy.

Another theory of the internal constitution of stars became widespread in the 1920s and 1930s thanks to James Jeans [5, 6]. This is a *model of liquid stars*. The public discussion between Jeans, who defended the liquid model, and Eddington, the follower of the gas model, was recorded in dozens of short messages published by them in scientific journals. Indeed, Eddington won in the end. Despite a lot of astrophysical evidence for liquid stars, Jeans' theory did not have a solid mathematical foundation. His theory was based on observational analysis and arguments rather than a mathematical model. On the contrary, the theory of gaseous stars was mathematically well founded by Eddington. In particular, the mathematical model of gaseous stars gives a theoretical derivation of the mass-luminosity relation, which is one of the main relations of observational astrophysics*. This was a "trump card": as soon as the gas model predicted the mass-luminosity relation, this model was declared correct in general, and all its inconsistencies with observational analysis were only some "difficulties" that had to be resolved in the future. Thus, the model of gaseous stars has become the generally accepted model for decades to come, up to the present.

We must now make an important remark. As is known, the core of the mathematical theory of the internal constitution of stars consists of two equations: the equation of mechanical equilibrium and the equation of thermal equilibrium. Mechanical equilibrium means that the weight of any unit volume of stellar substance is brought into equilibrium with the pressure from within the star. Thermal equilibrium (energy balance) means that the energy produced in any unit volume of stellar substance is brought into balance with the flow of energy (radiation, convection or heat conduction) from within the star to its surface. These two basic equations of the theory are taken from general physics and are *independent* of whether stars are composed of gas, liquid, or anything else. Only then, by introducing the equation of state of an ideal gas (and many other partial assumptions) into the basic equations, does the traditional theory lead to gaseous stars and other conclusions, including the mass-luminosity relation.

Jeans' theory of liquid stars cannot follow this path. The equation of state of an ideal liquid, given by classical physics, is so simple that it does not contain the characteristics of stellar substance necessary for further derivation using the equilibrium equations.

Instead of all these considerations using classical mechanics and thermodynamics, we propose a completely different approach to the problem. It is based on the joint consideration of a star and its field according to General Relativity. We are considering liquid stars: this corresponds to some new observational evidence for the state of condensed matter inside stars; in particular, that the Sun is composed of high-temperature liquid metallic hydrogen [7–10].

---

*The most comprehensive derivation of the mass-luminosity relation in the framework of the model of gaseous stars is given in Part I of Kozyrev's paper [4].

In the framework of General Relativity, the structure, substance and field of such a star are characterized by the Schwarzschild metric of a sphere filled with an incompressible liquid. A recent theoretical result obtained by L. Borisova [11, 12] showed that if the Sun is represented as a liquid sphere according to the Schwarzschild metric, then the Sun's field has a space breaking (discontinuity) in the asteroid belt: this means that the space breaking prevents the formation of substance in the form of a planet in this orbit. Therefore, we are confident that we are on the right path.

First, we deduce Einstein's field equations in a form that models stars as liquid spheres. This is a particular form of the field equations that may or may not satisfy a particular space metric. Therefore, we then prove that the resulting particular form of the field equations satisfies the Schwarzschild metric of a liquid sphere.

Then, based on the energy-momentum tensor of an ideal liquid (contained in the right hand side of the field equations), we deduce the conservation law for the liquid substance of ordinary stars. Solving the obtained energy-momentum conservation equations, we obtain the pressure and density of the liquid substance inside stars. Then we obtain a formula for the luminosity of stars according to the liquid model. Next, we study how this theoretical formula can be compatible with the mass-luminosity relation (one of the basic empirical relations of observational astrophysics). As a result, we obtain the physical characteristics of the mechanism that produces energy inside stars.

Concerning the stellar energy mechanism itself, we conclude that it is the transformation of substance into radiation on the surface of a tiny central "core" inside each star. The core may have a density different from the density of the rest of the star's substance (a liquid sphere inside is homogeneous) and is distinguished by a collapse surface with a radius determined by the star's mass. Despite the fact that almost all the mass of the star is outside the core (since the core is not a black hole), the gravitational force tends to infinity on the surface of the core due to the space breaking in the star's field on this surface. The super-strong gravitational force is sufficient to transfer the necessary kinetic energy to the light atomic nuclei of stellar substance in order to start the process of thermonuclear fusion. The energy produced by thermonuclear fusion is the same energy that stars emit. In other words, the tiny core of each star is its luminous "inner sun", and the produced stellar energy is then transferred to the physical surface of the star by thermal conduction (usual in liquid media).

Neutron stars and pulsars, being rapidly rotating objects, are a special type of stars. The structure, substance, and field of such stars must be described by another metric, namely, — the metric of a rotating liquid sphere under special physical conditions (specific for neutron stars and pulsars). We will introduce such a metric. According to the metric, the liquid substance of neutron stars and pulsars is in the same state as the high-density physical vacuum. We then deduce a particular form of Einstein's field equations, which satisfies the metric. We show that the energy-momentum tensor of the obtained field equations satisfies the conservation law only in the case, where the energy flow from within the star is highly anisotropic and directed towards the South and North Poles, and the magnetic field axis does not coincide with the star's rotation axis. This coincides with the known observational data on neutron stars and pulsars.

This is our plan for the upcoming Chapters. As a result, we obtain a mathematical theory of liquid stars and sources of stellar energy based on General Relativity.

Before moving on to these steps, in the next §1.2 we will consider the space metrics that we use in our theory. Then we introduce a new classification of stars. This classification is based on the location of the space breaking in the star's field relative to its surface (the space breaking is calculated based on the space metric and the proper parameters of the star).

At the end of this Chapter, in §1.3 we will give a detailed overview of the mathematical apparatus of physically observable quantities in the space-time of General Relativity, which we will need for our further calculations.

1.2 Modelling a star in terms of General Relativity

Stars are spherical bodies filled with substance and light. Their fields are also spherically symmetrical. Therefore, when considering a star in terms of General Relativity, the structure, substance and field of such an object must be given by a spherically symmetric space metric.

Among the space (space-time) metrics known in General Relativity, three basic metrics describe spherically symmetric fields. These are the

Schwarzschild metric of a material point, the Schwarzschild metric of a sphere filled with an incompressible liquid, and the de Sitter metric describing the spherical distribution of the physical vacuum (λ-field, determined by the λ-term in Einstein's field equations). All three of these metrics will be used when considering stars.

1.2.1 The *mass-point metric*

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad (1.1)$$

was introduced in 1916 by Karl Schwarzschild [13]. The metric describes the field of a spherically symmetric massive body at such a large distance from it that the physical sizes of the body are neglected (assuming that the body is a material point). The metric is written in the spherical coordinates r, ϕ, θ, the origin of which coincides with the mass-point. Here $r_g = \frac{2GM}{c^2}$ is the Hilbert radius of the massive body*, and M is the body's mass (i.e, the mass of the field source).

According to the metric (1.1), such a space does not rotate or deform. The gravitational inertial force (see §1.3 for detail) in such a space is due only to the g_{00} component of the fundamental metric tensor $g_{\alpha\beta}$. As is seen from the mass-point metric (1.1),

$$g_{00} = 1 - \frac{r_g}{r}\,. \qquad (1.2)$$

Differentiating the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ with respect to x^i, we obtain the gradient of the potential

$$\frac{\partial \mathrm{w}}{\partial x^i} = -\frac{c^2}{2\sqrt{g_{00}}}\frac{\partial g_{00}}{\partial x^i}\,. \qquad (1.3)$$

Then substitute it into the general formula for the gravitational inertial force (1.42), taking the absence of rotation of the space into account.

*This is not the same as the physical radius of the body. At a distance of the Hilbert radius from the centre of gravity of a massive body ($r = r_g$), a gravitational collapse occurs: in a space without rotation, this is the state in which the component g_{00} of the fundamental metric tensor $g_{\alpha\beta}$ is zero ($g_{00} = 0$). See §5.1 and §5.2 for detail. The *Hilbert radius* was introduced by David Hilbert (1862–1944), who considered it in 1917 [15] based on the Schwarzschild mass-point metric. It is also known as the *Schwarzschild radius*, despite the fact that Karl Schwarzschild (1873–1916) never considered gravitational collapse in his works [13, 14].

We obtain the formulae for the covariant and contravariant components of the gravitational inertial force

$$F_1 = -\frac{c^2 r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\,, \qquad F^1 = -\frac{c^2 r_g}{2r^2}\,. \tag{1.4}$$

As is seen from the formulae, the gravitational inertial force in the space of a mass-point is due only to the Newtonian gravitational field created by the mass and is inversely proportional to the square of the distance r from it.

The curvature of the space of a mass-point is due to the Newtonian gravitational field created by a massive body located at the coordinate origin. This is not a constant curvature space; its curvature decreases with distance from the massive body (source of the field). At an infinitely large distance from the body, the space is flat.

1.2.2 A space filled with a spherically symmetric homogeneous distribution of the physical vacuum (determined by the λ-field in Einstein's field equations) without an island of mass represented in it is described by the *de Sitter metric*

$$ds^2 = \left(1-\frac{\lambda r^2}{3}\right)c^2dt^2 - \frac{dr^2}{1-\frac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{1.5}$$

The metric was introduced in 1918 by Willem de Sitter [16] as a static model of the Universe. It is assumed that $\lambda < 10^{-56}$ in such a space, so the physical vacuum has a very low density in it. The modern version of the static model of the Universe is presented in [17].

The fundamental metric tensor through its components according to the de Sitter metric (1.5) shows that such a space does not rotate or deform. Therefore, the gravitational inertial force (1.42) in such a space is due only to the g_{00} component of $g_{\alpha\beta}$, which is

$$g_{00} = 1 - \frac{\lambda r^2}{3}\,. \tag{1.6}$$

Accordingly, after the same algebra as previously, we obtain

$$F_1 = \frac{\lambda c^2}{3}\frac{r}{1-\frac{\lambda r^2}{3}}\,, \qquad F^1 = \frac{\lambda c^2}{3}\,r\,. \tag{1.7}$$

This is a non-Newtonian force of gravitation proportional to distance: the force (λ-force) increases with the distance r over which it acts. If $\lambda < 0$ (the observed vacuum density is positive), then this is a force of attraction. If $\lambda > 0$ (the observed vacuum density is negative), then this is a force of repulsion. See Chapter 5 of our book [18], where we considered the physical vacuum in detail.

The curvature of a de Sitter space is due to the non-Newtonian gravitational field created by the physical vacuum (λ-field) homogeneously filling the space. The curvature is the same everywhere in the space. In other words, it is a constant curvature space.

1.2.3 The *metric of a sphere filled with an incompressible liquid* was first introduced in 1916 by Karl Schwarzschild [14] in a truncated form containing significant limitations. He artificially pre-constrained the derivation of the metric to free the field from a breaking*. The true metric of a sphere filled with an incompressible liquid remained unknown until 2009, when L. Borisova deduced it in the most complete form [11, 12], which is free of any limitations and thus takes a space breaking into account.

The model of stars as liquid spheres plays a key rôle in our theory. Therefore, we consider the metric of a sphere filled with an incompressible liquid in the complete form [11, 12]

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1-\frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \qquad (1.8)$$

where $a = const$ is the physical radius of the liquid sphere, and $r_g = \frac{2GM}{c^2}$ is the Hilbert radius, calculated from the liquid sphere's mass M (i.e., the mass of the field source). The derivation of this metric, containing all the necessary details, will be reproduced in §2.1 of the book, where we apply this metric to ordinary stars.

The metric (1.8) is written for distances $r < a$. This is the "internal metric" of a liquid sphere. On the surface of the sphere ($r = a$) the metric

*In fact, as soon as any limitation is pre-imposed on the metric of a space, the geometry of the metric space is artificially truncated. In this sense, the Schwarzschild metric, introduced in 1916, is not a true metric of the space of a liquid sphere.

coincides with the mass-point metric. Moreover, as was proved in [11] (this derivation will be reproduced in §2.1 of this book), the "external metric" of a liquid sphere ($r > a$) also coincides with the mass-point metric: the external field of a liquid sphere coincides with the Newtonian gravitational field of a material point.

As is seen from the liquid sphere metric (1.8), such a space does not rotate or deform. Therefore, according to the definition of the gravitational inertial force (1.42), the force in such a space is due only to g_{00}. Thus, in the metric (1.8) we have

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 . \tag{1.9}$$

After the same algebra as previously, we obtain

$$F_1 = -\frac{c^2 r_g r}{a^3}\,\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{1.10}$$

$$F^1 = -\frac{c^2 r_g r}{a^3}\,\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{1.11}$$

Since $r < a$ inside the sphere, $F_1 < 0$ in it. Therefore, this is a force of attraction. Its numerical value is proportional to the distance r over which the force acts. The force is zero at the centre of the sphere and reaches its limit at the surface of the sphere.

It can be shown that the curvature of such a space, due to the mentioned field of gravitation, increases with distance from the centre of the liquid sphere to its surface. In other words, the space inside a liquid sphere is not a constant curvature space. We will give a proof and discuss both the four-dimensional curvature and the observable three-dimensional curvature of such a space in §2.3.

1.2.4 Here we propose a new method for modelling stars, which is based on the mathematical methods of General Relativity.

Let us consider stars as spherical bodies consisting of a liquid. According to the model of liquid stars, the internal structure, substance, and field of a star are described by the metric of a sphere filled with an incompressible liquid. This is the formula (1.8). As shown above, the gravitational force in this case increases with distance from the centre of the star. The external field of such a star is described by the mass-point metric (1.1). In the external field, the ordinary Newtonian gravitational force acts: the force is inversely proportional to the square of the distance from the star.

The field of a liquid sphere as such is not continuous everywhere. According to the external metric (1.1) and the internal metric (1.8) of a liquid sphere, its field has a *space breaking* that appears at two distances from its centre. In this regard, we are now introducing a new classification of stars based on General Relativity. We hereby explain how to build this classification.

A space breaking occurs due to the violation of the *signature conditions* characteristic of the space metric. This means that the space has a singularity in that region (surface or volume) in which at least one of the signature conditions is violated. The signature conditions for the sign-alternating diagonal metric (+−−−) such as the metric of the four-dimensional pseudo-Riemannian space (which is the basic space-time of General Relativity) have the form

$$\left.\begin{aligned} & g_{00} > 0 \\ & g_{00}\, g_{11} < 0 \\ & g_{00}\, g_{11}\, g_{22} > 0 \\ & g = g_{00}\, g_{11}\, g_{22}\, g_{33} < 0 \end{aligned}\right\} . \qquad (1.12)$$

The first three are known as the *weak signature conditions*. The fourth is known as the *strong signature condition*. If one or all of the weak signature conditions are violated, but the strong signature condition remains valid, then this is a *removable singularity*. If the strong signature condition is violated, then the space-time has an *irremovable singularity*: in this case, the solution is usually dropped from consideration, since it "does not have a physical sense". Yes, perhaps someone could not see the physical sense in this. However, these cases are of great mathematical significance. Therefore, we will consider any space singularity (space breaking).

Consider now the space of a liquid sphere. The external metric (1.1) of the sphere violates the first signature condition ($g_{00}=0$) at the distance $r = r_g$ from the centre

$$\left.\begin{aligned} g_{00} &= 1 - \frac{r_g}{r} = 0 \\ g_{00}\, g_{11} &= -1 < 0 \\ g_{00}\, g_{11}\, g_{22} &= r^2 > 0 \\ g &= -r^4 \sin^2\theta < 0 \end{aligned}\right\}. \tag{1.13}$$

The internal metric (1.8) of the sphere shows that the second, third and fourth signature conditions are violated at the distance

$$r = r_{br} = \sqrt{\frac{a^3}{r_g}} \tag{1.14}$$

from the centre, at which the above three functions tend to infinity

$$\left.\begin{aligned} g_{00} &= \frac{9}{4}\left(1 - \frac{r_g}{a}\right) > 0 \\ g_{00}\, g_{11} &\to -\infty \\ g_{00}\, g_{11}\, g_{22} &\to \infty \\ g = g_{00}\, g_{11}\, g_{22}\, g_{33} &\to -\infty \end{aligned}\right\}. \tag{1.15}$$

This means that the field of a liquid sphere has a space breaking at two distances from its centre:

1. The first space breaking occurs on a spherical surface around the centre of gravity of the liquid sphere at a distance of the Hilbert radius $r = r_g$. This is the surface of gravitational collapse according to the condition $g_{00} = 0$ in this space breaking. In other words, although the liquid sphere itself may not be a collapsar, it always contains a central "core", which is separated from the other liquid substance by the surface of gravitational collapse. In the case, where the liquid sphere is a star (as in the model of liquid stars), each star contains such a core. The core is much smaller than the physical radius of ordinary stars: while the radius of the collapsed core (Hilbert radius) of the Sun is $r_g = 2.9 \times 10^5$ cm (2.9 km), the Sun's physical radius is 7.0×10^{10} cm (700,000 km). Therefore, we call the first space breaking the *inner space breaking*;

2. The second space breaking occurs on a spherical surface around the liquid body at the distance $r_{br}=\sqrt{a^3/r_g}$ from it. This distance is much greater than the physical radius of ordinary stars. Therefore, we call it the *outer space breaking* (as opposed to the inner space breaking at the Hilbert radius). For example, the outer space breaking in the Sun's field occurs at the distance $r_{br}=3,4\times 10^{13}$ cm = 340,000,000 km = 2.3 AU from the Sun. This space breaking is located in the asteroid belt, near the orbit of the maximum concentration of asteroids (the asteroid belt extends from 2.1 AU to 4.3 AU from the Sun). This means that the outer space breaking in the Sun's field does not allow substance to form into a joint physical body (such as a planet) in this orbit.

If the physical radius a of a liquid star coincides with its Hilbert radius $r_g=\frac{2GM}{c^2}$, the star is a gravitational collapsar. In this case ($r_g=a$), the internal metric of a liquid sphere (1.8) takes the form

$$ds^2=\frac{1}{4}\left(1-\frac{r^2}{a^2}\right)c^2dt^2-\frac{dr^2}{1-\frac{r^2}{a^2}}-r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right). \quad (1.16)$$

This metric under the particular condition $a^2=\frac{3}{\lambda}>0$ (where $\lambda>0$) has the same form as the de Sitter metric (1.5)

$$ds^2=\left(1-\frac{\lambda r^2}{3}\right)c^2dt^2-\frac{dr^2}{1-\frac{\lambda r^2}{3}}-r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right) \quad (1.17)$$

that describes a spherical distribution of the physical vacuum (λ-field). This means that such an object, which is a liquid sphere in the state of gravitational collapse, consists of a liquid, the state of which is close to the high-density state of the physical vacuum.

As a result, the new model of liquid stars allows us to introduce a new classification of stars according to the location of the space breakings in the star's field relative to its physical surface:

Type I: Ordinary stars including the Sun

The radius of the collapsed core, i.e., the Hilbert radius r_g of an ordinary star is many orders of magnitude smaller than its physical radius ($r_g \ll a$). The outer space breaking r_{br} is far from an ordinary star, in the outer cosmos ($r_{br}\gg a$). These are almost all

Object	Mass M, gram	Radius a, cm	Hilbert radius r_g, cm	$\frac{r_g}{a}$	Space breaking r_{br}, cm	$\frac{r_{br}}{a}$	Type
Red super-giant*	4.0×10^{34}	7.0×10^{13}	5.9×10^{6}	8.4×10^{-8}	2.4×10^{17}	3.4×10^{3}	I
White super-giant†	3.4×10^{34}	4.8×10^{12}	5.0×10^{6}	1.0×10^{-6}	4.7×10^{15}	9.8×10^{2}	I
Sun	2.0×10^{33}	7.0×10^{10}	2.9×10^{5}	4.1×10^{-6}	3.4×10^{13}	4.9×10^{2}	I
Jupiter (proto-star)	1.9×10^{30}	7.1×10^{9}	2.8×10^{2}	4.0×10^{-8}	3.4×10^{13}	4.8×10^{3}	I
White dwarf‡	2.0×10^{33}	6.4×10^{8}	3.0×10^{5}	4.7×10^{-4}	2.9×10^{10}	0.45×10^{2}	I
Red dwarfs	6.7×10^{32}	2.3×10^{10}	9.9×10^{4}	4.3×10^{-6}	1.1×10^{13}	4.8×10^{2}	I
Brown dwarfs	1.5×10^{32}	7.0×10^{9}	2.2×10^{4}	3.1×10^{-6}	4.0×10^{14}	5.7×10^{4}	I
Wolf-Rayet stars	1.0×10^{35}	1.4×10^{12}	1.5×10^{7}	1.1×10^{-5}	4.3×10^{14}	3.1×10^{2}	Ia
Neutron stars	2.6×10^{33}	1.0×10^{6}	3.9×10^{5}	0.39	1.6×10^{6}	1.6	II
Pulsar§	3.9×10^{33}	1.6×10^{6}	5.8×10^{5}	0.36	2.7×10^{6}	1.7	II
Black holes	various	various	various	1	1	1	III

*Betelgeuse. †Rigel. ‡Sirius B. §Radio-pulsar J1903+0327.

Table 1.1: The classification of stars according to General Relativity. This classification is represented by the numerical values of the parameters of stars, which we have calculated for typical members of the known families of stars.

visible stars: supergiants, the Sun, brown dwarfs and even white dwarfs. Ordinary stars will be considered in Chapter 2;

Type Ia: Wolf-Rayet stars

They are almost the same as ordinary stars, except that a powerful stellar wind must be taken into account, consisting of the particles of stellar substance permanently flying out of the stars (this is a property that characterizes Wolf-Rayet stars). Such stars and their stellar wind will be considered in Chapter 3;

Type II: Neutron stars and pulsars

The radius of the Hilbert core is close to the physical radius of such a star ($r_g \lesssim a$), but does not reach it (otherwise the star would be invisible to observation). The outer space breaking r_{br} is also close to the physical surface of such a star, but does not reach it ($r_{br} \gtrsim a$). In addition, pulsars rotate at high velocities close to relativistic. As a result, the metric and energy-momentum tensor of such a star differ from those of ordinary stars. We will consider neutron stars and pulsars in Chapter 4;

Type III: Black holes

The Hilbert radius r_g (radius of the inner space breaking) and the outer space breaking radius r_{br} for such an object coincide on its physical surface ($r_g = r_{br} = a$). These are gravitational collapsars (black holes): on the physical surface of such an object, the state of gravitational collapse occurs ($g_{00} = 0$), so all its mass is concentrated under the collapsed surface. Black holes will be the focus of Chapter 5 of the book.

This classification is presented in Table 1.1 with the numerical values of the parameters calculated for typical members of the known families of stars.

The new model of liquid stars, which we have just introduced on the basis of General Relativity and considered in the new classification of stars, will be developed in the following Chapters.

1.3 Physically observable quantities

Before considering stars from the point of view of General Relativity, it is necessary to explain the basics of the mathematical apparatus of physically observable quantities in the four-dimensional curved pseudo-Riemannian space (space-time). A detailed overview of this theory had already been given in the corresponding Chapters of our books [18,19]. We now give only the necessary foundations of this theory, with some additions necessary for our current research study*.

To draw a visual picture of any physical theory, we must express the obtained results in terms of real physical quantities that can be measured in an experiment (they are called *physically observable quantities*). In General Relativity, this problem is not at all trivial, because we consider objects in the four-dimensional space (space-time) and therefore must determine which components of four-dimensional tensor quantities are actually physically observable.

Here is the problem in a nutshell. All equations of the General Theory of Relativity are given in the *general covariant form*, which does not depend on our choice of a reference frame. The equations, like the variables they contain, are four-dimensional. Thus, we ask what components of these four-dimensional variables are actually observable in real physical experiments, i.e., what components are truly physically observable quantities? Intuitively, we could, at first glance, easily assume that the three-dimensional components of a four-dimensional tensor constitute a physically observable quantity. But at the same time, we cannot be absolutely sure that we observe only three-dimensional components of four-dimensional quantities, and not more complex variables that depend on other factors, such as the properties of the physical standards of our reference space.

As is known, a four-dimensional vector (tensor of the 1st rank) has only 4 components: 1 time component and 3 spatial components. A tensor of the 2 rank, such as a rotation tensor or a deformation tensor, has 16 components: 1 time component, 9 spatial components and 6 mixed (space-time) components. Now, are the mixed components really physically observable quantities? Higher rank tensors have even more components; for example, the Riemann-Christoffel curvature tensor has 256 components, so the problem of heuristically recognizing truly physically observable components becomes much more difficult. In addition, there

---

*To date, the most complete description (compendium) of the mathematical apparatus of physically observable quantities in General Relativity is given in our recent article. In this article, we have collected everything (or almost everything) that we know on this topic from Zelmanov and what has been obtained over the past decades: Rabounski D. and Borissova L. Physical observables in General Relativity and the Zelmanov chronometric invariants. *Progress in Physics*, 2023, vol. 19, no. 1, 3–29.

is an obstacle associated with recognizing the observable components of covariant tensors (which have lower indices) and mixed-type tensors with both lower and upper indices.

We see that the recognition of physically observable quantities in General Relativity is not a trivial task. Ideally, we would like to have a mathematical method for *unambiguously* calculating physically observable quantities for any tensors of any given rank.

Numerous attempts to develop such a mathematical method were made in the 1930s by some researchers of that time. A certain contribution was made by L. D. Landau and E. M. Lifshitz in their famous *The Classical Theory of Fields* [20], first published in 1939. In addition to discussing the problem of physically observable quantities, in §84 of their book they introduced the physically observable time interval along with the physically observable three-dimensional interval, which depend on the physical properties (physical standards) of the observer's reference space. But all such attempts made in the 1930s were very limited to solving some particular problems. None of them led to a complete mathematical apparatus.

The complete mathematical apparatus for calculating physically observable quantities in the four-dimensional pseudo-Riemannian space was first introduced by Abraham L. Zelmanov and is known as the *theory of chronometric invariants*, or the *chronometrically invariant formalism*. It was first presented in 1944 in his PhD thesis [21], then — in his short articles of 1956–1957 [22, 23].

The essence of Zelmanov's mathematical apparatus of physically observable quantities (chronometric invariants), developed specifically for the four-dimensional curved inhomogeneous pseudo-Riemannian space (space-time), is as follows.

At any point in the space-time, we can place a three-dimensional spatial section $x^0 = ct = const$ (three-dimensional space), orthogonal to a given time line $x^i = const$. If a spatial section is everywhere orthogonal to the time lines piercing it at every point, then such a space is called *holonomic*. Otherwise, if the spatial section is non-orthogonal to the above time lines, then the space is said to be *non-holonomic*.

The reference frame of a real observer includes a coordinate grid spanned over a real physical body (the reference body of the observer near him) and real clocks located at each point of the coordinate grid. Both the coordinate grid and the clocks are a set of real references with which the observer compares the results of his measurements. Therefore, the physically observable quantities registered by the observer must be the result of projecting four-dimensional quantities onto the time line and the spatial section associated with him.

The operator projecting onto the time line of an observer is the vector of the four-dimensional velocity

$$b^{\alpha}=\frac{dx^{\alpha}}{ds} \tag{1.18}$$

of the observer's reference body with respect to him. This vector is tangential to the world line of the observer at every point. Therefore, it is a unit length vector

$$b_{\alpha}b^{\alpha}=g_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}=\frac{g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}}{ds^{2}}=+1\,. \tag{1.19}$$

The operator projecting onto the spatial section of the observer (his local three-dimensional space) is defined as a four-dimensional symmetric tensor $h_{\alpha\beta}$, which has the form

$$\left.\begin{aligned} h_{\alpha\beta} &= -g_{\alpha\beta}+b_{\alpha}b_{\beta} \\ h^{\alpha\beta} &= -g^{\alpha\beta}+b^{\alpha}b^{\beta} \\ h_{\alpha}^{\beta} &= -g_{\alpha}^{\beta}+b_{\alpha}b^{\beta} \end{aligned}\right\}. \tag{1.20}$$

The vector b^{α} and the tensor $h_{\alpha\beta}$ are orthogonal to each other. Mathematically this means that their contraction with each other is zero, i.e., $h_{\alpha\beta}b^{\alpha}=0$, $h^{\alpha\beta}b_{\alpha}=0$, $h_{\beta}^{\alpha}b_{\alpha}=0$, $h_{\alpha}^{\beta}b^{\alpha}=0$. Therefore, the main properties of the operators projecting onto the time line and the spatial section of an observer are expressed, obviously, as follows

$$b_{\alpha}b^{\alpha}=+1\,, \qquad h_{\alpha}^{\beta}b^{\alpha}=0\,. \tag{1.21}$$

If the observer is at rest with respect to his reference body, his reference frame is called the *accompanying reference frame*. In this case, $b^{i}=0$ in his reference frame, and the coordinate grids of his spatial section are connected with each other by the transformations

$$\left.\begin{aligned} \tilde{x}^{0} &= \tilde{x}^{0}\left(x^{0},x^{1},x^{2},x^{3}\right) \\ \tilde{x}^{i} &= \tilde{x}^{i}\left(x^{1},x^{2},x^{3}\right), \quad \frac{\partial\tilde{x}^{i}}{\partial x^{0}}=0 \end{aligned}\right\}, \tag{1.22}$$

where the third equation shows the fact that the spatial coordinates of the tilde-marked grid are independent of time in the non-tilded grid, which is equivalent to a coordinate grid in which the time lines are fixed ($x^i = const$) at each point. A transformation of the spatial coordinates is nothing more than the transition from one coordinate grid to another within the same spatial section. A transformation of time means the change of the entire set of clocks, i.e., the transition to another spatial section (another three-dimensional reference space). In practice, this means replacing one reference body with all its physical standards by another reference body having its own physical standards. But when using different standards, the observer will obtain different results (other observed values). Therefore, physically observable projections in the accompanying reference frame must be invariant under time transformations, which entails the invariance under the transformations (1.22). In other words, such quantities must have the property of *chronometric invariance*.

Therefore, we call physically observable quantities determined in the accompanying reference frame *chronometrically invariant quantities*, or *chronometric invariants* in short.

The projection tensor $h_{\alpha\beta}$, considered in the reference space accompanying an observer, has all the properties attributed to the fundamental metric tensor, namely

$$h_i^\alpha h_\alpha^k = \delta_i^k - b_i b^k = \delta_i^k\,, \qquad \delta_i^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.23}$$

where δ_i^k is the unit three-dimensional tensor*. Therefore, in the accompanying reference frame the three-dimensional tensor h_{ik} can lift and lower indices in chronometrically invariant quantities.

Thus, in the accompanying reference frame the main properties of the projection operators are

$$b_\alpha b^\alpha = +1\,, \qquad h_\alpha^i b^\alpha = 0\,, \qquad h_i^\alpha h_\alpha^k = \delta_i^k\,. \tag{1.24}$$

Calculate the components of the projection operators in the accompanying reference frame. The component b^0 is obtained from the ob-

*The tensor δ_i^k is the three-dimensional part of the four-dimensional unit tensor δ_β^α, which can be used to replace indices in four-dimensional quantities.

vious condition $b_\alpha b^\alpha = g_{\alpha\beta} b^\alpha b^\beta = 1$, which in the accompanying reference frame ($b^i = 0$) has the form $b_\alpha b^\alpha = g_{00} b^0 b^0 = 1$. This component and the remaining components of b^α are

$$\left.\begin{aligned} b^0 &= \frac{1}{\sqrt{g_{00}}}\,, & b^i &= 0 \\ b_0 &= g_{0\alpha} b^\alpha = \sqrt{g_{00}}\,, & b_i &= g_{i\alpha} b^\alpha = \frac{g_{i0}}{\sqrt{g_{00}}} \end{aligned}\right\}, \tag{1.25}$$

and the components of $h_{\alpha\beta}$ are

$$\left.\begin{aligned} h_{00} &= 0\,, & h^{00} &= -g^{00} + \frac{1}{g_{00}}\,, & h^0_0 &= 0 \\ h_{0i} &= 0\,, & h^{0i} &= -g^{0i}, & h^i_0 &= \delta^i_0 = 0 \\ h_{i0} &= 0\,, & h^{i0} &= -g^{i0}, & h^0_i &= \frac{g_{i0}}{g_{00}} \\ h_{ik} &= -g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}}\,, & h^{ik} &= -g^{ik}, & h^i_k &= -g^i_k = \delta^i_k \end{aligned}\right\}. \tag{1.26}$$

Zelmanov had created a mathematical method for calculating the chronometrically invariant (physically observable) projections of any general covariant (four-dimensional) tensor quantity. He had formulated this method as a theorem, which we call *Zelmanov's theorem*:

Zelmanov's theorem

Let there be a four-dimensional tensor $Q^{\mu\nu\ldots\rho}_{\alpha\beta\ldots\sigma}$ of the r-th rank, where $Q^{ik\ldots p}_{00\ldots0}$ is the three-dimensional part of $Q^{\mu\nu\ldots\rho}_{00\ldots0}$, in which all upper indices are non-zero and all m lower indices are zeroes. Then,

$$T^{ik\ldots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\ldots p}_{00\ldots0} \tag{1.27}$$

is a chronometrically invariant three-dimensional contravariant tensor of the $(r-m)$-th rank. This means that the chr.inv.-tensor $T^{ik\ldots p}$ is the result of m-fold projection of the initial tensor $Q^{\mu\nu\ldots\rho}_{\alpha\beta\ldots\sigma}$ onto the time line by the indices $\alpha, \beta \ldots \sigma$ and onto the spatial section by $r-m$ indices $\mu, \nu \ldots \rho$.

According to this theorem, the chronometrically invariant (physically observable) projections of a four-dimensional vector Q^α are

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha Q^\alpha = Q^i, \tag{1.28}$$

and the chr.inv.-projections of a symmetric tensor of the 2nd rank $Q^{\alpha\beta}$ are the following quantities

$$b^{\alpha}b^{\beta}Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}\,, \quad h^{i\alpha}b^{\beta}Q_{\alpha\beta} = \frac{Q_{0}^{i}}{\sqrt{g_{00}}}\,, \quad h_{\alpha}^{i}h_{\beta}^{k}Q^{\alpha\beta} = Q^{ik}. \tag{1.29}$$

The chr.inv.-projections of a four-dimensional coordinate interval dx^{α} are the physically observable time interval

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^{i} \tag{1.30}$$

and the intervals dx^{i} of each of the three-dimensional (spatial) coordinates. Accordingly, the physically observable velocity of a particle is the three-dimensional chr.inv.-vector

$$\mathrm{v}^{i} = \frac{dx^{i}}{d\tau}\,, \qquad \mathrm{v}_{i}\mathrm{v}^{i} = h_{ik}\,\mathrm{v}^{i}\mathrm{v}^{k} = \mathrm{v}^{2}, \tag{1.31}$$

which at isotropic trajectories becomes the three-dimensional chr.inv.-vector of the physically observable velocity of light

$$c^{i} = \mathrm{v}^{i} = \frac{dx^{i}}{d\tau}\,, \quad c_{i}c^{i} = h_{ik}c^{i}c^{k} = c^{2}. \tag{1.32}$$

Projecting the covariant and contravariant fundamental metric tensor onto the spatial section associated with an observer, in the reference frame accompanying him ($b^{i}=0$) we have

$$\left.\begin{aligned} h_{i}^{\alpha}h_{k}^{\beta}g_{\alpha\beta} &= g_{ik} - b_{i}b_{k} = -h_{ik} \\ h_{\alpha}^{i}h_{\beta}^{k}g^{\alpha\beta} &= g^{ik} - b^{i}b^{k} = g^{ik} = -h^{ik} \end{aligned}\right\}, \tag{1.33}$$

which means that the chr.inv.-quantity

$$h_{ik} = -g_{ik} + b_{i}b_{k} \tag{1.34}$$

is the *chr.inv.-metric tensor*, i.e., the physically observable metric tensor, using which we can lift and lower indices in any three-dimensional chr.inv.-object. The contravariant and mixed components of the chr.inv-metric tensor are, obviously,

$$h^{ik} = -g^{ik}, \qquad h_{k}^{i} = -g_{k}^{i} = \delta_{k}^{i}\,. \tag{1.35}$$

Formulating $g_{\alpha\beta}$ through the definition of $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$, we obtain the formula for the four-dimensional interval

$$ds^2 = b_\alpha b_\beta \, dx^\alpha dx^\beta - h_{\alpha\beta} \, dx^\alpha dx^\beta, \tag{1.36}$$

expressed through the projection operators b_α and $h_{\alpha\beta}$. In this formula, we have $b_\alpha dx^\alpha = c\, d\tau$. Therefore, the first term is $b_\alpha b_\beta \, dx^\alpha dx^\beta = c^2 d\tau^2$. The second term $h_{\alpha\beta} \, dx^\alpha dx^\beta = d\sigma^2$ is the square of the physically observable three-dimensional interval*

$$d\sigma^2 = h_{ik} \, dx^i dx^k. \tag{1.37}$$

Thus, the four-dimensional interval, represented through physically observable quantities, is

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{1.38}$$

Zelmanov had also deduced the main physically observable properties characteristic of the accompanying reference space associated with an observer. He proceeded from the property of non-commutativity (non-zero difference) of the mixed second chr.inv.-derivatives

$$\frac{{}^*\partial^2}{\partial x^i \partial t} - \frac{{}^*\partial^2}{\partial t \, \partial x^i} = \frac{1}{c^2} F_i \frac{{}^*\partial}{\partial t}\,, \tag{1.39}$$

$$\frac{{}^*\partial^2}{\partial x^i \partial x^k} - \frac{{}^*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{{}^*\partial}{\partial t}\,, \tag{1.40}$$

where the chr.inv.-derivation operators that he had introduced are

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0}\,. \tag{1.41}$$

The first two physically observable properties of the observer's reference space are characterized by the chr.inv.-vector F_i of the gravitational inertial force and the antisymmetric chr.inv.-tensor A_{ik} of the angular velocity with which the reference space rotates

$$F_i = \frac{1}{\sqrt{g_{00}}} \left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \tag{1.42}$$

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} \left(F_i v_k - F_k v_i \right), \tag{1.43}$$

*Since $h_{\alpha\beta}$ in the accompanying reference frame has all properties of $g_{\alpha\beta}$.

where the quantities w and v_i characterize the reference body and its reference space. These are the gravitational potential

$$\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right), \qquad 1 - \frac{\mathrm{w}}{c^2} = \sqrt{g_{00}} \tag{1.44}$$

and the linear velocity with which the reference space rotates

$$\left.\begin{aligned} v_i &= -c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, & v^i &= -c\,g^{0i}\sqrt{g_{00}} \\ v_i &= h_{ik}\,v^k, & v^2 &= v_k\,v^k = h_{ik}\,v^i v^k \end{aligned}\right\}. \tag{1.45}$$

It should be noted that the quantities w and v_i do not have the property of chronometric invariance, despite the fact that $v_i = h_{ik}\,v^k$ can be obtained as for any chr.inv.-quantity through lowering the index by the chr.inv.-metric tensor h_{ik}.

Zelmanov had also found that the chr.inv.-quantities F_i and A_{ik} are connected by two identities, which we call *Zelmanov's identities*

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{1.46}$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0\,. \tag{1.47}$$

In the framework of quasi-Newtonian approximation, i.e., in a weak gravitational field at velocities much lower than the velocity of light and in the absence of rotation of the space, F_i (1.42) becomes an ordinary non-relativistic gravitational force $F_i = \frac{\partial \mathrm{w}}{\partial x^i}$.

Zelmanov had also introduced the following theorem setting up the space holonomity condition:

Zelmanov's theorem on the space holonomity condition

> For a four-dimensional region of a space (space-time), the identical equality to zero of the tensor A_{ik} is the necessary and sufficient condition for the orthogonality of the spatial sections to the time lines everywhere in this region.

In other words, the necessary and sufficient condition for a space to be holonomic is achieved by setting the tensor A_{ik} equal to zero. Naturally, if the spatial sections are everywhere orthogonal to the time lines (in this case the space is holonomic), then the quantities g_{0i} are zero.

Since $g_{0i}=0$, we have $v_i=0$ and $A_{ik}=0$. Therefore, we call the tensor A_{ik} the *space non-holonomity tensor*.

If the conditions $F_i=0$ and $A_{ik}=0$ are satisfied in a space region, then the conditions $g_{00}=1$ and $g_{0i}=0$ are also satisfied there. In such a region, according to (1.30), $d\tau=dt$: the difference between the coordinate time t and the physically observable time τ disappears, since the space is free from gravitational fields and rotation. In other words, according to the theory of chronometric invariants, the difference between the coordinate time t and the physically observable time τ comes from both the gravitational field and rotation of the observer's reference space (which is the local space of the Earth for an Earth-bound observer), or each of these physical factors separately.

On the other hand, it is unrealistic to find such a region in the Universe, where the background space would have neither gravitational fields nor rotation. Therefore, in practice, the physically observable time τ and the coordinate time t differ from each other. This means that the real space of our Universe is non-holonomic, and a holonomic space can only be its local approximation.

The space holonomity condition is directly related to the problem of integrability of time. The formula for the physically observable time interval (1.30) does not have an integrating factor. In other words, this formula cannot be reduced to the form

$$d\tau = A\,dt\,, \tag{1.48}$$

where the multiplier A depends on only t and x^i. This is because in a non-holonomic space, the formula (1.30) has a non-zero second term depending on the coordinate interval dx^i and also on g_{0i}. In a holonomic space, we have $A_{ik}=0$ and $g_{0i}=0$, so the second term of (1.30) is zero, while the first term is the elementary time interval dt with an integrating multiplier

$$A = \sqrt{g_{00}} = f\left(x^0, x^i\right), \tag{1.49}$$

which allows us to write the integral

$$d\tau = \int \sqrt{g_{00}}\,dt\,. \tag{1.50}$$

Therefore, time is globally integrable in a holonomic space ($A_{ik}=0$), but cannot be globally integrated in a non-holonomic space ($A_{ik}\neq 0$). In the case, where time is integrable (a holonomic space), we can synchronize clocks at two distant points in the space by moving a control clock along the path between these two points. In the case, where time cannot be globally integrated (a non-holonomic space), the clock synchronization at two distant points is impossible: the greater the distance between these two points, the greater the time deviation on these clocks.

The space of our planet Earth is non-holonomic due to its daily rotation around the Earth's axis. Therefore, two clocks located at different points on the Earth's surface must show a deviation between the time intervals registered on each of them. The greater the distance between these clocks, the greater the deviation of the physically observed time registered on them. This effect was undoubtedly verified by the Hafele-Keating experiment [24–29] on moving a set of standard atomic clocks by a jet airplane around the globe. In this experiment, the rotation of the Earth's space significantly changed the measured time. When flying along the Earth's rotation, the local space of an observer on board the airplane had a greater rotation than the local space of another observer, who remained motionless on the ground. During the flight against the Earth's rotation, it was the other way around. All the atomic clocks on board the airplane showed a significant deviation of the observed time depending on the velocity with which the observer's space rotates.

Since the synchronization of clocks at various points on the Earth's surface is the most important task of maritime navigation, as well as aviation, in the old days, desynchronization corrections were introduced in the form of tables containing empirically obtained corrections that take the Earth's rotation into account. Now, thanks to the theory of chronometric invariants, we know the origin of these corrections and can calculate them on the basis of General Relativity.

In addition to gravitation and rotation, the reference body can deform thereby changing its coordinate grids over time. This factor must also be taken into account in measurements. This can be done by selecting in the equations the three-dimensional symmetric chr.inv.-tensor of the deformation rate of the reference space

$$\left.\begin{aligned} D_{ik} &= \frac{1}{2}\frac{^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\frac{1}{2}\frac{^*\partial h^{ik}}{\partial t} \\ D &= h^{ik}D_{ik} = \frac{^*\partial \ln\sqrt{h}}{\partial t}\,, \qquad h = \det\|h_{ik}\| \end{aligned}\right\}. \tag{1.51}$$

The Christoffel symbols characterize the *inhomogeneity* of the observer's reference space. The regular Christoffel symbols of the 2nd rank $\Gamma^{\alpha}_{\mu\nu}$ and those of the 1st rank $\Gamma_{\mu\nu,\sigma}$, i.e.

$$\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\sigma}\,\Gamma_{\mu\nu,\sigma} = \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right), \tag{1.52}$$

are related to the corresponding chr.inv.-Christoffel symbols

$$\Delta^{i}_{jk} = h^{im}\Delta_{jk,m} = \frac{1}{2}\,h^{im}\left(\frac{{}^{*}\partial h_{jm}}{\partial x^{k}} + \frac{{}^{*}\partial h_{km}}{\partial x^{j}} - \frac{{}^{*}\partial h_{jk}}{\partial x^{m}}\right), \tag{1.53}$$

which are determined similarly to the $\Gamma^{\alpha}_{\mu\nu}$ and $\Gamma_{\mu\nu,\sigma}$. The only difference is that here, instead of the fundamental metric tensor $g_{\alpha\beta}$, the chr.inv.-metric tensor h_{ik} is used.

The components of the regular Christoffel symbols can be expressed through the chr.inv.-properties of the observer's reference space. Expressing the $g^{\alpha\beta}$ components and the first derivatives of $g_{\alpha\beta}$ in terms of F_i, A_{ik}, D_{ik}, w and v_i, after some algebra we obtain

$$\Gamma_{00,0} = -\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.54}$$

$$\Gamma_{00,i} = \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F_i + \frac{1}{c^4}\,v_i\,\frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.55}$$

$$\Gamma_{0i,0} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.56}$$

$$\Gamma_{0i,j} = -\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_{ij} + A_{ij} + \frac{1}{c^2}\,F_j v_i\right) + \frac{1}{c^3}\,v_j\,\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.57}$$

$$\Gamma_{ij,0} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[D_{ij} - \frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) + \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right)\right], \tag{1.58}$$

$$\Gamma_{ij,k} = -\Delta_{ij,k} + \frac{1}{c^2}\left[v_i A_{jk} + v_j A_{ik} + \frac{1}{2}\,v_k\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) - \right.$$
$$\left. - \frac{1}{2c^2}\,v_k\left(F_i v_j + F_j v_i\right)\right] + \frac{1}{c^4}F_k v_i v_j\,, \tag{1.59}$$

$$\Gamma^{0}_{00} = -\frac{1}{c^3}\left[\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)v_k F^k\right], \tag{1.60}$$

$$\Gamma^k_{00} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F^k, \tag{1.61}$$

$$\Gamma^0_{0i} = \frac{1}{c^2}\left[-\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial x^i} + v_k\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2}\,v_i F^k\right)\right], \tag{1.62}$$

$$\Gamma^k_{0i} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2}\,v_i F^k\right), \tag{1.63}$$

$$\begin{aligned}\Gamma^0_{ij} = -\frac{1}{c\left(1-\frac{\mathrm{w}}{c^2}\right)}\Bigg\{ -D_{ij} + \frac{1}{c^2}\,v_n \times \\ \times\left[v_j\left(D^n_i + A^{\cdot n}_{i\cdot}\right) + v_i\left(D^n_j + A^{\cdot n}_{j\cdot}\right) + \frac{1}{c^2}\,v_i v_j F^n\right] + \\ + \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) - \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right) - \Delta^n_{ij}\,v_n\Bigg\},\end{aligned} \tag{1.64}$$

$$\Gamma^k_{ij} = \Delta^k_{ij} - \frac{1}{c^2}\left[v_i\left(D^k_j + A^{\cdot k}_{j\cdot}\right) + v_j\left(D^k_i + A^{\cdot k}_{i\cdot}\right) + \frac{1}{c^2}\,v_i v_j F^k\right], \tag{1.65}$$

from which we obtain

$$D^i_k + A^{\cdot i}_{k\cdot} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma^i_{0k} - \frac{g_{0k}\Gamma^i_{00}}{g_{00}}\right), \tag{1.66}$$

$$F^k = -\frac{c^2\,\Gamma^k_{00}}{g_{00}}\,, \tag{1.67}$$

$$h^{iq}h^{ks}\Delta^m_{qs} = g^{i\alpha}g^{k\beta}\,\Gamma^m_{\alpha\beta}\,. \tag{1.68}$$

By analogy with the respective absolute derivatives, Zelmanov had also introduced the chr.inv.-derivatives

$${}^*\nabla_i\, Q_k = \frac{{}^*\partial Q_k}{dx^i} - \Delta^l_{ik}\, Q_l\,, \tag{1.69}$$

$${}^*\nabla_i\, Q^k = \frac{{}^*\partial Q^k}{dx^i} + \Delta^k_{il}\, Q^l, \tag{1.70}$$

$${}^*\nabla_i\, Q_{jk} = \frac{{}^*\partial Q_{jk}}{dx^i} - \Delta^l_{ij}\, Q_{lk} - \Delta^l_{ik}\, Q_{jl}\,, \tag{1.71}$$

$${}^*\nabla_i\, Q^k_j = \frac{{}^*\partial Q^k_j}{dx^i} - \Delta^l_{ij}\, Q^k_l + \Delta^k_{il}\, Q^l_j\,, \tag{1.72}$$

$$^*\nabla_i \, Q^{jk} = \frac{^*\partial Q^{jk}}{dx^i} + \Delta^j_{il} \, Q^{lk} + \Delta^k_{il} \, Q^{jl}, \tag{1.73}$$

$$^*\nabla_i \, Q^i = \frac{^*\partial Q^i}{\partial x^i} + \Delta^j_{ji} \, Q^i, \qquad \Delta^j_{ji} = \frac{^*\partial \ln\sqrt{h}}{\partial x^i}, \tag{1.74}$$

$$^*\nabla_i \, Q^{ji} = \frac{^*\partial Q^{ji}}{\partial x^i} + \Delta^j_{il} \, Q^{il} + \Delta^l_{li} \, Q^{ji}, \quad \Delta^l_{li} = \frac{^*\partial \ln\sqrt{h}}{\partial x^i}. \tag{1.75}$$

Zelmanov had also introduced the chr.inv.-curvature tensor. He followed the same procedure by which the Riemann-Christoffel curvature tensor was constructed, based on the non-commutativity of the second derivatives of an arbitrary vector Q^α taken in a given space, the geometry of which is Riemannian.

Taking into account the non-commutativity of the second chr.inv.-derivatives of an arbitrary three-dimensional vector

$$^*\nabla_i {}^*\nabla_k \, Q_l - {}^*\nabla_k {}^*\nabla_i \, Q_l = \frac{2A_{ik}}{c^2} \frac{^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdot\cdot\cdot j} \, Q_j \,, \tag{1.76}$$

where the chr.inv.-covariant differential of the vector is

$$^*\nabla_k \, Q^i dx^k = dQ^i + \Delta^i_{kl} \, Q^k dx^l, \tag{1.77}$$

Zelmanov obtained the chr.inv.-tensor

$$H_{lki\cdot}^{\cdot\cdot\cdot j} = \frac{^*\partial \Delta^j_{il}}{\partial x^k} - \frac{^*\partial \Delta^j_{kl}}{\partial x^i} + \Delta^m_{il} \, \Delta^j_{km} - \Delta^m_{kl} \, \Delta^j_{im} \,, \tag{1.78}$$

which is similar to Schouten's tensor from the theory of non-holonomic manifolds [30]. The tensor $H_{lki\cdot}^{\cdot\cdot\cdot j}$ differs from the Riemann-Christoffel tensor $R_{\beta\gamma\delta\cdot}^{\cdot\cdot\cdot\alpha}$ due to the presence of the space rotation tensor A_{ik} in the formula (1.76). Its generalization gives the chr.inv.-tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \tag{1.79}$$

which has all the algebraic properties of the Riemann-Christoffel tensor in the three-dimensional space of the observer (his spatial section). Since the chr.inv.-tensor C_{iklj} is in fact the physically observable curvature tensor of the observer's spatial section, Zelmanov called it the *chr.inv.-curvature tensor*. Contracting it step-by-step

$$C_{kj} = C_{kij\cdot}^{\cdot\cdot\cdot i} = h^{im} C_{kimj} \,, \qquad C = C^j_j = h^{lj} C_{lj} \,, \tag{1.80}$$

we obtain the chr.inv.-curvature scalar C, which is the *observable three-dimensional curvature* of the space.

The tensor H_{lkij} is related to the curvature tensor C_{lkij} by

$$H_{lkij} = C_{lkij} + \frac{1}{c^2}\Big(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + \\ + A_{kl}D_{ij} + A_{li}D_{jk}\Big), \qquad (1.81)$$

and their contractions $H_{lk} = H_{lki\cdot}^{\cdots i}$ and $C_{lk} = C_{lki\cdot}^{\cdots i}$ are related as

$$H_{lk} = C_{lk} + \frac{1}{c^2}\Big(A_{kj}D_l^j + A_{lj}D_k^j + A_{kl}D\Big). \qquad (1.82)$$

In a particular case, where the space does not rotate, the H_{lkij} and C_{lkij} are the same. This is as well true for the H_{lk} and C_{lk}. In this particular case, the tensor $C_{lk} = h^{ij}C_{ilkj}$ has the form

$$C_{lk} = \frac{{}^*\partial}{\partial x^k}\left(\frac{{}^*\partial \ln\sqrt{h}}{\partial x^l}\right) - \frac{{}^*\partial \Delta_{kl}^i}{\partial x^i} + \Delta_{il}^m \Delta_{km}^i - \Delta_{kl}^m \frac{{}^*\partial \ln\sqrt{h}}{\partial x^m}. \qquad (1.83)$$

Zelmanov had also deduced the chr.inv.-projections of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$. Being a two-pair symmetric tensor (its paired indices are non-symmetric inside each pair, while the pairs are symmetric with respect to each other), it has three chr.inv.-projections according to (1.29). They have the form

$$X^{ik} = -c^2\frac{R_{0\cdot 0\cdot}^{\cdot i\cdot k}}{g_{00}}, \quad Y^{ijk} = -c\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}, \quad Z^{ijkl} = c^2 R^{ijkl}. \qquad (1.84)$$

Substituting the necessary components of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ into the formulae for its chr.inv.-projections (1.84) and then lowering the indices, Zelmanov had obtained the formulae

$$X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} - \Big(D_i^l + A_{i\cdot}^{\cdot l}\Big)\Big(D_{jl} + A_{jl}\Big) + \\ + \Big({}^*\nabla_i F_j + {}^*\nabla_j F_i\Big) - \frac{1}{c^2}F_i F_j, \qquad (1.85)$$

$$Y_{ijk} = {}^*\nabla_i\Big(D_{jk} + A_{jk}\Big) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \frac{2}{c^2}A_{ij}F_k, \qquad (1.86)$$

$$Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - \\ - A_{il}A_{kj} + 2A_{ij}A_{kl} - c^2 C_{iklj}, \qquad (1.87)$$

where $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$ just like in the Riemann-Christoffel tensor. Contraction of the observable spatial projection Z_{iklj} step-by-step as $Z_{il} = h^{kj} Z_{iklj}$ and $Z = h^{il} Z_{il}$ gives

$$Z_{il} = D_{ik} D_l^k - D_{il} D + A_{ik} A_{l\cdot}^{\cdot k} + 2A_{ik} A_{\cdot l}^{k\cdot} - c^2 C_{il}\,, \tag{1.88}$$

$$Z = h^{il} Z_{il} = D_{ik} D^{ik} - D^2 - A_{ik} A^{ik} - c^2 C\,. \tag{1.89}$$

At the end of our overview of the chronometrically invariant formalism, consider Einstein's field equations*

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,. \tag{1.90}$$

Einstein's field equations, in addition to the fundamental metric tensor $g_{\alpha\beta}$, include: Ricci's tensor $R_{\alpha\beta} = R_{\alpha\sigma\beta\cdot}^{\cdots\sigma}$ (the 2nd rank symmetric tensor resulting from contraction of the Riemann-Christoffel curvature tensor), the Riemann curvature scalar $R = g^{\alpha\beta} R_{\alpha\beta}$, Einstein's gravitational constant $\varkappa = \frac{8\pi G}{c^2} = 18.6 \times 10^{-28}$ cm/gram, Gauss' gravitational constant $G = 6.672 \times 10^{-8}$ cm^{3}gram^{-1}sec^{-2}, the energy-momentum tensor $T_{\alpha\beta}$ of a distributed matter that fills the space, and also the λ-term [cm^{-2}] that describes the physical vacuum. See §5.2 of the book [18].

Landau and Lifshitz [20] used $\varkappa = \frac{8\pi G}{c^4}$ instead of $\varkappa = \frac{8\pi G}{c^2}$ as used by Zelmanov. To understand the reason, assume $\varkappa = \frac{8\pi G}{c^2}$ as in Zelmanov's theory of chronometric invariants and in our papers. Consider then the chr.inv.-projections of the energy-momentum tensor

$$\rho = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c T_0^i}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}, \tag{1.91}$$

which are calculated according to the formula (1.29) as the chr.inv.-projections of any 2nd rank symmetric tensor. They have the following physical sense: ρ is the observable mass density, J^i is the observable momentum density, and U^{ik} is the observable stress tensor. Ricci's tensor has the dimension [cm^{-2}]. Therefore, the scalar chr.inv.-projection of the field equations, $\frac{G_{00}}{g_{00}} = -\frac{\varkappa T_{00}}{g_{00}} + \lambda$, as well as $\frac{\varkappa T_{00}}{g_{00}} = \frac{8\pi G\rho}{c^2}$ have the same dimension [cm^{-2}]. Hence, the energy-momentum tensor $T_{\alpha\beta}$ has the same dimension as mass density [gram/cm^3]. Therefore, if we would

*The left hand side of the field equations (1.90) is often referred to as the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R$, in the notation $G_{\alpha\beta} = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$.

use $\varkappa = \frac{8\pi G}{c^4}$ on the right hand side of the field equations, then we used not the energy-momentum tensor $T_{\alpha\beta}$ but rather $c^2 T_{\alpha\beta}$.

The chr.inv.-projections of Einstein's field equations (1.90) are calculated as for any tensor of the 2nd rank (1.29). They have the form

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{lj} + A_{jl}A^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}F_j\right)F^j = \\ = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \lambda c^2, \quad (1.92)$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}F_j A^{ij} = \varkappa J^i, \quad (1.93)$$

$$\frac{{}^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D^j_k + A_{k\cdot}^{\cdot j}\right) + DD_{ik} - D_{ij}D^j_k + \\ + 3A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}F_iF_k - c^2 C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right) + \lambda c^2 h_{ik}\,, \quad (1.94)$$

which we call the *chr.inv.-Einstein equations*. Here $U = h^{ik}U_{ik}$ is the trace of the stress tensor U_{ik}.

In addition, the energy-momentum tensor $T_{\alpha\beta}$ of a distributed matter must satisfy the conservation law

$$\nabla_\sigma T^{\alpha\sigma} = 0\,. \quad (1.95)$$

The chr.inv.-projections of the conservation law are calculated as for any tensor of the 1st rank (1.28). We call them the *chr.inv.-conservation law equations*. They have the form

$$\frac{{}^*\partial\rho}{\partial t} + D\rho + \frac{1}{c^2}D_{ij}U^{ij} + {}^*\widetilde{\nabla}_i J^i - \frac{1}{c^2}F_i J^i = 0\,, \quad (1.96)$$

$$\frac{{}^*\partial J^k}{\partial t} + DJ^k + 2\left(D^k_i + A_{i\cdot}^{\cdot k}\right)J^i + {}^*\widetilde{\nabla}_i U^{ik} - \rho F^k = 0\,, \quad (1.97)$$

where the chr.inv.-operator ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}F_i$ is created on the basis of the chr.inv.-derivative operator ${}^*\nabla_i$.

With these definitions we can find out how any geometric object of the four-dimensional pseudo-Riemannian space (space-time of General Relativity) looks like from the point of view of any observer, located in this space. For example, having any equation obtained in the general covariant tensor analysis, we can calculate its chr.inv.-projections onto the time line and the spatial section associated with any particular reference frame, and then formulate the corresponding chr.inv.-projections in terms of the physically observable properties of this reference space. Following this way, we will come to the equations containing only quantities measurable in practice.

So, now we have all the mathematical "tools" necessary for our further mathematical theory of the internal constitution of stars and sources of stellar energy based on General Relativity.

Chapter 2 Ordinary Stars and the Sun

2.1 Introducing the space metric of an ordinary star. Einstein's equations in the form satisfying the metric

In this Chapter, we present our mathematical theory of liquid stars being applied to ordinary stars. This means Type I of stars according to the new classification we have just introduced based on General Relativity (see §1.2 and Table 1.1 therein). Type I covers the widest variety of stars, which includes super-giants, sun-like stars (including the Sun), ordinary dwarfs and also white dwarfs*.

The structure, substance and field of a liquid star are characterized by the Schwarzschild metric of a sphere filled with an incompressible liquid. The metric was originally introduced in 1916 by Karl Schwarzschild [14]. He, however, introduced it in a truncated form containing substantial limitations: he artificially pre-imposed these limitations in his derivation in order to free the field of a breaking; this leads to an artificial truncation of the geometry of this metric space. In other words, the metric introduced by Karl Schwarzschild is not exactly the space metric of a liquid sphere. The true metric of a sphere filled with an incompressible liquid, which is free of the mentioned limitations and, thus, takes a space breaking into account, was deduced in 2009 by L. Borissova [11, 12]. Let us now reproduce her derivation, following her most detailed explanation [11] along with some recent additions and comments.

Consider an empty space containing a spherical island, which is a liquid. The structure, substance and field of such a massive island should be characterized by a space metric with spherical symmetry. As

*In the framework of Eddington's theory of gaseous stars, white dwarfs are considered separately.

is known, all spherically symmetric metrics have the following general form

$$ds^2 = e^{\nu}c^2dt^2 - e^{\lambda}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{2.1}$$

where e^{ν} and e^{λ} are functions of r and t.

The substance and field of the spherical liquid island must satisfy Einstein's field equations (1.90), which in the case under consideration have the λ-field neglected, i.e.

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\, T_{\alpha\beta}\,, \tag{2.2}$$

where $R_{\alpha\beta}$ is Ricci's curvature tensor, R is the curvature scalar, $\varkappa = \frac{8\pi G}{c^2}$ $= 18.6 \times 10^{-28}$ cm/gram is Einstein's gravitational constant, and $T_{\alpha\beta}$ is the energy-momentum tensor of a matter (liquid) distributed over the space. Note that the energy-momentum tensor of any distributed matter must satisfy the conservation law

$$\nabla_{\sigma} T^{\alpha\sigma} = 0\,, \tag{2.3}$$

where ∇_{σ} is the general covariant derivative symbol.

Einstein's field equations connect the components of the fundamental metric tensor, the space curvature and the distributed matter according to Riemannian geometry. In other words, the invariant square form of Riemannian metric, $ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = inv$, together with Einstein's field equations characterize Riemannian spaces (i.e., spaces, the geometry of which is Riemannian). Concerning the General Theory of Relativity, this means the following. Let us have a Riemannian space with a specific metric ds^2. Assume that a matter is distributed over the space (thereby we assume a specific formula for the energy-momentum tensor $T_{\alpha\beta}$). Then, the components of the fundamental metric tensor $g_{\alpha\beta}$ (known from the specific formula for the metric ds^2) and the components of the specific energy-momentum tensor $T_{\alpha\beta}$, when substituted into (respectively) the left hand side and the right hand side of Einstein's field equations should transform these equations into identities.

Here is how, based on the general formula for the spherically symmetric metric (2.1), we can deduce the metric of a sphere filled with an ideal liquid. First, we take the energy-momentum tensor of an ideal liquid and substitute its components into the right hand side of the field equations. Then we find the components of the fundamental metric tensor from the spherically symmetric metric (2.1) in their general form, containing the coefficients e^{ν} and e^{λ}. We substitute these components into the left hand side of the field equations. Then we look at what kind of the coefficients e^{ν} and e^{λ} make the left hand side of the field equations the same as the right hand side (thus turning the field equations into identities). Finally, we substitute the resulting specific formulae for the coefficients e^{ν} and e^{λ} back into the general formula for the spherically symmetric metric. As a result, we obtain the true metric of a sphere filled with an ideal liquid. Voilà!

One might as well ask why Schwarzschild himself did not do just that? Instead, why did he go down another complicated path full of speculations? There is no answer to this question… Let us come back to our derivation.

As is known, the energy-momentum tensor of an ideal liquid (which is incompressible and non-viscous) has the form

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p}{c^2}\right) U^{\alpha} U^{\beta} - \frac{p}{c^2}\, g^{\alpha\beta}, \tag{2.4}$$

where $\rho = \rho_0 = const$ is the density of the liquid (which is constant), p is the pressure inside the liquid, and

$$U^{\alpha} = \frac{dx^{\alpha}}{ds}\,, \qquad U_{\alpha} U^{\alpha} = 1 \tag{2.5}$$

is the four-dimensional velocity of the liquid flow with respect to the observer (his reference space coincides with the space of the liquid sphere, at the centre of which the coordinate origin is located).

Let us formulate the field equations in component notation, taking into account the physically observable properties of the space associated with the liquid sphere.

First, we see that

$$\left.\begin{aligned} & g_{00} = e^{\nu}, \qquad g_{0i} = 0 \\ & g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta \end{aligned}\right\} \tag{2.6}$$

in the metric of spherically symmetric spaces (2.1). According to the chronometrically invariant formalism (see §1.3), the gravitational potential in such a space has the following formula

$$\mathrm{w} = c^2 \left(1 - e^{\frac{\nu}{2}}\right). \tag{2.7}$$

Since $g_{0i}=0$ in the metric, such a space does not rotate: the linear velocity of its rotation is $v_i=0$. Therefore, the chr.inv.-tensor of the angular velocity associated with the space is zero

$$A_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\frac{1}{2c^2}\left(F_i v_k-F_k v_i\right)=0\,, \tag{2.8}$$

and the chr.inv.-vector of the gravitational inertial force has the form

$$F_i=\frac{c^2}{c^2-\mathrm{w}}\left(\frac{\partial \mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}\right)=-\frac{c^2}{2}\,\nu'\,, \tag{2.9}$$

where the prime denotes differentiation along the radial coordinate r. With the above, the chr.inv.-metric tensor h_{ik} of the space has the following non-zero components

$$h_{11}=e^{\lambda}\,,\quad h_{22}=r^2\,,\quad h_{33}=r^2\sin^2\theta\,, \tag{2.10}$$

$$h^{11}=e^{-\lambda}\,,\quad h^{22}=\frac{1}{r^2}\,,\quad h_{33}=\frac{1}{r^2\sin^2\theta}\,, \tag{2.11}$$

$$h=\det\|h_{ik}\|=e^{\lambda}r^4\sin^2\theta\,. \tag{2.12}$$

Since the chr.inv.-tensor D_{ik} of the deformation rate of the space is determined through the chr.inv.-derivatives of the h_{ik}, it has only the following non-zero components

$$D_{11}=\frac{\dot{\lambda}}{2}\,e^{\lambda-\frac{\nu}{2}}\,,\quad D^{11}=\frac{\dot{\lambda}}{2}\,e^{-\lambda-\frac{\nu}{2}}\,,\quad D=\frac{\dot{\lambda}}{2}\,e^{-\frac{\nu}{2}}\,, \tag{2.13}$$

where the upper dot means differentiation along the time coordinate t.

The chr.inv.-Christoffel symbols (they characterize the physically observable inhomogeneity of the space) are calculated according to their definition given in §1.3, using the components of the chr.inv.-metric tensor h_{ik}. After some algebra, we obtain formulae for the non-zero components of $\Delta_{ij,m}$, which have the form

$$\Delta_{11,1}=\frac{\lambda'}{2}\,e^{\lambda}\,,\quad \Delta_{22,1}=-r\,,\quad \Delta_{33,1}=-r\sin^2\theta\,, \tag{2.14}$$

$$\Delta_{12,2}=r\,,\qquad \Delta_{33,2}=-r^2\sin\theta\cos\theta\,, \tag{2.15}$$

$$\Delta_{13,3}=r\sin^2\theta\,,\qquad \Delta_{23,3}=r^2\sin\theta\cos\theta\,, \tag{2.16}$$

then we obtain the non-zero components of Δ^k_{ij}

$$\Delta^1_{11} = \frac{\lambda'}{2}\,, \quad \Delta^1_{22} = -re^{-\lambda}, \quad \Delta^1_{33} = -r\sin^2\theta\, e^{-\lambda}, \tag{2.17}$$

$$\Delta^2_{12} = \frac{1}{r}\,, \quad \Delta^2_{33} = -\sin\theta\cos\theta\,, \tag{2.18}$$

$$\Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^3_{23} = \cot\theta\,. \tag{2.19}$$

As was shown in §1.3, in a space without rotation, the 2nd rank chr.inv.-curvature tensor $C_{lk} = h^{ij}C_{ilkj}$ (physically observable curvature tensor) has the form (1.83). We are considering a space that does not rotate. Thus, after some algebra, we obtain the non-zero components of the chr.inv.-curvature tensor C_{lk} for the spherically symmetric metric (2.1). They have the form

$$C_{11} = -\frac{\lambda'}{r}\,, \qquad C_{22} = \frac{C_{33}}{\sin^2\theta} = e^{-\lambda}\left(1 - \frac{r\lambda'}{2}\right) - 1. \tag{2.20}$$

Calculate the chr.inv.-projections of the energy-momentum tensor of an ideal liquid (2.4) according to the general formulae (1.91). The projections are the observable mass density ρ, the observable momentum density J^i and the observable stress tensor U^{ik} of the liquid. Using the conditions $b^i = 0$ and $b^0 = \frac{1}{\sqrt{g_{00}}}$ (1.25) characteristic of the accompanying reference frame (since, in the case under consideration, the observer accompanies the liquid sphere), we obtain

$$\rho = \frac{T_{00}}{g_{00}} = \rho_0\,, \quad J^i = \frac{cT^i_0}{\sqrt{g_{00}}} = 0\,, \quad U^{ik} = c^2 T^{ik} = ph^{ik}. \tag{2.21}$$

According to the first chr.inv.-component, the liquid medium has a density $\rho = \rho_0$, which is constant everywhere inside the sphere.

The obtained condition $J^i = 0$ means that the liquid medium has no flows, and $U^{ik} = ph^{ik}$ means that the observer's reference space accompanies the liquid.

Also, according to the third chr.inv.-component, the trace $U = h^{ik}U_{ik}$ of the observable stress tensor U^{ik} of the liquid medium is expressed through the pressure p inside it as follows

$$U = 3p\,. \tag{2.22}$$

The chr.inv.-Einstein equations (1.92–1.94) in a space without rotation take the simplified form

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}F_j\right)F^j = -\frac{\varkappa}{2}\left(\rho_0 c^2 + U\right), \qquad (2.23)$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij}\right) = 0\,, \qquad (2.24)$$

$$\frac{{}^*\partial D_{ik}}{\partial t} - D_{ij}D^j_k + DD_{ik} - D_{ij}D^j_k + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \\ - \frac{1}{c^2}F_iF_k - c^2 C_{ik} = \frac{\varkappa}{2}\left(\rho_0 c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right), \qquad (2.25)$$

where ${}^*\nabla_i$ is the chr.inv.-derivative symbol. The chr.inv.-conservation law equations (1.96, 1.97) are also simplified as

$$D\rho_0 + \frac{1}{c^2}D_{ij}U^{ij} = 0\,, \qquad (2.26)$$

$${}^*\widetilde{\nabla}_i U^{ik} - \rho_0 F^k = 0\,, \qquad (2.27)$$

where we denote ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}F_i$.

Substitute, into the chr.inv.-Einstein equations (2.23–2.25), the obtained chr.inv.-characteristics of a space with the spherically symmetric metric (2.1), as well as the obtained chr.inv.-components of the energy-momentum tensor of an ideal liquid. After some algebra, we obtain the chr.inv.-Einstein equations (2.23–2.25) in component notation

$$e^{-\nu}\left(\ddot{\lambda} - \frac{\dot{\lambda}\dot{\nu}}{2} + \frac{\dot{\lambda}^2}{2}\right) - c^2 e^{-\lambda}\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{r} + \frac{(\nu')^2}{2}\right] = \\ = -\varkappa\left(\rho_0 c^2 + 3p\right)e^{\lambda}, \qquad (2.28)$$

$$\frac{\dot{\lambda}}{r}\,e^{-\lambda-\frac{\nu}{2}} = 0\,, \qquad (2.29)$$

$$e^{\lambda-\nu}\left(\ddot{\lambda} - \frac{\dot{\lambda}\dot{\nu}}{2} + \frac{\dot{\lambda}^2}{2}\right) - c^2\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{(\nu')^2}{2}\right] + \frac{2c^2\lambda'}{r} = \\ = \varkappa\left(\rho_0 c^2 - p\right)e^{\lambda}, \qquad (2.30)$$

$$\frac{c^2\left(\lambda' - \nu'\right)}{r}\,e^{-\lambda} + \frac{2c^2}{r^2}\left(1 - e^{-\lambda}\right) = \varkappa\left(\rho_0 c^2 - p\right). \qquad (2.31)$$

The second equation shows that $\dot{\lambda}=0$ in this case. This means that the internal space of the liquid sphere does not deform: using $\dot{\lambda}=0$ we obtain $D_{11}=0$, $D^{11}=0$ and $D=0$ according to (2.13). Taking this circumstance into account, as well as the stationarity of the λ, we reduce the field equations (2.28–2.31) to the final form

$$c^2 e^{-\lambda}\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{r} + \frac{(\nu')^2}{2}\right] = \varkappa\left(\rho_0 c^2 + 3p\right)e^{\lambda}, \qquad (2.32)$$

$$\frac{2c^2\lambda'}{r} - c^2\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{(\nu')^2}{2}\right] = \varkappa\left(\rho_0 c^2 - p\right)e^{\lambda}, \qquad (2.33)$$

$$\frac{c^2\left(\lambda' - \nu'\right)}{r}\, e^{-\lambda} + \frac{2c^2}{r^2}\left(1 - e^{-\lambda}\right) = \varkappa\left(\rho_0 c^2 - p\right). \qquad (2.34)$$

To solve the field equations (2.32–2.34), we need a formula for the pressure p. To find this formula, consider the conservation equations (2.26, 2.27). Since the space does not deform ($D_{ik}=0$) in the case under consideration, the chr.inv.-scalar conservation equation (2.26) vanishes. Only the chr.inv.-vector conservation equation (2.27) remains non-zero. Under the above conditions that we have assumed, it takes the form

$${}^*\nabla_i\left(p h^{ik}\right) - \left(\rho_0 + \frac{p}{c^2}\right)F^k = 0\,. \qquad (2.35)$$

Since ${}^*\nabla_i\, h^{ik}=0$ is true always for the chr.inv.-metric tensor (as well as $\nabla_\sigma\, g^{\alpha\sigma}=0$ for the fundamental metric tensor), the remaining conservation equation (2.35) takes the form

$$h^{ik}\frac{{}^*\partial p}{\partial x^i} - \left(\rho_0 + \frac{p}{c^2}\right)F^k = 0\,. \qquad (2.36)$$

Because $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i}$ in a space without rotation, the above formula reduces to the non-trivial equation

$$p' e^{-\lambda} + \left(\rho_0 c^2 + p\right)\frac{\nu'}{2}\, e^{-\lambda} = 0\,, \qquad (2.37)$$

where $p' = \frac{dp}{dr}$, $\nu' = \frac{d\nu}{dr}$, and $e^{\lambda} \neq 0$. Dividing both sides of this formula by $e^{-\lambda}$, we obtain the equation

$$\frac{dp}{\rho_0 c^2 + p} = -\frac{d\nu}{2}\,, \qquad (2.38)$$

which is an ordinary differential equation with separable variables. It is integrated easily as

$$\rho_0 c^2 + p = B e^{-\frac{\nu}{2}}\,, \qquad B = const. \tag{2.39}$$

Thus, we obtain the pressure p as a function of the ν

$$p = B e^{-\frac{\nu}{2}} - \rho_0 c^2. \tag{2.40}$$

When looking for the function $p(r)$, we integrate the field equations (2.32–2.34). Summing up (2.32) and (2.33), we find

$$\frac{c^2\left(\lambda' + \nu'\right)}{r} = \varkappa B e^{\lambda - \frac{\nu}{2}}. \tag{2.41}$$

Express ν' from here, then substitute the obtained result into the third field equation (2.34). We obtain

$$\frac{2c^2}{r}\,\lambda' + \frac{2c^2}{r^2}\left(e^{\lambda} - 1\right) - \varkappa B e^{-\lambda - \frac{\nu}{2}} = \varkappa\left(\rho_0 c^2 - p\right)e^{\lambda}. \tag{2.42}$$

Substituting the pressure p from (2.40) into (2.42), we obtain the following differential equation with respect to λ

$$\lambda' + \frac{e^{\lambda} - 1}{r} - \varkappa\rho_0 r e^{\lambda} = 0\,. \tag{2.43}$$

Introduce a new variable $y = e^{\lambda}$. Thus, we have $\lambda' = \frac{y'}{y}$. Substituting these y and y' into the original equation, we obtain the Bernoulli equation (see Kamke [31], Part III, Chapter I, §1.34)

$$y' + f(r)y^2 + g(r)y = 0\,, \tag{2.44}$$

where

$$f(r) = \frac{1}{r} - \varkappa\rho_0 r\,, \qquad g(r) = -\frac{1}{r}\,. \tag{2.45}$$

It has the following solution

$$\frac{1}{y} = E(r)\int\frac{f(r)\,dr}{E(r)}\,, \tag{2.46}$$

where

$$E(r) = e^{\int g(r)dr}. \tag{2.47}$$

Integrating (2.47), we obtain the function $E(r)$

$$E(r) = e^{-\int \frac{dr}{r}} = e^{\ln \frac{L}{r}} = \frac{L}{r}\,, \qquad L = const > 0\,, \tag{2.48}$$

so we obtain $\frac{1}{y} = e^{-\lambda}$, which is

$$e^{-\lambda} = \frac{L}{r}\int \frac{r}{L}\left(\frac{1}{r} - \varkappa\rho_0 r\right)dr = 1 - \frac{\varkappa\rho_0 r^2}{3} + \frac{Q}{r}\,, \quad Q = const. \tag{2.49}$$

To find the integration constant Q, re-write the equation (2.42) as

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \varkappa\rho_0\,. \tag{2.50}$$

This equation has a singularity at the point $r=0$, i.e., at the centre of the sphere, where the numerical value of the right hand side (i.e., the liquid density) tends to infinity. This contradicts the initially assumed condition $\rho_0 = const$ characteristic of incompressible liquids. In fact, this contradiction should not exist in the theory. We resolve this contradiction (and the singularity) by re-writing (2.50) as

$$e^{-\lambda}\left(1 - r\lambda'\right) = \frac{d}{dr}\left(re^{-\lambda}\right) = 1 - \varkappa\rho_0 r^2. \tag{2.51}$$

After integration, we obtain

$$re^{-\lambda} = r - \frac{\varkappa\rho_0 r^3}{3} + A\,, \qquad A = const. \tag{2.52}$$

Since $A=0$ at the central point $r=0$, it must be zero at any other point as well. Dividing this equation by $r \neq 0$, we obtain

$$e^{-\lambda} = 1 - \frac{\varkappa\rho_0 r^2}{3}\,. \tag{2.53}$$

Comparing this solution with the formula for $e^{-\lambda}$ obtained earlier (2.49), we see that they meet each other if $Q=0$. Besides, we must assume that $e^{\lambda_0} = 1$ at the central point $r=0$, hence $\lambda_0 = 0$.

So, we have the components $h^{11} = e^{-\lambda}$ and $h_{11} = e^{\lambda}$ of the chr.inv.-metric tensor h_{ik} expressed through the radial coordinate r, i.e.

$$h^{11} = e^{-\lambda} = 1 - \frac{\varkappa\rho_0 r^2}{3}\,, \qquad h_{11} = e^{\lambda} = \frac{1}{1 - \frac{\varkappa\rho_0 r^2}{3}}\,. \tag{2.54}$$

Introduce the limit condition $r = a$ on the surface of the sphere (since a is its radius). In this case, we have

$$e^{-\lambda_a} = 1 - \frac{\varkappa \rho_0 a^2}{3} \,. \tag{2.55}$$

On the other hand, the solution to this equation is also the mass-point solution in emptiness. Hence, we have

$$e^{-\lambda_a} = 1 - \frac{2GM}{c^2 a} \,, \tag{2.56}$$

where M is the mass of the liquid sphere. Comparing both of these formulae for $e^{-\lambda_a}$ and taking into account that Einstein's gravitational constant is $\varkappa = \frac{8\pi G}{c^2}$, we obtain

$$M = \frac{4\pi a^3 \rho_0}{3} = \rho_0 V, \tag{2.57}$$

where $V = \frac{4\pi a^3}{3}$ is the volume of the sphere. We have obtained the usual relation between the mass and volume of a homogeneous sphere.

Our next step is to find a solution for $e^{-\lambda}$ outside the sphere, where $r > a$. Since outside the liquid sphere the density of substance is $\rho_0 = 0$, after integrating (2.51) we obtain

$$r e^{-\lambda} = \int_0^r dr - \int_0^a \varkappa \rho_0 r^2 dr = r - \frac{\varkappa \rho_0 a^3}{3} \,. \tag{2.58}$$

From this formula we obtain that

$$e^{-\lambda} = 1 - \frac{\varkappa \rho_0 a^3}{3r} \,. \tag{2.59}$$

Taking (2.55) and (2.56) into account, we arrive at the same solution as the mass-point solution in emptiness, i.e.

$$e^{-\lambda} = 1 - \frac{2GM}{c^2 r} \,. \tag{2.60}$$

To obtain the variable ν, we use the equation (2.41). Substituting

$$\lambda' = \frac{\frac{2\varkappa \rho_0 r}{3}}{1 - \frac{\varkappa \rho_0 r^2}{3}} \tag{2.61}$$

and the obtained formula for e^{λ} into (2.41), after transformations we obtain

$$\nu' + \frac{\frac{2\varkappa\rho_0 r^2}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} - \frac{\varkappa B}{c^2}\frac{r e^{-\frac{\nu}{2}}}{1-\frac{\varkappa\rho_0 r^2}{3}} = 0\,. \tag{2.62}$$

Introduce a new variable $e^{-\frac{\nu}{2}} = y$. Thus, we have $\nu' = -\frac{2y'}{y}$. Substituting the above into (2.62), we obtain the Bernoulli equation

$$y' + \frac{\varkappa B}{2c^2}\frac{ry^2}{1-\frac{\varkappa\rho_0 r^2}{3}} - \frac{\frac{\varkappa\rho_0 r}{3}\,y}{1-\frac{\varkappa\rho_0 r^2}{3}} = 0\,, \tag{2.63}$$

where

$$f(r) = \frac{\varkappa B}{2c^2}\frac{r}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \qquad g(r) = -\frac{\frac{\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}}\,. \tag{2.64}$$

Thus, we have the integral

$$\int g(r)dr = -\int \frac{\frac{\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} = \ln N\sqrt{\left|1-\frac{\varkappa\rho_0 r^2}{3}\right|}, \quad N = const, \tag{2.65}$$

where

$$E(r) = N\sqrt{\left|1-\frac{\varkappa\rho_0 r^2}{3}\right|}\,. \tag{2.66}$$

In a region, where the signature condition $h_{11} = e^{\lambda} > 0$ is satisfied, we have

$$1-\frac{\varkappa\rho_0 r^2}{3} > 0\,, \tag{2.67}$$

therefore here we must use the modulus of the function.

Next, we look for $\frac{1}{y} = e^{\frac{\nu}{2}}$, which is

$$e^{\frac{\nu}{2}} = \frac{\varkappa B}{2c^2}\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\int\frac{r\,dr}{\sqrt{\left(1-\frac{\varkappa\rho_0 r^2}{3}\right)^3}}\,. \tag{2.68}$$

After integration, we obtain

$$e^{\frac{\nu}{2}} = \frac{\varkappa B}{2c^2}\left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right), \qquad K = const. \tag{2.69}$$

Find the integration constants B and K. To find the B, re-write the formula for the pressure p (2.40) using the condition that $p=0$ on the surface of the sphere ($r=a$). Thus, we have

$$B = \rho_0 c^2 e^{\frac{\nu_a}{2}}, \tag{2.70}$$

where $e^{\frac{\nu_a}{2}}$ is the value of the function $e^{\frac{\nu}{2}}$ on the surface of the sphere. As a result, we obtain

$$e^{\frac{\nu}{2}} = \frac{\varkappa\rho_0}{2}\, e^{\frac{\nu_a}{2}} \left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right). \tag{2.71}$$

To find the K, consider the $e^{\frac{\nu}{2}}$ on the surface of the sphere ($r=a$)

$$e^{\frac{\nu_a}{2}} = \frac{\varkappa\rho_0\, e^{\frac{\nu_a}{2}}}{2} \left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}\right), \tag{2.72}$$

from which we obtain that

$$K = -\frac{1}{\varkappa\rho_0}\frac{1}{\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}\,. \tag{2.73}$$

The quantity $e^{\frac{\nu_a}{2}}$ means the numerical value of the function $e^{\frac{\nu}{2}}$ at $r=a$, i.e., on the surface of the sphere. Therefore, we can apply it to the mass-point solution in emptiness at $r=a$, i.e.

$$e^{\frac{\nu_a}{2}} = \sqrt{1-\frac{2GM}{c^2 a}}\,. \tag{2.74}$$

Taking the formulae (2.55) and (2.56) into account, we obtain

$$e^{\frac{\nu}{2}} = \frac{1}{2}\, e^{\frac{\nu_a}{2}} \left(3 - \sqrt{\frac{1-\frac{\varkappa\rho_0 r^2}{3}}{1-\frac{\varkappa\rho_0 a^2}{3}}}\right) = \frac{1}{2}\left(3\sqrt{1-\frac{2GM}{c^2 a}} - \sqrt{1-\frac{2GMr^2}{c^2 a^3}}\right). \tag{2.75}$$

This solution on the surface of the sphere ($r=a$) meets the mass-point solution in emptiness: $e^{\frac{\nu_a}{2}} = \sqrt{1-\frac{2GM}{c^2 a}} = \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}$.

With the obtained formulae for the coefficients e^{ν} and e^{λ}, the space metric of a sphere filled with an ideal liquid takes the form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{\varkappa\rho_0 r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \quad (2.76)$$

Taking (2.55) and (2.56) into account, we can re-write the obtained space metric (2.76) in the form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{2GM}{c^2a}} - \sqrt{1-\frac{2GMr^2}{c^2a^3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{2GMr^2}{c^2a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \quad (2.77)$$

Finally, because $\frac{2GM}{c^2} = r_g$ is the Hilbert radius calculated from the mass M of the liquid sphere and taking the obtained formula for $e^{\frac{\nu_a}{2}}$ into account, we re-write the resulting metric in the final form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2r_g}{a^3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{r^2r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \quad (2.78)$$

This is the final formula for the “internal” space metric of a sphere filled with an ideal liquid. As you can see, on the surface of the liquid sphere ($r = a$) its “internal” metric completely coincides with the metric of a material point in emptiness.

From here we can obtain the metric of the space outside the liquid sphere ($r > a$). Let us do it.

We have already obtained the “external” solution for $e^{-\lambda}$ (2.59), which turned out to be the same as the “external” mass-point solution for this function (2.60). Outside the sphere we have $B = 0$ (2.39). Therefore, (2.41) becomes

$$\lambda' + \nu' = 0\,, \quad (2.79)$$

where, according to (2.60),

$$\lambda' = \frac{2GM}{c^2r^2}\frac{1}{1-\frac{2GM}{c^2r}}\,. \tag{2.80}$$

Substituting (2.80) into (2.79) then integrating the resulting equation, we obtain

$$\nu = \ln\left(1-\frac{2GM}{c^2r}\right)+P, \qquad P = const, \tag{2.81}$$

therefore

$$e^{\nu} = P\left(1-\frac{2GM}{c^2r}\right). \tag{2.82}$$

Since this function has also the form

$$e^{\nu} = 1-\frac{2GM}{c^2a}\,, \tag{2.83}$$

then on the surface ($r=a$) of the liquid sphere we have $P=1$. Substituting the obtained formulae for e^{ν} (2.83) and e^{λ} (2.60) into the spherically symmetric metric (2.1), we obtain that the "external" space of a sphere filled with an ideal liquid is described by the metric of a mass-point in emptiness (1.1), i.e.

$$ds^2 = \left(1-\frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1-\frac{r_g}{r}} - r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right). \tag{2.84}$$

2.2 The outer space breaking in the Sun's field coincides with the asteroid belt

Here we propose a new model of the Solar System based on General Relativity. Namely, — the Sun and the planets will be considered as liquid spheres according to the liquid sphere metric (2.78) obtained above. The metric was also shown in the formula (1.8) of §1.2, where we considered the formulation of the star modelling problem in terms of General Relativity. In addition, as was proved in the previous §2.1, the external space of a liquid sphere is described by the metric of a mass-point in emptiness (1.1).

Note that we are not discussing here whether the internal planets can be represented as liquid spheres or not. Astrophysicists and geologists can simply refer to magma because it is in the state of liquid stone.

However, the Jovian planets (Jupiter, Saturn, Uranus and Neptune) in terms of their density and other parameters may well be considered stars. Here we limit ourselves to theoretical modeling of the Sun and the planets without analysing their origin. Let us dwell in detail on the location of the "inner" and "outer" space breakings of their fields: the space breaking in the field deep inside and far beyond the physical body (liquid sphere) of each of them. We will then compare the result with the observed distribution of the planets in the Solar System.

Our approach to the Solar System is simple. As is known, in a four-dimensional Riemannian space with a sign-alternating diagonal metric (+−−−), a breaking occurs in that region (point or surface) wherein at least one of the four signature conditions

$$\left.\begin{aligned} &g_{00} > 0 \\ &g_{00}\, g_{11} < 0 \\ &g_{00}\, g_{11}\, g_{22} > 0 \\ &g = g_{00}\, g_{11}\, g_{22}\, g_{33} < 0 \end{aligned}\right\} \tag{2.85}$$

is violated. The space (space-time) of General Relativity is one of the above type of Riemannian spaces. Therefore, we consider the signature conditions in the space inside and outside the liquid Sun.

2.2.1 In the "internal" space metric of a liquid sphere (2.78), taking into account that

$$\frac{\varkappa\rho_0 a^3}{3r} = \frac{2GM}{c^2 r} = \frac{r_g}{r} \tag{2.86}$$

therein*, the fundamental metric tensor has the following non-zero components

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 = \\ = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2, \tag{2.87}$$

$$g_{11} = -\frac{1}{1-\frac{r^2 r_g}{a^3}} = -\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.88}$$

*See formulae (2.59) and (2.60) in §2.1.

$$g_{22} = -r^2, \tag{2.89}$$

$$g_{33} = -r^2 \sin^2\theta. \tag{2.90}$$

From these components, we obtain that at a distance of

$$r = r_{br} = \sqrt{\frac{a^3}{r_g}} = \sqrt{\frac{3}{\varkappa\rho_0}}\,, \tag{2.91}$$

from the centre of the sphere, the second, third and fourth signature conditions are violated*

$$\left.\begin{aligned} & g_{00} = \frac{9}{4}\left(1-\frac{r_g}{a}\right) > 0 \\ & g_{00}\,g_{11} \rightarrow -\infty \\ & g_{00}\,g_{11}\,g_{22} \rightarrow \infty \\ & g = g_{00}\,g_{11}\,g_{22}\,g_{33} \rightarrow -\infty \end{aligned}\right\}. \tag{2.92}$$

This means that at the distance $r_{br} = \sqrt{a^3/r_g}$ from the centre of the liquid spherical body, its field has a space breaking on the surface of the mentioned radius r_{br}.

The Hilbert radius $r_g = \frac{2GM}{c^2}$ (gravitational collapse radius) calculated for ordinary physical bodies is many orders of magnitude smaller than their physical sizes. Hence, $a \gg r_g$ for an ordinary liquid sphere (such a body is not a collapsar). In this case, we have $r_{br} = \sqrt{a^3/r_g} \gg a$: the spherical surface on which the field has a space breaking is far beyond the physical surface of the liquid sphere (field source) and, hence, far from its internal field. In other words, the internal field and substance of a liquid sphere form a space breaking in its external field.

What does the outer space breaking in a star's field mean from a physical point of view? Does such a space breaking a real action on a physical body appearing in it, or is it just a mathematical fiction? As will be shown in the next §2.3, the space (space-time) of a liquid sphere has a breaking in its four-dimensional curvature tensor $R_{\alpha\beta\gamma\delta}$ under the condition $r = r_{br}$. Namely, — the component R_{0101} (2.113), which is the

---

*Namely, — these three functions tend to infinity. As is known, a function has a breaking as it tends to infinity.

four-dimensional curvature of the space in the (r-t)-direction 0101, has a breaking at the distance $r = r_{br}$ from the centre of the liquid sphere, i.e., the curvature becomes infinite ($R_{0101} \to \infty$) on a surface of the radius $r = r_{br}$. Since the four-dimensional curvature is determined by the gravitational field that fills the space (and vice versa), the breaking at $r = r_{br}$ means a breaking in the gravitational field of the liquid sphere.

This is the physical sense of the outer space breaking in the field of a liquid sphere.

2.2.2 The external field of a liquid sphere is due to the same liquid substance that fills the sphere and produces the field inside the sphere itself (its internal field). According to the formula for the "external" space metric (2.84), we see that its fundamental metric tensor has the following non-zero components

$$g_{00} = 1 - \frac{r_g}{r}\,, \tag{2.93}$$

$$g_{11} = -\frac{1}{1 - \frac{r_g}{r}}\,, \tag{2.94}$$

$$g_{22} = -r^2, \tag{2.95}$$

$$g_{33} = -r^2 \sin^2\theta. \tag{2.96}$$

We see that at the distance

$$r = r_g = \frac{2GM}{c^2} \tag{2.97}$$

from the centre of the body, the first signature condition ($g_{00} > 0$) is violated

$$\left.\begin{aligned} g_{00} &= 1 - \frac{r_g}{r} = 0 \\ g_{00}\, g_{11} &= -1 < 0 \\ g_{00}\, g_{11}\, g_{22} &= r^2 > 0 \\ g &= -r^4 \sin^2\theta < 0 \end{aligned}\right\}. \tag{2.98}$$

In other words, the external field of a liquid sphere produces a space breaking deep inside the sphere itself, in its internal space close to the centre. For example, the calculated Hilbert radius $r_g = \frac{2GM}{c^2}$ is only 2.9 km for the Sun, and for the Earth it is nothing but only 0.88 cm.

2.2.3 So, according to our model of liquid stars based on General Relativity, concerning ordinary stars and the Sun in particular, the above conclusions mean the following:

1. At the centre of every star there is a small core of the Hilbert radius r_g, on the surface of which the aforementioned inner space breaking in the star's field occurs. The inner space breaking physically means that the liquid substance of the star has a singularity on a spherical surface of the Hilbert radius r_g around the centre, thereby the mentioned small core is physically separated from the main substance of the star (the physical sense of this phenomenon will be clearer using the example of the outer space breaking in the Sun's field);
2. The field of every star has an outer space breaking on a spherical surface around the star. This is a "bubble" with a very large radius $r_{br} = \sqrt{a^3/r_g}$, which is many orders of magnitude greater than the physical radius a of the star. Physically, the outer space breaking prevents the formation of nearby substance, such as small stones or dust, rotating around the star, into a single planet in an orbit of the radius r_{br}.

Calculate the radius $r_{br} = \sqrt{a^3/r_g} = \sqrt{3/\varkappa\rho_0}$ (2.91) of the outer space breaking in the Sun's field. With the Sun's density $\rho_0 = 1.41$ gram/cm^3 or its mass $M = 2.0 \times 10^{33}$ gram and radius $a = 6.95 \times 10^{10}$ cm,

$$r_{br} = 3.4 \times 10^{13} \text{ cm} = 340{,}000{,}000 \text{ km} = 2.3 \text{ AU}. \qquad (2.99)$$

We obtain that the spherical surface (bubble) of the outer space breaking in the Sun's field is located in the asteroid belt, very close to the orbit of the maximum concentration of asteroids (the asteroid belt extends, approximately, from 2.1 to 4.3 AU from the Sun).

This truly amazing theoretical discovery leads us to the conclusion that the internal structure of the Solar System can be calculated according to the liquid model. Namely, — we consider the Sun and the planets as liquid spheres, then we calculate the space breaking r_{br} in the field of each of these cosmic bodies. The results of this calculation are summarized in Table 2.1.

These results related to the planets and the Sun, according to Table 2.1, lead to the following conclusions:

Object	Mass M, gram	Density ρ_0, gram/cm^3	Radius a, cm	Hilbert radius r_g, cm	Orbit, AU	Space breaking r_{br}, AU	Distance of r_{br} from the Sun, AU
Sun	1.98×10^{33}	1.41	6.95×10^{10}	2.9×10^{5}	—	2.3	2.3
Internal planets							
Mercury	2.21×10^{26}	4.10	2.36×10^{8}	0.03	0.39	1.3	−0.9 – 1.7
Venus	4.93×10^{27}	5.10	6.19×10^{8}	0.73	0.72	1.2	−0.5 – 1.9
Earth	5.97×10^{27}	5.52	6.38×10^{8}	0.88	1.00	1.1	−0.1 – 2.1
Mars	6.45×10^{26}	3.80	3.44×10^{8}	0.10	1.52	1.4	0.1 – 2.9
Asteroid belt	—	—	—	—	2.5*	—	—
Jovian planets							
Jupiter	1.90×10^{30}	1.38	7.11×10^{9}	280	5.20	2.3	2.9 – 7.5
Saturn	5.68×10^{29}	0.72	6.00×10^{9}	84	9.54	3.2	6.3 – 12.7
Uranus	8.72×10^{28}	1.30	2.55×10^{9}	13	19.2	2.4	16.8 – 21.6
Neptune	1.03×10^{29}	1.20	2.74×10^{9}	15	30.1	2.4	27.7 – 32.5
Pluto	1.31×10^{25}	2.00	1.20×10^{8}	0.002	39.5	1.9	37.6 – 41.4
Kuiper belt	—	—	—	—	30 – 100	—	—

*The maximum concentration of asteroids in the asteroid belt is registered at ~2.5 AU from the Sun, while the asteroid belt continues from 2.1 to 4.3 AU (approximately).

Table 2.1: The internal constitution of the Solar System according to General Relativity.

1. The outer space breaking in the Sun's field is located at the distance $r = 2{,}3$ AU from the Sun, which is near the maximum concentration of asteroids in the asteroid belt;
2. The internal planets of the Solar System (Mars, the Earth, Venus and Mercury) are located inside the "bubble" of the outer space breaking in the Sun's field;
3. For each of the internal planets, the "bubble" of the outer space breaking in its field is as well located inside the "bubble" of the outer space breaking in the Sun's field;
4. The outer space breaking in Mars' field and the outer space breaking in the Earth's field reach the asteroid belt;
5. The outer space breaking in Mars' field is located at 2.9 AU from the Sun. It is in the asteroid belt near the orbit of Phaeton, the hypothetical planet which was once orbiting the Sun according to the Titius-Bode law at $r = 2.8$ AU and whose distraction in the ancient ages gave birth to the asteroid belt;
6. The "bubble" of the outer space breaking in Jupiter's field meets, from its internal side, that of Mars at $r = 2.9$ AU from the Sun (in the case of a "parade of the planets"). It is very near 2.8 AU, which is the theoretical orbit of Phaeton according to the Titius-Bode law;
7. For each of the other Jovian planets (Saturn, Uranus and Neptune), the "bubble" of the outer space breaking in its field is located inside the inner boundary of the Kuiper belt (the belt of the aphelia of the comets orbiting the Sun);
8. The outer space breaking in Neptune's field meets, from the external side of this "bubble", the inner boundary of the Kuiper belt;
9. For Pluto, the "bubble" of the outer space breaking in its field is located entirely inside the Kuiper belt.

The fact that the outer space breaking in the Sun's field is located in the asteroid belt, near the maximum concentration of asteroids, allows us to say: yes, the space breaking considered in this study has a real physical sense. It is most likely that the outer space breaking in the Sun's field prevents the asteroids to merge into a single physical body (called Phaeton). Alternatively, if Phaeton was an already existing planet that was orbiting the Sun near the "space breaking orbit" in the past, the force of gravitation of another massive cosmic body, emerging near the Solar

System in the ancient ages (for example, another star passing near it), has displaced Phaeton to the "space breaking orbit" near it, thus leading to the distraction of Phaeton's body.

Thus, we arrive at the conclusion that the internal constitution of the Solar System is formed by the geometric structure of the Sun's field according to Riemannian geometry that is manifested in the laws of the General Theory of Relativity.

2.3 The geometric sense of the outer space breaking

Let us consider the properties attributed to the curvature of the space of a liquid sphere. To do this, we first need to calculate the components of the chr.inv.-curvature tensor C_{lkij}, which is the physically observable curvature tensor.

In the space of a non-rotating liquid sphere under consideration, $A_{ik}=0$ and, hence, $C_{lkij}=H_{lkij}$ according to the definition of H_{lkij} (1.81). Therefore, we calculate $C_{lkij}=H_{lkij}=h_{jm}H_{lki\cdot}^{\cdot\cdot\cdot m}$ from the formula for $H_{lki\cdot}^{\cdot\cdot\cdot m}$ (1.78), where we substitute the respective chr.inv.-Christoffel symbols Δ^i_{jk} (2.17–2.19) obtained for the metric of a liquid sphere (2.78). After some algebra, we obtain that the chr.inv.-curvature tensor C_{lkij} in the space of a liquid sphere has the following non-zero components

$$C_{1212} = H_{1212} = -\frac{\varkappa\rho_0}{3}\,\frac{r^2}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.100}$$

$$C_{1313} = H_{1313} = -\frac{\varkappa\rho_0}{3}\,\frac{r^2\sin^2\theta}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.101}$$

$$C_{2323} = H_{2323} = -\frac{\varkappa\rho_0}{3}\,r^4\sin^2\theta. \tag{2.102}$$

We see that in the space of a liquid sphere the non-zero components of the observable space curvature tensor C_{iklj} satisfy the condition

$$C_{iklj} = -\frac{\varkappa\rho_0}{3}\left(h_{kl}h_{ij}-h_{il}h_{kj}\right), \tag{2.103}$$

where the negative constant $-\frac{\varkappa\rho_0}{3}$ is the observable three-dimensional curvature of the space in the respective two-dimensional direction. This means that the three-dimensional space of a non-rotating liquid sphere has a *constant negative curvature*.

Calculating the observable curvature scalar $C = h^{ik}C_{ik}$, where the non-zero components of C_{ik} are

$$C_{11} = -\frac{2\varkappa\rho_0}{3}\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.104}$$

$$C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2\varkappa\rho_0 r^2}{3}\,, \tag{2.105}$$

we obtain

$$C = -2\varkappa\rho_0 = const < 0\,. \tag{2.106}$$

Hence, according to (2.103), the chr.inv.-curvature tensor C_{iklj} is expressed through the observable curvature scalar C as

$$C_{iklj} = \frac{C}{6}\left(h_{kl}h_{ij} - h_{il}h_{kj}\right). \tag{2.107}$$

Therefore, the observable three-dimensional space of a non-rotating liquid sphere is a *constant negative curvature space*, and its curvature radius $\mathfrak{R}$ is imaginary: the $\mathfrak{R}$ is formulated in terms of the observable curvature scalar C as

$$C = -2\varkappa\rho_0 = \frac{1}{\mathfrak{R}^2}\,, \tag{2.108}$$

thus we obtain, finally,

$$\mathfrak{R} = \frac{i}{2\varkappa\rho_0}\,. \tag{2.109}$$

So forth we calculate the components of the Riemann-Christoffel curvature tensor. As is known, the tensor is determined as

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial x^\beta\partial x^\gamma} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha\partial x^\delta} - \frac{\partial g_{\beta\delta}}{\partial x^\alpha\partial x^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x^\beta\partial x^\delta}\right) + \\ + g^{\sigma\tau}\left(\Gamma_{\alpha\delta,\sigma}\Gamma_{\beta\gamma,\tau} - \Gamma_{\beta\delta,\sigma}\Gamma_{\alpha\gamma,\tau}\right). \tag{2.110}$$

According to the metric of a liquid sphere (2.78), we have $g_{ik} = -h_{ik}$ and $\Gamma_{ik,j} = -\Delta_{ik,j}$. Thus, calculating the non-zero components of $\Gamma_{\alpha\beta,\delta}$,

$$\Gamma_{01,0} = -\Gamma_{00,1} = \frac{\varkappa\rho_0 r}{12}\,\frac{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.111}$$

$$\Gamma_{11,1} = -\frac{\varkappa\rho_0 r}{3}\,\frac{1}{\left(1-\frac{\varkappa\rho_0 r^2}{3}\right)^2}\,, \tag{2.112}$$

then substituting these into (2.110), we obtain

$$R_{0101} = -\frac{\varkappa\rho_0}{12}\,\frac{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.113}$$

$$R_{1212} = \frac{\varkappa\rho_0}{3}\,\frac{r^2}{1-\frac{\varkappa\rho_0 r^2}{3}} = -C_{1212}\,, \tag{2.114}$$

$$R_{1313} = \frac{\varkappa\rho_0}{3}\,\frac{r^2\sin^2\theta}{1-\frac{\varkappa\rho_0 r^2}{3}} = -C_{1313}\,, \tag{2.115}$$

$$R_{2323} = \frac{\varkappa\rho_0}{3}\,r^4\sin^2\theta = -C_{2323}\,. \tag{2.116}$$

We see that the component R_{0101} determining the four-dimensional curvature in the (r-t)-direction 0101 does not satisfy the condition

$$R_{\alpha\beta\gamma\delta} = Q\left(g_{\beta\gamma}\,g_{\alpha\delta} - g_{\beta\delta}\,g_{\alpha\gamma}\right), \qquad Q = const, \tag{2.117}$$

which is specific to four-dimensional constant curvature spaces.

As a result of the above calculations, we arrive at the following conclusion about the space of a non-rotating liquid sphere:

> The four-dimensional space (space-time) of a non-rotating liquid sphere *is not* a constant curvature space. This is in contrast to its observable three-dimensional space, which, as proven above, is a *constant negative curvature space*.

In addition, based on the obtained formulae for C_{1212} (2.100) and C_{1313} (2.101), we also see that the observable three-dimensional curvature C_{iklj} has a space breaking

$$C_{1212} \rightarrow -\infty, \qquad C_{1313} \rightarrow -\infty \tag{2.118}$$

under the condition $r = r_{br} = \sqrt{3/\varkappa\rho_0} = \sqrt{a^3/r_g}$. By the same condition $r = r_{br}$, according to the formula for R_{0101} (2.113), we have

$$R_{0101} \rightarrow -\infty. \tag{2.119}$$

In other words, the three-dimensional chr.inv.-curvature C_{iklj} and the four-dimensional Riemannian curvature $R_{\alpha\beta\gamma\delta}$ have a common space breaking under the condition $r = r_{br}$. Concerning the model of liquid stars, this means:

> Both the observable three-dimensional space curvature C_{iklj} and the four-dimensional Riemannian curvature $R_{\alpha\beta\gamma\delta}$ in the field of any star have a common space breaking on a spherical surface at the distance $r = r_{br} = \sqrt{3/\varkappa\rho_0} = \sqrt{a^3/r_g}$ from the star.

This is the geometric sense of the outer space breaking in a star's field (according to the considered model of liquid stars).

2.4 The gravitational force acting inside a liquid star

In a space without rotation, the gravitational inertial force F_i (1.42) is due only to g_{00} determined by the gravitational potential w. Let us calculate this force. Since the gravitational potential is $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$, we have

$$F_i = \frac{\partial \mathrm{w}}{\partial x^i} = -\frac{c^2}{2\sqrt{g_{00}}}\frac{\partial g_{00}}{\partial x^i}\,. \tag{2.120}$$

According to the "internal" metric of a non-rotating liquid sphere (2.76), we have

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2, \tag{2.121}$$

or, in the same metric written in the other form (2.78),

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2, \tag{2.122}$$

hence the force acting inside the sphere is

$$F_1 = -\frac{\varkappa\rho_0 c^2 r}{3}\,\frac{1}{\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.123}$$

$$F^1 = -\frac{\varkappa\rho_0 c^2 r}{3}\,\frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.124}$$

or, in the other form,

$$F_1 = -\frac{c^2 r_g r}{a^3} \frac{1}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{2.125}$$

$$F^1 = -\frac{c^2 r_g r}{a^3} \frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{2.126}$$

This is a force of attraction: since $r < a$ inside the sphere, $F_1 < 0$ in it. This force is proportional to distance. Its numerical value is zero at the centre of the sphere (where $r = 0$), then increases with distance to a maximum value on the surface of the star (where $r = a$)

$$(F_1)_{r=a} = -\frac{\varkappa \rho_0 c^2 a}{6} \frac{1}{1-\frac{\varkappa \rho_0 a^2}{3}} = -\frac{c^2 r_g}{2a^2} \frac{1}{1-\frac{r_g}{a}}\,, \tag{2.127}$$

$$(F^1)_{r=a} = -\frac{\varkappa \rho_0 c^2 a}{6} = -\frac{c^2 r_g}{2a^2}\,. \tag{2.128}$$

2.5 Solving the conservation law equations: pressure and density inside the stars

Consider now the pressure p and the density ρ_0 inside an ordinary liquid star. The formula relating pressure and density in a medium is called the equation of state. It follows as a solution to the conservation law equations.

We have already obtained almost everything that is needed for this formula. In §2.1, we solved the conservation law equations with the energy-momentum tensor of an ideal liquid (2.4), which is characteristic of the substance of liquid stars. After substituting the physically observable components (2.21) of the energy-momentum tensor, the general form (1.96–1.97) of the conservation law equations takes the particular form (2.26, 2.27). In a non-deforming space such as the space of an ordinary star, only the vector conservation equation remains non-zero. It has the form (2.36). This equation is solved as (2.40)

$$p = B e^{-\frac{\nu}{2}} - \rho_0 c^2. \tag{2.129}$$

Substituting the already found integration constant B (2.70) and the function $e^{\frac{\nu}{2}}$ (2.75) into the p (2.129), we obtain the following formula connecting the pressure p and the density ρ_0 inside an ordinary star

$$p = \rho_0 c^2 \frac{\sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}}}{3\sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}}. \tag{2.130}$$

Find the pressure in the near-surface layer of a star. The constant $\varkappa = 18.6 \times 10^{-28}$ cm/gram is a very small value, while $\rho_0 = 1.4$ gram/cm^3 for the Sun (yellow dwarf) is much smaller than the ρ_0 for larger stars. Therefore, $\varkappa \rho_0 a^2$ is much smaller than 1 even for very large stars. For example, Betelgeuse, which is one of the largest red super-giants, has $M = 4.0 \times 10^{34}$ gram, $a = 7.0 \times 10^{13}$ cm and $\rho_0 = 2.8 \times 10^{-8}$ gram/cm^3. In this case, we have $\varkappa \rho_0 a^2 = 2.6 \times 10^{-7}$. As a result, we have

$$\sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \approx 1 - \frac{\varkappa \rho_0 r^2}{6}, \qquad \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} \approx 1 - \frac{\varkappa \rho_0 a^2}{6}. \tag{2.131}$$

After some algebra, we obtain an approximate formula for the pressure p inside an ordinary star, which has the form

$$p \approx \frac{\varkappa \rho_0^2 c^2 \left(a^2 - r^2\right)}{12} = \frac{\rho_0 G M}{2a^2} \left(\frac{a^2 - r^2}{a}\right). \tag{2.132}$$

Let $h = a - r$ be the distance from the surface of the sphere to the point of measurement. Since $h \ll r$ in the near-surface layer, we have

$$a^2 - r^2 = (a - r)(a + r) = h(2a + h) \approx 2ah. \tag{2.133}$$

Thus, from (2.132), we obtain the ordinary formula for the pressure in the near-surface layer

$$p = \rho_0 g h, \tag{2.134}$$

where $\frac{GM}{a^2} = g$ is the free-fall acceleration in the near-surface layer.

The pressure $p_0 = p_{r=0}$ in the central region of an ordinary star can easily be found by assuming $r = 0$ in the general formula (2.130)

$$p_0 = \rho_0 c^2 \frac{1 - \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}}}{3\sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - 1} \approx \frac{\varkappa \rho_0^2 a^2 c^2}{12}. \tag{2.135}$$

Since $\varkappa = \frac{8\pi G}{c^2}$, we can also re-write this formula in the form

$$p_0 \approx \frac{3GM^2}{8\pi a^4}\,. \tag{2.136}$$

Table 2.2 gives the numerical values of the central pressure p_0, which we have calculated according to the above formula for typical members of the known families of ordinary stars.

We see that, according to our model of liquid stars, the pressure in the central region of Betelgeuse, which is one of the largest stars, is only 0.53 atmosphere (1 atm = 10^6 dynes/cm^2). The smaller the star, the higher the pressure inside it. The pressure in the central region of the white supergiant Rigel, the radius of which is 14.6 times smaller than Betelgeuse's radius, is 1.7×10^4 atm. Sun-like dwarfs have a central pressure of $\sim 10^9$ atm. However, the central pressure in white dwarfs reaches 10^{17} atm.

Note that the temperature of a condensed matter does not depend on the pressure in it. The incompressible liquid of stars is a kind of condensed matter. Therefore, the temperature inside stars depends solely on the formula of the particular mechanism that produces stellar energy.

This remark is important for understanding the physical conditions inside stars and sources of stellar energy.

Object	Mass M, gram	Radius a, cm	Density ρ_0, gram/cm^3	Pressure p_0, dynes/cm^2
Red super-giant*	4.0×10^{34}	7.0×10^{13}	2.8×10^{-8}	5.3×10^{5}
White super-giant†	3.4×10^{34}	4.8×10^{12}	7.3×10^{-5}	1.7×10^{10}
Sun	2.0×10^{33}	7.0×10^{10}	1.4	1.3×10^{15}
Jupiter (proto-star)	1.9×10^{30}	7.1×10^{9}	1.3	1.2×10^{15}
Red dwarfs	6.7×10^{32}	2.3×10^{10}	13	1.3×10^{16}
Brown dwarf‡	4.1×10^{31}	7.0×10^{9}	29	5.7×10^{15}
White dwarf§	2.0×10^{33}	6.4×10^{8}	1.8×10^{6}	1.9×10^{23}

*Betelgeuse. †Rigel. ‡Corot-Exo-3. §Sirius B.

Table 2.2: The main characteristics of ordinary stars.

2.6 The stellar energy mechanism according to the model of liquid stars and the mass-luminosity relation

Let us turn to the dimensionless characteristics of stars, which are expressed in fractions of the corresponding characteristics of the Sun

$$\bar{M} = \frac{M}{M_\odot}\,, \quad \bar{a} = \frac{a}{a_\odot}\,, \quad \bar{\rho} = \frac{\rho}{\rho_\odot}\,, \ \dots \ \text{etc.}, \tag{2.137}$$

where $\bar{M} = \bar{\rho}_0 \bar{a}^3$ for a liquid sphere*. For the luminosity L of a star, which is the energy radiated from the entire surface of the star to the cosmos per one second, we have

$$\bar{L} = \frac{L}{L_\odot}\,. \tag{2.138}$$

With the above dimensionless representation of the characteristics of stars, the analysis is greatly simplified. This is due to the fact that only significant factors remain in the formulae, and all constant coefficients disappear.

Let us study what mechanism for the production of stellar energy can now be proposed based on General Relativity, so that its productivity satisfies the observed luminosity of stars. In other words, to be a real mechanism generating energy in stars, the calculated energy production of the proposed mechanism must correspond to the mass-luminosity relation, which is the main empirical relation of observational astrophysics.

Consider the space metric of a liquid star. As we already know, the space of a liquid star has two primary regions, described by different space metrics:

1. The internal space metric (metric of a liquid sphere) is valid from the centre of the star to its surface, except for a singular spherical surface of the tiny radius $r_g = \frac{2GM}{c^2}$ around the centre of the star (see below). The internal metric is also valid on a singular spherical surface of the radius $r_{br} = \sqrt{a^3/r_g} = \sqrt{3/\varkappa\rho_0}$ in the outer cosmos: on this spherical surface around the star in the outer cosmos, the star's gravitational field has a space breaking produced due to its internal metric;

*A liquid star has the same density $\rho = \rho_0 = const$ throughout its volume, so its mass is $M = \frac{4}{3}\pi\rho_0 a^3$. In fractions of the Sun's mass, it is $\bar{M} = \bar{\rho}_0 \bar{a}^3$.

2. The external space metric (mass-point metric) is valid from the surface of the star to infinity, except for a singular spherical surface of the radius $r_{br} = \sqrt{a^3/r_g} = \sqrt{3/\varkappa\rho_0}$ around the star in the outer cosmos (see above). The external metric is also valid deep inside the star, on a singular spherical surface of the tiny radius $r_g = \frac{2GM}{c^2}$ from the centre of the star: on this spherical surface deep inside the star, the star's gravitational field has a space breaking produced due to its external metric.

As was shown in §2.3, the outer space breaking in the outer cosmos only implies a breaking of the space curvature. In addition, it can be shown based on §2.3 that this does not lead to an anomaly in the acting gravitational force.

However, now we will show that the gravitational force has a very strong anomaly on the singular spherical surface of the inner space breaking. Indeed, inside a star at the Hilbert radius r_g from its centre, the external space metric is valid (and the internal metric is valid both inside the Hilbert radius and outside it). Therefore, all calculations for the inner singular surface are performed with the external space metric (mass-point metric) despite the fact that this singular surface is located deep inside the star near its centre.

According to the fundamental metric tensor of the external metric of a liquid star (1.1), the physically observable chr.inv.-vector of the gravitational force F_i has the form (1.4). On the singular spherical surface of the Hilbert radius $r = r_g$, deep inside the star, the observed force of gravity (1.4) reaches an infinitely large value

$$F_1 = -\frac{c^2 r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}} \rightarrow -\infty\,, \tag{2.139}$$

which means that the gravitational field of the star has a space breaking on this surface.

Due to its infinitely large magnitude, the force of gravity there, by definition, is sufficient to transfer the necessary kinetic energy to the light atomic nuclei of stellar substance in order to start the process of thermonuclear fusion. The energy released by thermonuclear fusion is the energy radiated by stars.

The singular spherical surface of the Hilbert radius $r_g = \frac{2GM}{c^2}$ surrounds the geometric centre of each star. This means that at the centre of each star there is a luminous "inner sun". This "inner sun" is tiny compared to the size of the star. For example, the Hilbert radius of the Sun is only 2.9 km, while the physical radius of the Sun is 700,000 km. Thus, the thermonuclear fusion zone is not only a surface layer of the radius r_g, but the entire volume of the "inner sun". In other words, the "inner sun" of the radius r_g is the very place where, as a result of thermonuclear fusion, helium is formed from hydrogen, which provides the luminosity of the star with energy. Further, the energy is transferred from the "inner sun" of the star to its surface due to thermal conduction (the usual heat transfer in liquids); the transmitted energy is then radiated from the surface of the star into the outer cosmos.

Since the "inner sun" of a star has a radius equal to the Hilbert radius r_g for the star, we will further refer to it as the *luminous Hilbert core*, or merely — the *Hilbert core*.

The luminosity of a star shining due to the proposed mechanism of stellar energy depends only on two factors: the volume $V = \frac{4}{3}\pi r_g^3$ of the Hilbert core, in which stellar energy is released, and also on the density ρ_g of stellar substance in it (which may differ from the density ρ_0 of the main mass of the star, see the explanation below). This means

$$\bar{L} = \bar{\rho}_g \bar{r}_g^3 = \bar{\rho}_g \bar{M}^3. \tag{2.140}$$

Recall that the proposed mechanism of stellar energy does not depend on the pressure in the central region of a star: the super-strong force of gravity (2.139) acting in the central region provides the necessary conditions for thermonuclear fusion. But its productivity depends on the density of stellar substance in the Hilbert core.

Let us calculate such a density of stellar substance in the Hilbert core, with which the proposed mechanism of stellar energy satisfies the observed mass-luminosity relation.

We start from the facts of observational astronomy. It shows the mass-luminosity relation $\bar{L} = \bar{M}^{2.6}$ for the stars, the masses of which are in the range between $0.2M_\odot$ and $0.5M_\odot$, $\bar{L} = \bar{M}^{4.5}$ for the star masses between $0.5M_\odot$ and $2M_\odot$, $\bar{L} = \bar{M}^{3.6}$ in the range between $2M_\odot$ and $10M_\odot$, and also $\bar{L} = \bar{M}$ for the stars much heavier than $10M_\odot$. See Table 2.3.

The above empirical data from observational astronomy are consistent with our theoretical formula for the luminosity of stars L (2.140), if the stellar substance of the Hilbert core (in which stellar energy is released) has a density as shown in Table 2.4.

Observed mass-luminosity relation $\bar{L}=\bar{M}^x$	Scale of the stellar masses, in fractions of the Sun's mass $M_\odot$
$\bar{L} = \bar{M}^{2.6}$	$\bar{M} = 0.2 \ldots 0.5$
$\bar{L} = \bar{M}^{4.5}$	$\bar{M} = 0.5 \ldots 2$
$\bar{L} = \bar{M}^{3.6}$	$\bar{M} = 2 \ldots 10$
$\bar{L} = \bar{M}$	$\bar{M} > 10$

Table 2.3: The observed mass-luminosity relation $\bar{L}=\bar{M}^x$.

Density of the Hilbert core $\bar{\rho}_g$	Scale of the stellar masses, in fractions of the Sun's mass $M_\odot$
$\bar{\rho}_g = \bar{M}^{0.4}$	$\bar{M} = 0.2 \ldots 0.5$
$\bar{\rho}_g = \bar{M}^{1.5}$	$\bar{M} = 0.5 \ldots 2$
$\bar{\rho}_g = \bar{M}^{0.6}$	$\bar{M} = 2 \ldots 10$
$\bar{\rho}_g = \bar{M}^{-2}$	$\bar{M} > 10$

Table 2.4: The density of stellar substance inside the Hilbert core.

Object	Mass $\bar{M}$	Density $\bar{\rho}_0$	Ratio $\bar{\rho}_g/\bar{\rho}_0$
Betelgeuse (red super-giant)	20	2.0×10^{-8}	1.3×10^{9}
Rigel (white super-giant)	17	5.2×10^{-5}	6.7×10^{7}
Jupiter (proto-star)	9.5×10^{-4}	0.9	0.069
Red dwarfs	0.34	9	0.072
Brown dwarf (Corot-Exo-3)	0.021	21	0.010
White dwarf (Sirius B)	1	1.3×10^{6}	7.7×10^{-7}

Table 2.5: The ratio $\bar{\rho}_g/\bar{\rho}_0$ for some typical stars.

Based on the function $\bar{\rho}_g = \bar{M}^y$ according to Table 2.4, we can find out how dense the Hilbert core of a star is compared to the main substance of the star (known from astronomical observations). Thus, we calculate the following ratio for stars

$$\frac{\bar{\rho}_g}{\bar{\rho}_0} = \frac{\bar{M}^y}{\bar{\rho}_0}\,. \tag{2.141}$$

The calculation results are shown in Table 2.5. Based on the calculated ratio $\bar{\rho}_g/\bar{\rho}_0$ shown in Table 2.5, we arrive at the following conclusion. The luminous Hilbert core of a star — its "inner sun" — can have a density different from that of the main substance of the star. It depends on the particular type of star. For example, the Hilbert core of a giant or supergiant is many orders of magnitude denser than the main substance of these stars. The Hilbert core of a star similar to the Sun has about the same density as the star itself. As for dwarf stars, the Hilbert core of such a star is more rarefied than the main substance of the star. The greater the density of a dwarf star, the lower the density of its core compared to the density of the entire star. In a star such as a white dwarf, the Hilbert core is many orders of magnitude more rarefied than the main substance of the star.

Accordingly, the following question arises. All physical bodies have masses, so every body has a core of the Hilbert radius. Not only stars, but also planets and even individual elementary particles should have such a core. But why do not they shine like stars?

The answer comes from the state of the substance of which these physical bodies are composed. Stars are made up of liquid substance, consisting mainly of light chemical elements such as hydrogen and helium. Therefore, thermonuclear fusion of such light atomic nuclei is possible in the Hilbert core of every star. Due to the fact that stellar substance is liquid, more and more "nuclear fuel" is delivered to the luminous Hilbert core of a star from its other regions, thereby supporting combustion inside the "nuclear boiler", until the moment when all the nuclear fuel of the star runs out. Another thing are planets. They consist mainly of heavy elements with negligible hydrogen content. Therefore, as soon as the "nuclear boiler" of the Hilbert core has used up the entire supply of hydrogen fuel in the central region of a planet, it ceases to produce energy, but continues to exist in the centre of the planet, in a latent state.

Astronomers know that the energy radiated by Jupiter exceeds the solar energy absorbed by the entire surface of this planet. The same is true for Saturn. This means, according to our theory, that the Hilbert core of each of these planets is still converting hydrogen into helium, thereby releasing nuclear energy.

Concerning individual elementary particles, such as protons, neurons and electrons: as you know, they are stable and indifferent for a long time until they interact with other particles. In fact, this means that the Hilbert core of the proton (as well as the neutron and the electron) does not interact with the main mass of the particle. Why is this happening? One can only guess that either the substance inside the particles is in a super-solid state, or there is a layer of very strong vacuum between the nucleus and the rest of the particle's mass. On the other hand, the Hilbert core of the proton (and the core of the neutron) has a tiny radius $(r_g)_p = \frac{2Gm_p}{c^2} = 2.48 \times 10^{-52}$ cm, while the Hilbert core of the electron has an even smaller radius $(r_g)_e = \frac{2Gm_e}{c^2} = 1.35 \times 10^{-55}$ cm. As noted by Albert Einstein, the geometric laws (space-time geometry) of General Relativity are probably true up to the scale of elementary particles. On a subnuclear scale, another geometry may work, asserting its own laws, different from the laws of General Relativity. Therefore, we cannot now say something definite about the physical conditions and processes inside elementary particles.

But as for the world of ordinary stars and planets, experimental physics and observational astronomy show that Einstein's theory is correct and works on these scales with high accuracy. Therefore, all our conclusions about the internal constitution of stars and about the mechanism of energy production in stars must be taken into account.

The specific details of the proposed mechanism of stellar energy is a separate topic that is outside the scope of this book (which is mainly about the internal constitution of stars).

2.7 Conclusion

All theoretical conclusions about the source of stellar energy and about the internal constitution of stars presented in this Chapter were obtained in the framework of our model of liquid stars. Our model is based on the concept of stars as space-time objects according to General Relativity. Below we list the most important conclusions that we thus arrived at:

1. The field of each star has an outer space breaking on a spherical surface around the star. The "bubble" of the outer space breaking in the field of each star has a radius of
$$r_{br} = \sqrt{\frac{3}{\varkappa\rho_0}} = \sqrt{\frac{a^3}{r_g}}\,, \tag{2.142}$$
which exceeds the physical radius a of the star by many orders of magnitude. The observable three-dimensional space curvature C_{iklj} and the four-dimensional Riemannian curvature $R_{\alpha\beta\gamma\delta}$ have a common space breaking on this surface. The outer space breaking prevents the formation of nearby substance into a planet in this orbit. The outer space breaking in the Sun's field is located in the asteroid belt, near the maximum concentration of asteroids;
2. The field of each star has an inner space breaking, deep inside the physical body of the star, on a surface of the Hilbert radius
$$r_g = \frac{2GM}{c^2} \tag{2.143}$$
from its centre. This means that there is a small core separated by a singular surface from the main substance of the star. On the surface of this core, the force of gravity reaches an infinitely large value. By definition, the super-strong force of gravity is sufficient to transfer the necessary kinetic energy to the light atomic nuclei of stellar substance in order for thermonuclear fusion to begin. Thus, nuclear energy is released. Liquid "nuclear fuel" is delivered from other regions of the star to the core, maintaining combustion inside this "nuclear boiler";
3. Every star has a mass. Therefore, a luminous core of the Hilbert radius $r_g = \frac{2GM}{c^2}$ — an "inner sun" — exists at the centre of every star. We call it the *Hilbert core*. This is where thermonuclear fusion produces helium from hydrogen, thus making stars luminous. The energy is then transferred from the "inner sun" of the star to its surface by thermal conduction (the usual heat transfer in liquids) to then be radiated into the cosmos;
4. The Hilbert core is tiny compared to the size of stars. For example, for the Sun, $r_g = 2.9$ km;
5. The observed mass-luminosity relation of stars is satisfied if the density of the Hilbert core depends on the particular type of star.

The Hilbert core of a giant or supergiant must be many orders of magnitude denser than the main substance of these stars. The Hilbert core of a star like the Sun should be about the same density as the star itself. In a dwarf star, the Hilbert core must be more rarefied than the main substance of the star (the core of a white dwarf must be extremely rarefied);

6. Every planet has a mass. Therefore, the Hilbert core exists at the centre of every planet. But planets are made up mostly of heavy elements with a small amount of hydrogen. As soon as the "nuclear boiler" in the Hilbert core of a planet uses up the entire supply of hydrogen fuel in its central region, the "nuclear boiler" will cease to produce energy, but will still exist in the centre of the planet, in a latent state.

Chapter 3 Description of Ordinary Stars

3.1 Problem statement. The internal space metric of an ordinary non-rotating star

To understand the description of an ordinary star, recall that in §2.1 we reproduced the derivation of the true space metric of a liquid sphere, originally obtained by L. Borissova [11, 12], following the "historical path" as Schwarzschild did it. Namely, — we considered the metric of a spherically symmetric space in a general form, then applied the particular conditions characteristic of a sphere filled with an ideal liquid. The only difference from Schwarzschild's derivation was that we did not assume any artificial limitations. When following this derivation, we obtained the observable characteristics of the space in the implicit form, as an auxiliary result. Then, using the obtained results, we have deduced the space metric of a liquid sphere in the final form.

Now, we express the observable characteristics of the space in the explicit form, through the components of the fundamental metric tensor of the metric obtained in Chapter 2. So, the true space metric of a liquid sphere obtained by L. Borissova [11, 12] has the form (1.8)

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1-\frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \qquad (3.1)$$

Let us calculate the chr.inv.-characteristics of the space, according to their definitions given in §1.3 and taking into account the respective components of the fundamental metric tensor according to the space metric (3.1).

The chr.inv.-metric tensor h_{ik} of the metric (3.1) has the following non-zero components

$$h_{11} = \frac{1}{1 - \frac{r^2 r_g}{a^3}}\,, \qquad h_{22} = r^2, \qquad h_{33} = r^2 \sin^2\theta, \tag{3.2}$$

$$h^{11} = 1 - \frac{r^2 r_g}{a^3}\,, \qquad h^{22} = \frac{1}{r^2}\,, \qquad h^{33} = \frac{1}{r^2 \sin^2\theta}\,, \tag{3.3}$$

and, hence, its determinant $h = \det\|h_{ik}\|$ and the non-zero spatial derivatives of $\ln\sqrt{h}$ have the form

$$h = \det\|h_{ik}\| = \frac{r^4 \sin^2\theta}{1 - \frac{r^2 r_g}{a^3}}\,, \tag{3.4}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial r} = \frac{2}{r} + \frac{r_g r}{a^3}\frac{1}{1 - \frac{r^2 r_g}{a^3}}\,, \qquad \frac{{}^*\partial \ln\sqrt{h}}{\partial \theta} = \cot\theta. \tag{3.5}$$

After some algebra according to the chronometrically invariant formalism (see §1.3), we obtain the following. The chr.inv.-vector of the gravitational inertial force acting in the space has the form

$$F_1 = -\frac{c^2 r_g}{a^3}\frac{r}{\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}\right)\sqrt{1 - \frac{r_g r^2}{a^3}}}\,, \tag{3.6}$$

$$F^1 = -\frac{c^2 r_g}{a^3}\frac{r\sqrt{1 - \frac{r_g r^2}{a^3}}}{3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}}\,, \tag{3.7}$$

where $r < a$ since all of this is inside the sphere. Therefore, $F_1 < 0$, i.e., this is a force of attraction.

Calculate the non-zero components of the chr.inv.-Christoffel symbols. After some algebra, we obtain

$$\Delta^1_{11} = \frac{r_g r}{a^3}\frac{1}{1 - \frac{r_g r^2}{a^3}}\,, \qquad \Delta^1_{22} = -\frac{\Delta^1_{33}}{\sin^2\theta} = -r\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.8}$$

$$\Delta^2_{12} = \Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^2_{33} = -\sin\theta\cos\theta, \qquad \Delta^3_{23} = \cot\theta. \tag{3.9}$$

Based on the above, we calculate the non-zero components of the chr.inv.-tensor of the observable three-dimensional curvature C_{iklj} and of its contraction C_{ik}. We obtain

$$C_{1212} = \frac{C_{1313}}{\sin^2\theta} = -\frac{r_g r^2}{a^3}\frac{1}{1-\frac{r_g r^2}{a^3}}\,, \quad C_{2323} = -\frac{r_g r^4}{a^3}\sin^2\theta, \quad (3.10)$$

$$C_{11} = -\frac{2r_g}{a^3}\frac{1}{1-\frac{r_g r^2}{a^3}}\,, \qquad C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2r_g r^2}{a^3}\,. \qquad (3.11)$$

So, with the obtained physically observable chr.inv.characteristics of the internal space of a liquid sphere, we now have everything we need to consider Einstein's equations in the internal field of an ordinary non-rotating star.

3.2 Einstein's equations in the internal field of an ordinary non-rotating star

Let us consider Einstein's field equations in the internal space of a liquid sphere, the metric of which is (3.1).

As is known, the energy-momentum tensor of an ideal liquid has the following form (2.4)

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p}{c^2}\right)U^\alpha U^\beta - \frac{p}{c^2}\,g^{\alpha\beta}, \qquad (3.12)$$

where $\rho_0 = const$ is the density of the liquid, p is the pressure inside the liquid, and U^α is the four-dimensional velocity of the liquid flow with respect to the observer (the U^α is a unit four-dimensional vector, therefore $U_\alpha U^\alpha = 1$). The chr.inv.-projections of the energy-momentum tensor have the form (2.21)

$$\rho = \frac{T_{00}}{g_{00}} = \rho_0\,, \quad J^i = \frac{cT_0^i}{\sqrt{g_{00}}} = 0\,, \quad U^{ik} = c^2T^{ik} = ph^{ik}, \quad (3.13)$$

where ρ is the observable mass density, J^i is the observable momentum density, and U^{ik} is the observable stress tensor.

Using the above formulae and taking into account the fact that the space of the liquid sphere under consideration does not rotate or deform ($A_{ik} = 0$, $D_{ik} = 0$), we obtain the chr.inv.-Einstein equations (1.92–1.94) in the simplified form

$$ {}^*\nabla_j F^j - \frac{1}{c^2} F_j F^j = -\frac{\varkappa}{2}\left(\rho_0 c^2 + U\right), \tag{3.14} $$

$$ J^i = 0\,, \tag{3.15} $$

$$ \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2} F_i F_k - c^2 C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho_0 c^2 h_{ik} + 2U_{ik} - U h_{ik}\right), \tag{3.16} $$

where ${}^*\nabla_i$ is the chr.inv.-derivative symbol, $U_{ik} = p h_{ik}$ and $U = 3p$.

Substitute the formulae for F_i, C_{ik} and h_{ik} calculated for the metric (3.1) into the above Einstein field equations. We obtain that only two equations remain non-zero

$$ \frac{3c^2 r_g}{a^3}\,\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}} = \frac{\varkappa}{2}\left(\rho_0 c^2 + 3p\right), \tag{3.17} $$

$$ \frac{3c^2 r_g}{a^3}\,\frac{2\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}} = \frac{\varkappa}{2}\left(\rho_0 c^2 - p\right). \tag{3.18} $$

Multiplying (3.18) by 3 then summing up the product with (3.17), we obtain

$$ \varkappa\rho_0 c^2 = \frac{3c^2 r_g}{a^3}\,. \tag{3.19} $$

Substituting this result back into (3.18), we obtain the *equation of state** for the liquid substance of ordinary stars

$$ p = \rho_0 c^2\,\frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,. \tag{3.20} $$

This formula completely coincides with the formula for the pressure p (2.130), which we have obtained in Chapter 2 as a result of following Schwarzschild's derivation.

*The formula connecting pressure and density inside the medium.

The above formula for the pressure p can also be obtained from the conservation equations (2.26, 2.27). Since the space metric (3.1) does not deform (this means that $h_{ik} \neq f(t)$ and, hence, $D_{ik} = 0$), the chr.inv.-scalar conservation equation (2.26) vanishes. Only the chr.inv.-vector conservation equation (2.27) remains non-zero

$$^*\nabla_i \left(p h^{ik}\right) - \left(\rho_0 + \frac{p}{c^2}\right) F^k = 0\,. \tag{3.21}$$

Here $^*\nabla_i\, h^{ik} = 0$ is true always for h^{ik}, as well as $\nabla_\sigma\, g^{\alpha\sigma} = 0$ for the fundamental metric tensor. Therefore and since the chr.inv.-derivation operator with respect to the spatial coordinates coincides with the ordinary spatial derivation operator in a space without rotation, the remaining conservation equation (3.21) takes the form

$$h^{ik} \frac{\partial p}{\partial x^i} - \left(\rho_0 + \frac{p}{c^2}\right) F^k = 0\,. \tag{3.22}$$

Substituting the formulae for h^{11} and F^1, which we have obtained for the metric (3.1), we transform (3.22) into the differential equation

$$\frac{dp}{\rho_0 c^2 + p} = -\frac{r_g}{a^3} \frac{r\,dr}{\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}\right)\sqrt{1 - \frac{r_g r^2}{a^3}}}\,. \tag{3.23}$$

This equation can be re-written in the form

$$d \ln\left(\rho_0 c^2 + p\right) = -d \ln\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}\right), \tag{3.24}$$

which is easy to integrate. After integration, we have

$$p + \rho_0 c^2 = \frac{Q}{3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}}\,, \tag{3.25}$$

where the integration constant Q can be obtained from the obvious condition $p = 0$ on the star's surface (where $r = a$). Then

$$Q = 2\rho_0 c^2 \sqrt{1 - \frac{r_g}{a}} \tag{3.26}$$

and, thus, we obtain the solution

$$p + \rho_0 c^2 = 2\rho_0 c^2 \frac{\sqrt{1 - \frac{r_g}{a}}}{3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}} \,. \tag{3.27}$$

It is easy to see that this solution leads to the same formula for p as (3.20) that we have obtained from the Einstein field equations.

3.3 The internal space metric of an ordinary rotating star

Let us now consider the metric of an ordinary liquid star (3.1) with the only difference that the star rotates with an angular velocity ω along its equatorial axis — the ϕ axis in the spherical coordinates r, θ, ϕ. In this case, the metric of a non-rotating liquid sphere (3.1) takes the form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r^2 r_g}{a^3}}\right)^2 c^2 dt^2 + \\ + \frac{2\omega r^2 \cos\theta}{c}\, c\,dt\, d\varphi - \frac{dr^2}{1 - \frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{3.28}$$

Note that we are still considering ordinary stars. This is what we call such stars, the Hilbert radius of which is much smaller than their physical radius.

According to the metric (3.28) that we have obtained for an ordinary rotating space, the linear velocity with which the space rotates is

$$v_1 = v_2 = 0\,, \qquad v_3 = -\frac{2\omega r^2 \cos\theta}{3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}} \,. \tag{3.29}$$

As is known from observational astronomy, most stars rotate with linear velocities $v < 420$ km/sec. Hence, we have $v^2/c^2 < 2\times 10^{-6}$: most stars rotate slowly compared to the velocity of light.

According to the space metric of an ordinary slowly rotating star (3.28), represented as a rotating liquid sphere, we have

$$v^2 = h^{ik} v_i v_k = h^{33} v_3 v_3\,, \qquad h^{33} = -g^{33} = \frac{1}{r^2 \sin^2\theta}\,. \tag{3.30}$$

Thus, v^2/c^2 in the space metric (3.28) has the form

$$\frac{v^2}{c^2} = \frac{4\omega^2 r^2 \cot^2\theta}{c^2\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)^2}\,. \tag{3.31}$$

Expanding the radicands of this formula into series, after elementary transformations we obtain

$$\frac{v^2}{c^2} = \frac{\omega^2 r^2 \cot^2\theta}{c^2}\left(1 + \frac{3r_g}{2a} - \frac{r_g r^2}{2a^3}\right). \tag{3.32}$$

Further, we will neglect the higher-order terms of the series, since they are small due to $r_g \ll a$ for ordinary stars. Therefore,

$$v_3 = \omega r^2 \cos\theta\,, \qquad \frac{v^2}{c^2} = \frac{v_3^2}{c^2} = \frac{\omega^2 r^4 \cos^2\theta}{c^2}\,. \tag{3.33}$$

The non-zero components of the chr.inv.-metric tensor h_{ik} of the metric (3.28) have the form

$$h_{11} = \frac{1}{h^{11}} = \frac{1}{1-\frac{r^2 r_g}{a^3}}\,, \quad h_{22} = \frac{1}{h^{22}} = r^2, \tag{3.34}$$

$$h_{33} = \frac{1}{h^{33}} = r^2\sin^2\theta\left(1 + \frac{\omega^2 r^2\cot^2\theta}{c^2}\right), \tag{3.35}$$

while the determinant $h = \det\|h_{ik}\|$ of the chr.inv.-metric tensor h_{ik} and the non-zero spatial derivatives of $\ln\sqrt{h}$ have the form

$$h = \det\|h_{ik}\| = \frac{r^4\sin^2\theta}{1-\frac{r^2 r_g}{a^3}}\left(1 + \frac{\omega^2 r^2\cot^2\theta}{c^2}\right), \tag{3.36}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial r} = \frac{2}{r} + \frac{r_g r}{a^3}\frac{1}{1-\frac{r^2 r_g}{a^3}}\,, \tag{3.37}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial \theta} = \cot\theta\left(1 - \frac{\omega^2 r^2}{c^2\sin^2\theta}\,\frac{1}{1+\frac{\omega^2 r^2\cot^2\theta}{c^2}}\right). \tag{3.38}$$

Following the chronometrically invariant formalism (see §1.3), we also obtain formulae for the other chr.inv.-characteristics of the space.

The chr.inv.-vector of the gravitational inertial force F_i acting in the space takes the form

$$F_1 = -\frac{c^2 r_g}{a^3}\frac{r}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}} < 0\,, \tag{3.39}$$

$$F^1 = -\frac{c^2 r_g}{a^3}\frac{r\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}} < 0\,, \tag{3.40}$$

which is a non-Newtonian force of attraction acting inside the liquid sphere (ordinary star). The approximate formula for the force is

$$F_1 = F^1 \approx -\frac{c^2 r_g r}{2a^3}\,. \tag{3.41}$$

The chr.inv.-tensor A_{ik} of the angular velocity with which the space rotates has the following non-zero components

$$A_{13} = \frac{2\omega r\cos\theta}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\times \tag{3.42}$$

$$\times\left[\frac{r_g r^2}{a^3}\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}-1\right], \tag{3.43}$$

$$A^{13} = \frac{2\omega\left(1-\frac{r_g r^2}{a^3}\right)\cot\theta}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)r\sin\theta\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)}\times \tag{3.44}$$

$$\times\left[\frac{r_g r^2}{a^3}\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}-1\right], \tag{3.45}$$

$$A_{23} = \frac{\omega r^2\sin\theta}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{3.46}$$

$$A^{23} = \frac{\omega}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right) r^2 \sin\theta \left(1+\frac{\omega^2 r^2 \cot^2\theta}{c^2}\right)}\,, \tag{3.47}$$

the approximate formulae for which have the form

$$A_{13} = -\omega r \cos\theta \left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{a^3}\right), \tag{3.48}$$

$$A^{13} = -\frac{\omega \cot\theta}{r \sin\theta} \left(1 + \frac{3r_g}{4a} - \frac{2r_g r^2}{a^3} - \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right), \tag{3.49}$$

$$A_{23} = \frac{\omega r^2 \sin\theta}{2} \left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{4a^3}\right), \tag{3.50}$$

$$A^{23} = \frac{\omega}{2r^2 \sin\theta} \left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{4a^3} - \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right). \tag{3.51}$$

The non-zero chr.inv.-Christoffel symbols for the metric, with the high-order term $\omega^4 r^4/c^4$ withheld, have the form

$$\Delta^1_{11} = \frac{r_g r}{a^3} \frac{1}{1-\frac{r_g r^2}{a^3}}\,, \qquad \Delta^1_{22} = -r\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.52}$$

$$\Delta^1_{33} = -r \sin^2\theta \left(1 + \frac{2\omega^2 r^2 \cot^2\theta}{c^2}\right)\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.53}$$

$$\Delta^2_{12} = \frac{1}{r}\,, \qquad \Delta^3_{13} = \frac{1}{r}\left(1 + \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right), \tag{3.54}$$

$$\Delta^2_{33} = -\sin\theta\cos\theta\left(1 - \frac{\omega^2 r^2}{c^2}\right), \quad \Delta^3_{23} = \cot\theta\left(1 - \frac{\omega^2 r^2}{c^2 \sin^2\theta}\right). \tag{3.55}$$

For these components, neglecting r_g^2/a^2 and the product of $\omega^2 r^2/c^2$ by r_g/a, we obtain the approximate formulae

$$\Delta^1_{11} = \frac{r_g r}{a^3}\,, \qquad \Delta^1_{22} = -r\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.56}$$

$$\Delta^1_{33} = -r \sin^2\theta \left(1 + \frac{2\omega^2 r^2 \cot^2\theta}{c^2} - \frac{r_g r^2}{a^3}\right), \tag{3.57}$$

$$\Delta_{12}^{2}=\frac{1}{r}\,,\qquad \Delta_{13}^{3}=\frac{1}{r}\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right), \tag{3.58}$$

$$\Delta_{33}^{2}=-\sin\theta\cos\theta\left(1-\frac{\omega^2 r^2}{c^2}\right),\quad \Delta_{23}^{3}=\cot\theta\left(1-\frac{\omega^2 r^2}{c^2\sin^2\theta}\right), \tag{3.59}$$

as well as the non-zero components of the chr.inv.-curvature tensor C_{iklj} along with the non-zero components of its contraction C_{ik}

$$C_{1212}=-\frac{r_g r^2}{a^3}\,\frac{1}{1-\frac{r_g r^2}{a^3}}\,, \tag{3.60}$$

$$C_{1313}=r^2\sin^2\theta\left(\frac{3\,\omega^2\cot^2\theta}{c^2}-\frac{r_g}{a^3}\,\frac{1+\frac{\omega^2 r^2\cot^2\theta}{c^2}}{1-\frac{r_g r^2}{a^3}}\right), \tag{3.61}$$

$$C_{2323}=\left[-\frac{r_g r^2}{a^3}\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)+\right.$$
$$\left.+\,\frac{\omega^2 r^2}{c^2}\left(\cot^2\theta+\frac{1}{\sin^4\theta}\right)\right]r^2\sin^2\theta, \tag{3.62}$$

$$C_{11}=-\frac{2r_g}{a^3}\,\frac{1}{1-\frac{r_g r^2}{a^3}}+\frac{3\,\omega^2\cot^2\theta}{c^2}\,, \tag{3.63}$$

$$C_{22}=-\frac{2r_g r^2}{a^3}+\frac{\omega^2 r^2\cot^2\theta}{c^2}\left(\cot^2\theta+\frac{1}{\sin^4\theta}\right), \tag{3.64}$$

$$C_{33}=\left[-\frac{2r_g}{a^3}\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)+\left(1-\frac{r_g r^2}{a^3}\right)\times\right.$$
$$\left.\times\,\frac{3\,\omega^2\cot^2\theta}{c^2}+\frac{\omega^2}{c^2}\left(\cot^2\theta+\frac{1}{\sin^4\theta}\right)\right]r^2\sin^2\theta. \tag{3.65}$$

Since $r_g/a \ll 1$ and $\omega^2 r^2/c^2 \ll 1$ for ordinary stars, we neglect r_g^2/a^2 and the product of $\omega^2 r^2/c^2$ by r_g/a. As a result, we obtain

$$C_{1212}=-\frac{r_g r^2}{a^3}\,,\quad C_{1313}=r^2\sin^2\theta\left(\frac{3\,\omega^2\cot^2\theta}{c^2}-\frac{r_g}{a^3}\right), \tag{3.66}$$

$$C_{2323}=\left[-\frac{r_g r^2}{a^3}+\frac{\omega^2 r^2}{c^2}\left(\cot^2\theta+\frac{1}{\sin^4\theta}\right)\right]r^2\sin^2\theta, \tag{3.67}$$

$$C_{11} = -\frac{2r_g}{a^3} + \frac{3\omega^2 \cot^2\theta}{c^2}\,, \tag{3.68}$$

$$C_{22} = -\frac{2r_g r^2}{a^3} + \frac{\omega^2 r^2 \cot^2\theta}{c^2}\left(\cot^2\theta + \frac{1}{\sin^4\theta}\right), \tag{3.69}$$

$$C_{33} = \left(-\frac{2r_g}{a^3} + \frac{4\omega^2 \cot^2\theta}{c^2} + \frac{\omega^2}{c^2 \sin^4\theta}\right) r^2 \sin^2\theta. \tag{3.70}$$

3.4 Einstein's equations in the internal field of an ordinary rotating star

Let us now solve Einstein's field equations in the internal space of a rotating ordinary star, i.e., in accordance with the space metric (3.28). In the absence of rotation ($A_{ik} = 0$), this problem was considered earlier in §3.2 for a non-rotating ordinary star.

Consider the chr.inv.-Einstein equations (1.92–1.94) in the space of a liquid sphere that rotates ($A_{ik} \neq 0$), but does not deform ($D_{ik} = 0$). In this case, the mentioned chr.inv.-Einstein equations take the form

$$A_{jl}A^{lj} + {}^*\nabla_j F^j - \frac{1}{c^2} F_j F^j = -\frac{\varkappa}{2}\left(\rho c^2 + U\right), \tag{3.71}$$

$${}^*\nabla_j A^{ij} - \frac{2}{c^2} F_j A^{ij} = -\varkappa J^i, \tag{3.72}$$

$$2A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2} F_i F_k - c^2 C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right), \tag{3.73}$$

where ${}^*\nabla_i$ is the chr.inv.-derivative symbol, while ρ, J^i and U^{ik} are the physically observable chr.inv.-projections of the energy-momentum tensor $T_{\alpha\beta}$ of the liquid that fills the space.

With the obtained components of A_{ik} and F_i (see §3.2 for detail), the chr.inv.-Einstein equations (3.71–3.73) take the form

$$2\omega^2 \cot^2\theta\left(1 + \frac{3r_g}{2a} - \frac{3r_g r^2}{a^3} - \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right) + \\ + \frac{\omega^2}{2}\left(1 + \frac{3r_g}{2a} - \frac{r_g r^2}{2a^3} - \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right) + \frac{3c^2 r_g}{2a^3} = \\ = \frac{\varkappa}{2}\left(\rho c^2 + U\right), \tag{3.74}$$

$$\frac{\omega\cot\theta}{r^2\sin\theta}\left(1+\frac{3r_g}{4a}-\frac{4r_g r^2}{a^3}-\frac{3\omega^2r^2\cot^2\theta}{c^2}\right)=-\varkappa J^3, \tag{3.75}$$

$$2\omega^2\cot^2\theta\left(1+\frac{3r_g}{2a}-\frac{2r_g r^2}{a^3}-\frac{\omega^2r^2\cot^2\theta}{c^2}\right)+\frac{3c^2r_g}{2a^3}= \\ =\frac{\varkappa}{2}\left[\left(\rho c^2-U\right)\left(1-\frac{r_g r^2}{a^3}\right)+2U_{11}\right], \tag{3.76}$$

$$\omega^2 r\cot\theta\left(1+\frac{3r_g}{2a}-\frac{5r_g r^2}{4a^3}-\frac{\omega^2r^2\cot^2\theta}{c^2}\right)=-\varkappa U_{12}\,, \tag{3.77}$$

$$\frac{\omega^2r^2}{2}\left(1+\frac{3r_g}{2a}-\frac{r_g r^2}{2a^3}-\frac{\omega^2r^2\cot^2\theta}{c^2}\right)+\frac{3c^2r_g r^2}{2a^3}= \\ =\frac{\varkappa}{2}\left[\left(\rho c^2-U\right)r^2+2U_{22}\right], \tag{3.78}$$

$$\left[2\omega^2\cot^2\theta\left(1+\frac{3r_g}{2a}-\frac{3r_g r^2}{a^3}\right)+\frac{\omega^2}{2}\left(1+\frac{3r_g}{2a}-\frac{r_g r^2}{2a^3}\right)\right]\times \\ \times r^2\sin^2\theta+\frac{3c^2r_g r^2\sin^2\theta}{a^3}= \\ =\frac{\varkappa}{2}\left[\left(\rho c^2-U\right)r^2\sin^2\theta+2U_{33}\right]. \tag{3.79}$$

In the framework of our approximation ($r_g/a \ll 1$ and $\omega^2r^2/c^2 \ll 1$) specific for ordinary stars, we neglect r_g^2/a^2 and the product of ω^2r^2/c^2 by r_g/a. Thus, the chr.inv.-Einstein equations become simplified

$$2\omega^2\cot^2\theta+\frac{\omega^2}{2}=\frac{\varkappa}{2}\left(\rho c^2+U\right), \tag{3.80}$$

$$\frac{\omega\cot\theta}{r^2\sin\theta}=-\varkappa J^3, \tag{3.81}$$

$$2\omega^2\cot^2\theta-\frac{\varkappa}{2}\left(\rho c^2-U\right)=\varkappa U_1^1\,, \tag{3.82}$$

$$\omega^2 r\cot\theta=-\varkappa U_{12}\,, \tag{3.83}$$

$$\frac{\omega^2}{2}-\frac{\varkappa}{2}\left(\rho c^2-U\right)=\varkappa U_2^2\,, \tag{3.84}$$

$$2\omega^2\cot^2\theta+\frac{\omega^2}{2}-\frac{\varkappa}{2}\left(\rho c^2-U\right)=\varkappa U_3^3\,. \tag{3.85}$$

Summing up (3.82), (3.84), (3.85) and taking $U^1_1 + U^2_2 + U^3_3 = U$ into account, we have

$$4\omega^2 \cot^2\theta + \omega^2 = \frac{\varkappa}{2}\left(3\rho c^2 - U\right). \tag{3.86}$$

Summing up the above result (3.86) and (3.80), we obtain

$$3\omega^2 \cot^2\theta + \frac{3\omega^2}{4} = \varkappa\rho c^2. \tag{3.87}$$

Multiplying (3.80) by 3, then subtracting it from (3.86), we obtain

$$\omega^2 \cot^2\theta + \frac{\omega^2}{4} = \varkappa U. \tag{3.88}$$

As is seen from (3.87) and (3.88), we have

$$\rho c^2 = 3U. \tag{3.89}$$

The energy-momentum tensor $T_{\alpha\beta}$ must satisfy the chr.inv.-conservation equations (1.96, 1.97). In a space that does not deform they are

$$\frac{^*\partial\rho}{\partial t} + {}^*\nabla_i J^i - \frac{2}{c^2} F_i J^i = 0\,, \tag{3.90}$$

$$\frac{^*\partial J^k}{\partial t} + 2A_{i\cdot}^{\cdot k} J^i + {}^*\nabla_i U^{ik} - \frac{1}{c^2} F_i U^{ik} - \rho F^k = 0\,. \tag{3.91}$$

As follows from the chr.inv.-scalar conservation equation (3.90),

$$\frac{^*\partial\rho}{\partial t} = 0 \quad\Longrightarrow\quad \rho = const. \tag{3.92}$$

The vector chr.inv.-conservation equation (3.91) with the index $i = 3$ is satisfied identically. The equations with $i = 1$ and $i = 2$ take the form

$$2A_{3\cdot}^{\cdot 1} J^3 + \frac{\partial U^{11}}{\partial r} + \frac{\partial U^{12}}{\partial\theta} + \frac{\partial \ln\sqrt{h}}{\partial\theta} U^{12} + \Delta^1_{22} U^{22} + \\ + \Delta^1_{33} U^{33} + \left(\Delta^1_{11} + \frac{\partial \ln\sqrt{h}}{\partial r} - \frac{1}{c^2} F_1\right) U^{11} = \rho F^1, \tag{3.93}$$

$$2A_{3\cdot}^{\cdot 2} J^3 + \frac{\partial U^{12}}{\partial r} + \frac{\partial U^{22}}{\partial\theta} + \frac{\partial \ln\sqrt{h}}{\partial\theta} U^{22} + \\ + \Delta^2_{33} U^{33} + \left(2\Delta^2_{12} + \frac{\partial \ln\sqrt{h}}{\partial r} - \frac{1}{c^2} F_1\right) U^{12} = 0\,. \tag{3.94}$$

In the tensor chr.inv.-Einstein equations (3.82–3.85), we take into account that $\rho c^2 = 3U$ (3.89) and the formula for U (3.88). We obtain

$$\varkappa U^{11} = \omega^2 \cot^2\theta - \frac{\omega^2}{4}\,, \tag{3.95}$$

$$\varkappa U^{12} = -\frac{\omega^2 \cot\theta}{r}\,, \tag{3.96}$$

$$\varkappa U^{22} = \frac{\omega^2}{4r^2} - \frac{\omega^2 \cot^2\theta}{r^2}\,, \tag{3.97}$$

$$\varkappa U^{33} = \frac{\omega^2 \cot^2\theta}{r^2 \sin^2\theta} + \frac{\omega^2}{4r^2 \sin^2\theta}\,. \tag{3.98}$$

Substitute the above formulae and the other required quantities into the remaining conservation equations (3.93) and (3.94). After some algebra, we see that these equations are satisfied identically.

So, we have obtained that the Einstein field equations and the conservation equations satisfy the internal space metric of an ordinary rotating star, i.e., the space metric (3.28).

3.5 The stationary vortex-free electromagnetic field of an ordinary rotating star

A real star has its own electromagnetic field. Therefore, we must introduce an electromagnetic field into the theory of liquid stars. Electrodynamics in terms of the chronometrically invariant formalism was presented in Chapter 3 of our book [18]. We are following Chapter 3 of [18] in order to apply it to our theory of liquid stars.

So, as is known from the general covariant formulation of electrodynamics [20], the energy-momentum tensor of an arbitrary electromagnetic field has the form

$$T^{\alpha\beta}_{\text{em}} = \frac{1}{4\pi c^2}\left(-F^{\alpha\cdot}_{\cdot\sigma}F^{\beta\sigma} + \frac{1}{4}\,g^{\alpha\beta}F_{\mu\sigma}F^{\mu\sigma}\right), \tag{3.99}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor known also as the Maxwell tensor. The field tensor $F_{\alpha\beta}$ is defined as a curl of the four-dimensional electromagnetic field potential A^{α}, i.e.

$$F_{\alpha\beta} = \nabla_{\alpha}A_{\beta} - \nabla_{\beta}A_{\alpha} = \frac{\partial A_{\beta}}{\partial x^{\alpha}} - \frac{\partial A_{\alpha}}{\partial x^{\beta}}\,. \tag{3.100}$$

The physically observable projections of the four-dimensional electromagnetic potential A^α are the chr.inv.-scalar electromagnetic potential φ and the chr.inv.-vector electromagnetic potential q^i

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i. \tag{3.101}$$

The electromagnetic field tensor $F_{\alpha\beta}$ (5.44) has the following physically observable projections

$$\rho_{\text{em}} = \frac{T_{00}}{g_{00}} = \frac{E_i E^i + H_{*i} H^{*i}}{8\pi c^2}\,, \tag{3.102}$$

$$J^i_{\text{em}} = \frac{c T^i_0}{\sqrt{g_{00}}} = \frac{1}{4\pi c}\,\varepsilon^{ikm} E_k H_{*m}\,, \tag{3.103}$$

$$U^{ik}_{\text{em}} = c^2 T^{ik} = \rho_{\text{em}} c^2 h^{ik} - \frac{1}{4\pi}\left(E^i E^k + H^{*i} H^{*k}\right), \tag{3.104}$$

where E^i is the three-dimensional chr.inv.-electric field strength vector, H^{*i} is the three-dimensional chr.inv.-magnetic field strength pseudovector, and ε^{imn} is the unit completely antisymmetric three-dimensional chr.inv.-pseudotensor [18]

$$\left.\begin{aligned} E^{*ik} &= -\varepsilon^{ikn} E_n, \qquad E_n = \frac{^*\partial\varphi}{\partial x^n} + \frac{1}{c}\frac{^*\partial q_n}{\partial t} - \frac{\varphi}{c^2} F_n \\ H^{*i} &= \frac{1}{2}\,\varepsilon^{imn} H_{mn}, \quad H_{mn} = \frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c} A_{mn} \end{aligned}\right\}. \tag{3.105}$$

As is seen from the definitions (3.105), the chr.inv.-electric strength and the chr.inv.-magnetic strength depend on not only the electromagnetic field potentials φ and q^i, but also on the characteristics of the field's home space. These are the gravitational inertial force F_i acting in the space and the angular velocity A_{ik} with which the space rotates.

Assume that the scalar and vector potentials of the electromagnetic field are stationary and homogeneously distributed, i.e., the electromagnetic field under consideration is stationary and vortex-free

$$\left.\begin{aligned} &\frac{^*\partial\varphi}{\partial t} = 0\,, \quad \frac{^*\partial\varphi}{\partial x^i} = 0 \\ &\frac{^*\partial q_i}{\partial t} = 0\,, \quad q_{ik} = \frac{^*\partial q_i}{\partial x^k} - \frac{^*\partial q_k}{\partial x^i} = 0 \end{aligned}\right\}. \tag{3.106}$$

In this case, we have

$$\left.\begin{aligned} E^i &= -\frac{\varphi}{c^2}\,F^i, & E_i &= -\frac{\varphi}{c^2}\,F_i \\ H^{*i} &= -\frac{2\varphi}{c}\,\Omega^{*i}, & H_{*i} &= -\frac{2\varphi}{c}\,\Omega_{*i} \end{aligned}\right\}, \tag{3.107}$$

where Ω^{*i} is the three-dimensional chr.inv.-pseudovector of the angular velocity with which the space rotates

$$\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn}\,, \qquad \Omega_{*i} = \frac{1}{2}\,\varepsilon_{imn}A^{mn}. \tag{3.108}$$

It is easy to find that in the internal space of an ordinary rotating star, i.e., according to the space metric (3.28), we have

$$\Omega^{*1} = \Omega_{*1} = \frac{\omega}{2}\,, \quad \Omega^{*2} = \frac{\omega\cot\theta}{r}\,, \quad \Omega_{*2} = \omega\, r\cot\theta\,, \tag{3.109}$$

$$\Omega_{*j}\,\Omega^{*j} = \omega^2\left(\frac{1}{4} + \cot^2\theta\right). \tag{3.110}$$

As is seen from the formulae (3.107), in the stationary vortex-free electromagnetic field of an ordinary star, the electric field strength E^i is determined by the scalar electromagnetic potential φ and the gravitational inertial force F^i acting in the space, and the magnetic field strength H^{*i} is determined by the scalar electromagnetic potential φ and the angular velocity Ω^{*i} with which the space rotates.

Using the formulae for E^i and H^{*i} (3.107), we obtain the chr.inv.-components (3.102–3.104) of the electromagnetic field tensor $F_{\alpha\beta}$

$$\rho_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right), \tag{3.111}$$

$$J^i_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm}F_k\,\Omega_{*m}\,, \tag{3.112}$$

$$U^{ik}_{\text{em}} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right)h^{ik} - \frac{\varphi^2}{\pi c^2}\left(\frac{F^i F^k}{4c^2} + \Omega^{*i}\Omega^{*k}\right), \tag{3.113}$$

and the trace $U_{\text{em}} = h_{ik}U^{ik}_{\text{em}}$ of the electromagnetic field stress tensor

$$U_{\text{em}} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right) = \rho_{\text{em}}c^2. \tag{3.114}$$

As follows from the general form of the energy-momentum tensor $T_{\alpha\beta}$ satisfying the metric (3.28), this tensor must satisfy the condition $\rho c^2 = 3U$ (3.89). This formula differs from $\rho_{\text{em}} c^2 = U_{\text{em}}$ (3.114) that we have obtained for a stationary vortex-free electromagnetic field. Therefore, we must find such an electromagnetic field structure that makes the U_{em} satisfying $\rho c^2 = 3U$.

As follows from $\rho c^2 = 3U$ (3.89) and $\omega^2 \cot^2\theta + \frac{1}{4}\,\omega^2 = \varkappa U$ (3.88) obtained from the chr.inv.-Einstein equations, inside an ordinary rotating star we should have the following conditions

$$\rho = \frac{3\,\Omega_{*j}\,\Omega^{*j}}{\varkappa c^2}\,, \qquad U = \frac{\Omega_{*j}\,\Omega^{*j}}{\varkappa}\,. \tag{3.115}$$

Therefore, we substitute the required condition

$$U_{\text{em}} = \frac{\Omega_{*j}\,\Omega^{*j}}{\varkappa} \tag{3.116}$$

into (3.114) that we have obtained for a stationary vortex-free electromagnetic field. We obtain

$$\frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right) = \frac{\Omega_{*j}\,\Omega^{*j}}{\varkappa}\,, \tag{3.117}$$

or, expanding Einstein's gravitational constant $\varkappa = \frac{8\pi G}{c^2}$, we obtain an equivalent form of the above, which is

$$c^2\,\Omega_{*j}\,\Omega^{*j} = \frac{\frac{G\varphi^2}{c^4}}{1 - \frac{4G\varphi^2}{c^4}}\,F_j F^j. \tag{3.118}$$

We are considering a stationary electromagnetic field. In this case, the scalar and vector electromagnetic potentials remain unchanged, i.e., $\varphi = const$ and $q_i = const$. Therefore,

$$\frac{G\varphi^2}{c^4} = n\,, \qquad n < \frac{1}{4} \tag{3.119}$$

in the formula (3.118) is a dimensionless constant coefficient depending only on the scalar electromagnetic field potential φ.

Using the constant n (3.119), we re-write (3.118) as

$$c^2\,\Omega_{*j}\,\Omega^{*j} = \frac{n}{1 - 4n}\,F_i F^i, \quad n < \frac{1}{4}\,. \tag{3.120}$$

Substituting $\Omega_{*j}\,\Omega^{*j}$ (3.110) and $F_i F^i$ (3.41) into the above condition (3.120), we obtain it in the alternative (expanded) form

$$\omega^2\left(1+4\cot^2\theta\right)=\frac{n}{1-4n}\,\frac{c^2 r_g^2 r^2}{4a^6}\,. \tag{3.121}$$

When $n=n_{\max}=\frac{1}{4}$ and, therefore, $\varphi=\varphi_{\max}$, the angular velocity of the star's rotation is $\omega=\infty$ that is nonsense. Hence, $n<\frac{1}{4}$ for all real stars, including the Sun. With $n<\frac{1}{4}$ we obtain the upper limit for the value of the scalar electromagnetic potential of a real star

$$\varphi=\frac{c^2}{2\sqrt{G}}<1.74\times10^{24}\ \text{gram}^{1/2}\,\text{cm}^{1/2}\,\text{sec}^{-1}. \tag{3.122}$$

As a result, we obtain that under the condition (3.120) a stationary vortex-free electromagnetic field satisfies the Einstein equations and the internal space metric of an ordinary rotating star. In other words, under the condition (3.120) a stationary rotating ordinary star is a *permanent magnet*.

Since $n=const$ in a stationary electromagnetic field, the above condition (3.120) allows us to express the electromagnetic field characteristics through the geometric and physical characteristics of the space. This means that in this particular case we can *geometrize* the electromagnetic field.

To do this, we substitute the obtained formulae for the gravitational inertial force and the angular velocity with which the space rotates into the physically observable components (3.111–3.113) of the electromagnetic field tensor $F_{\alpha\beta}$. Taking into account the relations (3.117) and (3.119), we obtain the observable components of the $F_{\alpha\beta}$ in the form

$$\rho_{\text{em}}=\frac{n}{2\pi G}\left[\frac{c^2 r_g^2 r^2}{16a^6}+\omega^2\left(\frac{1}{4}+\cot^2\theta\right)\right], \tag{3.123}$$

$$J^3_{\text{em}}=-\frac{nc^2}{4\pi G}\,\frac{\omega r_g}{a^3}\,\frac{\cot\theta}{\sin\theta}\,, \tag{3.124}$$

$$U^{11}_{\text{em}}=-\frac{nc^2}{2\pi G}\left[\frac{c^2 r_g^2 r^2}{16a^6}+\omega^2\left(\frac{1}{4}-\cot^2\theta\right)\right], \tag{3.125}$$

$$U^{12}_{\text{em}}=-\frac{nc^2}{2\pi G}\,\frac{\omega^2\cot\theta}{r}\,, \tag{3.126}$$

$$U^{22}_{\rm em} = \frac{nc^2}{2\pi G r^2}\left[\frac{c^2 r_g^2 r^2}{16a^6} + \omega^2\left(\frac{1}{4} - \cot^2\theta\right)\right], \tag{3.127}$$

$$U^{33}_{\rm em} = \frac{nc^2}{2\pi G r^2 \sin^2\theta}\left[\frac{c^2 r_g^2 r^2}{16a^6} + \omega^2\left(\frac{1}{4} + \cot^2\theta\right)\right]. \tag{3.128}$$

From the above formulae we obtain, as previously,

$$U_{\rm em} = h_{ik}U^{ik}_{\rm em} = \rho_{\rm em}c^2. \tag{3.129}$$

The chr.inv.-Einstein equations (3.123–3.128) can be simplified. In the surface layer of a star ($r \approx a$), the first term in the brackets is

$$\frac{c^2 r_g^2 r^2}{16a^6} \simeq \frac{c^2 r_g^2}{16a^4}\,. \tag{3.130}$$

Consider the Sun as an example. Its surface layer makes one full revolution with a period of $\simeq 27$ days, which is equivalent to the angular rotation velocity $\omega_\odot \simeq 2.7\times10^{-6}$ sec^{-1}. Therefore, the second term in the brackets of the above formulae is

$$\frac{1}{4}\,\omega_\odot^2 \simeq 1.8\times10^{-12}\ \text{sec}^{-2}. \tag{3.131}$$

The first term in the brackets, taking the Hilbert radius for the Sun $r_{g\odot} = 2.9\times10^5$ cm and the Sun's physical radius $a_\odot = 7.0\times10^{10}$ cm into account, is ten times smaller

$$\frac{c^2 r_{g\odot}^2}{16a_\odot^4} \simeq 2.0\times10^{-13}\ \text{sec}^{-2}. \tag{3.132}$$

Therefore, we neglect the first term in the brackets for even slowly rotating stars such as the Sun. As a result, the chr.inv.-Einstein equations (3.123–3.128) take the simplified form

$$\rho_{\rm em} = \frac{n\omega^2}{2\pi G}\left(\frac{1}{4} + \cot^2\theta\right), \tag{3.133}$$

$$J^3_{\rm em} = -\frac{nc^2}{4\pi G}\,\frac{\omega r_g}{a^3}\,\frac{\cot\theta}{\sin\theta}\,, \tag{3.134}$$

$$U^{11}_{\rm em} = -\frac{nc^2\omega^2}{2\pi G}\left(\frac{1}{4} - \cot^2\theta\right), \tag{3.135}$$

$$U^{12}_{\rm em} = -\frac{nc^2}{2\pi G}\frac{\omega^2\cot\theta}{r}\,, \tag{3.136}$$

$$U^{22}_{\rm em} = \frac{nc^2\omega^2}{2\pi G r^2}\left(\frac{1}{4}-\cot^2\theta\right), \tag{3.137}$$

$$U^{33}_{\rm em} = \frac{nc^2\omega^2}{2\pi G r^2\sin^2\theta}\left(\frac{1}{4}+\cot^2\theta\right). \tag{3.138}$$

3.6 Solving Maxwell's equations in the vortex-free electromagnetic field of an ordinary rotating star

As is known, the electromagnetic field is described by Maxwell's field equations. They consist of two groups. The general covariant formulation of Maxwell's equations has the form [20]

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c}\,j^\mu, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,, \tag{3.139}$$

where the first equation expresses the Group I, and the second equation expresses the Group II. Here $F^{*\mu\sigma} = \varepsilon^{\mu\sigma\alpha\beta}F_{\alpha\beta}$ is the pseudotensor dual to the electromagnetic field tensor $F_{\alpha\beta}$, and j^μ is the four-dimensional current vector of the electromagnetic field.

In terms of the chronometrically invariant formalism, the general covariant Maxwell equations (3.139) have the following form

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c}H^{ik}A_{ik} = 4\pi\rho \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2}F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + DE^i\right) = \frac{4\pi}{c}\,j^i \end{aligned}\right\}\ \text{I}, \tag{3.140}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c}E^{*ik}A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2}F_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right) = 0 \end{aligned}\right\}\ \text{II}, \tag{3.141}$$

see Chapter 3 of the book [18]. Here $E^{*ik} = -\varepsilon^{ikn}E_k$ is the pseudotensor dual to the electric strength vector E_i, $H^{*i} = \frac{1}{2}\,\varepsilon^{imn}H_{mn}$ is the pseudovector dual to the magnetic strength tensor H_{mn}. See their definitions in (3.105). The trace $D = h^{ik}D_{ik}$ of the space deformation tensor D_{ik} is the space deformation rate.

In the chr.inv.-Maxwell equations, the physically observable charge density ρ and the physically observable current vector j^i are the chr.inv.-projections of the four-dimensional current vector j^μ, i.e.

$$\rho = \frac{1}{c}\frac{j_0}{\sqrt{g_{00}}}\,, \qquad j^i = h^i_\mu\, j^\mu. \tag{3.142}$$

Since the space under consideration is stationary (the metric of a liquid sphere does not depend on time), and the electromagnetic field is also stationary, then the terms containing the space deformation tensor D_{ik} and the time derivatives of the electric E^i and magnetic H^{*i} field strengths vanish. In this particular case, the chr.inv.-Maxwell equations (3.140–3.141) take the simplified form

$$\left.\begin{aligned} {}^*\nabla_j E^j - \frac{1}{c}H^{ik}A_{ik} &= 4\pi\rho \\ {}^*\nabla_k H^{ik} - \frac{1}{c^2}F_k H^{ik} &= \frac{4\pi}{c}\,j^i \end{aligned}\right\}\ \text{I}, \tag{3.143}$$

$$\left.\begin{aligned} {}^*\nabla_i H^{*i} - \frac{1}{c}E^{*ik}A_{ik} &= 0 \\ {}^*\nabla_k E^{*ik} - \frac{1}{c^2}F_k E^{*ik} &= 0 \end{aligned}\right\}\ \text{II}. \tag{3.144}$$

Substitute the formulae for the gravitational inertial force F_i (3.41) and the angular velocity of the space A_{ik} (3.48–3.51), obtained for the space metric of an ordinary rotating star (3.28), and also the electric strength E^i and the magnetic strength H^{*i} (3.107) of a stationary vortex-free electromagnetic field.

Let us simplify the algebra. Since for the Sun we have $\omega_\odot \simeq 2.7\times 10^{-6}$ sec^{-1}, $r_{g\odot} = 2.9\times 10^5$ cm, $a_\odot = 7.0\times 10^{10}$ cm, then

$$\left.\begin{aligned} \frac{r_g}{a} &= 4.1\times 10^{-6} \\ \frac{r_g}{a^3} &= 8.5\times 10^{-28}\ \text{cm}^{-2} \\ \frac{\omega^2}{c^2} &= 8.1\times 10^{-33}\ \text{cm}^{-2} \end{aligned}\right\}. \tag{3.145}$$

For other ordinary stars, these terms take numerical values within several orders of magnitude of those given above. Therefore, the above terms can be omitted in the equations for ordinary stars.

After some algebra, the Group II of the chr.inv.-Maxwell equations (3.143–3.144) vanishes, and the Group I equations take the form

$$\frac{3\varphi r_g}{2a^3} = 4\pi\rho\,, \qquad \frac{\omega\varphi\cot\theta}{r^2\sin\theta} = -2\pi j^3. \tag{3.146}$$

The considered Maxwell equations characterize an electromagnetic field arising due to electric charges and currents — the electromagnetic field sources, which determine the right hand side terms of the Group I equations. These terms are the charge density ρ and the current vector j^i, which are the chr.inv.-projections of the four-dimensional current vector j^μ (3.142) of the field. If the right hand side of the equations were zero, then it would be an electromagnetic field without sources (existing independently of sources).

The electromagnetic field sources, which are the charge density ρ and the current vector j^i, must satisfy the general covariant law of conservation of electric charge

$$\nabla_\sigma j^\sigma = 0\,, \tag{3.147}$$

which is also known as the continuity equation. This law means that the four-dimensional current vector j^σ and, hence, its chr.inv.-projections ρ and j^i (the electromagnetic field sources) are conserved in the four-dimensional field volume.

The four-dimensional electromagnetic field potential A^σ must satisfy the general covariant Lorenz condition

$$\nabla_\sigma A^\sigma = 0\,, \tag{3.148}$$

which means that the four-dimensional field potential A^σ and, hence, its chr.inv.-projections φ and q^i, which are the chr.inv.-scalar and chr.inv.-vector potentials of the field, are conserved in the four-dimensional field volume.

In an arbitrary electromagnetic field, the general covariant conservation law (3.147) and the general covariant Lorenz condition (3.148) have the following chr.inv.-formulation

$$\frac{^*\partial\rho}{\partial t} + \rho D + {^*\widetilde{\nabla}}_i\, j^i - \frac{1}{c^2}F_i\, j^i = 0\,, \tag{3.149}$$

$$\frac{1}{c}\frac{^*\partial\varphi}{\partial t} + \frac{\varphi}{c}D + {^*\widetilde{\nabla}}_i q^i - \frac{1}{c^2}F_i q^i = 0\,, \tag{3.150}$$

where we denote ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2} F_i$. See Chapter 3 of the book [18].

Recall that, according to our initial assumption, the electromagnetic field under consideration is stationary and vortex-free. This means that the conditions (3.106) must be true both for the field potentials φ and q^i and for the field sources ρ and j^i. It is easy to see, in this case and since we assumed that the space does not deform ($D_{ik} = 0$), the chr.inv.-conservation equation (3.149) and the chr.inv.-Lorenz condition (3.150) are satisfied as identities.

3.7 Solving Maxwell's equations in the vortical electromagnetic field of an ordinary rotating star

Let us now consider an ordinary rotating star, the electromagnetic field of which is vortical. In this case, the curl q_{ik} of the three-dimensional chr.inv.-vector potential q_i of the field is non-zero

$$q_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} \neq 0\,. \tag{3.151}$$

As shown in Chapter 3 of the book [18], where we considered relativistic electrodynamics, the four-dimensional electromagnetic field potential A^α has the form

$$A^\alpha = \varphi\,\frac{dx^\alpha}{ds}\,, \qquad g_{\alpha\beta}\,\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1, \tag{3.152}$$

and the chr.inv.-projections of the A^α are

$$\frac{A_0}{\sqrt{g_{00}}} = \widetilde{\varphi}, \qquad A^i = q^i = \frac{\widetilde{\varphi}}{c}\,\mathrm{v}^i, \tag{3.153}$$

where $\widetilde{\varphi}$ is the chr.inv.-scalar relativistic potential of the field, which is dependent on the velocity of the field electric charges

$$\widetilde{\varphi} = \frac{\varphi}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,, \qquad \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \qquad \mathrm{v}^2 = h_{ik}\,\mathrm{v}^i\mathrm{v}^k. \tag{3.154}$$

Assume that electric charges travel inside the star along only the equatorial coordinate ϕ, which is the geographical longitude of the star (in the spherical coordinates r, θ, ϕ). Assume also that the charges travel with small velocities ($\mathrm{v}^2 \ll c^2$). In this case, $\widetilde{\varphi} = \varphi$. In addition, following our previous consideration of ordinary rotating stars, we assume that

$\varphi = const$ and $q^1 = q^2 = 0$. Then the chr.inv.-components of the A^α take the form

$$\varphi = const, \qquad q^3 = \frac{\varphi}{c}\,\mathrm{v}^3, \tag{3.155}$$

where $\mathrm{v}^3 = \frac{d\phi}{d\tau}$ is the physically observable velocity of the charges along the equatorial coordinate ϕ.

Assuming that the electromagnetic field curl is due only to the star's rotation, we have

$$\frac{d\phi}{d\tau} = \omega. \tag{3.156}$$

Therefore, the non-zero components of the electromagnetic field vector potential q_i and its curl q_{ik} have the form

$$q^3 = \frac{\varphi\omega}{c}, \qquad q_3 = \frac{\varphi\omega}{c}\,r^2\sin^2\theta, \tag{3.157}$$

$$q_{31} = \frac{\partial q_3}{\partial r} = \frac{2\varphi\omega}{c}\,r\sin^2\theta, \tag{3.158}$$

$$q_{23} = -\frac{\partial q_3}{\partial\theta} = -\frac{2\varphi\omega}{c}\,r^2\sin\theta\cos\theta. \tag{3.159}$$

Substitute the formulae for A_{ik} (3.48–3.51) obtained for the space metric of an ordinary rotating star (3.28) into the definition of the magnetic strength tensor H_{ik} (3.105). We obtain the non-zero components of the magnetic strength

$$H_{23} = -\frac{2\varphi\omega r^2\sin\theta}{c}\left[\cos\theta + \frac{1}{2}\left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{a^3}\right)\right], \tag{3.160}$$

$$H_{31} = \frac{2\varphi\omega r}{c}\left[\sin^2\theta - \cos\theta\left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{a^3}\right)\right]. \tag{3.161}$$

So forth, according to the definition of the magnetic strength pseudovector H^{*i} (3.105), we have its contravariant components

$$H^{*1} = -\frac{2\varphi\omega}{c}\left[\left(1 - \frac{r_g r^2}{2a^3}\right)\cos\theta + \frac{1}{2}\left(1 + \frac{3r_g}{4a} - \frac{3r_g r^2}{2a^3}\right)\right], \tag{3.162}$$

$$H^{*2} = \frac{2\varphi\omega}{cr}\left[\left(1 - \frac{r_g r^2}{2a^3}\right)\sin\theta - \cot\theta\left(1 + \frac{3r_g}{4a} - \frac{3r_g r^2}{2a^3}\right)\right], \tag{3.163}$$

and its covariant (lower-index) versions can be obtained as $H_{*1} = h_{11}H^{*1}$ and $H_{*2} = h_{22}H^{*2}$. With higher-order terms withheld, we have

$$H^{*1} = -\frac{2\varphi\omega}{c}\left(\cos\theta + \frac{1}{2}\right), \tag{3.164}$$

$$H^{*2} = \frac{2\varphi\omega}{cr}\left(\sin\theta - \cot\theta\right). \tag{3.165}$$

As for the chr.inv.-Maxwell equations, in a stationary electromagnetic field they have the form (3.143–3.144)

$$\left.\begin{aligned} {}^*\nabla_j E^j - \frac{1}{c} H^{ik} A_{ik} &= 4\pi\rho \\ {}^*\nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} &= \frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{I}, \tag{3.166}$$

$$\left.\begin{aligned} {}^*\nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik} &= 0 \\ {}^*\nabla_k E^{*ik} - \frac{1}{c^2} F_k E^{*ik} &= 0 \end{aligned}\right\} \text{II}. \tag{3.167}$$

Substitute, into the above equations, the obtained formulae for the magnetic field strength and the electric field strength (3.105), adapted to a stationary electromagnetic field

$$E^{*ik} = -\varepsilon^{ikn} E_n\,, \qquad E_i = -\frac{\varphi}{c^2} F_i = \frac{\varphi r_g r}{2a^3}\,. \tag{3.168}$$

The Group II equations (3.167) are satisfied identically. The Group I equations (3.167) take the form

$$\frac{3\varphi r_g}{2a^3} = 4\pi\breve{\rho}, \qquad \frac{\omega\varphi\cot\theta}{r^2\sin\theta} = -2\pi\breve{j}^3, \tag{3.169}$$

where $\breve{\rho}$ and $\breve{j}^3$ are the charge density and the current of the vortical electromagnetic field.

The above solutions are identical to the solutions (3.146) obtained in the vortex-free electromagnetic field of an ordinary rotating star. In addition, it is easy to see that the conservation equation $\nabla_\sigma j^\sigma = 0$ (3.147) and the Lorenz condition $\nabla_\sigma A^\sigma = 0$ (3.148), the chr.inv.-formulae of which are (3.149) and (3.150), are satisfied in the vortical electromagnetic field as identities.

This means that all the results obtained earlier in the vortex-free electromagnetic field of an ordinary rotating star are as well true in the present case, where the electromagnetic field of the star is vortical.

This is because all the terms that appear in the equations due to the electromagnetic field curl vanish within the second-order approximation. In terms of physics, this means that the presence of a curl in the electromagnetic field of an ordinary rotating star does not change the field sources. The vortical electromagnetic field can be meaningful only in the case of exotic stars, the characteristics of which differ from those of ordinary stars. We will see the difference in Chapter 5 when considering rapidly rotating neutron stars (pulsars).

3.8 Geometrization of the electromagnetic field of an ordinary rotating star

Using the geometric formula for the scalar electromagnetic potential

$$\varphi = c^2\sqrt{\frac{n}{G}} \tag{3.170}$$

that follows from (3.119), we write down the non-vanishing chr.inv.-Maxwell equations (3.146), or (3.169) which is the same as

$$\breve{\rho} = \rho = \frac{3c^2 r_g}{8\pi a^3}\sqrt{\frac{n}{G}} = \frac{3c^2}{8\pi a^2}\sqrt{\frac{n}{G}}\,\frac{r_g}{a}\,, \tag{3.171}$$

$$\breve{j}^3 = j^3 = -\frac{\omega c^2}{2\pi r^2}\sqrt{\frac{n}{G}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{3.172}$$

$$\breve{j} = j = \sqrt{h_{ik}\, j^i j^k} = \sqrt{h_{33}\, j^3 j^3} = \frac{\omega c^2\cot\theta}{2\pi r}\sqrt{\frac{n}{G}}\,, \tag{3.173}$$

where $\breve{\rho}$ is the charge density of the vortical electromagnetic field, $\breve{j}^3$ is the field current, while ρ and j^3 denote those in a vortex-free field. The dimensionless coefficient $n = \frac{G\varphi^2}{c^4}$ (3.119) is in the range $0 < n < \frac{1}{4}$. To see why $\frac{1}{4}$, see formula (3.118).

The electromagnetic field sources are expressed here through only the geometric characteristics of the star's space and the fundamental constants. This means that we have completely geometrized the sources of the stationary electromagnetic field of an ordinary rotating star.

Express the electric and magnetic strengths using the geometric formula for the scalar electromagnetic potential φ (3.170). Using the formulae for the non-zero components E_1, H^{*1} and H^{*2} obtained for the stationary electromagnetic field of an ordinary rotating star, we obtain

$$E^1 = E_1 = \sqrt{\frac{n}{G}}\,\frac{c^2 r_g r}{2a^3}\,, \tag{3.174}$$

$$H^{*1} = -2\omega c\sqrt{\frac{n}{G}}\left(\cos\theta + \frac{1}{2}\right) = -2c\sqrt{\frac{n}{G}}\left(\omega\cos\theta + \Omega^{*1}\right), \tag{3.175}$$

$$H^{*2} = \frac{2\omega c}{r}\sqrt{\frac{n}{G}}\,(\sin\theta - \cot\theta) = 2c\sqrt{\frac{n}{G}}\left(\frac{\omega\sin\theta}{r} - \Omega^{*2}\right), \tag{3.176}$$

$$H_{*1} = h_{11}H^{*1} = H^{*1}, \qquad H_{*2} = h_{22}H^{*2} = r^2 H^{*2}. \tag{3.177}$$

Here, according to our calculation (3.109–3.110) made in the space metric of an ordinary rotating star (3.28), we have

$$\Omega^{*1} = \Omega_{*1} = \frac{\omega}{2}\,, \qquad \Omega^{*2} = \frac{\omega\cot\theta}{r}\,, \qquad \Omega_{*2} = \omega r\cot\theta\,, \tag{3.178}$$

$$\Omega_{*j}\,\Omega^{*j} = \omega^2\left(\frac{1}{4} + \cot^2\theta\right). \tag{3.179}$$

So forth, we express the field density $\breve{\rho}_{\text{em}}$ and the momentum flow $\breve{J}^3_{\text{em}}$ of the vortical electromagnetic field, which are the physically observable projections of the energy-momentum tensor of the field (see Einstein's equations). Using their general formulation (3.102–3.103) that is true for any arbitrary electromagnetic field, we obtain

$$\breve{\rho}_{\text{em}} = \frac{n}{2\pi G}\left(\frac{1}{4c^2}\,F_j F^j + \Omega_{*j}\,\Omega^{*j}\right) + \frac{\omega^2}{2\pi}\,\frac{n}{G} = \rho_{\text{em}} + \frac{\omega^2}{2\pi}\,\frac{n}{G}\,, \tag{3.180}$$

$$\breve{J}^3_{\text{em}} = \frac{c^2 r_g r}{4\pi a^3\sin\theta}\,\frac{n}{G}\left(\frac{\omega\sin\theta}{r} - \Omega^{*2}\right) = J^3_{\text{em}} + \frac{c^2 r_g\,\omega}{4\pi a^3}\,\frac{n}{G}\,. \tag{3.181}$$

As you can see, the observable characteristics of the electromagnetic field are expressed here through only the geometric characteristics of the star's space and the fundamental constants. This is true for both the vortical electromagnetic field and the vortex-free field of the star. From a mathematical point of view, this means that the electromagnetic field of an ordinary rotating star is completely geometrized.

So, in the case of an ordinary rotating star, both Maxwell's equations and Einstein's field equations are satisfied. This means that they consist a self-consistent system of the Einstein-Maxwell equations, which completely describes both gravitational and electromagnetic phenomena inside ordinary rotating stars.

Finally, we can deduce something very interesting for astrophysics by writing the formula for the charge density (3.171) in the form

$$\rho = \frac{3c^2}{8\pi G a^2}\sqrt{nG}\,\frac{r_g}{a}\,. \qquad (3.182)$$

Here the first multiplier coincides with the formula for a "critical density" of substance in the Universe

$$\rho_{\rm cr} = \frac{3c^2}{8\pi G a^2} = \frac{3H^2}{8\pi G}\,, \qquad (3.183)$$

which is known from observational cosmology. Here $H = \frac{c}{a}$ is the Hubble constant, and a is the radius of the observable Universe. By analogy with the Universe, a critical density can formally be introduced for any liquid star. Thus, we express the charge density ρ of a liquid star as

$$\rho = \rho_{\rm cr}\sqrt{nG}\,\frac{r_g}{a}\,, \qquad (3.184)$$

where $n < \frac{1}{4}$ and $\sqrt{G} = 2.6\times 10^{-4}$ cm$^{3/2}$ gram$^{-1/2}$ sec^{-1}.

If the charge density is $\rho = \rho_{\rm cr}\sqrt{nG}$, then the physical radius of the star coincides with its Hilbert radius $a = r_g$. Since $r_g \ll a$ for ordinary stars, we conclude that the charge density inside any ordinary rotating star is much smaller than $\rho_{\rm cr}\sqrt{nG}$, i.e.

$$\rho \ll \rho_{\rm cr}\sqrt{nG}\,. \qquad (3.185)$$

A few words should be said at the end. The General Theory of Relativity is a geometric theory of space-time-matter. Its primary task is to express all physical phenomena as manifestations of the space (space-time) geometry. The gravitational field was initially geometrized by Einstein thanks to his field equations. However, the electromagnetic field was not geometrized at that time: as was already shown by Einstein, mathematically this problem in a general case is very non-trivial. Nevertheless, it is possible to solve this problem in a particular case, where specific conditions simplify the mathematics. For example, as showed above, we have completely geometrized the electromagnetic field in the internal field of an ordinary rotating star.

3.9 Conclusion

This Chapter is complementary to the previous Chapter 2, wherein we considered ordinary stars including the Sun. Three primary tasks were achieved in this Chapter.

First. In Chapter 2, when we considered the internal space metric of a liquid star, we followed the historical path as Schwarzschild did when introduced the metric. Namely, — when we introduced the internal space metric of a liquid sphere in a complete form (taking singularities of the space into account), we used the Schwarzschild notations. These notations come from the general form of a spherically symmetric metric and thus contain the coefficients e^{ν} and e^{λ}, which are functions of r and t. This is the commonly accepted method of writing any spherically symmetric metric. Even when we calculated the physically observable characteristics of such a metric space, we obtained them in terms of the unknowns e^{ν} and e^{λ}. As a result, we have obtained the physically observable characteristics of the space in an incomplete form, which requires further calculation of the coefficients e^{ν} and e^{λ}. This creates enormous difficulties in solving particular problems in the space of such a metric. Therefore, in Chapter 3, we initially introduced the internal space metric of a liquid sphere in its final form, where the coefficients e^{ν} and e^{λ} are expressed through the main characteristics of the sphere, such as its physical radius and Hilbert radius, and also through the radial coordinate r and time t. As a result, we have obtained all components of the fundamental metric tensor in an explicit form, without unknown coefficients. It was the subject of §3.1 and §3.2. Therefore, if we (or someone else) further solve problems using the internal space metric of an ordinary liquid star, then we will initially have formulae for all physically observable characteristics of its internal space.

Second. We considered the space metric of a non-rotating liquid sphere. Nevertheless, we know that most stars rotate. Most likely, all stars rotate, but many of them rotate so slowly that the Doppler splitting of spectral lines due to their rotation cannot be detected by modern spectroscopy methods. In any case, if we consider a liquid star with an electromagnetic field, then we should consider the internal space metric of a rotating liquid sphere. This metric was introduced in §3.3, then we introduced Einstein's field equations and Maxwell's equations in a form that satisfies the metric. We have shown that the electric component of the electromagnetic field of a star is due to the gravitational field of the star, while the magnetic component is due to the star's rotation. We have also found that the vortical nature of the electromagnetic field does not play a significant rôle in ordinary rotating stars.

Third. Concerning the most important achievement of this Chapter. In §3.8 we showed that, in the internal space metric of a rotating liquid star, the physically observable characteristics of the electromagnetic field of the star are expressed through only the geometric characteristics of the star's space and the fundamental constants. This means that in the internal field of a rotating liquid star, the electromagnetic field is completely geometrized.

Chapter 4 Stellar Wind

4.1 Finding the escape velocity condition for a star

A stream of particles of stellar substance is permanently erupted from the surface of any star. A fraction of the stream consists of so rapid particles that they leave the gravitational field of the star forever, for the outer cosmos, thereby producing a stellar wind.* In terms of our mathematical theory of liquid stars, this means that such particles of the star's surface layer are faster than the escape velocity of the star.

Why do some particles of stellar substance leave the surface of a star? Can this process be likened to boiling water in a kettle, or is it completely different? Finding an answer to this question is our research task in this Chapter.

To answer this question we should study how particles of stellar substance travel inside a star. To do this, we first find a formula for the escape velocity, expressed through the components of the space metric of a liquid star. Then we deduce the equations of motion of the particles inside the star. Thus, we obtain the physical conditions under which the particles travelling in the surface layer of the star are faster than the escape velocity. After that, we will be able to solve the equations of motion for any particles of stellar substance.

The mentioned escape velocity, known also as the second cosmic velocity v_{II}, is the velocity at which a particle can "leave", forever, the gravitational field of the massive body.†

Let us assume that particles of stellar substance travel, radially, from the centre of the star to its surface. Let the particles reach the star's

---

*Wolf-Rayet stars differ from ordinary stars in an extremely huge stellar wind: this stream is so strong that any Wolf-Rayet star loses a significant part of its mass with the stellar wind.

†The first cosmic velocity v_I, known also as the orbital velocity, allows the particle to be orbiting the massive body without falling down onto its surface.

surface then leave the star, forever, for the outer cosmos, thus forming a stellar wind. Therefore, we call the formula for the velocity of a particle of stellar matter, which is expressed through the star's escape velocity, the *escape velocity condition*.

For a spherically symmetric body, the mass of which is M, the escape velocity at a distance r from the body's centre is

$$\mathrm{v}_{\mathrm{II}} = \sqrt{\frac{2GM}{r}}\,. \tag{4.1}$$

This formula comes from the mass-point metric (1.1),

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{4.2}$$

where $r_g = \frac{2GM}{c^2}$, while M is the body's mass.

As was shown in Chapter 2, the field of any liquid star has two primary regions. They are described by two different metrics. The metric of a liquid sphere is valid from the centre of the star ($r = 0$) up to its surface ($r = a$). The mass-point metric is valid from the surface of the star up to the outer cosmos. In other words, particles of stellar substance travel inside a star along the trajectories determined by the metric of a liquid sphere. As soon as the particles leave the star (in the case, where their velocities exceed the escape velocity of the star), they travel in the cosmos along the trajectories determined by the mass-point metric.

Therefore, the velocity of a particle of stellar substance, travelling from the surface of a star for the outer cosmos, is a solution to the equations of motion according to the mass-point metric. Expressed through the escape velocity of the star, this solution is the escape velocity condition for the star.

We deduce this formula as a solution to the chr.inv.-equations of non-isotropic geodesics [18, 19]

$$\left.\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2}F_i\mathrm{v}^i + \frac{m}{c^2}D_{ik}\mathrm{v}^i\mathrm{v}^k = 0 \\ &\frac{d(m\mathrm{v}^i)}{d\tau} + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0 \end{aligned}\right\}, \tag{4.3}$$

which are the equations of motion of a mass-bearing particle travelling with the observable velocity v^i. These equations are obtained as the chr.inv.-projections of the general covariant equations of non-isotropic geodesics. See [18, 19] for detail.

Let us solve the equations (4.3) for a particle of stellar substance travelling along the radial direction r. Therefore, we assume

$$\mathrm{v}^1 = \frac{dr}{d\tau} \neq 0\,, \qquad \mathrm{v}^2 = \mathrm{v}^3 = 0\,. \tag{4.4}$$

To solve the equations (4.3), we need to find specific formulae for the physically observable properties characteristic of a space of the mass-point metric (4.2). As is seen from the mass-point metric (4.2), such a space does not rotate or deform ($A_{ik}=0$, $D_{ik}=0$). Only the gravitational inertial force F_i and the Christoffel symbols Δ^i_{nk} remain non-zero. Calculating these quantities and also the components of the chr.inv.-metric tensor h_{ik} according to their definitions given in §1.3, we obtain that for the metric (4.2) they have the form

$$F_1 = -\frac{c^2 r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\,, \qquad F^1 = -\frac{c^2 r_g}{2r^2}\,, \tag{4.5}$$

$$h_{11} = \frac{1}{h^{11}} = 1-\frac{r_g}{r}\,, \;\; h_{22} = \frac{1}{h^{22}} = r^2, \;\; h_{33} = \frac{1}{h^{33}} = r^2\sin^2\theta, \tag{4.6}$$

$$\Delta^1_{11} = -\frac{r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\,, \qquad \Delta^1_{22} = \frac{\Delta^1_{33}}{\sin^2\theta} = -r\left(1-\frac{r_g}{r}\right), \tag{4.7}$$

$$\Delta^2_{12} = \Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^2_{33} = -\sin\theta\cos\theta\,, \qquad \Delta^3_{23} = \cot\theta\,. \tag{4.8}$$

With these, we obtain that the chr.inv.-equations of motion (4.3) in a space of the mass-point metric have the form

$$\left.\begin{aligned} &\frac{1}{m}\frac{dm}{d\tau} = -\frac{r_g}{2r^2}\,\frac{1}{1-\frac{r_g}{r}}\,\frac{dr}{d\tau} \\ &\frac{1}{m}\frac{d}{d\tau}\left(m\frac{dr}{d\tau}\right) - \frac{r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2 r_g}{2r^2} = 0 \end{aligned}\right\}, \tag{4.9}$$

where

$$m = \frac{m_0}{\sqrt{1-\frac{\dot{r}^2}{c^2\left(1-\frac{r_g}{r}\right)}}}\,, \qquad \dot{r} = \frac{dr}{d\tau}\,. \tag{4.10}$$

Denote the relativistic mass of the particle on the surface of the star ($r = a$) as $m_{(0)}$. This is the "start-mass" of the particle when leaving the star. Then, denoting the observable velocity of the particle in the moment of time, when it leaves the star's surface, as $\dot{r}_0$, we have

$$m = m_{(0)} \frac{\sqrt{1 - \frac{r_g}{a}}}{\sqrt{1 - \frac{r_g}{r}}}\,, \qquad m_{(0)} = \frac{m_0}{\sqrt{1 - \frac{\dot{r}_0^2}{c^2\left(1 - \frac{r_g^3}{a^3}\right)}}}\,. \tag{4.11}$$

Let us begin solving the chr.inv.-equations of motion (4.9). Substituting the scalar equation into the vector equation, we obtain the vector equation of motion along the radial coordinate r

$$\ddot{r} - \frac{r_g}{r^2}\frac{\dot{r}^2}{1 - \frac{r_g}{r}} + \frac{c^2 r_g}{2r^2} = 0\,. \tag{4.12}$$

Denote $\dot{r} = y$. Then we have

$$\ddot{r} = yy'\,, \qquad y' = \frac{dy}{dr}\,, \tag{4.13}$$

thus the equation (4.12) takes the form

$$yy' - \frac{r_g}{r^2}\frac{y^2}{1 - \frac{r_g}{r}} + \frac{c^2 r_g}{2r^2} = 0\,. \tag{4.14}$$

Assuming $u(r) = y^2$, we transform the previous equation into the ordinary linear differential equation

$$u' - \frac{2r_g}{r^2}\frac{u}{1 - \frac{r_g}{r}} + \frac{c^2 r_g}{r^2} = 0\,. \tag{4.15}$$

This equation has the following exact solution

$$u = e^{-F}\left(u_0 + \int_r^a g(r)\, e^F dr\right), \qquad u_0 = y_0^2 = \dot{r}_0^2\,, \tag{4.16}$$

where the functions contained in it have the form

$$F(r) = \int_r^a f(r)\,dr\,, \quad f(r) = -\frac{2r_g}{r^2}\frac{1}{1 - \frac{r_g}{r}}\,, \quad g(r) = \frac{c^2 r_g}{r^2}\,. \tag{4.17}$$

Integrating the function $f(r)$, we obtain

$$F(r) = \ln\left(\frac{1-\frac{r_g}{a}}{1-\frac{r_g}{r}}\right)^2, \qquad e^F = \left(\frac{1-\frac{r_g}{a}}{1-\frac{r_g}{r}}\right)^2, \tag{4.18}$$

$$\int_r^a \frac{c^2 r_g \left(1-\frac{r_g}{a}\right)^2 dr}{r^2\left(1-\frac{r_g}{r}\right)^2} = c^2\left(1-\frac{r_g}{a}\right)\left(1-\frac{1-\frac{r_g}{a}}{1-\frac{r_g}{r}}\right). \tag{4.19}$$

Substituting (4.17–4.19) into (4.16) then neglecting the higher-order terms of $\frac{r_g}{a}$ (since this ratio is tiny for ordinary stars), we obtain

$$\dot{r}^2 = \dot{r}_0^2\left(1+\frac{2r_g}{a}-\frac{2r_g}{r}\right)+c^2\left(\frac{r_g}{a}-\frac{r_g}{r}\right). \tag{4.20}$$

From here, we obtain a formula for the radial velocity of a particle of stellar substance leaving the star with a stellar wind. Since v_{II} (4.1) on the star's surface ($r=a$) is

$$\mathrm{v}_{\mathrm{II}} = \sqrt{\frac{2GM}{r}} = c\sqrt{\frac{r_g}{r}} = c\sqrt{\frac{r_g}{a}}\,, \tag{4.21}$$

the mentioned formula has the form

$$\dot{r} = \frac{dr}{d\tau} = c\sqrt{\frac{\dot{r}_0^2+\mathrm{v}_{\mathrm{II}}^2}{c^2}-\frac{r_g}{r}+\frac{2\dot{r}_0^2}{c^2}\left(\frac{\mathrm{v}_{\mathrm{II}}^2}{c^2}-\frac{c^2 r_g}{r}\right)}\,. \tag{4.22}$$

This is the escape velocity condition we were looking for. If $\dot{r}_0=0$, then the equation (4.22) transforms into the obvious condition

$$\frac{dr}{d\tau} = \sqrt{\mathrm{v}_{\mathrm{II}}^2-\frac{c^2 r_g}{r}} < \mathrm{v}_{\mathrm{II}}\,. \tag{4.23}$$

According to this condition, a particle of stellar substance cannot leave the gravitational field of a star, if its start-velocity on the surface of the star is zero. Therefore, in further consideration of the stellar wind, we always assume $\dot{r}_0 \neq 0$ in all equations of our theory.

Let us obtain the final simplification to the escape velocity condition (4.22). Denote the last term in the radicand as

$$q = \frac{2\dot{r}_0^2}{c^2}\left(\frac{\mathrm{v}_{\mathrm{II}}^2}{c^2}-\frac{c^2 r_g}{r}\right). \tag{4.24}$$

For the Sun, i.e., a typical ordinary star, we have: $\text{v}_{\text{II}} = 617$ km/sec, $r_g = 2.9$ km, $\dot{r}_0 = 750$ km/sec* and $a = 7.0\times10^5$ km. Since $q = 0$ at $r = a$, we assume $r > a$ as for a stellar wind. After some algebra, we obtain

$$\frac{\dot{r}_0^2 + \text{v}_{\text{II}}^2}{c^2} \simeq 10^{-5}, \qquad \frac{r_g}{r} < 4.1\times10^{-6}, \qquad q < 5.3\times10^{-11}. \tag{4.25}$$

For a typical star of the Wolf-Rayet family (see Table 1.1), we have: $\text{v}_{\text{II}} = 982$ km/sec, $r_g = 150$ km, $\dot{r}_0 = 2200$ km/sec and $a = 1.4\times10^7$ km. Therefore, for a typical Wolf-Rayet star, we obtain

$$\frac{\dot{r}_0^2 + \text{v}_{\text{II}}^2}{c^2} \simeq 6.4\times10^{-5}, \quad \frac{r_g}{r} < 1.1\times10^{-5}, \quad q < 1.2\times10^{-9}. \tag{4.26}$$

The term q has such a small numerical value (four orders of magnitude smaller, than the other terms in the formula) that it can be neglected for the stellar wind that comes from both an ordinary star and a Wolf-Rayet star. Therefore, the escape velocity condition has the form

$$\frac{dr}{d\tau} = c\sqrt{\frac{\dot{r}_0^2}{c^2} + \frac{\text{v}_{\text{II}}^2}{c^2} - \frac{r_g}{r}}\,. \tag{4.27}$$

As follows from the above formula, the velocity of a particle of stellar substance on the surface of an ordinary star ($r = a$) is $\dot{r}_0$.

4.2 Light-like particles inside an ordinary star

Let us now consider how particles of stellar substance and particles of light travel inside a star. (Stars are filled not only with substance, but also with light.) First, consider light-like (massless) particles inside an ordinary star. Such particles travel along isotropic geodesic lines. The chr.inv.-equations of isotropic geodesics have the form [18, 19]

$$\left.\begin{aligned} &\frac{d\omega}{d\tau} - \frac{\omega}{c^2}F_i c^i + \frac{\omega}{c^2}D_{ik}c^i c^k = 0 \\ &\frac{d(\omega c^i)}{d\tau} + 2\omega\left(D^i_k + A^{\cdot i}_{k\cdot}\right)c^k - \omega F^i + \omega\Delta^i_{nk}c^n c^k = 0 \end{aligned}\right\}. \tag{4.28}$$

*$\dot{r}_0 \simeq 750$ km/sec is typical for the particles of the fast component of the solar wind, the composition of which is that of the photosphere. In contrast, the slow component of the solar wind has a composition close to that of the corona. Its particles travel from the Sun with a velocity of about 400 km/sec.

These are the equations of motion of a light-like particle (such as a photon, the frequency of which is ω) travelling with the observable velocity of light c^i. These chr.inv.-equations are obtained as the chr.inv.-projections of the general covariant equations of isotropic geodesics. See [18, 19] for detail.

As previously, we assume that ordinary stars do not rotate or deform ($A_{ik}=0$, $D_{ik}=0$). Also, we consider a photon travelling strictly along the radial coordinate ($x^1=r$ direction) from the centre of the star to its surface. In this case, the isotropic geodesic equations (4.28) inside an ordinary star take the simplified form

$$\left.\begin{aligned} &\frac{d\omega}{d\tau}-\frac{\omega}{c^2}F_1c^1=0\\ &\frac{d(\omega c^1)}{d\tau}-\omega F^1+\omega\Delta^1_{11}c^1c^1=0 \end{aligned}\right\}, \tag{4.29}$$

where the observable (light) velocity of the photon is $c^1=\frac{dr}{d\tau}$.

Consider the chr.inv.-scalar geodesic equation of (4.29). Substituting F_1 (3.6), obtained for the metric of a liquid sphere, we have

$$\frac{1}{\omega}\frac{d\omega}{d\tau}=-\frac{r_g}{a^3}\frac{r}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_gr^2}{a^3}}\right)\sqrt{1-\frac{r_gr^2}{a^3}}}\frac{dr}{d\tau}. \tag{4.30}$$

Re-write this equation in a form, which can easily be integrated

$$d\ln\omega=-\frac{d\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_gr^2}{a^3}}\right|}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_gr^2}{a^3}}\right|}=$$

$$=d\ln\frac{1}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_gr^2}{a^3}}\right|}. \tag{4.31}$$

We are considering photons travelling inside the star. Therefore, the solution must in the range $r_g\leqslant r\leqslant a$. After integration, we obtain

$$\omega=\frac{B}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_gr^2}{a^3}}}, \qquad B=const. \tag{4.32}$$

We assume that a photon starts from the Hilbert surface of the star ($r_0 = r_g$). In this case, we obtain

$$B = \omega_0 \left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g^3}{a^3}} \right), \tag{4.33}$$

where ω_0 is the initial value of the photon's frequency on the Hilbert surface, from which it started deep inside the star. Since $r_g \ll a$ for ordinary stars, we neglect the higher-order terms of $\frac{r_g}{a}$. In this case, the solution to the chr.inv.-scalar geodesic equation, which is the photon's frequency (4.32), takes the form

$$\omega = \frac{\omega_0 \left(3\sqrt{1-\frac{r_g}{a}} - 1 \right)}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{4.34}$$

Next, consider the vector geodesic equation of (4.29). With our assumption of the radial motion of the photon, it has the form

$$\frac{d^2 r}{d\tau^2} + \frac{1}{\omega}\frac{d\omega}{d\tau}\frac{dr}{d\tau} + \Delta^1_{11}\left(\frac{dr}{d\tau}\right)^2 - F^1 = 0\,. \tag{4.35}$$

Denote $\ddot{r} = \frac{d^2 r}{d\tau^2}$ and $\dot{r} = \frac{dr}{d\tau}$, then substitute $\frac{1}{\omega}\frac{d\omega}{d\tau}$ (4.30), Δ^1_{11} (3.8) and F^1 (3.7). As a result, we transform the vector geodesic equation (4.35) into a second order non-linear differential equation with respect to r

$$\ddot{r} - \frac{r_g r}{a^3}\frac{\dot{r}^2}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}} + {}$$

$$+ \frac{r_g r}{a^3}\frac{\dot{r}^2}{1-\frac{r_g r^2}{a^3}} + \frac{c^2 r_g r}{a^3}\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}} = 0\,. \tag{4.36}$$

In this form, the equation cannot be solved. Therefore, we simplify it by the formula for $\dot{r}^2$ taken from the obvious relation $h_{ik}c^i c^k = c^2$, which in the present case has the form

$$\frac{\dot{r}^2}{1-\frac{r_g r^2}{a^3}} = c^2. \tag{4.37}$$

As a result, the initial equation (4.36) takes the form

$$\ddot{r} + \frac{c^2 r_g r}{a^3} = 0\,, \tag{4.38}$$

which is the equation of harmonic oscillation at the frequency

$$\Omega = \frac{c}{a}\sqrt{\frac{r_g}{a}} = \frac{\mathrm{v}_{\mathrm{II}}}{a} = \sqrt{\frac{2GM}{a^3}}\,, \tag{4.39}$$

which is dependent on the escape velocity v_{II} (4.21) of the star.

In general, the oscillation frequency Ω (4.39) depends only on the mass M and the radius a, which are the integral characteristics of the star. Therefore, we call the Ω the *proper frequency* of the star. Table 4.1 gives the numerical values of the proper frequency Ω for typical members of the known families of stars.

The proper frequency Ω of a star reaches its maximum magnitude $\Omega_{\max} = \frac{c}{a}$ by $r_g = a$. This is the case of gravitational collapsars (black holes), which is also applicable to the entire Universe. According to observational estimates, the radius of the Universe is $a = 1.3\times10^{28}$ cm that coincides with its Hilbert radius r_g. Hence, the Universe is a huge gravitational collapsar. Calculating the proper frequency Ω for the Universe,

Object	Mass M, gram	Radius a, cm	Proper frequency Ω, sec^{-1}
Wolf-Rayet stars	1.0×10^{35}	1.4×10^{12}	7.0×10^{-5}
Red super-giant*	4.0×10^{34}	7.0×10^{13}	1.6×10^{-7}
White super-giant†	3.4×10^{34}	4.8×10^{12}	6.4×10^{-6}
Sun	2.0×10^{33}	7.0×10^{10}	8.8×10^{-4}
Jupiter (proto-star)	1.9×10^{30}	7.1×10^{9}	8.4×10^{-4}
Red dwarfs	6.7×10^{32}	2.3×10^{10}	2.7×10^{-3}
Brown dwarf‡	4.1×10^{31}	7.0×10^{9}	7.4×10^{-2}
White dwarf§	2.0×10^{33}	6.4×10^{8}	1.0
Universe	8.8×10^{55}	1.3×10^{28}	2.3×10^{-18}

*Betelgeuse. †Rigel. ‡Corot-Exo-3. §Sirius B.

Table 4.1: The proper frequency Ω for typical members of the known families of stars and for the Universe.

we obtain

$$\Omega_{\max} = \frac{c}{a} = 2.3\times10^{-18}\ \text{sec}^{-1}, \tag{4.40}$$

which completely coincides with the numerical value of the Hubble constant $H = \frac{c}{a} = (2.3\pm0.3)\times10^{-18}\ \text{sec}^{-1}$. In this case, according to (4.39), the integral mass of the Universe should be

$$M = \frac{\Omega^2 a^3}{2G} = 8.8\times10^{55}\ \text{gram}, \tag{4.41}$$

which is consistent with the observed average substance density in the Universe, estimated to be in the range of 10^{-28} to 10^{-31} gram/cm^3.

The chr.inv.-vector equation of isotropic geodesics in its final form (4.38) is solved as

$$r = B_1\cos\left(\sqrt{\frac{r_g}{a}}\,\frac{c\tau}{a}\right) + B_2\sin\left(\sqrt{\frac{r_g}{a}}\,\frac{c\tau}{a}\right), \tag{4.42}$$

where B_1 and B_2 are integration constants. Assuming r and $\dot{r}$ at the initial moment of time $\tau_0 = 0$ to be $r_0 = r_g$ and $\dot{r}_0 = c$, we obtain

$$B_1 = r_g, \qquad B_2 = a\sqrt{\frac{a}{r_g}}\,. \tag{4.43}$$

As a result, we obtain the final solution for r

$$r = r_g\cos\Omega\tau + a\sqrt{\frac{a}{r_g}}\,\sin\Omega\tau, \qquad \Omega = \frac{c}{a}\sqrt{\frac{r_g}{a}}\,, \tag{4.44}$$

which is the harmonic oscillation equation $r = A_1\cos\Omega\tau + A_2\sin\Omega\tau$. Differentiating (4.44), we obtain the oscillation velocity of the photon

$$\dot{r} = c\cos\Omega\tau - \frac{c r_g}{a}\,\sin\Omega\tau, \qquad \Omega = \frac{c}{a}\sqrt{\frac{r_g}{a}}\,. \tag{4.45}$$

As is seen from the solution (4.44), the entire light-like matter of each star oscillates at the frequency Ω (4.39), which is the proper frequency of that particular star and is determined by its mass and radius. This oscillation occurs with two amplitudes:

a) The amplitude $A_1 = r_g$ coincides with the radius of the inner space breaking in the star's field on the surface of the Hilbert core of the star, where stellar energy is released;

b) The amplitude $A_2 = \sqrt{a^3/r_g}$ coincides with the outer space breaking in the star's field (see Chapter 2 for detail). The outer space breaking is located outside the star, in the outer cosmos. For the Sun ($a = 7.0 \times 10^{10}$ cm, $r_g = 2.9 \times 10^5$ cm), we obtain $A_2 = 3.4 \times 10^{13}$ cm $= 2.3$ AU, which is the distance from the Sun to the maximum concentration of asteroids in the asteroid belt. This means that the light-like stellar matter of each star oscillates at the same frequency both on the spherical surface of the outer space breaking in the field of that particular star, in the outer cosmos, and on the surface of the Hilbert core deep inside the star.

The mentioned oscillation of the light-like matter of each star is due to the gravitational field of that particular star, created by its mass M.

How does this oscillation affect the frequency of stellar photons? To answer this question, consider the obtained solution for the photon's frequency ω (4.34) in two limiting cases corresponding to two oscillation amplitudes: $r = A_1 = r_g$ and $r = A_2 = \sqrt{a^3/r_g}$. Thus, the frequency takes the following numerical values

$$r = A_1 = r_g \,, \qquad \omega = \omega_0 \frac{3\sqrt{1-\frac{r_g}{a}} - 1}{3\sqrt{1-\frac{r_g}{a}} - 1} = \omega_0 \,, \tag{4.46}$$

$$r = A_2 = \frac{a^2}{r_g} \,, \qquad \omega = \omega_0 \frac{3\sqrt{1-\frac{r_g}{a}} - 1}{3\sqrt{1-\frac{r_g}{a}}} \,. \tag{4.47}$$

As you see, this oscillation does not change the frequency of stellar radiation near the Hilbert core (in the centre of the star), but affects its frequency at large distances from the Hilbert core.

4.3 Particles of stellar substance inside an ordinary star

Such particles travel along non-isotropic geodesics. The chr.inv.-equations of non-isotropic geodesics [18, 19] have the form (4.3)

$$\left.\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0 \\ &\frac{d(m\mathrm{v}^i)}{d\tau} + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0 \end{aligned}\right\} . \tag{4.48}$$

We assume that an ordinary star is a liquid sphere that does not rotate or deform ($A_{ik}=0$, $D_{ik}=0$). For a particle of stellar substance, which travels inside the star radially from the centre to the surface, the observable velocity is $\mathrm{v}^1=\frac{dr}{d\tau}$, while $\mathrm{v}^2=\mathrm{v}^3=0$. In this case, the chr.inv.-equations of non-isotropic geodesics (4.48) take the form

$$\left.\begin{aligned} &\frac{dm}{d\tau}-\frac{m}{c^2}F_1\mathrm{v}^1=0\\ &\frac{d(m\mathrm{v}^1)}{d\tau}-mF^1+m\Delta^1_{11}\mathrm{v}^1\mathrm{v}^1=0 \end{aligned}\right\}. \tag{4.49}$$

They have the same structure as the chr.inv.-equations of isotropic geodesics (4.29). Therefore, they are solved in the same way. But the light speed condition $h_{ik}c^ic^k=c^2$ (4.37) used in the isotropic geodesic equations does not hold for mass-bearing particles. Hence, the chr.inv.-equations of non-isotropic geodesics (4.49) will have a different solution than that of the chr.inv.-equations of isotropic geodesics (4.29).

Substitute, into the scalar equation of (4.49), the formula for F_1 (3.6), which we have obtained for the metric of a liquid sphere. Thus, we obtain the scalar geodesic equation in the form

$$\frac{1}{m}\frac{dm}{d\tau}=-\frac{r_g}{a^3}\frac{r}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\frac{dr}{d\tau}. \tag{4.50}$$

This equation can be re-written in the form

$$d\ln m=-\frac{d\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|}=$$

$$=d\ln\frac{1}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|}, \tag{4.51}$$

which is easy to integrate. Using $r_g\leqslant r\leqslant a$ (we are considering only particles inside the star), after integration we obtain

$$m=\frac{B}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}, \qquad B=const. \tag{4.52}$$

Let us assume that a mass-bearing particle starts from the Hilbert surface ($r_0 = r_g$), i.e., near the centre of the star. Then

$$B = m_{(0)}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g^3}{a^3}}\,\right), \tag{4.53}$$

where

$$m_{(0)} = \frac{m_0}{\sqrt{1-\frac{\dot{r}_0^2}{c^2\left(1-\frac{r_g^3}{a^3}\right)}}} \tag{4.54}$$

is the initial value of the relativistic mass of the particle on the Hilbert surface of the star. Since $r_g \ll a$ for ordinary stars, we neglect the higher-order terms of $\frac{r_g}{a}$. Taking into account all this, the solution (4.52) of the scalar geodesic equation takes the form

$$m = \frac{m_{(0)}\left(3\sqrt{1-\frac{r_g}{a}} - 1\right)}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}} =$$

$$= \frac{m_0\left(3\sqrt{1-\frac{r_g}{a}} - 1\right)}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{\dot{r}_0^2}{c^2}}}\,, \tag{4.55}$$

where

$$m_{(0)} = \frac{m_0}{\sqrt{1-\frac{\dot{r}_0^2}{c^2}}} \tag{4.56}$$

in the framework of our approximation mentioned above.

Now, we consider the vector geodesic equation of (4.49). With our assumption that particles of stellar substance travel radially, from the centre of the star to its surface, the equation has the form

$$\frac{d^2r}{d\tau^2} + \frac{1}{m}\frac{dm}{d\tau}\frac{dr}{d\tau} + \Delta^1_{11}\left(\frac{dr}{d\tau}\right)^2 - F^1 = 0\,. \tag{4.57}$$

Denote $\ddot{r} = \frac{d^2r}{d\tau^2}$ and $\dot{r} = \frac{dr}{d\tau}$. Substituting $\frac{1}{\omega}\frac{d\omega}{d\tau}$ (4.30), Δ^1_{11} (3.8) and F^1 (3.7), we transform the above equation into a non-linear differential equation of the second order with respect to r, which has the form

$$\ddot{r} - \frac{r_g r}{a^3} \frac{\dot{r}^2}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}} + $$

$$+ \frac{r_g r}{a^3} \frac{\dot{r}^2}{1-\frac{r_g r^2}{a^3}} + \frac{c^2 r_g r}{a^3} \frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}} = 0\,. \quad (4.58)$$

This equation is identical to the equation (4.36), which we have obtained for photons. It cannot be solved as well. To simplify the equation, we express $\dot{r}^2$ from the obvious relation $h_{11}\dot{r}\dot{r} = \dot{r}^2$. We obtain

$$c^2\left(1-\frac{r_g r^2}{a^3}\right)\left(1-\frac{m_0^2}{m^2}\right) = \dot{r}^2, \quad (4.59)$$

where

$$m = \frac{m_0}{\sqrt{1-\frac{\dot{r}^2}{c^2\left(1-\frac{r_g r^2}{a^3}\right)}}}\,. \quad (4.60)$$

It follows from (4.55) that

$$\frac{m_0}{m} = \frac{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{\dot{r}_0^2}{c^2}}}{3\sqrt{1-\frac{r_g}{a}} - 1}\,. \quad (4.61)$$

Therefore, from (4.59) we obtain

$$\dot{r}^2 = c^2\left(1-\frac{r_g r^2}{a^3}\right)\left[1 - \frac{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)^2\left(1-\frac{\dot{r}_0^2}{c^2}\right)}{\left(3\sqrt{1-\frac{r_g}{a}} - 1\right)^2}\right]. \quad (4.62)$$

Substituting this formula for $\dot{r}^2$ into the initial differential equation (4.58) and neglecting the higher-order terms of $\frac{r_g}{a}$, we obtain the vector geodesic equation (4.58) in the solvable form

$$\ddot{r} + \frac{\left(c^2 + \dot{r}_0^2\right) r_g r}{2a^3} = 0\,. \quad (4.63)$$

This is the equation of harmonic oscillation at the frequency

$$\Omega = \sqrt{\frac{\left(c^2 + \dot{r}_0^2\right) r_g}{2a^3}}\,. \tag{4.64}$$

This solution concerns particles of stellar substance. It is easy to see that this formula transforms into the formula for the oscillation frequency of stellar photons, Ω (4.39), by the limit condition $\dot{r} = c$.

The vector geodesic equation (4.63) is solved as

$$r = Q_1 \cos\Omega\tau + Q_2 \sin\Omega\tau\,, \tag{4.65}$$

where the integration constant Q_1 and Q_2 results from the conditions $r_0 = r_g$ and $\dot{r}_0 = 0$ at the initial moment of time $\tau_0 = 0$. We obtain

$$Q_1 = r_g\,, \qquad Q_2 = \frac{\dot{r}_0\, a\sqrt{2a}}{\sqrt{\left(c^2 + \dot{r}_0^2\right) r_g}}\,. \tag{4.66}$$

Therefore, the final solution for r has the form

$$r = r_g \cos\Omega\tau + \frac{\dot{r}_0\, a\sqrt{2a}}{\sqrt{\left(c^2 + \dot{r}_0^2\right) r_g}} \sin\Omega\tau\,, \tag{4.67}$$

which is the harmonic oscillation equation $r = A_1 \cos\Omega\tau + A_2 \sin\Omega\tau$. Differentiating (4.67), we obtain the velocity of the particle

$$\dot{r} = -\sqrt{\frac{\left(c^2 + \dot{r}_0^2\right) r_g^3}{2a^3}} \sin\Omega\tau + \dot{r}_0 \cos\Omega\tau\,. \tag{4.68}$$

The obtained solution (4.67) shows that particles of the liquid substance of each star oscillate at the frequency Ω (4.64), which depends on the mass and radius of that particular star, and also on the initial velocity of the particles. This oscillation occurs with two amplitudes:

a) The amplitude $A_1 = r_g$ is the same as that of the light-like matter of stars (see §4.2). That is, particles of the liquid substance of each star oscillate with the same amplitude as the light-like matter of that particular star. The amplitude coincides with the radius of the inner space breaking in the star's field on the surface of the Hilbert core of the star, where stellar energy is released;

b) The amplitude

$$A_2 = \frac{\dot{r}_0 \, a \sqrt{2a}}{\sqrt{\left(c^2 + \dot{r}_0^2\right) r_g}} \tag{4.69}$$

depends on the initial velocity $\dot{r}_0$ of the particles. If $\dot{r}_0 = c$, then $A_2 = \sqrt{a^3/r_g}$ coincides with the second oscillation amplitude of the light-like matter of the star (see §4.2). According to (4.69), the initial velocity of those of the particles, the oscillation amplitude of which reaches the star's surface ($A_2 = a$), is

$$\dot{r}_0 = \frac{c\sqrt{r_g}}{\sqrt{2a - r_g}} = \frac{\mathrm{v}_{\mathrm{II}}}{\sqrt{2 - \frac{r_g}{a}}} \simeq \frac{\mathrm{v}_{\mathrm{II}}}{\sqrt{2}}\,, \qquad \mathrm{v}_{\mathrm{II}} = \sqrt{\frac{2GM}{a}}\,, \tag{4.70}$$

where v_{II} (4.21) is the escape velocity of the star (with which particles near the star can leave, forever, the star's gravitational field). Applying the condition $A_2 \geqslant a$ to (4.69), we obtain the velocity $\dot{r}_0$ required for a particle of stellar substance to leave the star's surface, forever, for the outer cosmos

$$\dot{r}_0 \geqslant \sqrt{\frac{GM}{a}}\,, \tag{4.71}$$

which is different from the escape velocity v_{II} for a particle not bound to the star's substance.

Transform the proper frequency Ω (4.64) of a star to express it in terms of the orbital velocity v_{I} of a particle, calculated for the star

$$\Omega = \frac{c}{a}\sqrt{\frac{r_g}{2a}}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}} = \frac{\mathrm{v}_{\mathrm{II}}}{a\sqrt{2}}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}} = \frac{\mathrm{v}_{\mathrm{I}}}{a}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}}\,. \tag{4.72}$$

Using this formula, we express r (4.67) in the form

$$r = r_g \cos\Omega\tau + \frac{\dot{r}_0 \, a}{\mathrm{v}_{\mathrm{I}}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}}}\sin\Omega\tau\,, \tag{4.73}$$

which is $r = A_1 \cos\Omega\tau + A_2 \sin\Omega\tau$. Therefore, we have

$$A_1 = r_g\,, \qquad A_2 = \frac{\dot{r}_0 \, a}{\mathrm{v}_{\mathrm{I}}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}}}\,. \tag{4.74}$$

Thus, we transform $\dot{r}$ (4.68) to

$$\dot{r} = -\frac{r_g \mathrm{v}_{\mathrm{I}}}{a}\sqrt{1+\frac{\dot{r}_0^2}{c^2}}\,\sin\Omega\tau + \dot{r}_0\cos\Omega\tau\,. \tag{4.75}$$

Consider now the amplitude A_2 (4.74) for some special cases, where it takes different numerical values:

1. If $\dot{r}_0 = 0$, then we have $A_2 = 0$ according to the definition of A_2 (4.74). In this case, the particles of stellar substance oscillate at the amplitude r_g. In other words, if $\dot{r}_0 = 0$, then the particles cannot reach the surface of the star and, hence, leave the star;
2. If $\dot{r}_0 = \mathrm{v}_{\mathrm{I}}$, then the particles of stellar substance also cannot leave the star. This is because they oscillate with an amplitude, which is as well smaller than the physical radius of the star

$$A_2 = \frac{a}{\sqrt{1+\frac{\mathrm{v}_{\mathrm{I}}^2}{c^2}}} < a\,; \tag{4.76}$$

3. If $\dot{r}_0 = \mathrm{v}_{\mathrm{II}}$, then the particles of stellar substance leave the star. This is because if $\dot{r}_0 = \mathrm{v}_{\mathrm{II}}$, then we have

$$A_2 = \frac{a\sqrt{2}}{\sqrt{1+\frac{\mathrm{v}_{\mathrm{II}}^2}{c^2}}} \simeq \left(1-\frac{\mathrm{v}_{\mathrm{II}}^2}{2c^2}\right) a\sqrt{2} \simeq a\sqrt{2} > a\,; \tag{4.77}$$

4. If $A_2 = a$, then the amplitude is equal to the physical radius of the star. In this case, from the definition of A_2 (4.74), we obtain

$$\dot{r}_0 = \frac{\mathrm{v}_{\mathrm{I}}}{\sqrt{1+\frac{\mathrm{v}_{\mathrm{I}}^2}{c^2}}} \simeq \left(1-\frac{\mathrm{v}_{\mathrm{I}}^2}{2c^2}\right)\mathrm{v}_{\mathrm{I}} < \mathrm{v}_{\mathrm{I}}\,, \tag{4.78}$$

   i.e., the particles of stellar substance are a little slower than the orbital velocity for the star. That is, if the amplitude reaches the physical radius of the star ($A_2 = a$), then the particles can jump out from the surface of the star, but still cannot leave the star into its orbit (they always fall back down on the star).

Thus, our mathematical theory of liquid stars provides a solid theoretical foundation for the stellar wind emitted by a star as a wind consisting of two components. One of the components is slightly slower than the orbital velocity for the star, and the other is faster than the escape velocity of the star. This is consistent with observational data. For example, the solar wind has two components. The slow solar wind travels at about 400 km/sec (slower than the orbital velocity $v_{\rm I} = 440$ km/sec for the Sun). The fast solar wind travels at about 750 km/sec (faster than the escape velocity of the Sun, $v_{\rm II} = 617$ km/sec).

4.4 Conclusion

Let us summarize the main results on the origin of the stellar wind, which we have obtained. The results are as follows:

1. The light-like matter of each star oscillates at a certain frequency

$$\Omega = \frac{c}{a}\sqrt{\frac{r_g}{a}} = \frac{v_{\rm II}}{a} = \sqrt{\frac{2GM}{a^3}}\,, \tag{4.79}$$

characteristic of that particular star. This means that each star has its own characteristic frequency Ω determined according to its mass M and radius a. Therefore, we call the Ω the *proper frequency* of the star;

2. The mentioned oscillation occurs with two amplitudes. The amplitude $A_1 = r_g$ coincides with the radius of the Hilbert core of the star, on the surface of which stellar energy is released. The other amplitude $A_2 = \sqrt{a^3/r_g}$ coincides with the radius of the outer space breaking in the star's field, which is in the outer cosmos. For the Sun, $A_2 = 3.4 \times 10^{13}$ cm $= 2.3$ AU coincides with the maximum concentration of asteroids in the asteroid belt;

3. This is a common oscillation of the entire light-like matter of the star. Its origin is the gravitational field of the star, the source of which is the star's mass M. In other words, this oscillation is the own "breathing" of the star;

4. Particles of the liquid substance of each star oscillate with two amplitudes at a certain frequency

$$\Omega = \sqrt{\frac{\left(c^2 + \dot{r}_0^2\right) r_g}{2a^3}} = \frac{v_{\rm II}}{a\sqrt{2}}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}} = \frac{v_{\rm I}}{a}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}}\,, \tag{4.80}$$

which is determined by the mass and radius of that particular star, and also is dependent on the initial velocity $\dot{r}_0$ of the particles.

This frequency can be expressed through the escape velocity v_{II} and the orbital velocity v_{I}, calculated for the star;

5. Their oscillation amplitude $A_1 = r_g$ is the same as that of the light-like matter (photons) of the star. The other amplitude A_2 depends on the initial velocity of the particles

$$A_2 = \frac{\dot{r}_0 a \sqrt{2a}}{\sqrt{\left(c^2 + \dot{r}_0^2\right) r_g}} = \frac{\dot{r}_0 a}{\mathrm{v}_{\mathrm{I}} \sqrt{1 + \frac{\dot{r}_0^2}{c^2}}}\,; \tag{4.81}$$

6. Stars radiate light (photons) and erupt particles of stellar substance (stellar wind) not due to special physical conditions, but automatically. The three-dimensional equation of motion of the particles (both photons and particles of stellar substance), which travel radially from the centre of a liquid star to its surface, is the harmonic free-oscillation equation

$$\ddot{r} + \Omega^2 r = 0\,, \qquad \Omega^2 = -\frac{2F_1}{r} = \frac{c^2 r_g}{a^3}\,, \tag{4.82}$$

where $F_1 = -\frac{c^2 r_g r}{2a^3}$ is the linearized form (in the sense of $r_g \ll a$) of the force of gravity acting inside any liquid star. This is a non-Newtonian gravitational force proportional to distance, which is the cause of the mentioned common oscillation of both light-like stellar matter and stellar substance. Once the oscillation amplitude exceeds the physical radius of the star, the particles come out the star for the cosmos. Therefore, we arrive at a conclusion that the cause of both stellar radiation and stellar wind is the internal structure of the bodies of stars, which are liquid spheres in the weightless state in the cosmos;

7. According to the theory, the stellar wind emitted by a star consists of two components: a slow stellar wind and a fast stellar wind. The particles, the oscillation amplitude of which reaches the star's surface ($A_2 = a$), have the initial velocity

$$\dot{r}_0 = \frac{\mathrm{v}_{\mathrm{I}}}{\sqrt{1 + \frac{\mathrm{v}_{\mathrm{I}}^2}{c^2}}} \simeq \left(1 - \frac{\mathrm{v}_{\mathrm{I}}^2}{2c^2}\right) \mathrm{v}_{\mathrm{I}} < \mathrm{v}_{\mathrm{I}}\,, \tag{4.83}$$

which does not exceed the orbital velocity v_{I} for the star. The particles that are as fast as the escape velocity of the star ($\dot{r}_0 = \mathrm{v}_{\mathrm{II}}$)

have the oscillation amplitude

$$A_2 = \frac{a\sqrt{2}}{\sqrt{1+\frac{\mathrm{v}_{\mathrm{II}}^2}{c^2}}} \simeq \left(1 - \frac{\mathrm{v}_{\mathrm{II}}^2}{2c^2}\right) a\sqrt{2} \simeq a\sqrt{2} > a\,. \qquad (4.84)$$

This means that the slow stellar wind consists of the particles, the oscillation amplitude of which is in the range of $a \leqslant A_2 < a\sqrt{2}$. These particles leave the surface of the star, but not forever. They always fall back down on the star. The fast stellar wind consists of the particles, the oscillation amplitude of which is $A_2 \geqslant a\sqrt{2}$. They leave the gravitational field of the star, forever, for the outer cosmos. This theoretical result is consistent with observational data: the solar wind is divided into the slow solar wind travelling at ~ 400 km/sec (slower than $\mathrm{v}_{\mathrm{I}\odot} = 440$ km/sec) and the fast solar travelling at ~ 750 km/sec (faster than $\mathrm{v}_{\mathrm{II}\odot} = 617$ km/sec).

Chapter 5 Neutron Stars and Pulsars

5.1 Introducing the space metric of a rotating neutron star

This Chapter is the shortest and most mathematically complicate of the other Chapters in this book. Here we will apply our liquid star model to neutron stars and pulsars. The high level of complexity is due to the fact that as soon as we introduce a rotation even around one coordinate axis into the metric of a space, further calculations become very problematic. Anyhow, let us begin.

Neutron stars and pulsars are Type II of our classification of stars based on General Relativity (see Table 1.1 in §1.2). This means that the physical radius a of such a star is slightly larger than its Hilbert radius r_g: the star is almost a collapsar, but still has the ability to glow like an ordinary star. In §1.2, we showed that the space metric of a liquid sphere transforms into the de Sitter metric of a vacuum sphere under the condition of gravitational collapse $a = r_g$ (when the liquid sphere is a collapsar). This metric has the form (1.16)

$$ds^2 = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \qquad (5.1)$$

The physical parameters of neutron stars and pulsars are close to those of collapsars, but do not coincide (see Table 1.1). Therefore, the metric (5.1), which includes the collapse condition, is close to the metric of a neutron star or pulsar, but is not.

How to modify the space metric of a collapsar (5.1) to obtain the metric of a neutron star or pulsar? To get out of the state of collapse, but at the same time be close to it. Easy.

Recall that the particular condition of gravitational collapse ($g_{00} = 0$) follows from the general condition of gravitational collapse, according to which the physically observable time τ (1.30) stops on the surface of the object

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = 0\,. \tag{5.2}$$

If the space of the object does not rotate (all $g_{0i} = 0$), then the mentioned particular condition of collapse ($g_{00} = 0$) occurs.

But if the object rotates (i.e., if at least one of the three quantities g_{0i} is non-zero), then the condition $g_{00} = 0$ can remain true on the surface of the object but does not mean gravitational collapse. This is due to the second term of the complete condition of collapse (5.2), which is non-zero in this case. Therefore, if a rotation is introduced into the metric (5.1), then the metric describes a liquid sphere that is outside the state of gravitational collapse. The faster the sphere rotates, the more its state is different from the state of a collapsed sphere.

If we add a rotation to the space metric of a collapsed liquid sphere, and also find Einstein's field equations in a form that contains a strong magnetic field and at the same time satisfies this metric, then we get a complete description of a rotating neutron star or pulsar. This is our research plan for this Chapter.

First, we add a rotation to the space metric of a collapsed liquid sphere (5.1) according to the chronometric invariant formalism: see the formulae (1.45) of §1.3. Assume that the object — a liquid sphere of a radius a — rotates with an angular velocity ω along its equatorial axis (the axis ϕ in the spherical coordinates r, θ, ϕ). In this case, the initially metric of a collapsed liquid sphere (5.1) takes the following form

$$ds^2 = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right)c^2dt^2 + \frac{2\omega r^2\cos\theta}{c}\,c\,dt\,d\phi - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \tag{5.3}$$

which means that the sphere is not a collapsar due to its rotation.

The linear velocity of its rotation is determined by g_{0i} of the space metric according to the general formula (1.45). In the present case, i.e., in the metric (5.3), it has the form

$$v_1 = v_2 = 0\,, \qquad v_3 = -\frac{2\omega a r^2\cos\theta}{\sqrt{a^2 - r^2}}\,. \tag{5.4}$$

The maximum rotation velocity of neutron stars, registered in astronomical observations, is about 1,000 km/sec. Therefore, we neglect the term $\frac{v^2}{c^2}$, where $v^2 = h^{ik} v_i v_k \ll c^2$.

The three-dimensional observable chr.inv.-metric tensor h_{ik} (1.34) of the space metric (5.3) has the components

$$h_{11} = \frac{1}{h^{11}} = \frac{a^2}{a^2 - r^2}, \quad h_{22} = \frac{1}{h^{22}} = r^2, \quad h_{33} = \frac{1}{h^{33}} = r^2 \sin^2\theta, \tag{5.5}$$

and the determinant $h = \det \|h_{ik}\|$ of the chr.inv.-metric tensor h_{ik} and the non-zero spatial derivatives of $\ln\sqrt{h}$ have the form

$$h = \det \|h_{ik}\| = \frac{a^2 r^4 \sin^2\theta}{a^2 - r^2}, \tag{5.6}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial r} = \frac{2a^2 - r^2}{r\,(a^2 - r^2)}, \tag{5.7}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial \theta} = \cot\theta. \tag{5.8}$$

In addition, due to the assumed condition $v^2 \ll c^2$ (non-relativistic rotation of the object), the chr.inv.-derivation operator along the spatial coordinates (1.41) is the same as the ordinary derivation operator.

Using the formulae for g_{00} and g_{0i} of the metric (5.3), we now obtain the chr.inv.-vector F_i of the gravitational inertial force and the chr.inv.-tensor A_{ik} of the angular velocity with which the space rotates. According to the definitions of these quantities (see §1.3), we obtain

$$F_1 = \frac{c^2 r}{a^2 - r^2}, \qquad F^1 = \frac{c^2 r}{a^2}, \tag{5.9}$$

$$\left.\begin{aligned} A_{13} &= -\frac{2\omega a^3 r \cos\theta}{(a^2 - r^2)^{3/2}}, \quad & A^{13} &= -\frac{2\omega a \cos\theta}{r\sqrt{a^2 - r^2}\,\sin^2\theta} \\ A_{23} &= \frac{\omega a r^2 \sin\theta}{\sqrt{a^2 - r^2}}, \quad & A^{23} &= \frac{\omega a}{r^2\sqrt{a^2 - r^2}\,\sin\theta} \end{aligned}\right\}. \tag{5.10}$$

After some algebra according to the space metric (5.3), we obtain formulae for the chr.inv.-Christoffel symbols Δ^i_{kn}, the physically observable chr.inv.-curvature tensor C_{iklj} and the contraction $C_{ik} = h^{mn} C_{imkn}$

(which is the chr.inv.-analogy of Ricci's tensor). Their non-zero components have the following form

$$\Delta^1_{11} = \frac{r}{a^2 - r^2}\,, \qquad \Delta^1_{22} = -\frac{r\left(a^2 - r^2\right)}{a^2}\,, \tag{5.11}$$

$$\Delta^1_{33} = -\frac{r\left(a^2 - r^2\right)}{a^2}\sin^2\theta, \quad \Delta^2_{12} = \Delta^3_{13} = \frac{1}{r}\,, \tag{5.12}$$

$$\Delta^2_{33} = -\sin\theta\cos\theta, \qquad \Delta^3_{23} = \cot\theta, \tag{5.13}$$

$$C_{1212} = -\frac{r^2}{a^2 - r^2}\,, \tag{5.14}$$

$$C_{1313} = -\frac{r^2}{a^2 - r^2}\sin^2\theta\,, \tag{5.15}$$

$$C_{2323} = -\frac{r^4}{a^2}\sin^2\theta\,, \tag{5.16}$$

$$C_{11} = -\frac{2}{a^2 - r^2}\,, \;\; C_{22} = -\frac{2r^2}{a^2}\,, \;\; C_{33} = -\frac{2r^2}{a^2}\sin^2\theta. \tag{5.17}$$

Using these characteristics of the space metric (5.3), we will deduce Einstein's field equations in a form satisfying the metric. This is the next step in our research into neutron stars and pulsars.

5.2 Einstein's equations and the conservation law equations satisfying the metric

Let us consider the chr.inv.-Einstein field equations in the general form (1.92–1.94). In a stationary space (which means that the space does not deform), such as a space of the metric (5.3), which we suggest to neutron stars and pulsars, the chr.inv.-Einstein equations are simplified

$$A_{jl}A^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}F_j\right)F^j = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \lambda c^2, \tag{5.18}$$

$$\frac{2}{c^2}F_jA^{ij} - {}^*\nabla_j A^{ij} = \varkappa J^i, \tag{5.19}$$

$$2A_{ij}A^{\cdot j}_{k\cdot} + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}F_iF_k - c^2C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right) + \lambda c^2 h_{ik}\,. \tag{5.20}$$

The right hand side of the equations contains the chr.inv.-projections (1.91) of the energy-momentum tensor of a matter that fills the space: the observable mass density ρ, the observable momentum density J^i and the observable stress tensor U^{ik}, while $U = h^{ik}U_{ik}$ is the trace of the observable stress tensor. Note that the energy-momentum tensor has an arbitrary form here. This means that the kind of distributed matter is not specified for yet.

Substitute, into the above equations, the chr.inv.-characteristics of the metric (5.3). While doing so we should take into account the fact that the initially (non-rotating) metric (5.1) was deduced under the obvious conditions $a^2 = \frac{3}{\lambda} > 0$ and $\lambda > 0$ (see §1.2 for detail). As a result, we transform the chr.inv.-Einstein equations (5.18–5.20) to the form satisfying the conditions

$$\frac{8\omega^2 a^4 \cot^2\theta}{(a^2 - r^2)^2} + \frac{2\omega^2 a^2}{a^2 - r^2} = \frac{\varkappa}{2}\left(\rho c^2 + U\right), \tag{5.21}$$

$$\frac{2\omega a \cot\theta}{r^2\sqrt{a^2 - r^2}\,\sin\theta} = -\varkappa J^3, \tag{5.22}$$

$$\frac{8\omega^2 a^4 \cot^2\theta}{(a^2 - r^2)^2} - \frac{\varkappa}{2}\left(\rho c^2 - U\right) = \frac{\varkappa U_{11}\left(a^2 - r^2\right)}{a^2}, \tag{5.23}$$

$$\frac{4\omega^2 a^4 r \cot\theta}{(a^2 - r^2)^2} = -\varkappa U_{12}, \tag{5.24}$$

$$\frac{2\omega^2 a^2}{a^2 - r^2} - \frac{\varkappa}{2}\left(\rho c^2 - U\right) = \frac{\varkappa U_{22}}{r^2}, \tag{5.25}$$

$$\frac{2\omega^2 a^2}{a^2 - r^2} + \frac{8\omega^2 a^4 \cot^2\theta}{(a^2 - r^2)^2} - \frac{\varkappa}{2}\left(\rho c^2 - U\right) = \frac{\varkappa U_{33}}{r^2 \sin^2\theta}. \tag{5.26}$$

Taking into account that $U = h^{ik}U_{ik} = h^{11}U_{11} + h^{22}U_{22} + h^{33}U_{33}$, we use the three respective tensor equations of these to obtain the relation connecting the quantities ρ and U

$$\frac{16\omega^2 a^4 \cot^2\theta}{(a^2 - r^2)^2} + \frac{4\omega^2 a^2}{a^2 - r^2} = \frac{\varkappa}{2}\left(3\rho c^2 - U\right). \tag{5.27}$$

Summing up (5.21) and (5.27), we obtain a formula for the density of the distributed matter that fills the space inside a rotating neutron star or pulsar. The formula is

$$\frac{12\,\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} + \frac{3\,\omega^2 a^2}{a^2-r^2} = \varkappa\rho c^2. \tag{5.28}$$

Multiplying (5.21) by 3, then subtracting (5.27) from the obtained product, we obtain the formula for U

$$\frac{4\,\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} + \frac{\omega^2 a^2}{a^2-r^2} = \varkappa U. \tag{5.29}$$

Comparing the obtained formulae (5.28) and (5.29), we see that the ρ and U specific for the matter filling a rotating neutron star or pulsar are connected by the following relation

$$U = \frac{1}{3}\rho c^2. \tag{5.30}$$

Finally, we transform the tensor equations of the chr.inv.-Einstein equations (5.21–5.26) so that they express the non-zero contravariant components of the stress tensor: $U^{11} = h^{1m}h^{1n}U_{mn}$, $U^{12} = h^{1m}h^{2n}U_{mn}$, $U^{22} = h^{2m}h^{2n}U_{mn}$, $U^{33} = h^{3m}h^{3n}U_{mn}$. Taking into account the obtained formulae for $U = \frac{1}{3}\rho c^2$ (5.30) and ρ (5.28), we have

$$\varkappa U^{11} = \frac{8\,\omega^2 a^2 \cot^2\theta}{a^2-r^2} - \frac{\varkappa\rho c^2\left(a^2-r^2\right)}{3a^2}, \tag{5.31}$$

$$\varkappa U^{12} = -\frac{4\,\omega^2 a^2 \cot\theta}{r\,(a^2-r^2)}, \tag{5.32}$$

$$\varkappa U^{22} = \frac{1}{r^2}\left(\frac{2\,\omega^2 a^2}{a^2-r^2} - \frac{\varkappa\rho c^2}{3}\right), \tag{5.33}$$

$$\varkappa U^{33} = \frac{1}{r^2\sin^2\theta}\left[\frac{2\,\omega^2 a^2}{a^2-r^2} + \frac{8\,\omega^2 a^4}{(a^2-r^2)^2} - \frac{\varkappa\rho c^2}{3}\right]. \tag{5.34}$$

Now, we have to check whether the obtained chr.inv.-Einstein equations (i.e. the given particular type of distributed matter) satisfying the metric (5.3) or not.

How to do it? The terms consisting Einstein's field equations are of two sorts. These are the characteristics of a particular space and the characteristics of a matter that fills the space (the latter are the chr.inv.-components of the energy-momentum tensor of the matter). Suppose that we have obtained, in another way, the components of the energy-momentum tensor for the matter, which are expressed through the characteristics of the space. Then, substituting the energy-momentum tensor components into the Einstein equations, we will see: if the equations become identities, then they satisfy the particular space; and if not, then they do not satisfy.

To find how the ρ, J^i and U^{ik} of the obtained chr.inv.-Einstein equations are expressed through the characteristics of the space, we consider the conservation law equations (1.96–1.97). They are the chr.inv.-notation of the conservation law $\nabla_\sigma T^{\alpha\sigma}=0$ for the energy-momentum tensor of a distributed matter.

In a non-deforming space, such as a space of the metric (5.3) that we have proposed to neutron stars and pulsars, the chr.inv.-conservation law equations (1.96–1.97) take the simplified form

$$\frac{^*\partial\rho}{\partial t} + {}^*\widetilde{\nabla}_i J^i - \frac{1}{c^2}F_i J^i = 0\,, \tag{5.35}$$

$$\frac{^*\partial J^k}{\partial t} + 2A_{i\cdot}^{\cdot k} J^i + {}^*\widetilde{\nabla}_i U^{ik} - \rho F^k = 0\,, \tag{5.36}$$

where we denote ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}F_i$. From the obtained chr.inv.-Einstein equations, we see that only $J^3\neq 0$ of the observable momentum density J^i in the rotating liquid sphere. In addition, as was shown in §4.2, only $F_1\neq 0$ in the sphere. Therefore, for the chr.inv.-scalar conservation equation (5.35), we have

$$\begin{aligned} {}^*\widetilde{\nabla}_i J^i - \frac{1}{c^2}F_i J^i = {}^*\widetilde{\nabla}_3 J^3 - \frac{1}{c^2}F_3 J^3 = \\ = \left(\frac{^*\partial J^3}{\partial\phi} + J^3\Delta^j_{j3} - \frac{1}{c^2}F_3 J^3\right) - \frac{1}{c^2}F_3 J^3 = 0\,. \end{aligned} \tag{5.37}$$

As a result, the chr.inv.-scalar conservation equation (5.35) transforms into the condition

$$\frac{^*\partial\rho}{\partial t} = 0\,, \tag{5.38}$$

which means that the observable density of the matter (liquid substance and fields) that fills the sphere is stationary.

Of the three vector equations of the conservation law (5.36), the equation with the index $k=3$ vanishes. The remaining two vector equations (with the indices $k=1,2$) take the form, respectively

$$\frac{2A_{31}\left(a^2-r^2\right)}{a^2}J^3+\frac{\partial U^{11}}{\partial r}+\frac{\partial U^{12}}{\partial\theta}+\left(\frac{\partial\ln\sqrt{h}}{\partial\theta}\right)U^{12}+ \\ +\Delta^1_{22}U^{22}+\Delta^1_{33}U^{33}+\left(\Delta^1_{11}+\frac{\partial\ln\sqrt{h}}{\partial r}-\frac{1}{c^2}F_1\right)U^{11}=\rho F^1, \quad (5.39)$$

$$\frac{2A_{32}}{r^2}J^3+\frac{\partial U^{12}}{\partial r}+\frac{\partial U^{22}}{\partial\theta}+\left(\frac{\partial\ln\sqrt{h}}{\partial\theta}\right)U^{22}+ \\ +\Delta^2_{33}U^{33}+\left(2\Delta^2_{12}+\frac{\partial\ln\sqrt{h}}{\partial r}-\frac{1}{c^2}F_1\right)U^{12}=0\,. \quad (5.40)$$

Apply the characteristics of the space of a rotating liquid sphere and the characteristics of the matter that fills it. The formulae for U^{ik} (5.31–5.34) and J^3 (5.22) come from the chr.inv.-Einstein equations. The formulae for the logarithmic derivatives have the form (5.7, 5.8). The formula for ρ has the form (5.28). The acting gravitational inertial force F_1 has the form (5.9), and the non-zero components A_{13} and A_{23} of the angular velocity with which the space rotates have the form (5.10). When all these formulae are substituted into the remaining conservation law equations (5.39, 5.40), after some algebra we see that these equations also vanish.

So, the common solution to Einstein's field equations and the conservation law equations in the space of a rotating liquid sphere showed that the proposed equations are valid in the space. In other words, the space metric (5.3) that we have proposed to neutron stars or pulsars satisfies Einstein's field equations (and vice versa).

5.3 Introducing the electromagnetic field

As is known, every rotating neutron star or pulsar has a strong magnetic field. Therefore, we move on to the next stage of this research. We need to introduce such an energy-momentum tensor that describes the electromagnetic field and satisfies the relation $U=\frac{1}{3}\rho c^2$ (5.30) that follows from the obtained chr.inv.-Einstein equations. As soon as the energy-momentum tensor is obtained, it will be possible to deduce the equations of the electromagnetic field. Then we will conclude how the electromagnetic field is distributed inside a rotating neutron star or pulsar according to our theory. This is our plan for now.

The energy-momentum tensor of an arbitrary electromagnetic field has the following general form

$$T^{\alpha\beta}_{\text{em}} = \frac{1}{4\pi c^2}\left(-F^{\alpha\cdot}_{\cdot\sigma}F^{\beta\sigma} + \frac{1}{4}\, g^{\alpha\beta}F_{\mu\sigma}F^{\mu\sigma}\right). \tag{5.41}$$

Here $F_{\alpha\beta}$ is the electromagnetic field tensor — the curl of the four-dimensional electromagnetic field potential A^{α}

$$F_{\alpha\beta} = \nabla_{\alpha}A_{\beta} - \nabla_{\beta}A_{\alpha} = \frac{\partial A_{\beta}}{\partial x^{\alpha}} - \frac{\partial A_{\alpha}}{\partial x^{\beta}}\,, \tag{5.42}$$

the physically observable chr.inv.-projections of which are the scalar potential φ and the vector potential q^i of the electromagnetic field

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i. \tag{5.43}$$

The electromagnetic field theory, expressed in terms of chronometric invariants, is well developed in our book [18]. Here we follow the theory and refer everyone who is interested in the details to [18].

The physically observable chr.inv.-projections of the electromagnetic field tensor $F_{\alpha\beta}$ (5.42) have the form

$$\rho_{\text{em}} = \frac{T_{00}}{g_{00}} = \frac{E_i E^i + H_{*i}H^{*i}}{8\pi c^2}\,, \tag{5.44}$$

$$J^i_{\text{em}} = \frac{c\,T^i_0}{\sqrt{g_{00}}} = \frac{1}{4\pi c}\,\varepsilon^{ikm}E_k H_{*m}\,, \tag{5.45}$$

$$U^{ik}_{\text{em}} = c^2 T^{ik} = \rho_{\text{em}}c^2 h^{ik} - \frac{1}{4\pi}\left(E^i E^k + H^{*i}H^{*k}\right), \tag{5.46}$$

where E^i is the three-dimensional chr.inv.-electric strength vector, H^{*i} is the three-dimensional chr.inv.-magnetic strength pseudovector, and ε^{imn} is the unit completely antisymmetric three-dimensional chr.inv.-pseudotensor. They are expressed as [18]

$$\left.\begin{aligned} E^{*ik} &= -\varepsilon^{ikn}E_n, \quad & E_n &= \frac{{}^*\partial\varphi}{\partial x^n} + \frac{1}{c}\frac{{}^*\partial q_n}{\partial t} - \frac{\varphi}{c^2}F_n \\ H^{*i} &= \frac{1}{2}\,\varepsilon^{imn}H_{mn}, \quad & H_{mn} &= \frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}A_{mn} \end{aligned}\right\}. \tag{5.47}$$

We see that the observable electric and magnetic strengths depend on not only the electromagnetic field itself (the scalar and vector electromagnetic field potentials), but also on the acting gravitational inertial force F_i and the angular velocity A_{ik} with which the space rotates.

Pulsars are massive objects with a strong electromagnetic field and rapid rotation. Therefore, the factors of F_i and A_{ik} are significant in our consideration. At the same time, we neglect time variations and spatial inhomogeneity of the electromagnetic field potentials assuming the conditions (3.106)

$$\left.\begin{array}{l} \dfrac{{}^*\partial\varphi}{\partial t} = 0\,, \quad \dfrac{{}^*\partial\varphi}{\partial x^i} = 0 \\ \dfrac{{}^*\partial q_i}{\partial t} = 0\,, \quad q_{ik} = \dfrac{{}^*\partial q_i}{\partial x^k} - \dfrac{{}^*\partial q_k}{\partial x^i} = 0 \end{array}\right\}, \tag{5.48}$$

i.e., we assume that the electromagnetic field is stationary and vortex-free. Under the above assumptions, the electric and magnetic strengths of the field take the simplified form

$$E_i = -\frac{\varphi}{c^2}\,F_i\,, \qquad H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}, \tag{5.49}$$

where

$$\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn}\,, \qquad \Omega_{*i} = \frac{1}{2}\,\varepsilon_{imn}A^{mn} \tag{5.50}$$

is the three-dimensional chr.inv.-pseudovector of the angular velocity with which the star rotates.

The chr.inv.-tensor A_{ik} of the angular velocity with which a space rotates is determined by the metric of that particular rotating space. Its components, calculated for the metric of a rotating neutron star or pulsar, are presented in the formula (5.10).

Note that the formulae for the electric and magnetic strengths (5.49), which we finally propose to neutron stars and pulsars, show that the electromagnetic field of such a star depends on its gravitational field and rotation. Namely, — even if the electromagnetic field potential φ is presented in the star, the electric field strength E^i is manifested only due to the gravitational field of the star, and the magnetic field strength H^{*i} is manifested only if the star rotates.

Using the formulae for E^i and H^{*i} (5.49) and all the mentioned assumptions that we have proposed to neutron stars and pulsars, we transform the physically observable components of the electromagnetic field tensor $F_{\alpha\beta}$ (5.44–5.45) to the form

$$\rho_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right), \tag{5.51}$$

$$J^i_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm} F_k\,\Omega_{*m}, \tag{5.52}$$

$$U^{ik}_{\text{em}} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right)h^{ik} - \frac{\varphi^2}{\pi c^2}\left(\frac{F^i F^k}{4c^2} + \Omega^{*i}\Omega^{*k}\right), \tag{5.53}$$

which corresponds to a vortex-free electromagnetic field. From here, we obtain the formula for $U_{\text{em}} = h_{ik} U^{ik}_{\text{em}}$

$$U_{\text{em}} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right) = \rho_{\text{em}} c^2. \tag{5.54}$$

As is seen from this formula, we have $U = \rho c^2$ in the framework of the assumed conditions related to the particular electromagnetic field. However, as we have obtained earlier for the space metric of a rotating neutron star or pulsar, there should be $U = \frac{1}{3}\rho c^2$ (5.30). In other words, according to the metric, we should have

$$U_{\text{em}} = \frac{1}{3}\,\rho_{\text{em}} c^2, \tag{5.55}$$

where

$$U_{\text{em}} = \frac{\Omega_{*j}\,\Omega^{*j}}{\varkappa}, \qquad \rho_{\text{em}} = \frac{3\,\Omega_{*j}\,\Omega^{*j}}{\varkappa c^2}. \tag{5.56}$$

Therefore, our task now is to find such a physical condition under which the electromagnetic field satisfies the conditions (5.56) and, therefore, (5.55).

Let us find this condition. Using the obtained relation $U_{\text{em}} = \rho_{\text{em}} c^2$ (5.54), we re-write the formula $U_{\text{em}} = \frac{1}{\varkappa}\,\Omega_{*j}\,\Omega^{*j}$ (5.56) in the form

$$\frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right) = \frac{\Omega_{*j}\,\Omega^{*j}}{\varkappa}, \tag{5.57}$$

or, since $\varkappa = \frac{8\pi G}{c^2}$, in the equivalent form

$$c^2\,\Omega_{*j}\,\Omega^{*j} = \frac{\frac{G\varphi^2}{c^4}}{1 - \frac{4G\varphi^2}{c^4}}\,F_j F^j. \tag{5.58}$$

Note that the quantity $\frac{G\varphi^2}{c^4}$ is dimensionless. The scalar electromagnetic potential is constant $\varphi = const$ according to our initial assumptions. Therefore and since the magnetic strength is $H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}$ (5.49), we obtain that any stationary rotating star is a *permanent magnet*.

Denote

$$\frac{G\varphi^2}{c^4} = n\,, \tag{5.59}$$

where $n < \frac{1}{4}$, while c and G are the fundamental constants. Therefore, we obtain the numerical value

$$\varphi = \frac{c^2}{2\sqrt{G}} < 1.74\times 10^{24}\ \text{gram}^{1/2}\,\text{cm}^{1/2}\text{sec}^{-1}. \tag{5.60}$$

With the scalar electromagnetic field potential φ within this scale of magnitudes, the electromagnetic field satisfies the space metric of a rotating neutron star or pulsar.

As a result, we re-write the obtained formula (5.58) in the form

$$c^2\,\Omega_{*j}\,\Omega^{*j} = \frac{n}{1-4n}\,F_i F^i, \quad n < \frac{1}{4}\,. \tag{5.61}$$

Under this particular condition, which links the acting force of gravity to the angular velocity with which the space rotates, the electromagnetic field satisfies the space metric and the Einstein field equations, which we proposed for rotating neutron stars and pulsars.

5.4 Distribution of the magnetic field of a pulsar

To find out how the magnetic field strength is distributed over the surface of a rotating neutron star or pulsar, consider Maxwell's equations. The general covariant formulation of the two groups of Maxwell's equations is as follows

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c}\,j^\mu, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,, \tag{5.62}$$

where $F^{*\mu\sigma} = \varepsilon^{\mu\sigma\alpha\beta}F_{\alpha\beta}$ is the pseudotensor dual to the electromagnetic field tensor $F_{\alpha\beta}$, while j^μ is the four-dimensional current vector.

This formulation of Maxwell's equations implies an arbitrary electromagnetic field. Let us transform the equations, taking into account our assumptions specific to rotating neutron stars and pulsars. As before, we neglect time variations and spatial inhomogeneity of the electromagnetic field potentials, assuming the conditions (5.48).

Since we are considering the electromagnetic field of a star on its surface, the four-dimensional current vector is zero: $j^{\mu}=0$. This means that the electromagnetic field of the star does not contain sources such as electric charges and currents on the star's surface (all the electromagnetic field sources are inside the star).

In this case, Maxwell's equations (5.62) take the simplified form

$$\nabla_{\sigma} F^{\mu\sigma} = 0\,, \qquad \nabla_{\sigma} F^{*\mu\sigma} = 0\,. \tag{5.63}$$

Write down Maxwell's equations (5.63) in terms of the chronometrically invariant formalism. In an electromagnetic field without sources, the chr.inv.-Maxwell equations take the form

$$\left.\begin{aligned} &{}^{*}\nabla_{j} E^{j} - \frac{1}{c} H^{ik} A_{ik} = 0 \\ &{}^{*}\nabla_{k} H^{ik} - \frac{1}{c^2} F_{k} H^{ik} - \frac{1}{c}\left(\frac{{}^{*}\partial E^{i}}{\partial t} + D E^{i}\right) = 0 \end{aligned}\right\} \text{I}, \tag{5.64}$$

$$\left.\begin{aligned} &{}^{*}\nabla_{i} H^{*i} - \frac{1}{c} E^{*ik} A_{ik} = 0 \\ &{}^{*}\nabla_{k} E^{*ik} - \frac{1}{c^2} F_{k} E^{*ik} - \frac{1}{c}\left(\frac{{}^{*}\partial H^{*i}}{\partial t} + D H^{*i}\right) = 0 \end{aligned}\right\} \text{II}, \tag{5.65}$$

see Chapter 3 of the book [18]. Here $E^{*ik}=-\varepsilon^{ikn}E_k$ is the pseudotensor dual to the electric strength vector E_i, $H^{*i}=\frac{1}{2}\,\varepsilon^{imn}H_{mn}$ is the pseudovector dual to the magnetic strength tensor H_{mn}, and $D=h^{ik}D_{ik}$ is the deformation rate of the space.

Since the space of a rotating liquid sphere under consideration does not deform and also, according to our initial assumptions, the electric and magnetic strengths are stationary, the above chr.inv.-Maxwell equations without the field sources take the simplified form

$$\left.\begin{aligned} &{}^{*}\nabla_{j} E^{j} - \frac{1}{c} H^{ik} A_{ik} = 0 \\ &{}^{*}\nabla_{k} H^{ik} - \frac{1}{c^2} F_{k} H^{ik} = 0 \end{aligned}\right\} \text{I}, \tag{5.66}$$

$$\left.\begin{aligned} &{}^{*}\nabla_{i} H^{*i} - \frac{1}{c} E^{*ik} A_{ik} = 0 \\ &{}^{*}\nabla_{k} E^{*ik} - \frac{1}{c^2} F_{k} E^{*ik} = 0 \end{aligned}\right\} \text{II}. \tag{5.67}$$

Substitute, into the chr.inv.-Maxwell equation (5.66, 5.67) which have already been adapted to the space metric of a rotating neutron star or pulsar, the respective formulae for E^i and H^{ik} (5.49) (and those for their dual pseudotensors) and also the respective characteristics of the space which we have obtained in §5.1.

The first (scalar) equation of the Group I (5.66) takes the form

$$\frac{c^2}{a^2}\,\frac{3a^2-2r^2}{a^2-r^2} = 4\,\Omega_{*j}\,\Omega^{*j}. \tag{5.68}$$

Two of the vector equations of the Group I vanish, while the third vector equation takes the form

$$\frac{2\omega a^3}{r^3\,(a^2-r^2)\sqrt{a^2-r^2}}\,\frac{\cot\theta}{\sin\theta} = 0\,, \tag{5.69}$$

where ω, according to the space metric of the star (5.3), is the angular velocity with which the star rotates along its equatorial axis ϕ. Both the scalar and vector equations of the Group II (5.67) vanish. Therefore, the dry rest, which we have from the chr.inv.-Maxwell equations adapted to neutron stars and pulsars, are only the equations (5.68) and (5.69).

Due to the obvious assumption that stars are not point-like objects (since $a>0$), and that the radial coordinate is positive ($r>0$), we arrive at the solely valid solution to the equation (5.69)

$$\theta = \pm\frac{\pi}{2}\,. \tag{5.70}$$

The obtained solution means: the only non-vanished vector equation of the Group I has a solution only at the poles of a rotating neutron star or pulsar.

Generally speaking, the vector equation of the Group I determines the chr.inv.-function ${}^*\nabla_k H^{ik}$, the physical sense of which is the physically observable three-dimensional distribution of the magnetic field strength H^{ik} over the surface of the star. Therefore, the solution (5.70) that we have obtained according to our theory of liquid stars means that the magnetic field of a rotating neutron star or pulsar manifests itself only at the South Pole and North Pole of the star.

Calculate the magnetic strength $H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}$ (5.49) for this case. The unit antisymmetric chr.inv.-pseudotensor ε^{ikm} has components explained in detail in Chapter 2 of the book [18]. Thus, after some algebra, we obtain the components of the chr.inv.-pseudovector Ω^{*i} (5.50) of the angular velocity with which the star rotates

$$\Omega^{*1} = \frac{A_{23}}{\sqrt{h}} = \omega, \qquad \Omega_{*1} = A^{23}\sqrt{h} = \frac{\omega a^2}{a^2 - r^2}, \tag{5.71}$$

$$\Omega^{*2} = \frac{A_{31}}{\sqrt{h}} = \frac{2\omega a^2 \cot\theta}{r\,(a^2 - r^2)}, \quad \Omega_{*2} = A^{31}\sqrt{h} = \frac{2\omega a^2 r \cot\theta}{a^2 - r^2}. \tag{5.72}$$

Using the obtained solution $\theta = \pm\frac{\pi}{2}$ (5.70) of the chr.inv.-Maxwell equations, we obtain $\cot\theta = 0$ and, hence, $\Omega^{*2} = \Omega_{*2} = 0$. This means that the magnetic field of a rotating neutron star or pulsar has the solely non-zero component

$$H^{*1} = -\frac{2\varphi}{c}\,\Omega^{*1}, \tag{5.73}$$

which is the radial r-component directed from the centre of the star to its South Pole and North Pole, then — along the respective polar directions from the star into the outer cosmos.

The above result and the solution $\theta = \pm\frac{\pi}{2}$ were obtained on the basis of our mathematical theory of liquid neutron stars and pulsars. These purely theoretical results completely coincide with the well-known observational data about pulsars.

5.5 The frequency and the magnetic field strength of a pulsar

Individual pulses of the electromagnetic radiation emitted by a pulsar (rapidly rotating neutron star) are repeated at a frequency equal to the rotation frequency of the pulsar itself. Let us calculate the pulse frequency of a typical pulsar, based on our theory of liquid neutron stars and pulsars.

Calculating $\Omega_{*j}\,\Omega^{*j}$ at the South Pole and North Pole of a rotating neutron star (pulsar), where $\theta = \pm\frac{\pi}{2}$, we obtain

$$\Omega_{*j}\,\Omega^{*j} = \Omega_{*1}\,\Omega^{*1} = \frac{\omega^2 a^2}{a^2 - r^2}. \tag{5.74}$$

Then the relation (5.58) between the angular velocity with which the star rotates and the acting gravitational force takes the form

$$\frac{\omega^2 a^2}{a^2 - r^2} = \frac{n}{1 - 4n}\,\frac{c^2 r^2}{a^2\,(a^2 - r^2)}. \tag{5.75}$$

The magnetic field strength of a rotating neutron star or pulsar is $H^{*1} = -\frac{2\varphi}{c}\Omega^{*1}$ (5.73). It is due to the star's rotation. Hence, by studying the obtained relation (5.75), we can draw a conclusion about the electromagnetic radiation of the star.

The relation (5.75) has a breaking at the surface of the star ($r = a$). Therefore, we assume $r \neq a$. Thus, the relation (5.75) takes the form

$$r^2 = \frac{1-4n}{n}\,\frac{\omega^2 a^4}{c^2}\,, \tag{5.76}$$

where r, taking the solution $\theta = \pm\frac{\pi}{2}$ (5.70) into account, is the radial distance from the centre of the star along the polar axis of its rotation.

In the surface layer of the star, from where electromagnetic radiation is emitted to the outer cosmos, we have $r \simeq a$. Thus, after trivial transformations, we obtain the following formula for the pulse frequency of the star's magnetic field

$$\omega = \omega_0\sqrt{\frac{n}{1-4n}}\,, \qquad \omega_0 = \frac{c}{a}\,, \tag{5.77}$$

where ω_0 is the maximum rotation frequency of the star, at which the star rotates with the velocity of light.

Assume that $a = 10^6$ cm, which is the typical radius of a neutron star. With this radius, we obtain

$$\omega_0 = 3\times10^4\ \text{sec}^{-1}. \tag{5.78}$$

It follows from (5.77) that the n is expressed through the pulse frequency of the star's magnetic field as

$$n = \frac{\omega^2}{\omega_0^2 + 4\omega^2}\,. \tag{5.79}$$

The observed frequencies of radio-pulsars are in the range between $\omega_{\min} = 0.53$ and $\omega_{\max} = 448.57\ \text{sec}^{-1}$. This means that $\omega^2 \ll \omega_0^2$. Therefore, we neglect ω in the denominator of (5.79). We obtain

$$n = \frac{\omega^2}{\omega_0^2} = \frac{\omega^2 a^2}{c^2}\,. \tag{5.80}$$

Therefore, for real pulsars, the number n lies in the range

$$3.1\times10^{-10} < n < 2.2\times10^{-4}. \tag{5.81}$$

Also, according to the formula (5.59) obtained in the framework of our theory, the scalar electromagnetic field potential of a pulsar is

$$\varphi = c^2 \sqrt{\frac{n}{G}}\,. \tag{5.82}$$

Consequently, for real pulsars we have

$$6.1 \times 10^{19} < \varphi < 5.2 \times 10^{22} \text{ gram}^{1/2}\,\text{cm}^{1/2}\,\text{sec}^{-1}, \tag{5.83}$$

which is within the upper theoretical limit on the potential φ, which, according to our theory, is $\varphi < 1.74 \times 10^{24}$ (5.60).

Finally, we now calculate, based on our theory, the expected range of the magnetic field strength for pulsars. According to our theory of liquid neutron stars and pulsars, $H^{*1} = -\frac{2\varphi}{c}\,\Omega^{*1}$ (5.73). According to the calculated range of the scalar electromagnetic potential φ and with the estimated range of the rotation frequencies ω of pulsars, we obtain the expected range of the magnetic field strengths for pulsars

$$2.1 \times 10^{9} < H^{*1} < 1.5 \times 10^{15} \text{ gram}^{1/2}\,\text{cm}^{-1/2}\,\text{sec}^{-1}, \tag{5.84}$$

which is very consistent with the magnetic field magnitudes of radio-pulsars, known from radio-astronomical observations.

5.6 Solving Maxwell's equations in the stationary vortex-free electromagnetic field of a pulsar

Previously, in §5.4–§5.5, we solved Maxwell's equations in the electromagnetic field of a rotating neutron star or pulsar, assuming the four-dimensional current vector j^{α} in the field equal to zero ($j^{\alpha} = 0$), which is true on the surface of the star and above, in the cosmos. See (5.63) and so forth. In other words, we assumed that the electromagnetic field does not contain sources (charges and currents).

This assumption creates the following problem. Look at the formula for the observable electromagnetic field momentum J^{i}_{em} (5.52)

$$J^{i}_{\text{em}} = \frac{1}{4\pi c}\,\varepsilon^{ikm} E_k H_{*m} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm} F_k\,\Omega_{*m}\,, \tag{5.85}$$

which is the Poynting vector of the field. Assuming $j^{\alpha} = 0$ in §5.4, we have obtained that only the component $H^{*1} = -\frac{2\varphi}{c}\,\Omega^{*1}$ of the magnetic field strength H^{*i}, i.e., only at the South Pole and North Pole of the star, is non-zero. On the other hand, in this case, the circular momentum J^3_{em} of the field, which should generate the magnetic component H^{*1}, is zero: $J^3_{\text{em}} = \frac{1}{4\pi c}\,\varepsilon^{312}E_1 H_{*2} = 0$. This creates a problem, because a model that satisfies the astronomical evidence for pulsars should obviously show both $H^{*1} = -\frac{2\varphi}{c}\,\Omega^{*1} \neq 0$ and $J^3_{\text{em}} = \frac{1}{4\pi c}\,\varepsilon^{312}E_1 H_{*2} \neq 0$.

Recall that we came to the problem that $H^{*1} \neq 0$ but $J^3_{\text{em}} = 0$ as a result of our assumption that the electromagnetic field does not contain currents ($j^\alpha = 0$). Therefore, we will now solve Maxwell's equations under the condition $j^\alpha \neq 0$.

Let us first solve Maxwell's equations in the vortex-free electromagnetic field of a rotating neutron star of pulsar. In the next §5.7, Maxwell's equations will be solved in the vortical electromagnetic field of such a star.

The space (space-time) metric of a rotating liquid neutron star or pulsar has the form (5.3). This metric means that the liquid sphere is not a collapsar due to its rotation. See the necessary explanation above in §5.1. The physical and geometric characteristics of such a space were calculated and presented in §5.1. In addition to them, it should only be added that the pseudovector of the angular velocity $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn}$ with which the space rotates has the following components

$$\left.\begin{aligned} \Omega^{*1} &= \omega\,, & \Omega_{*1} &= \frac{\omega a^2}{a^2 - r^2} \\ \Omega^{*2} &= \frac{2\omega a^2 \cot\theta}{r\,(a^2 - r^2)}\,, & \Omega_{*2} &= \frac{2\omega a^2 r \cot\theta}{a^2 - r^2} \end{aligned}\right\}, \tag{5.86}$$

hence, the square of the angular velocity pseudovector is

$$\Omega_{*j}\,\Omega^{*j} = \frac{\omega^2 a^2}{a^2 - r^2}\left(1 + \frac{4a^2 \cot^2\theta}{a^2 - r^2}\right). \tag{5.87}$$

As before, we will assume that the scalar and vector electromagnetic potentials are constant and homogeneously distributed (5.48)

$$\left.\begin{aligned} &\frac{{}^*\partial\varphi}{\partial t} = 0\,, \quad \frac{{}^*\partial\varphi}{\partial x^i} = 0 \\ &\frac{{}^*\partial q_i}{\partial t} = 0\,, \quad q_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} = 0 \end{aligned}\right\}, \tag{5.88}$$

i.e., the electromagnetic field is stationary and vortex-free.

In this case, the components of the electric and magnetic strengths (5.47) take the form

$$\left.\begin{aligned} E^i &= -\frac{\varphi}{c^2}\,F^i, & E_i &= -\frac{\varphi}{c^2}\,F_i \\ H^{*i} &= \frac{1}{2}\,\varepsilon^{imn}H_{mn}\,, & H_{mn} &= -\frac{2\varphi}{c}\,A_{mn} \end{aligned}\right\}. \tag{5.89}$$

From here, since $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn}$, we re-write the H^{*i} as

$$H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}, \qquad H_{*i} = -\frac{2\varphi}{c}\,\Omega_{*i}\,. \tag{5.90}$$

Using the formula for F_1 (5.9), then calculating Ω^{*1} and Ω^{*2} from (5.86), we obtain the non-zero components of the E^i and H^{*i}

$$E_1 = -\frac{\varphi r}{a^2-r^2}\,, \qquad E^1 = -\frac{\varphi r}{a^2}\,, \tag{5.91}$$

$$H_{*1} = -\frac{2\varphi\omega a^2}{c\,(a^2-r^2)}\,, \qquad H^{*1} = -\frac{2\varphi\omega}{c}\,, \tag{5.92}$$

$$H_{*2} = -\frac{4\varphi\omega a^2 r\cot\theta}{c\,(a^2-r^2)}\,, \quad H^{*2} = -\frac{4\varphi\omega a^2\cot\theta}{cr\,(a^2-r^2)}\,. \tag{5.93}$$

Let us find how the magnetic field strength is distributed over the surface of a rotating liquid sphere in this case. Consider Maxwell's equations in their complete form (5.62)

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c}\,j^\mu, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,, \tag{5.94}$$

which means the presence of the field current ($j^\alpha \neq 0$). Their physically observable chr.inv.-projections have the form

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c}\,H^{ik}A_{ik} = 4\pi\rho \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2}\,F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + DE^i\right) = \frac{4\pi}{c}\,j^i \end{aligned}\right\} \text{ I}, \tag{5.95}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c}\,E^{*ik}A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2}\,F_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right) = 0 \end{aligned}\right\} \text{ II}, \tag{5.96}$$

see Chapter 3 of the book [18]. Here $E^{*ik} = -\varepsilon^{ikn} E_k$ is the pseudotensor dual to the electric strength vector E_i, $H^{*i} = \frac{1}{2}\,\varepsilon^{imn} H_{mn}$ is the pseudovector dual to the magnetic strength tensor H_{mn}, and $D = h^{ik} D_{ik}$ is the deformation rate of the space.

Since the space of a rotating liquid sphere does not deform, and the electric and magnetic strengths are stationary (according to our initial assumptions), the chr.inv.-Maxwell equations are simplified

$$\left.\begin{aligned} {}^*\nabla_j E^j - \frac{1}{c} H^{ik} A_{ik} &= 4\pi\rho \\ {}^*\nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} &= \frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{ I,} \tag{5.97}$$

$$\left.\begin{aligned} {}^*\nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik} &= 0 \\ {}^*\nabla_k E^{*ik} - \frac{1}{c^2} F_k E^{*ik} &= 0 \end{aligned}\right\} \text{ II.} \tag{5.98}$$

The first equation of the Group I (5.97) takes the form

$$\frac{4\varphi\omega^2 a^2}{c^2\left(a^2 - r^2\right)}\left(1 + \frac{4a^2 \cot^2\theta}{a^2 - r^2}\right) - \frac{\varphi\left(3a^2 - 2r^2\right)}{a^2\left(a^2 - r^2\right)} = 4\pi\rho\,. \tag{5.99}$$

In the second equation of the Group I, $j^1 = j^2 = 0$ in the framework of our model, while the equation for j^3 takes the form

$$\frac{\varphi\omega a^3}{r^2\left(a^2 - r^2\right)\sqrt{a^2 - r^2}}\,\frac{\cot\theta}{\sin\theta} = -\pi j^3, \tag{5.100}$$

and the absolute value of the chr.inv.-current vector j^i is

$$j = \sqrt{j_k j^k} = \frac{\varphi\omega a^3 \cot\theta}{\pi r\left(a^2 - r^2\right)\sqrt{a^2 - r^2}}\,. \tag{5.101}$$

The Group II equations (5.98) are satisfied as identities. Thus, the formulae for ρ, j^3 and j (5.99–5.101) are exact solutions to the chr.inv.-Maxwell equations which we have just considered.

So, we have obtained exact solutions to Maxwell's equations in the internal electromagnetic field of a rotating neutron star or pulsar, where the field arises due to its sources, which are the distributed charges ρ and the currents j^i.

The law of conservation of electric charge determines a connexion between the sources of a electromagnetic field. This law, also known as the continuity equation, has the following general covariant form

$$\nabla_\sigma j^\sigma = 0\,. \tag{5.102}$$

The above law means that the distributed electric charges ρ and the currents j^i (the physically observable chr.inv.-projections of the four-dimensional current vector j^α) are conserved in the four-dimensional volume of the field.

The four-dimensional electromagnetic field potential A^σ must satisfy the general covariant Lorenz condition

$$\nabla_\sigma A^\sigma = 0\,, \tag{5.103}$$

according to which the four-dimensional field potential A^σ and, therefore, its chr.inv.-projections φ and q^i, which are the chr.inv.-scalar and chr.inv.-vector field potentials, are conserved in the four-dimensional field volume.

In a general case, the conservation law (5.102) and the Lorenz condition (5.103) written in terms of the chronometrically invariant formalism, have the form, respectively

$$\frac{^*\partial\rho}{\partial t} + \rho D + {}^*\widetilde{\nabla}_i\, j^i - \frac{1}{c^2}\, F_i\, j^i = 0\,, \tag{5.104}$$

$$\frac{1}{c}\frac{^*\partial\varphi}{\partial t} + \frac{\varphi}{c}\, D + {}^*\widetilde{\nabla}_i\, q^i - \frac{1}{c^2}\, F_i q^i = 0\,, \tag{5.105}$$

where we denote ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}\, F_i$. See Chapter 3 of the book [18].

It is easy to show that, under the particular conditions of the problem that we are considering, the chr.inv.-continuity equation (5.104) and the chr.inv.-Lorenz condition (5.105) are satisfied as identities.

Now, based on the obtained solutions (5.99–5.101) of Maxwell's equations, we look for the chr.inv.-Poynting vector J^i_{em} that is the observable momentum of the electromagnetic field. We need to know how the Poynting vector is distributed over the surface of the sphere, which is the surface of a rotating neutron star or pulsar.

The Poynting vector J^i_{em} is the second of the three physically observable projections of the electromagnetic field tensor $F_{\alpha\beta}$ (5.42), which are ρ_{em} (5.44), J^i_{em} (5.45) and U^{ik}_{em} (5.46). We are looking for the Poynting vector J^i_{em} (5.45) in the framework of our assumptions (5.88), according to which the scalar φ and vector q_i electromagnetic potentials are constant and homogeneously distributed, i.e., the electromagnetic field is stationary and vortex-free.

Substituting the non-zero components (5.91) of the electric strength E^i and the non-zero components (5.93) of the magnetic strength H^{*i} into (5.44–5.46), we obtain

$$\rho_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\,\Omega^{*j}\right) = = \frac{\varphi^2}{2\pi c^4}\left[\frac{\omega^2 a^2}{a^2 - r^2} + \frac{4\,\omega^2 a^4 \cot^2\theta}{(a^2 - r^2)^2} + \frac{c^2 r^2}{4a^2\,(a^2 - r^2)}\right], \qquad (5.106)$$

$$J^3_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm} F_k\,\Omega_{*m} = \frac{\varphi^2 F_1\,\Omega_{*2}}{2\pi c^4\sqrt{h}} = = \frac{\varphi^2 \omega a}{\pi c^2\,(a^2 - r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \qquad (5.107)$$

$$J_{\text{em}} = \left|\sqrt{h_{33} J^3_{\text{em}} J^3_{\text{em}}}\right| = \frac{\varphi}{\pi c^2}\,\frac{\omega a r \cot\theta}{(a^2 - r^2)^{3/2}}\,. \qquad (5.108)$$

Based on the equations, we can draw a conclusion about the rotating neutron stars and pulsars that have a vortex-free electromagnetic field.

We see that the electromagnetic field density ρ_{em} of such a star is due to the star's gravitational force, which is the non-Newtonian gravitational force F_i acting inside the star, and also due to the star's rotation. The electromagnetic field density ρ_{em} of the star is non-zero by the condition $F_i \neq 0$ or $A_{ik} \neq 0$, and also by these conditions together. The field momentum density J^i_{em} is non-zero only by the joint condition $F_i \neq 0$ and $A_{ik} \neq 0$.

As follows from (5.106), the density ρ_{em} of the vortex-free electromagnetic field of a rotating neutron star (pulsar) is zero at the equator of the star ($\theta = 0$). Then the electromagnetic field density ρ_{em} increases with the geographic latitude θ to the South Pole and North Pole, where $\theta = \frac{\pi}{2}$ and, thus, it takes the maximum magnitude $\rho_{\text{em}} = (\rho_{\text{em}})_{\max}$.

On the contrary, the electromagnetic field momentum density J^i_{em} (5.108) has a maximum at the equator, where $\theta = 0$. Then the magnitude of the field momentum J^i_{em} decreases with the geographic latitude θ to the South Pole and North Pole, where it is $J^i_{\text{em}} = 0$.

In addition to the above, we can also draw a conclusion about the charge density ρ and the currents j^i of the vortex-free electromagnetic field of a rotating neutron star or pulsar.

Re-write the formulae for the charge density ρ (5.99) and the current j^3 (5.100), obtained from the Group I of the chr.inv.-Maxwell equations, as follows

$$\rho = \frac{\varphi}{\pi c^4}\left(\Omega_{*j}\,\Omega^{*j} - \frac{1}{4}\,\nabla_j F^j\right), \tag{5.109}$$

$$j^3 = -\frac{\varphi}{\pi}\,\frac{\omega a^3}{r^2\left(a^2 - r^2\right)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{5.110}$$

where

$$\nabla_j F^j = \frac{c^2\left(3a^2 - 2r^2\right)}{a^2\left(a^2 - r^2\right)} > 0\,, \tag{5.111}$$

$$j = \left|\sqrt{h_{33}\, j^3_{\text{em}}\, j^3_{\text{em}}}\right| = \frac{\varphi\omega a^3 \cot\theta}{\pi r\left(a^2 - r^2\right)^{3/2}}\,. \tag{5.112}$$

As a result, we see that the charge density in a rotating neutron star or pulsar is positive $\rho > 0$ (that should be according to the physical sense of a physical field) if

$$\Omega_{*j}\,\Omega^{*j} > \frac{1}{4}\,\nabla_j F^j. \tag{5.113}$$

Re-write this inequality with the formula for ρ (5.99). We obtain the condition

$$\frac{4\omega^2 a^2}{c^2}\left(1 + \frac{4\cot^2\theta}{a^2 - r^2}\right) > \frac{3a^2 - 2r^2}{a^2}\,, \tag{5.114}$$

which is necessary according to the physical sense.

Compare the obtained formulae for the electromagnetic field current j^3 (5.110) and its power j (5.112) with the formulae for the electromagnetic field momentum density J^3_{em} (5.107) and the momentum power J_{em} (5.108). As a result, we have

$$c^2 J^3_{\text{em}} = -\frac{\varphi r^2}{a^2}\, j^3, \qquad c^2 J_{\text{em}} = \frac{\varphi r^2}{a^2}\, j\,. \tag{5.115}$$

Taking (5.58) into account, express the scalar electromagnetic field potential φ, which is $\varphi = const$ according to our initial assumptions, through the dimensionless constant $n = \frac{G\varphi^2}{c^4}$ (5.59), which is $n < \frac{1}{4}$ (see in the end of §5.3). Thus, we have

$$\varphi = c^2 \sqrt{\frac{n}{G}}\,, \qquad \varphi^2 = \frac{nc^4}{G}\,, \qquad n < \frac{1}{4}\,. \tag{5.116}$$

With these, we obtain

$$\rho_{\mathrm{em}} = \frac{n}{2\pi G}\left(\Omega_{*j}\,\Omega^{*j} + \frac{1}{4c^2}\,F_j F^j\right), \tag{5.117}$$

$$J^3_{\mathrm{em}} = \frac{E_1 H_{*2}}{\sqrt{h}} = \frac{4nc^3}{G}\,\frac{\omega a}{\left(a^2 - r^2\right)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{5.118}$$

$$\rho = \frac{1}{\pi}\sqrt{\frac{n}{G}}\left(\Omega_{*j}\,\Omega^{*j} - \frac{1}{4}\,\nabla_j F^j\right), \tag{5.119}$$

$$j^3 = -c^2 \sqrt{\frac{n}{G}}\,\frac{\omega a^3}{r^2\left(a^2 - r^2\right)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{5.120}$$

$$J^3_{\mathrm{em}} = -\frac{4\pi r^2 c}{a^2}\sqrt{\frac{n}{G}}\; j^3. \tag{5.121}$$

We see that the greater the scalar electromagnetic potential φ (5.116) of a rotating neutron star or pulsar, the stronger the three-dimensional circular current j^3 and the three-dimensional circular momentum J^3_{em} of its electromagnetic field. Moreover, the electromagnetic field current and momentum flow exist in the star only if it rotates on the equatorial plane (x^1, x^3), i.e., only if $\Omega_{*2} \neq 0$. If the neutron star does not rotate $(\Omega_{*j}\,\Omega^{*j} = 0)$, the electric charge density of its internal electromagnetic field would be negative $(\rho < 0)$.

So, we have arrived at a non-satisfactory result. Both the circular electromagnetic field current j^3 (that goes along the longitudinal coordinate ϕ) and the electromagnetic field momentum J^3_{em} are zero at the South Pole and North Pole of the star, where the geographical latitude is $\theta = \frac{\pi}{2}$. They reach their maximum power at the equator, where the latitude is $\theta = 0$.

Here we have assumed that the electromagnetic field of a rotating neutron star is vortex-free. The final step of matching observational data will be done with the vortical electromagnetic field of a rotating neutron star (pulsar). We will do it next, in §5.7.

5.7 Solving Maxwell's equations in the stationary vortical electromagnetic field of a pulsar

By analogy with §3.7, consider a rotating neutron star (pulsar), the electromagnetic field of which is vortical. In this case, the curl q_{ik} of the three-dimensional chr.inv.-vector potential q_i of the field is non-zero

$$q_{ik} = \frac{^*\partial q_i}{\partial x^k} - \frac{^*\partial q_k}{\partial x^i} \neq 0\,. \tag{5.122}$$

The four-dimensional electromagnetic field potential

$$A^{\alpha} = \varphi\,\frac{dx^{\alpha}}{ds}\,, \qquad g_{\alpha\beta}\,\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 1 \tag{5.123}$$

has two chr.inv.-projections

$$\frac{A_0}{\sqrt{g_{00}}} = \widetilde{\varphi}, \qquad A^i = q^i = \frac{\widetilde{\varphi}}{c}\,\mathrm{v}^i, \tag{5.124}$$

where $\widetilde{\varphi}$ is the relativistic chr.inv.-scalar potential of the field

$$\widetilde{\varphi} = \frac{\varphi}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} \simeq \varphi\,, \;\; \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \;\; \mathrm{v}^2 = h_{ik}\mathrm{v}^i\mathrm{v}^k \ll c^2. \tag{5.125}$$

Assume $\varphi = const$ and $q^1 = q^2 = 0$ for the electromagnetic field under consideration. Therefore, we have $\mathrm{v}^3 = \frac{d\phi}{d\tau} = \omega$, and the non-zero components of the electromagnetic vector potential and its curl are

$$q^3 = \frac{\varphi\omega}{c}\,, \qquad q_3 = \frac{\varphi\omega}{c}\,r^2\sin^2\theta, \tag{5.126}$$

$$q_{31} = \frac{\partial q_3}{\partial r} = \frac{2\varphi\omega}{c}\,r\sin\theta, \tag{5.127}$$

$$q_{23} = -\frac{\partial q_3}{\partial\theta} = -\frac{2\varphi\omega}{c}\,r^2\sin\theta\cos\theta. \tag{5.128}$$

Calculate the non-zero components of the magnetic strength of the vortical field. Using its definition (5.47), we obtain

$$H_{23} = -\frac{2\varphi\omega r^2\sin\theta}{c}\left(\frac{a}{\sqrt{a^2-r^2}} + \cos\theta\right), \tag{5.129}$$

$$H_{31} = \frac{2\varphi\omega}{c}\left[\sin^2\theta - \frac{2a^3\cos\theta}{(a^2-r^2)\sqrt{a^2-r^2}}\right]. \tag{5.130}$$

Using the definition

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn} = \frac{1}{2}\,\varepsilon^{imn}q_{mn} - \frac{2\varphi}{c}\,\Omega^{*i}, \qquad (5.131)$$

we re-write the magnetic field strength components H_{23} (5.129) and H_{31} (5.130) in the following form

$$H^{*1} = -\frac{2\varphi\omega}{c}\left(1 + \frac{\sqrt{a^2 - r^2}}{a}\,\cos\theta\right), \qquad (5.132)$$

$$H^{*2} = \frac{2\varphi\omega}{cr}\left(\frac{\sqrt{a^2 - r^2}\,\sin\theta}{a} - \frac{2a^2\cot\theta}{a^2 - r^2}\right), \qquad (5.133)$$

while their covariant (lower-index) versions can easily be obtained as $H_{*1} = h_{11}H^{*1}$ and $H_{*2} = h_{22}H^{*2}$.

Let us find a solution to the chr.inv.-Maxwell equations. In the case, where the electromagnetic field is stationary, and the space does not deform, the chr.inv.-Maxwell equations have the form (5.97–5.98), i.e.

$$\left.\begin{aligned} {}^*\nabla_j E^j - \frac{1}{c}\,H^{ik}A_{ik} &= 4\pi\rho \\ {}^*\nabla_k H^{ik} - \frac{1}{c^2}\,F_k H^{ik} &= \frac{4\pi}{c}\,j^i \end{aligned}\right\}\ \text{I}, \qquad (5.134)$$

$$\left.\begin{aligned} {}^*\nabla_i H^{*i} - \frac{1}{c}\,E^{*ik}A_{ik} &= 0 \\ {}^*\nabla_k E^{*ik} - \frac{1}{c^2}\,F_k E^{*ik} &= 0 \end{aligned}\right\}\ \text{II}. \qquad (5.135)$$

Substituting the electric and magnetic strengths of the vortical electromagnetic field, we see that the Group II equations (5.135) are satisfied as identities. The Group I equations take the form

$$\frac{4\varphi\omega^2}{c^2}\left[\frac{a^2}{a^2 - r^2}\left(1 + \frac{4a^2\cot^2\theta}{a^2 - r^2}\right) - \frac{a\cos\theta}{\sqrt{a^2 - r^2}}\right] - \frac{\varphi\left(3a^2 - 2r^2\right)}{a^2\left(a^2 - r^2\right)} = 4\pi\breve{\rho}\,, \qquad (5.136)$$

$$\frac{3}{2}\frac{\varphi\omega}{a^2} + \frac{\varphi\omega a^3}{r^2\left(a^2 - r^2\right)\sqrt{a^2 - r^2}}\frac{\cot\theta}{\sin\theta} = -\pi\breve{j}^3, \qquad (5.137)$$

where $\breve{\rho}$ and $\breve{j}^3$ are the charge density and current of the vortical electromagnetic field, and $\omega = \Omega^{*1}$. The physical sense of the equations looks more understandable if they are re-written in the form

$$\breve{\rho} = \frac{\varphi}{\pi c^2}\left(\Omega_{*j}\,\Omega^{*j} - \frac{1}{4}\,\nabla_j F^j\right) - \frac{\varphi\omega^2}{\pi c^2}\frac{a\cos\theta}{\sqrt{a^2-r^2}}\,, \tag{5.138}$$

$$\breve{j}^3 = -\frac{\varphi\omega a^3}{\pi r^2\left(a^2-r^2\right)^{3/2}}\frac{\cot\theta}{\sin\theta} - \frac{3\varphi\omega}{2\pi a^2}\,. \tag{5.139}$$

Expressing the charge density $\breve{\rho}$ and the current $\breve{j}^3$ of the vortical electromagnetic field through the same characteristics ρ (5.109) and j^3 (5.110), which we have calculated in the vortex-free field, we obtain

$$\breve{\rho} = \rho - \frac{\varphi\omega^2 a\cos\theta}{\pi c^2\sqrt{a^2-r^2}}\,, \tag{5.140}$$

$$\breve{j}^3 = j^3 - \frac{3\varphi\omega}{2\pi a^2}\,. \tag{5.141}$$

As is seen from the equations (5.140) and (5.141), in the vortical electromagnetic field of a rotating neutron star (pulsar), the charge density and field currents differ from those of a vortex-free electromagnetic field by the terms depending on the rotation of the star.

Accordingly, the field density $\rho_{\rm em}$ (5.44) and the circular momentum flow $J^3_{\rm em}$ (5.45) of the vortical electromagnetic field have the form

$$\breve{\rho}_{\rm em} = \frac{\varphi^2}{2\pi c^4}\left(\frac{1}{4c^2}\,F_j F^j + \Omega_{*j}\,\Omega^{*j}\right) + \\ + \frac{\varphi^2}{2\pi c^4}\left[\omega^2\left(1 - \frac{r^2\sin^2\!\theta}{a^2}\right) - \frac{a\omega^2\cos\theta}{\sqrt{a^2-r^2}}\right], \tag{5.142}$$

$$\breve{J}^3_{\rm em} = \frac{\varphi^2}{2\pi c^2 a^2}\left[\frac{2\omega a^3}{\left(a^2-r^2\right)^{3/2}}\frac{\cot\theta}{\sin\theta} - \omega\right], \tag{5.143}$$

or, in the other form,

$$\breve{\rho}_{\rm em} = \rho_{\rm em} + \frac{\varphi^2}{2\pi c^4}\left[\omega^2\left(1 - \frac{r^2\sin^2\!\theta}{a^2}\right) - \frac{\omega^2 a\cos\theta}{\sqrt{a^2-r^2}}\right], \tag{5.144}$$

$$\breve{J}^3_{\rm em} = J^3_{\rm em} - \frac{\varphi^2\,\omega}{2\pi c^2 a^2}\,. \tag{5.145}$$

To understand the sense of the resulting formulae, recall that, as follows from the formulae for A_{31} (5.10),

$$A_{31} = \frac{2\omega a^3 r \cos\theta}{\left(a^2 - r^2\right)^{3/2}}\,, \qquad A^{31} = \frac{2\omega a \cot\theta}{r\sqrt{a^2 - r^2}\,\sin\theta}\,, \tag{5.146}$$

this component and, hence, $\Omega^{*2} = \frac{1}{2}\,\varepsilon^{231} A_{31}$ depend on the geographical latitude θ of the star, and the component $\Omega^{*1} = \frac{1}{2}\,\varepsilon^{123} A_{23} = \omega$ does not.

The obtained formulae for the current vector $\breve{j}^3$ (5.141) and the Poynting vector $\breve{J}^3_{\rm em}$ (5.145) of the vortical electromagnetic field of a rotating neutron star (pulsar) contain a term which does not depend on the geographical latitude. This means that, in contrast to a rotating neutron star (pulsar) with a vortex-free electromagnetic field, the current vector $\breve{j}^3$ and the momentum flow $\breve{J}^3_{\rm em}$ of the vortical electromagnetic field are non-zero at the South Pole and North Pole of the star.

The obtained $\breve{J}^3_{\rm em} \neq 0$ at the South Pole and North Pole means that a rotating neutron star (pulsar), the electromagnetic field of which is vortical, emits electromagnetic radiation along its polar axis, while a rotating neutron star with a vortex-free electromagnetic field does not.

We also have to make one more important conclusion in the framework of our mathematical theory of pulsars. Look at the formula for the magnetic field strength $H^{*i} = \frac{1}{2}\,\varepsilon^{imn} H_{mn}$ (5.47)

$$\begin{aligned} H^{*i} &= \frac{1}{2}\,\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}\,A_{mn}\right) = \\ &= \frac{1}{2}\,\varepsilon^{imn}\left(\frac{^*\partial q_m}{\partial x^n} - \frac{^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}\,\Omega^{*i} = \\ &= \frac{1}{2}\,\varepsilon^{imn} q_{mn} - \frac{2\varphi}{c}\,\Omega^{*i}. \end{aligned} \tag{5.147}$$

We see that the magnetic field strength pseudovector H^{*i} is the sum of the electromagnetic field curl pseudovector $q^{*i} = \frac{1}{2}\,\varepsilon^{imn} q_{mn}$ and the pseudovector Ω^{*i} of the angular velocity with which the star rotates. The magnetic field strength pseudovector H^{*i} coincides with the pseudovector Ω^{*i} of the star's rotation only if the electromagnetic field curl q_{mn} is zero. If $q_{mn} \neq 0$, i.e., if the star's electromagnetic field is vortical, then the H^{*i} deviates from the Ω^{*i}. The stronger the electromagnetic field curl q_{mn}, the more the magnetic axis deviates from the axis of the star's rotation.

Astronomers tell us that the electromagnetic fields of observed pulsars are very strong. They also tell that electromagnetic radiation leaves such a star only at the polar regions, where the latitudinal and longitudinal electromagnetic field components are not so strong as at the equatorial latitudes. Also, analysing the oscillation of the signals registered from pulsars, astronomers conclude that the axis of emission of radiation from a pulsar does not coincide with and the axis of its rotation. All these facts of observational astronomy are in complete agreement with our theoretical results on pulsars.

As a result, our mathematical theory of pulsars leads us to the conclusions that are consistent with observational data:

> A rotating neutron star can be a pulsar only if its electromagnetic field is vortical. Moreover, the electromagnetic field curl means that the magnetic axis does not coincide with the axis of the star's rotation. Otherwise, in a rotating neutron star with a vortex-free electromagnetic field, electromagnetic radiation would not come from the South and North Poles.

All these theoretical results were obtained in the framework of the assumption that the scalar and vector electromagnetic field potentials of the star do not depend on time. Of course, some time variations of the potentials should pose an effect on the Poynting vector (momentum flow of the field) and thus on the electromagnetic radiation emitted by the pulsar. But we neglect these effects here.

5.8 Geometrization of the electromagnetic field of a pulsar

Geometrization of the electromagnetic field is one of the primary tasks in the General Theory of Relativity. As was shown already by Einstein, mathematically this problem in a general case is very non-trivial. For this reason, the problem is still not resolved in general. Nevertheless, the electromagnetic field can be geometrized in particular cases, under certain conditions that simplify the mathematics.

Let us now show that in the case of a pulsar the electromagnetic field is geometrized. In the language of mathematics, this means that once we have Einstein's field equations and Maxwell's equations, the electromagnetic field characteristics can be expressed in only terms of the geometric characteristics of the space.

Consider the Einstein field equations (5.18–5.20) and the Maxwell equations (5.134–5.135) in the internal field of a rotating neutron star (pulsar). Note that in the de Sitter-like metric that we applied to neutron stars, the λ-term describes the physical vacuum in the inflation state $\lambda = \varkappa\rho$ (see Chapter 1). Moreover, as we showed in §5.2, this form of the Einstein equations satisfies the conservation equations in the space of the above metric.

We will consider a vortical electromagnetic field. This is because we have shown that only the vortical field gives a result consistent with astronomical observations of pulsars, i.e., the fact that a pulsar emits electromagnetic radiation only from its polar regions.

As before, we assume that the scalar electromagnetic field potential φ is constant and is expressed through the fundamental constants as $\varphi = c^2\sqrt{\frac{n}{G}}$ (5.59), where $n < \frac{1}{4}$ (see the end of §5.3). With it, we obtain the electric and magnetic strengths of the vortical electromagnetic field (see §5.7) in the form

$$E^1 = -\sqrt{\frac{n}{G}}\,\frac{c^2 r}{a^2}\,, \tag{5.148}$$

$$E_1 = h_{11}E^1 = -\sqrt{\frac{n}{G}}\,\frac{c^2 r}{a^2 - r^2}\,, \tag{5.149}$$

$$\begin{aligned} H^{*1} &= -2\omega c\sqrt{\frac{n}{G}}\left(1 + \frac{\sqrt{a^2 - r^2}}{a}\cos\theta\right) = \\ &= -2c\sqrt{\frac{n}{G}}\left(\Omega^{*1} + \frac{\omega\sqrt{a^2 - r^2}}{a}\cos\theta\right), \end{aligned} \tag{5.150}$$

$$\begin{aligned} H^{*2} &= \frac{2\omega c}{r}\sqrt{\frac{n}{G}}\left(\frac{\sqrt{a^2 - r^2}}{a}\sin\theta - \frac{2a^2\cot\theta}{a^2 - r^2}\right) = \\ &= -2c\sqrt{\frac{n}{G}}\left(\Omega^{*2} - \frac{\omega\sqrt{a^2 - r^2}}{ar}\sin\theta\right), \end{aligned} \tag{5.151}$$

$$H_{*1} = h_{11}H^{*1} = \frac{a^2}{a^2 - r^2}\,H^{*1}, \tag{5.152}$$

$$H_{*2} = h_{22}H^{*2} = r^2 H^{*2}. \tag{5.153}$$

As is seen from the above formulae, both the electric and magnetic strengths are expressed here through only the geometric characteristics of the internal space of the star.

According to the internal space metric of a rotating neutron star or pulsar, we have Ω^{*1} (5.71), Ω^{*2} (5.72), $\Omega_{*j}\,\Omega^{*j}$ (5.87), which are

$$\Omega^{*1} = \frac{A_{23}}{\sqrt{h}} = \omega, \qquad \Omega_{*1} = A^{23}\sqrt{h} = \frac{\omega a^2}{a^2 - r^2}\,, \tag{5.154}$$

$$\Omega^{*2} = \frac{A_{31}}{\sqrt{h}} = \frac{2\omega a^2 \cot\theta}{r\,(a^2 - r^2)}\,, \quad \Omega_{*2} = A^{31}\sqrt{h} = \frac{2\omega a^2 r \cot\theta}{a^2 - r^2}\,, \tag{5.155}$$

$$\Omega_{*j}\,\Omega^{*j} = \frac{\omega^2 a^2}{a^2 - r^2}\left(1 + \frac{4a^2 \cot^2\theta}{a^2 - r^2}\right). \tag{5.156}$$

Thus, the charge density $\breve{\rho}$ (5.138) and the current vector $\breve{j}^3$ (5.139) of the vortical electromagnetic field of a rotating neutron star of pulsar, obtained from Maxwell's equations, are expressed as

$$\breve{\rho} = \frac{1}{\pi}\sqrt{\frac{n}{G}}\left(\Omega_{*j}\,\Omega^{*j} - \frac{1}{4}\,\nabla_j F^j\right) - \frac{1}{\pi}\sqrt{\frac{n}{G}}\,\frac{\omega^2 a\cos\theta}{\sqrt{a^2 - r^2}} = \\ = \rho - \frac{1}{\pi}\sqrt{\frac{n}{G}}\,\frac{\omega^2 a\cos\theta}{\sqrt{a^2 - r^2}}\,, \tag{5.157}$$

$$\breve{j}^3 = -\frac{c^2}{\pi r^2}\sqrt{\frac{n}{G}}\,\frac{\omega a^3}{(a^2 - r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta} + \frac{3c^2\omega}{2a^2}\sqrt{\frac{n}{G}} = \\ = j^3 + \frac{3c^2\omega}{2a^2}\sqrt{\frac{n}{G}}\,. \tag{5.158}$$

Accordingly, the density $\breve{\rho}_{\rm em}$ (5.142) and the momentum flow $\breve{J}^3_{\rm em}$ (5.143) of the vortical electromagnetic field, obtained from the energy-momentum tensor of the field, are expressed as

$$\breve{\rho}_{\rm em} = \frac{n}{2\pi G}\left(\frac{1}{4c^2}\,F_j F^j + \Omega_{*j}\,\Omega^{*j}\right) + \\ + \frac{n}{2\pi G}\left[\,\omega^2\left(1 - \frac{r^2\sin^2\theta}{a^2}\right) - \frac{a\omega^2\cos\theta}{\sqrt{a^2 - r^2}}\,\right] = \\ = \rho_{\rm em} + \frac{n}{2\pi G}\left[\,\omega^2\left(1 - \frac{r^2\sin^2\theta}{a^2}\right) - \frac{a\omega^2\cos\theta}{\sqrt{a^2 - r^2}}\,\right], \tag{5.159}$$

$$\breve{J}^3_{\rm em} = \frac{nc^2}{2\pi G a^2}\left[\,\frac{2\omega a^3}{(a^2 - r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta} - \omega\,\right] = J^3_{\rm em} - \frac{nc^2\omega}{2\pi G a^2}\,. \tag{5.160}$$

We see that all the characteristics of the vortical magnetic field are uniquely expressed through only the geometric characteristics of the space inside the pulsar. Therefore, the vortical electromagnetic field of a rotating neutron star (pulsar) is hereby geometrized.

This fact also means that the system of Einstein's equations and Maxwell's equations in the internal space of a pulsar is a self-consistent system of equations. This self-consistent system of Einstein-Maxwell equations completely describes both gravitational and electromagnetic phenomena inside the pulsar.

However, if the electromagnetic field of a rotating neutron star is vortex-free, then Einstein's equations and Maxwell's equations do not comprise a self-consistent system: the electromagnetic field is not geometrized inside such a star. As was shown in §5.7, such a neutron star cannot emit electromagnetic radiation from its polar regions. Therefore, it cannot be a pulsar.

5.9 Boundaries of the physical space of a pulsar

Let us consider an observer, whose reference frame is connected to the internal space of a star. Where, from his point of view, does the observable physical space of the star end? At which distance from the star?

These questions are answered by the theory of physically observables (chronometric invariants). In terms of physical observables, the real physical space of an observer "ends" at that distance from him, where the physically observable time stops. The physically observable time τ is calculated according to the metric of the observer's space. Therefore, the real physical boundaries of his space are determined by the stopped time condition $d\tau = 0$ according to his space metric.

Let us calculate the boundary of the observable space of a pulsar. This is the distance from the centre of the pulsar at which, according to the space metric of the pulsar, the observable time stops for an observer, whose reference frame is associated with the pulsar.

The physically observable time interval is formulated as (1.30)

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = \sqrt{g_{00}}\,dt - \frac{1}{c^2}\,v_i\,dx^i. \qquad (5.161)$$

It consists of two terms. The first term is due to the gravitational field potential $\mathrm{w} = c^2(1-\sqrt{g_{00}})$ (1.44). The second term is due to the fact that the space rotates and is dependent on the linear velocity of its rotation $v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}$ (1.45).

Therefore, the condition $d\tau = 0$ by which the observable time stops in the space of a gravitating and rotating body is expressed as

$$\sqrt{g_{00}}\,dt = \frac{1}{c^2}\,v_i\,dx^i. \tag{5.162}$$

The space metric of a rotating neutron star or pulsar has the form (5.3). See §5.1 for detail. In this metric, we have

$$g_{00} = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right), \tag{5.163}$$

$$v_1 = v_2 = 0\,, \qquad v_3 = -\frac{2\omega a r^2\cos\theta}{\sqrt{a^2 - r^2}}\,. \tag{5.164}$$

In this case, the stopped time condition (5.162) takes the form

$$g_{00} = \frac{1}{c^4}\left(v_3\frac{dx^3}{dt}\right)^2, \tag{5.165}$$

where $\frac{dx^3}{dt} = \frac{d\phi}{dt} = \omega$. Substituting the g_{00} (5.163) and v_3 (5.164) of the space metric (5.3) into the stopped time condition (5.165), we obtain the distance r at which the observable time stops in such a space

$$r = \frac{a}{\sqrt{1 + \frac{4a^2\omega^2\cos\theta}{c^2}}}\,. \tag{5.166}$$

This formula determines the physical boundary at which the physically observable space of a rotating neutron star or pulsar ends. As you can see, the boundary r is the same as the star's physical radius a at the South Pole and North Pole, where the geographical latitude is $\theta = \frac{\pi}{2}$, i.e., $\cos\theta = 0$ and, therefore, $r = a$ according to (5.166). Then the boundary r of the physically observable space decreases to the equator, where it takes the minimum numerical value

$$r_{\min} = \frac{a}{\sqrt{1 + \frac{4a^2\omega^2}{c^2}}}\,, \tag{5.167}$$

which depends only on the radius a of the star and the angular velocity of its rotation ω.

The faster a neutron star rotates, the more oblate its physical space at its equator. According to our formula (5.167), this oblateness manifests itself only at relativistic speeds of rotation, i.e., in pulsars.

Consider PSR J1748-2446ad that is the fastest-known pulsar discovered in 2004 [32]. It rotates with a period of 0.00139595482(6) sec that means the angular rotation velocity $\omega = \frac{2\pi}{T} = 4501 \text{ sec}^{-1}$. Its radius a is estimated to be smaller than 16 km. Proceeding from these observational data, we calculate the oblateness of the physical space of the pulsar J1748-2446ad at its equator

$$\frac{r_{\min}}{a} = \frac{1}{\sqrt{1 + \frac{4a^2\omega^2}{c^2}}} \simeq 0.90\,. \tag{5.168}$$

5.10 Conclusion

So, the complete mathematical theory of rotating liquid neutron stars and pulsars is presented in this Chapter. Let us repeat the most important conclusions we have arrived at on the basis of our theory:

1. As follows from our mathematical theory, the electromagnetic field of a rotating neutron star or pulsar is due to its rotation and gravitation. The faster the star rotates, the stronger the magnetic field strength H^{*i} of the star;
2. The magnetic field strength H^{*i} of a pulsar is directed strictly along the polar axis of its rotation. Electromagnetic radiation is emitted only from the poles of the star, then comes into the outer cosmos strictly along its rotation axis;
3. The electric field strength E_i depends on the spatial distribution of the scalar electromagnetic field potential and on the time variation of the vector electromagnetic field potential. The magnetic field strength H^{*i} depends on the curl of the vector electromagnetic field potential and on the angular velocity of the star. Therefore, the time and spatial variations of the electromagnetic field potentials should affect the outcoming electromagnetic pulses emitted by a pulsar;
4. The Poynting vector (electromagnetic field momentum) is nonzero at the South Pole and North Pole of a rotating neutron star only if its electromagnetic field is vortical. Therefore, a rotating neutron star is a pulsar, thus emitting electromagnetic radiation

from the polar regions, only if its electromagnetic field is vortical; a rotating neutron star, the electromagnetic field of which is vortex-free, cannot emit electromagnetic radiation from its polar regions, so it cannot be a pulsar. In addition, our theory shows that, due to the electromagnetic field curls, the magnetic axis of a rotating neutron star does not coincide with its axis of rotation. This theoretical conclusion is completely consistent with the astronomical evidence for the fast periodic pulses of electromagnetic radiation registered from observed pulsars.

All the conclusions are valid only for a rotating star, the physical radius of which is close to its Hilbert radius. These are rotating neutron stars and also pulsars, not the ordinary stars such as the Sun etc.

Chapter 6 Black Holes

6.1 Non-rotating liquid collapsars

We are now going to study the collapse condition of a non-rotating sphere filled with an ideal liquid, i.e., a collapsed liquid star without rotation (in terms of our model of liquid stars). At first glance, this formulation of the problem sounds meaningless: an ideal liquid is incompressible, therefore, such a liquid body cannot be compressed. Yes, it would be meaningless, if the collapse were considered a process of compression of a liquid cosmic body. We do not do that: we do not discuss cosmogony. We merely treat a liquid collapsar as an already existing object. Thus, the above problem is reduced to the consideration of physical conditions, and not the evolutionary compression of a liquid cosmic body.

A cosmic body is a gravitational collapsar, if the parameters of its field on its physical surface correspond to the condition of gravitational collapse. Namely, — the gravitational field of the body is so strong on its surface that light signals cannot leave it into the cosmos. In terms of General Relativity, this means that the physically observable time stops on the surface of the body.

According to the theory of physically observable quantities (chronometric invariants), the physically observable time interval $d\tau$ (1.30) is formulated through the gravitational potential w and the linear velocity v_i with which the space rotates as follows

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = \left(1 - \frac{\mathrm{w}}{c^2}\right)dt - \frac{1}{c^2}\,v_i\,dx^i. \qquad (6.1)$$

Therefore, the general condition of gravitational collapse has the below form consisting of two terms

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = 0\,. \qquad (6.2)$$

In a space without rotation (wherein $v_i = 0$), the general condition of gravitational collapse is simpler

$$d\tau = \sqrt{g_{00}}\, dt = 0\,, \tag{6.3}$$

or merely

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 = 0\,. \tag{6.4}$$

Therefore, a non-rotating cosmic object is a collapsar, if the three-dimensional gravitational potential w on its surface takes the value

$$\mathrm{w} = c^2. \tag{6.5}$$

Let us consider the collapse condition for a non-rotating star consisting of an ideal liquid. According to the above, the collapse condition in this case has the form $g_{00} = 0$. As is seen from the space metric of a non-rotating liquid star (2.76)

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1 - \frac{\varkappa\rho_0 r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{6.6}$$

which in terms of the Hilbert radius r_g has the form (2.78)

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r^2 r_g}{a^3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1 - \frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{6.7}$$

the collapse condition ($g_{00} = 0$) in such a space has the form

$$3\sqrt{1 - \frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}} = 0\,, \tag{6.8}$$

or, which is the same,

$$3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r^2 r_g}{a^3}} = 0\,. \tag{6.9}$$

Thus, we obtain the radial coordinate r, at which a non-rotating liquid star of a radius a meets the state of gravitational collapse

$$r_c = \sqrt{9a^2 - \frac{8a^3}{r_g}} \,. \tag{6.10}$$

Since we keep in mind real cosmic objects, the numerical value of the r_c must be real (as well as a and r_g). This requirement is satisfied by the following obvious condition

$$a \leqslant 1.125\, r_g \,. \tag{6.11}$$

If this condition is not satisfied (i.e., the physical radius a of a liquid body is $a \geqslant 1.125\, r_g$), then the non-rotating liquid body (star) cannot be in the state of gravitational collapse.

As you can see, the general collapse condition (6.11) includes the particular condition $a = r_g$. In this particular case of a collapsed non-rotating liquid star, we see that the physical radius a of the star's surface, the Hilbert radius r_g and the radius of the outer space breaking $r_{br} = \sqrt{a^3/r_g}$ in the star's field coincide

$$r_c = r_{br} = r_g = a \,. \tag{6.12}$$

The obtained collapse condition $a = r_g$ (6.12) is only a particular case of the general collapse condition (6.11). The general collapse condition (6.11) includes three particular cases, concerning the location of the physical surface of the collapsed liquid star:

1. The collapsed liquid star is larger than the Hilbert radius calculated for the star ($a > r_g$), but smaller than $1.125\, r_g$;
2. The surface of the collapsed liquid star coincides with its Hilbert radius ($a = r_g$);
3. The collapsed liquid star is completely located inside its Hilbert radius ($a < r_g$).

It is obvious that r_c (6.10) is imaginary for $r_g \ll a$, hence the state of gravitational collapse is impossible for such a star. For example, considering the Sun ($a = 7 \times 10^7$ cm, $M = 2 \times 10^{33}$ gram, $r_g = 3 \times 10^5$ cm), we see that the r_c takes an imaginary numerical value. The same is as well true for other ordinary stars, ranging from super-giants to white dwarfs. Hence, *ordinary stars cannot collapse*.

In fact, the particular collapse condition $r_c = r_{br} = r_g = a$ (6.12) formulates the collapse radius r_c as follows*

$$r_c = a = \sqrt{\frac{3}{\varkappa\rho_0}} = \frac{4.0 \times 10^{13}}{\sqrt{\rho_0}} \text{ cm.} \tag{6.13}$$

For example, if a collapsed liquid sphere consists of an ordinary water ($\rho_0 = 1.0$ gram/cm^3), then its radius is $r_c = 4.0{\times}10^{13}$ cm= 3.1 AU, i.e. is located in the asteroid belt (the asteroids are located, approximately, from 2.1 AU to 4.3 AU from the Sun).

Another example: a neutron star or pulsar. Such a star has a radius of $a = (8–16){\times}10^5$ cm = 8–16 km. Hence, to be a collapsar, such a liquid star must have $\rho_0 = 2.5 \times 10^{15} - 6.3 \times 10^{14}$ gram/cm^3 according to the obtained formula for r_c (6.13).

Calculate the mass of a non-rotating liquid collapsar using the usual formula $M = \frac{4}{3}\pi a^3\rho_0$ and the obtained formula $a = r_c = \sqrt{3/\varkappa\rho_0}$ (6.13). We obtain the following dependencies

$$M = \frac{4\pi a}{\varkappa} = 6.8 \times 10^{27}\, a \text{ gram,} \tag{6.14}$$

$$M = \frac{4\sqrt{3}\,\pi}{\varkappa^{3/2}\sqrt{\rho_0}} = \frac{2.7 \times 10^{41}}{\sqrt{\rho_0}} \text{ gram,} \tag{6.15}$$

which we call the *mass-radius relation* and *mass-density relation* for non-rotating liquid collapsars.

For example, if a collapsed liquid sphere has a size typical of a neutron star or pulsar, which is $a = (8–16) \times 10^5$ cm = 8–16 km, then its mass should theoretically be $M = (5.4–11) \times 10^{33}$ gram that is 2.7–5.5 masses of the Sun.

6.2 The Universe as a huge liquid collapsar

Here is another example: the Universe itself. Astronomers estimate the average density of substance in the Universe to be in the range of 10^{-28} to 10^{-31} gram/cm^3. In addition, according to the observational estimates, the Hubble constant is $H = \frac{c}{a} = (2.3 \pm 0.3) \times 10^{-18}$ sec^{-1}, and the Universe's radius is $a = 1.3 \times 10^{28}$ cm. At the upper limit of the estimated density $\rho_0 = 10^{-28}$ gram/cm^3, the collapse radius r_c (6.10) falls into real

*$\varkappa = \frac{8\pi G}{c^2} = 18.6 \times 10^{-28}$ cm/gram is Einstein's gravitational constant.

numerical values. Thus, according to the observational estimates, we obtain the following characteristics of the Universe

$$\left.\begin{array}{ll} a &= 1.3\times 10^{28}\text{ cm} \\ \rho_0 &= 10^{-28}\text{ gram/cm}^3 \\ M &= 9.2\times 10^{56}\text{ gram} \\ r_g &= 1.4\times 10^{28}\text{ cm} \\ r_{br} &= 1.3\times 10^{28}\text{ cm} \\ r_c &= 1.5\times 10^{28}\text{ cm} \end{array}\right\}. \tag{6.16}$$

This is a reason to think that the Universe can be considered as a sphere of an ideal liquid, which is in the state of gravitational collapse. We call this the *liquid model of the Universe*. In this case, we should have $r_c = r_{br} = r_g = a$ (6.12). Based on this condition and the numerical value of the Universe's radius $a = 1.3\times 10^{28}$ cm, obtained from the Hubble constant, we calculate the mass and density that should be associates with the Universe in the framework of the liquid model (according to $a = r_g = \frac{2GM}{c^2}$ and $M = \frac{4}{3}\pi a^3\rho_0$). We obtain

$$\left.\begin{array}{ll} a &= 1.3\times 10^{28}\text{ cm} \\ \rho_0 &= 9.6\times 10^{-31}\text{ gram/cm}^3 \\ M &= 8.8\times 10^{55}\text{ gram} \\ r_g &= 1.3\times 10^{28}\text{ cm} \\ r_{br} &= 1.3\times 10^{28}\text{ cm} \\ r_c &= 1.3\times 10^{28}\text{ cm} \end{array}\right\}. \tag{6.17}$$

The calculated theoretical values (6.17) are compared with the estimates of observational astronomy (6.16) in Table 6.1. Since these observational estimates are known very approximately, we conclude that the observable Universe is a huge collapsar. Therefore, all the world that we observe, including ourselves, is located inside a huge black hole.

In particular, this conclusion meets another one made in 1965 by Kyril P. Stanyukovich [33]. He neither studied the geometric properties of a liquid sphere nor introduced a particular space metric. His analysis was based on the properties of elementary particles. Following this way,

	M, gram	ρ_0, g/cm^3	a, cm	r_g, cm	r_{br}, cm	r_c, cm
Astron. esteems	9.2×10^{56}	$\sim10^{-28}$	1.3×10^{28}	1.4×10^{28}	1.3×10^{28}	1.5×10^{28}
Liquid model	8.8×10^{55}	9.6×10^{-31}	1.3×10^{28}	1.3×10^{28}	1.3×10^{28}	1.3×10^{28}

Table 6.1: The model of the observable Universe as a non-rotating liquid sphere in the state of gravitational collapse. The calculated parameters of the liquid model are compared with the observational esteems.

Stanyukovich obtained that the Hilbert radius of the Universe is the same as the observed event horizon: the observable Universe is a collapsar. So, despite the fact that Stanyukovich used a different theoretical base from ours, he had arrived at the same conclusion.

6.3 Pressure and density inside a liquid collapsar

Let us calculate pressure and density inside non-rotating liquid collapsars. The formula (2.130) that we have obtained for the pressure p inside a sphere filled with an ideal liquid

$$p = \rho_0 c^2 \frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}} \tag{6.18}$$

under the collapse condition $a = \sqrt{3/\varkappa\rho_0}$ takes the simplest form

$$p = -\rho_0 c^2 = const, \tag{6.19}$$

where $\rho_0 = const$ by definition inside any liquid sphere. This formula is the *equation of state of a liquid*. Such a state is known as *inflation*: at a positive density of a substance the pressure from within it is negative, so the internal pressure of the substance tends to expand the body from within (despite the fact that any liquid body is incompressible).

As is seen from this formula, the pressure inside a non-rotating liquid collapsar is constant as well as the density. This means that the liquid substance that fills a non-rotating collapsar is in the state of inflation and has the same pressure and density throughout the entire volume of the collapsar, from its centre to the surface.

6.4 The internal forces of gravitation. The internal redshift

The force of gravitation acting inside a non-rotating liquid collapsar can be found based on the force acting inside a non-rotating liquid sphere, if the sphere is in the state of gravitational collapse (in this case, its physical radius is $a = r_g = \sqrt{3/\varkappa\rho_0}$).

Based on the formulae for the components F_1 (2.123, 2.125) and F^1 (2.124, 2.126) acting inside a non-rotating liquid sphere, we obtain

$$F_1 = \frac{\varkappa\rho_0 c^2 r}{3}\,\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}} = \frac{c^2 r}{a^2}\,\frac{1}{1-\frac{r^2}{a^2}}\,, \tag{6.20}$$

$$F^1 = \frac{\varkappa\rho_0 c^2 r}{3} = \frac{c^2 r}{a^2}\,. \tag{6.21}$$

Since $r < a$ inside the sphere, we have $F_1 > 0$. Therefore, this is a force of repulsion. This force increases with the distance r, from zero at the centre of the liquid collapsar to its maximum value on the surface.

If the observable Universe is really a huge liquid collapsar (at least astronomical data evidence it, as was shown above), then the repulsive radial force acting inside the collapsar may cause a frequency shift in photons. To investigate this problem, we consider the chr.inv.-equations of isotropic geodesics

$$\left.\begin{aligned} &\frac{d\omega}{d\tau} - \frac{\omega}{c^2}F_i c^i + \frac{\omega}{c^2}D_{ik}c^i c^k = 0 \\ &\frac{d(\omega c^i)}{d\tau} + 2\omega\left(D^i_k + A_{k\cdot}^{\cdot i}\right)c^k - \omega F^i + \omega\Delta^i_{nk}c^n c^k = 0 \end{aligned}\right\}, \tag{6.22}$$

which are the equations of motion of a light-like massless particle (such as a photon, the frequency of which is ω), which travels with the observable velocity of light c^i. These chr.inv.-equations are obtained as the chr.inv.-projections of the general covariant equations of isotropic geodesics. See [18, 19] for detail.

If the space of a non-rotating liquid collapsar does not rotate or deform ($A_{ik} = 0$, $D_{ik} = 0$), then the equations (6.22) take the form

$$\left.\begin{aligned} &\frac{d\omega}{d\tau} - \frac{\omega}{c^2}F_i c^i = 0 \\ &\frac{d(\omega c^i)}{d\tau} - \omega F^i + \omega\Delta^i_{nk}c^n c^k = 0 \end{aligned}\right\}. \tag{6.23}$$

Let a photon travels only along the radial direction $x^1 = r$. Consider the chr.inv.-scalar geodesic equation of the photon. Substitute the obtained formula for F_1 (6.20). Because the photon's observable velocity is the observable velocity of light along the radial direction, $c^1 = \frac{dr}{d\tau}$, the chr.inv.-scalar geodesic equation of the photon takes the form

$$\frac{1}{\omega}\frac{d\omega}{d\tau} = \frac{r}{a^2 - r^2}\frac{dr}{d\tau}\,. \tag{6.24}$$

This equation is solved as $d\ln\omega = -\frac{1}{2}\,d\ln|a^2 - r^2|$, or

$$d\ln\omega = d\ln\frac{1}{\sqrt{a^2 - r^2}}\,, \tag{6.25}$$

whence we obtain the function

$$\omega\,(r) = \frac{Q}{\sqrt{a^2 - r^2}}\,, \qquad Q = const. \tag{6.26}$$

The integration constant Q is found from the obvious limit condition $\omega_{(r=0)} = \omega_0$. It is $Q = a^2\omega_0$. Finally, we obtain the solution

$$\omega = \frac{\omega_0}{\sqrt{1 - \frac{r^2}{a^2}}}\,. \tag{6.27}$$

At the distances travelled by the photon, which are small to the physical radius of the collapsar ($r \ll a$), this formula transforms into

$$\omega \simeq \omega_0\left(1 + \frac{r^2}{2a^2}\right). \tag{6.28}$$

This causes a *square redshift* of the photon's frequency

$$z = \frac{\omega - \omega_0}{\omega_0} = \frac{1}{\sqrt{1 - \frac{r^2}{a^2}}} - 1 > 0\,, \tag{6.29}$$

which we call a *parabolic redshift* due to the parabolic square function. That is, the force of repulsion F_1 acting along the radial coordinate from the observer decelerates photons travelling inside the star to him. At small distances of the photon's travel ($r \ll a$), the redshift is

$$z \simeq \frac{r^2}{2a^2}\,, \tag{6.30}$$

or, formulating this result through the Hubble constant $H = \frac{c}{a}$,

$$z \simeq \frac{H^2 r^2}{2c^2}\,. \tag{6.31}$$

So, the observed parameters of the Universe indicate that it is a huge collapsar. This conclusion coincides with the calculations according to the theory of non-rotating liquid collapsars presented here. Therefore, astronomers should expect a non-linear parabolic redshift on the photons coming to us from the farthest regions of the Universe. The greater the distance travelled by the photon, the greater the non-linearity of the redshift function, which is expected to be registered in astronomical observations.

6.5 The state of a collapsed liquid substance

Let us now discuss the state of the substance that fills non-rotating liquid collapsars. It is easy to see that once a non-rotating liquid star is in the state of gravitational collapse ($r_g = a$), the space metric (6.7) of such a star takes the form

$$ds^2 = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{6.32}$$

This metric under the particular condition $a^2 = \frac{3}{\lambda} > 0$ (thus $\lambda > 0$) has the same form as the de Sitter metric (1.5)

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{6.33}$$

which describes a spherical distribution of the physical vacuum (determined by the λ-field in Einstein's field equations).

This means that liquid collapsars consist of an ideal liquid, the state of which is similar to the state of the physical vacuum. The only difference is that the liquid filling collapsars has a positive density, while the density of the physical vacuum at $\lambda > 0$ is negative; see §5.2 and §5.3 of our book [18] for detail. In addition, ordinary liquid collapsars have a small size and high density (in contrast to the Universe as a whole). Therefore, the liquid that fills ordinary (compact) collapsars is in a state similar to the state of the high-density physical vacuum.

What is the physical vacuum, known also as the λ-field? The physical vacuum is known due to the general formulation of Einstein's field equations containing the λ-term on the right hand side

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,, \tag{6.34}$$

where the right hand side determines a distributed matter that fills the space, and the left hand side determines the space geometry that is Riemannian according to the formulation.

Let us re-write the field equations in the form

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa \widetilde{T}_{\alpha\beta}\,, \tag{6.35}$$

where the joint energy-momentum tensor $\widetilde{T}_{\alpha\beta} = T_{\alpha\beta} + \breve{T}_{\alpha\beta}$ characterizes both the distributed substance and the physical vacuum (λ-field).

The energy-momentum tensor of the physical vacuum

$$\breve{T}_{\alpha\beta} = -\frac{\lambda}{\varkappa}\, g_{\alpha\beta} \tag{6.36}$$

was first deduced in 1995 by us and published in §5.2 and §5.3 of the book [18]. It has the physically observable chr.inv.-projections

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = -\frac{\lambda}{\varkappa} = const < 0\,, \tag{6.37}$$

$$\breve{J}^{i} = \frac{c\,\breve{T}^{i}_{0}}{\sqrt{g_{00}}} = 0\,, \tag{6.38}$$

$$\breve{U}^{ik} = c^2 \breve{T}^{ik} = \frac{\lambda}{\varkappa}\, c^2 h^{ik} = -\breve{\rho} c^2 h^{ik}, \tag{6.39}$$

calculated in the same way as the observable chr.inv.-projections (1.91) of any energy-momentum tensor.

The scalar chr.inv.-projection $\breve{\rho} = -\frac{\lambda}{\varkappa} = const$ means that the physical vacuum is homogeneously distributed over the space, i.e., is a *homogeneous medium*. The vector chr.inv.-projection $\breve{J}^{i} = 0$ means that the physical vacuum does not contain energy flows, i.e., is a *non-radiating medium*.

Let us find the equation of state of the physical vacuum. According to the chronometrically invariant formalism, the chr.inv.-stress tensor

U^{ik} of a distributed medium is expressed through the pressure inside the medium as follows [18, 23]

$$U_{ik} = p_0 h_{ik} - \alpha_{ik} = p h_{ik} - \beta_{ik}\,, \tag{6.40}$$

where p_0 is the equilibrium pressure known due to the equation of state, p is the true pressure inside the medium, α_{ik} is the chr.inv.-viscous stress tensor, $\beta_{ik} = \alpha_{ik} - \frac{1}{3}\alpha h_{ik}$ is its anisotropic part that manifests itself in anisotropic deformations, and $\alpha = h^{ik}\alpha_{ik}$ is the trace of the viscous stress tensor α_{ik}. Since a spherically symmetric space is isotropic by definition, we have $\beta_{ik} = 0$ in the present case. In addition, according to the initial assumption, the vacuum medium is non-viscous ($\alpha_{ik} = 0$). Therefore, for the physical vacuum, we have

$$\breve{U}_{ik} = \breve{p} h_{ik} = -\breve{\rho} c^2 h_{ik}\,. \tag{6.41}$$

Thus, using the formula for the trace of the observable stress tensor $U = h^{ik} U_{ik}$, we obtain the equation of state of the physical vacuum

$$\breve{p} = -\breve{\rho} c^2 \tag{6.42}$$

that at a negative density $\breve{\rho} = -\frac{\lambda}{\varkappa} < 0$ is a manifestation of the state of deflation, which means that the pressure from within the medium tends to compress the sphere.

Deduce the components of the gravitational force acting inside a vacuum collapsar (we call it a *de Sitter collapsar*). Following the same way of derivation as that for the force acting inside a liquid collapsar (6.20, 6.21), we obtain the force

$$F_1 = \frac{\lambda c^2 r}{3} \frac{1}{1 - \frac{\lambda r^2}{3}}\,, \qquad F^1 = \frac{\lambda c^2 r}{3}\,, \tag{6.43}$$

and for the frequency and frequency shift of a photon we obtain

$$\omega = \frac{\omega_0}{\sqrt{1 - \frac{\lambda r^2}{3}}} \simeq \omega_0 \left(1 + \frac{\lambda r^2}{6}\right), \tag{6.44}$$

$$z = \frac{\omega - \omega_0}{\omega_0} = \frac{1}{\sqrt{1 - \frac{\lambda r^2}{3}}} - 1 \simeq \frac{\lambda r^2}{6} > 0\,. \tag{6.45}$$

To understand the results that we have obtained, let us recall that we were able to transform the space metric of a collapsed liquid sphere (6.32) into the de Sitter space metric (6.33) by only the particular condition $a^2 = \frac{3}{\lambda} > 0$. Hence, we have assumed $\lambda > 0$. With $\lambda > 0$, we have obtained a negative density of the physical vacuum $\breve{\rho} = -\frac{\lambda}{\varkappa} < 0$ (6.37), the state of inflation $\breve{p} = -\breve{\rho}c^2$ (6.42), the repulsing force $F_1 > 0$ (6.43) and the square (parabolic) redshift (6.45).

These are the same results as those we have obtained for a liquid collapsar, except for the negative density $\breve{\rho} = -\frac{\lambda}{\varkappa} < 0$ (and, hence, the positive pressure $\breve{p} = -\breve{\rho}c^2 > 0$, which gives the state of deflation) that creates a problem.

To remove this problem, we could assume a negative value of the λ, i.e., $\lambda < 0$ to get a positive density of the physical vacuum. But if so, then the collapsar's radius a would be imaginary, which is nonsense in the observed Universe.

On the other hand, there is another way to remove this problem. Now we will show you how to do it.

Consider Einstein's field equations (6.34) in the form, where the energy-momentum tensor of a distributed substance and the λ-term are taken with the same sign

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} - \lambda g_{\alpha\beta}\,. \tag{6.46}$$

In this case, the energy-momentum tensor of the physical vacuum has the form

$$\breve{T}_{\alpha\beta} = \frac{\lambda}{\varkappa}\, g_{\alpha\beta}\,, \tag{6.47}$$

and its physically observable chr.inv.-projections are

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = \frac{\lambda}{\varkappa} = const > 0\,, \tag{6.48}$$

$$\breve{J}^i = \frac{c\breve{T}^i_0}{\sqrt{g_{00}}} = 0\,, \tag{6.49}$$

$$\breve{U}^{ik} = c^2 \breve{T}^{ik} = -\frac{\lambda}{\varkappa}\, c^2 h^{ik} = -\breve{\rho} c^2 h^{ik}. \tag{6.50}$$

In this case, the physical vacuum (λ-field) is in the state of inflation ($\breve{p} = -\breve{\rho}c^2$), but its density is positive: $\breve{\rho} = \frac{\lambda}{\varkappa} > 0$. Therefore, the modified form (6.46) of Einstein's field equations removes the aforementioned contradiction between the theory of liquid collapsars and the observed positive density of substance in the Universe.

Hence, we have obtained that the physical vacuum (λ-field) is a homogeneous, non-viscous, non-radiating medium, which in the state of inflation.

Concerning the deduced redshift formula (6.45), it depends only on the formula for the force of repulsion deduced from the specific g_{00} of the de Sitter metric (6.33). Since we did not change the space metric, the redshift formula (6.45) remains unchanged.

6.6 Time flows in the opposite direction inside collapsars

In a space without rotation, the observable time interval $d\tau$ (1.30) has a simplified formula: $d\tau = \sqrt{g_{00}}\,dt$. Therefore, according to the specific g_{00} of the metric of a non-rotating liquid sphere (6.6), $d\tau$ in the field of a non-rotating liquid star has the form

$$d\tau = \pm\frac{1}{2}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)dt\,. \tag{6.51}$$

This formula under the condition $a = r_g = \sqrt{3/\varkappa\rho_0}$ characterizing a star in the state of gravitational collapse transforms into

$$d\tau = \mp\frac{1}{2}\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\,dt\,. \tag{6.52}$$

We see that the sign of the observable time interval $d\tau$ inside a regular liquid star is opposite to the $d\tau$ inside a liquid star in the state of gravitational collapse. In other words, the observable time inside ordinary stars flows in the opposite direction than the observable time inside collapsars.

Just one illustration: we usually assume that the observable time flows from the past to the future. If so, then the observable time inside collapsars flows from the future to the past.

6.7 The boundary conditions in a liquid collapsar

Under the condition $a = r_g = \sqrt{3/\varkappa\rho_0}$ characterizing liquid collapsars, the non-zero components of the Riemann-Christoffel curvature tensor

$R_{\alpha\beta\gamma\delta}$ (2.113–2.116) obtained in §2.3 take the form

$$R_{0101} = \frac{\varkappa\rho_0}{12} = \frac{1}{4a^2} = const\,, \tag{6.53}$$

$$R_{1212} = -C_{1212} = \frac{\varkappa\rho_0}{3}\,\frac{r^2}{1-\frac{\varkappa\rho_0 r^2}{3}} = \frac{r^2}{a^2}\,\frac{1}{1-\frac{r^2}{a^2}}\,, \tag{6.54}$$

$$R_{1313} = -C_{1313} = \frac{\varkappa\rho_0}{3}\,\frac{r^2\sin^2\theta}{1-\frac{\varkappa\rho_0 r^2}{3}} = \frac{r^2}{a^2}\,\frac{\sin^2\theta}{1-\frac{r^2}{a^2}}\,, \tag{6.55}$$

$$R_{2323} = -C_{2323} = \frac{\varkappa\rho_0}{3}\,r^4\sin^2\theta = \frac{r^4}{a^2}\,\sin^2\theta\,. \tag{6.56}$$

Since $R_{0101} = \frac{\varkappa\rho_0}{12} = const$ and $R_{0101} > 0$ in the case of a positive density $\rho_0 > 0$ of the liquid, the internal space of a liquid collapsar is a four-dimensional *positive constant curvature space*. This is in contrast to our result of §2.3, where we showed that the space inside a regular liquid sphere has a *variable four-dimensional negative curvature*. Hence:

> The state of gravitational collapse is a "bridge" connecting the world of a varying four-dimensional negative curvature inside ordinary stars and the world of a four-dimensional positive constant curvature inside those stars that are in the state of gravitational collapse.

Calculate the observable three-dimensional curvature of the space inside non-rotating liquid collapsars. We calculate C_{11} (2.104), C_{22} (2.105) and the observable curvature scalar $C = h^{ik}C_{ik}$ under the condition $a = r_g = \sqrt{3/\varkappa\rho_0}$ characterizing liquid collapsars. We obtain

$$C_{11} = -\frac{2\varkappa\rho_0}{3}\,\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}} = -\frac{2}{a^2}\,\frac{1}{1-\frac{r^2}{a^2}}\,, \tag{6.57}$$

$$C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2\varkappa\rho_0 r^2}{3} = -\frac{2r^2}{a^2}\,, \tag{6.58}$$

$$C = -2\varkappa\rho_0 = -\frac{6}{a^2} = const < 0\,. \tag{6.59}$$

This is a *three-dimensional negative constant curvature space* as well as the space inside ordinary liquid stars.

So forth, we express the force of gravitation acting in the internal space of a non-rotating liquid collapsar through the observable three-dimensional curvature of the internal space. From the formulae for F_1 (6.20) and F^1 (6.21), we obtain

$$F_1 = -\frac{c^2 r}{2}\, C_{11}\,, \qquad F^1 = -\frac{c^2}{2r}\, C_{22}\,. \tag{6.60}$$

We see that both the observable three-dimensional curvature and the force of gravitation have a space breaking

$$C_{11} \to -\infty, \qquad F_1 \to \infty \tag{6.61}$$

by the limit condition $r = a$ on the surface of the collapsar. This result is, however, trivial.

6.8 Rotating liquid collapsars

Here we complicate our task by considering rotating liquid collapsars. Let the space of the metric (6.32) characteristic of a liquid collapsar rotates with an angular velocity ω around the polar axis of the collapsar. In this case, among the g_{0i}-th components of the fundamental metric tensor $g_{\alpha\beta}$, only the non-zero component

$$g_{03} = -\frac{2\omega r^2 \cos\theta}{c} \tag{6.62}$$

characterizes the rotation, while $g_{01} = g_{02} = 0$. Therefore, the linear velocity v_i (1.45) with which the space rotates has the form

$$v_3 = \frac{2\omega r^2 \cos\theta}{\sqrt{1-\frac{r^2}{a^2}}}\,, \qquad v_1 = v_2 = 0\,. \tag{6.63}$$

As a result, we get the space metric of a rotating liquid collapsar

$$ds^2 = \frac{1}{4}\left(1-\frac{r^2}{a^2}\right)c^2 dt^2 - \frac{dr^2}{1-\frac{r^2}{a^2}} - {} - \frac{2\omega r^2 \cos\theta}{c}\, c\,dt\, d\phi - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{6.64}$$

It is possible to prove that this space metric satisfies Einstein's field equations containing the energy-momentum tensor of an ideal liquid

(2.4). If we substitute the specific $g_{\alpha\beta}$ components of the metric (6.64), then the left hand side and right hand side of the equations become the same: the field equations are satisfied as identities.

The general condition of gravitational collapse means that the physically observable time stops ($d\tau = 0$) on the collapse surface. The definition of $d\tau$ (1.30) takes both the factors g_{00} and g_{0i} into account

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = \left(1 - \frac{\mathrm{w}}{c^2}\right)dt - \frac{1}{c^2}\,v_i\,dx^i. \tag{6.65}$$

Therefore, the collapse condition with $v_i \neq 0$ is not $d\tau = \sqrt{g_{00}}\,dt = 0$ as that for non-rotating collapsars, but takes the complete form

$$\sqrt{g_{00}} - \frac{1}{c^2}\,v_3\,u^3 = 0\,, \tag{6.66}$$

where $u^3 = \frac{d\phi}{dt} = \omega$. Using the g_{00} and v_3 (6.63) of the metric (6.64), we obtain the collapse surface radius of a rotating liquid collapsar

$$r_c = \frac{a}{\sqrt{1 + \frac{4\,\omega^2 a^2 \cos\theta}{c^2}}} \leqslant a\,, \tag{6.67}$$

and, hence,

$$r_c \simeq a\left(1 - \frac{2\,\omega^2 a^2 \cos\theta}{c^2}\right) = a - \Delta a\,. \tag{6.68}$$

Assuming $\omega = 10^3$ sec^{-1} and $a = 10^6$ cm for example, we obtain $\Delta a \simeq 22\cos\theta$, i.e. $\Delta a \simeq 22$ metres at the equator of the star and $\Delta a = 0$ at the South Pole and North Pole.

We see that the collapse surface meets the radius a of the star only at the poles of the star's rotation, where the latitude is $\theta = \pm\frac{\pi}{2}$ and, therefore, $\cos\theta = 0$. In other words, rotating liquid collapsars are not spheres, but have an *elliptic form* flattened on the equatorial plane (which is orthogonal to the axis of rotation).

If a collapsar does not rotate ($\omega = 0$), then its form is spherically symmetric ($r_c = a$). At a maximum relativistic rotation speed, the collapsar's elliptic form is highly flattened on the equatorial plane. In the limiting case, when the collapsar rotates with a velocity close to the velocity of light ($\omega^2 a^2 \to c^2$), its form is determined by the equation

$$r_c = \frac{a}{\sqrt{1 + 4\cos\theta}}\,. \tag{6.69}$$

The other parameters of rotating liquid collapsars, which we have obtained in the framework of our theory do not change the fundamental results obtained in §5.1 for non-rotating liquid collapsars. The only difference is that the formulae contain a correction for the angular velocity of the collapsar ω. Therefore, we omit these results from consideration.

6.9 Conclusion

Let us recall everything we have obtained here on liquid collapsars:

1. The radial coordinate at which a non-rotating liquid sphere of a radius a meets the state of gravitational collapse, is r_c (6.10)

$$r_c = \sqrt{9a^2 - \frac{8a^3}{r_g}}\,. \tag{6.70}$$

   For ordinary stars, r_c takes imaginary numerical values. Therefore, ordinary stars ranging from super-giants to dwarfs and white dwarfs cannot collapse;

2. Since the collapse radius r_c must be real for real objects, the physical radius a of a non-rotating liquid collapsar must be

$$a \leqslant 1.125\, r_g\,. \tag{6.71}$$

   If a non-rotating liquid star has a radius of $a \geqslant 1.125\, r_g$, then this star cannot be in the state of gravitational collapse;

3. Density is the primary characteristic of non-rotating liquid collapsars. The physical radius a of such a collapsar is inversely proportional to the square root of its density ρ_0 (6.13)

$$a = \sqrt{\frac{3}{\varkappa\rho_0}} = \frac{4.0\times10^{13}}{\sqrt{\rho_0}}\ \text{cm}; \tag{6.72}$$

4. The mass M of a non-rotating liquid collapsar is proportional to its physical radius a (6.14) and is inversely proportional to the square root of its density ρ_0 (6.15)

$$M = \frac{4\pi a}{\varkappa} = 6.8\times10^{27}\,a\ \ \text{gram}, \tag{6.73}$$

$$M = \frac{4\sqrt{3}\,\pi}{\varkappa^{3/2}\sqrt{\rho_0}} = \frac{2.7\times10^{41}}{\sqrt{\rho_0}}\ \ \text{gram}; \tag{6.74}$$

5. The observable Universe is completely located inside its collapse radius. Therefore, we conclude that the Universe is a gravitational collapsar: all stars and galaxies, including ourselves, exist inside a huge black hole. Its parameters theoretically calculated according to the model of liquid collapsars are

$$\left.\begin{aligned} a &= 1.3\times 10^{28}\ \text{cm} \\ \rho_0 &= 9.6\times 10^{-31}\ \text{gram/cm}^3 \\ M &= 8.8\times 10^{55}\ \text{gram} \end{aligned}\right\}; \tag{6.75}$$

6. A liquid substance that fills liquid collapsars is in the state of inflation. Its equation of state is

$$p = -\rho_0 c^2 = const, \tag{6.76}$$

which means that at a positive density of substance the pressure is negative, so the pressure from within tends to expand the body (but the collapsar does not expand, because a liquid body is incompressible). The pressure and density remain unchanged from the centre of the collapsar up to its surface;
7. The gravitational inertial force acting inside a non-rotating liquid collapsar is a force of repulsion. It increases with distance, from zero at the centre of the collapsar to its maximum value on the surface;
8. The internal force of repulsion produces a square (parabolic) redshift on photons travelling inside the collapsar;
9. The state of a liquid substance that fills ordinary (compact) collapsars is similar to the state of the high-density physical vacuum (high-density λ-field), which is a homogeneous, non-viscous, non-radiating medium in the state of inflation;
10. The observable time flows in the opposite directions inside and outside collapsars: if we assume that the observable time of our world flows from the past to the future, then the observable time flows from the future to the past inside collapsars;
11. The state of gravitational collapse is a "bridge" connecting the world of a varying four-dimensional negative curvature inside ordinary stars and the world of a four-dimensional positive constant curvature inside gravitational collapsars (black holes);

12. Rotating liquid collapsars are not spheres, but have an *elliptic form* flattened on the equatorial plane. The radius r_c of a rotating liquid collapsar is formulated through the sphere's radius a, the latitude θ and the angular velocity of its rotation ω as

$$r_c = \frac{a}{\sqrt{1 + \frac{4\,\omega^2 a^2 \cos\theta}{c^2}}}\,. \tag{6.77}$$

Bibliography

1. Eddington A. S. The Internal Constitution of the Stars. Cambridge University Press, Cambridge, 1926.
2. Bethe H. A. Energy production in stars. *Physical Review*, 1939, vol. 55, no. 5, 434–456.
3. Milne E. A. The analysis of stellar structure. *Monthly Notices of the Royal Astronomical Society*, 1930, vol. 91, no. 1, 4–55.
4. Kozyrev N. A. Sources of stellar energy and the theory of the internal constitution of stars. *Progress in Physics*, 2005, vol. 1, no. 3, 61–99. Translated from: Kozyrev N. A. Sources of stellar energy and the theory of the internal constitution of stars. Part I. *Proceedings of Crimean Astronomical Observatory*, 1948, vol. 2, 3–43; Part II, *ibid.*, 1951, vol. 6, 54–83.
5. Jeans J. H. Recent developments of cosmical physics. *Nature*, December 4, 1926, vol. 118, no. 2979, 29–40 (in the *Supplement* to the issue).
6. Jeans J. H. Astronomy and Cosmogony. Cambridge University Press, Cambridge, 1928.
7. Robitaille P.-M. A high temperature liquid plasma model of the Sun. *Progress in Physics*, 2007, vol. 3, no. 1, 70–81.
8. Robitaille P.-M. A thermodynamic history of the solar constitution — I: the journey to a gaseous sun. *Progress in Physics*, 2011, vol. 7, no. 3, 3–25.
9. Robitaille P.-M. A thermodynamic history of the solar constitution — II: the theory of a gaseous sun and Jeans' failed liquid alternative. *Progress in Physics*, 2011, vol. 7, no. 3, 41–59.
10. Robitaille P.-M. Forty lines of evidence for condensed matter — the Sun on trial: liquid metallic hydrogen as a Solar building block. *Progress in Physics*, 2013, vol. 9, no. 4, 90–142.
11. Borissova L. The gravitational field of a condensed matter model of the Sun: the space breaking meets the Asteroid strip. *The Abraham Zelmanov Journal*, 2009, vol. 2, 224–260.
12. Borissova L. The Solar System according to General Relativity: the Sun's space breaking meets the Asteroid strip. *Progress in Physics*, 2010, vol. 6, no. 2, 43–47.

13. Schwarzschild K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, 189–196.
English translation: Schwarzschild K. On the gravitational field of a point mass according to Einstein's theory. *The Abraham Zelmanov Journal*, 2008, vol. 1, 10–19.

14. Schwarzschild K. Über das Gravitationsfeld einer Kugel aus incompressiebler Flüssigkeit nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, 424–435.
English translation: Schwarzschild K. On the gravitational field of a sphere of incompressible liquid, according to Einstein's theory. *The Abraham Zelmanov Journal*, 2008, vol. 1, 20–32.

15. Hilbert D. Die Grundlagen der Physik. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, *Mathematisch-Physikalische Klasse*, 1917, 53–76.

16. De Sitter W. On the curvature of space. *Koninklijke Nederlandsche Akad. van Wetenschappen*, *Proceedings*, 1918, vol. XX, part I, no. 2, 229–243.

17. Borissova L. De Sitter bubble as a model of the observable Universe. *The Abraham Zelmanov Journal*, 2010, vol. 3, 3–24.

18. Borissova L. and Rabounski D. Fields, Vacuum and the Mirror Universe. The 3rd revised edition, New Scientific Frontiers, London, 2023.
First published in 1999, in Russian; published in English — 2001, 2009, 2023; in French — 2010, 2023.
French translation: Rabounski D. et Borissova L. Champs, vide, et Univers miroir. La 2ème édition révisée, New Scientific Frontiers, Londres, 2023.

19. Rabounski D. and Borissova L. Particles Here and Beyond the Mirror. The 4th revised edition, New Scientific Frontiers, London, 2023.
Published in English — 2001, 2008, 2012, 2023; in French — 2010, 2023; in Russian — 2023.
French translation: Rabounski D. et Borissova L. Particules de l'Univers et au delà du miroir. La 2ème édition révisée, New Scientific Frontiers, Londres, 2023.

20. Landau L. D. and Lifshitz E. M. The Classical Theory of Fields. Pergamon Press, Oxford, 1951.
Translated from the 1st Russian edition published in 1939. Section references are given from the final 4th English edition, expanded twicely from the 1st edition, Butterworth-Heinemann, 1979.

21. Zelmanov A. L. Chronometric Invariants. Translated from the 1944 PhD thesis, American Research Press, Rehoboth, New Mexico, 2006.

22. Zelmanov A. L. Chronometric invariants and accompanying frames of reference in the General Theory of Relativity. *Soviet Physics Doklady*,

1956, vol. 1, 227–230.
Translated from: *Doklady Academii Nauk SSSR*, 1956, vol. 107, no. 6, 815–818.

23. Zelmanov A. L. On the relativistic theory of an anisotropic inhomogeneous universe. *The Abraham Zelmanov Journal*, 2008, vol. 1, 33–63.
Translated from: *Proceedings of the 6th Soviet Conference on the Problems of Cosmogony*, held in 1957 in Moscow, USSR Acad. Science Publishers, Moscow, 1959, 144–174.
24. Hafele J. C. Relativistic behaviour of moving terrestrial clocks. *Nature*, 18 July 1970, vol. 227, 270–271.
25. Hafele J. Performance and results of portable clocks in aircraft. PTTI 3rd Annual Meeting, November 16–18, 1971, 261–288.
26. Hafele J. C. Relativistic time for terrestrial circumnavigations. *American Journal of Physics*, 1972, vol. 40, 81–85.
27. Hafele J. and Keating R. Around the world atomic clocks: predicted relativistic time gains. *Science*, July 14, 1972, vol. 177, 166–168.
28. Hafele J. and Keating R. Around the world atomic clocks: observed relativistic time gains. *Science*, July 14, 1972, vol. 177, 168–170.
29. Demonstrating relativity by flying atomic clocks. *Metromnia*, the UK's National Measurement Laboratory Newsletter, issue 18, Spring 2005.
30. Schouten J. A. und Struik D. J. Einführung in die neuren Methoden der Differentialgeometrie. Noordhoff, Groningen, 1938.
First published in *Zentralblatt für Mathematik*, 1935, Bd. 11 und Bd. 19.
31. Kamke E. Differentialgleichungen: Lösungsmethoden und Lösungen. Chelsea Publishing Co., New York, 1959.
First published as: Kamke E. Differentialgleichungen. Lösungsmethoden und Lösungen. Gewöhnliche Differentialgleichungen, S. Hirzel Verlag, Leipzig 1942.
32. Hessels J. W. T., Ransom S. M., Stairs I. H., Freire P. C., Kaspi V. M., Camilo F. A radio pulsar spinning at 716 Hz. *Science*, March 31, 2006, vol. 311, no. 5769, 1901–1904.
33. Stanyukovich K. On the problem of the existence of stable particles in the Metagalaxy. *The Abraham Zelmanov Journal*, 2008, vol. 1, 99–110.
Translated from: *Problems of the Theory of Gravitation and Elementary Particles*, vol. 1, Atomizdat, Moscow, 1966, 267–279.

About the authors

Larissa Borissova (b. 1944, Moscow, Russia) is a graduate of the Sternberg Astronomical Institute (Faculty of Physics, Moscow University). Since 1964 she was trained by Dr. Abraham Zelmanov (1913–1987), a famous cosmologist and researcher in General Relativity. Since 1968 she worked with Prof. Kyril Stanyukovich (1916–1989), a prominent scientist in gas dynamics and General Relativity. In 1975, Larissa Borissova got a "Candidate of Science" degree (Soviet PhD) in gravitational waves. She has published about 50 scientific papers and 3 books on General Relativity. In 2005, Larissa Borissova became a co-founder and Editor of *Progress in Physics* and in 2008, an Editor of *The Abraham Zelmanov Journal*. She has since continued her scientific studies as an independent researcher.

Dmitri Rabounski (b. 1965, Moscow, Russia). Since 1983 he was trained by Prof. Kyril Stanyukovich (1916–1989), a prominent scientist in gas dynamics and General Relativity. He was also trained with Dr. Abraham Zelmanov (1913–1987), a famous cosmologist and researcher in General Relativity. He was also trained by Dr. Vitaly Bronshten (1918–2004), the well-known expert in the physics of destruction of bodies in the atmosphere. Dmitri Rabounski has published about 50 scientific papers and 3 books on General Relativity. In 2005, he started a new journal on physics, *Progress in Physics*, where he is the Editor-in-Chief. In 2008, he started a new journal on General Relativity, *The Abraham Zelmanov Journal*, while continuing his scientific studies as an independent researcher.

Cover image: An SDO/NASA photo image of the Sun. Courtesy of SDO/NASA and the AIA, EVE, and HMI science teams. SDO images and movies are not copyrighted unless explicitly noted: the use of SDO images for non-commercial purposes and public education and information efforts is strongly encouraged and requires no expressed authorization. See http://sdo.gsfc.nasa.gov/data/rules.php for detail.

Titlepage image: The enigmatic woodcut by an unknown artist of the Middle Ages. It is referred to as the *Flammarion Woodcut* because its appearance in page 163 of Camille Flammarion's *L'Atmosphère*: *Météorologie populaire* (Paris, 1888), a work on meteorology for a general audience. The woodcut depicts a man peering through the Earth's atmosphere as if it were a curtain to look at the inner workings of the Universe. The caption "Un missionnaire du moyen àge raconte qu'il avait trouvé le point où le ciel et la Terre se touchent…" translates to "A medieval missionary tells that he has found the point where heaven [the sense here is "sky"] and Earth meet…"

Book format: 60 × 90 1/16
Typeface family: Tempora TLF
Default font size: 11 pt
Typos corrected on: 20.12.2023

Inside Stars

A theory of the internal constitution of stars and sources of stellar energy based on the General Theory of Relativity

by L. Borissova and D. Rabounski

The 3rd revised edition

New Scientific Frontiers
London, 2023

ISBN 978-1-7392930-6-2

9 781739 293062