On the Gravitational Field of a Sphere of Incompressible Liquid, According to Einstein's Theory

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Abstract: This is a translation of the paper *Über das Gravitationsfeld einer Kugel aus incompressiebler Flüssigkeit nach der Einsteinschen Theorie* published by Karl Schwarzschild, in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 424–435. Here Schwarzschild expounds his previously obtained metric for the spherically symmetric gravitational field produced by a point-mass, to the case where the source of the field is represented by a sphere of incompressible fluid. Schwarzschild formulates the physical condition of degeneration of such a field. Translated from the German in 2008 by Larissa Borissova and Dmitri Rabounski.

§1. As the next step of my study concerning Einstein's theory of gravitation, I calculated the gravitational field of a homogeneous sphere of a finite radius, consisting of incompressible fluid. This clarification, "consisting of incompressible fluid", is necessary to be added, due to the fact that gravitation, in the framework of the relativistic theory, depends on not only the quantity of the matter, but also on its energy. For instance, a solid body having a specific state of internal stress would produce a gravitation other than that of a liquid.

This calculation is a direct continuation of my presentation concerning the gravitational field of a point-mass (see *Sitzungsberichte*, 1916, S. 189[*]), to which I will refer here in short[†].

§2. Einstein's equations of gravitation (see *Sitzungsberichte*, 1915, S. 845[‡]) in the general form manifest that

$$\sum_\alpha \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} + \sum_{\alpha\beta} \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha} = G_{\mu\nu} \,. \tag{1}$$

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[*]Schwarzschild K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

[†]Schwarzschild means that, somewhere in this paper, he will refer to his formulae deduced in his first publication of 1916. — Editor's comment. D.R.

[‡]Einstein A. Die Feldgleichungen der Gravitation. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1915, S. 844–847. — Editor's comment. D.R.

The quantities $G_{\mu\nu}$ vanish where is no matter. Inside an incompressible liquid they are determined in the following way: the "mixed tensor of the energy" of an incompressible liquid, according to Einstein (see *Sitzungsberichte*, 1914, S. 1062[*]) is equal to

$$T_1^1 = T_2^2 = T_3^3 = -p\,, \qquad T_4^4 = \rho_0\,, \tag{2}$$

while the rest of the T_μ^ν are zero. Here p is the pressure, ρ is the constant density of the liquid.

The "covariant tensor of the energy" will be

$$T_{\mu\nu} = \sum_\tau T_\mu^\tau\, g_{\nu\tau}\,. \tag{3}$$

Besides

$$T = \sum_\tau T_\tau^\tau = \rho_0 - 3p \tag{4}$$

and also

$$\varkappa = 8\pi k^2,$$

where $\varkappa$ is Gauss' gravitational constant. Then, according to Einstein (see *Sitzungsberichte*, 1915, S. 845, Gleichung 2a[†]), the right sides of the equations have the form

$$G_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,T\right). \tag{5}$$

To be in the state of equilibrium, such a liquid should satisfy the conditions (see equation 7a ibidem[†])

$$\sum_\alpha \frac{\partial T_\tau^\alpha}{\partial x^\alpha} + \sum_{\mu\nu} \Gamma_{\tau\nu}^\mu\, T_\mu^\nu = 0\,. \tag{6}$$

§3. In the case of such a sphere, as well as in the case of a point-mass, these general equations should be normalized for the symmetrical rotation around the origin of the coordinates. As in the case of a point-mass, it is recommended to move to the spherical coordinates chosen

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[*]Einstein A. Die formale Grundlage der allgemeinen Relativitätstheorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1914, S. 1030–1085. This is a bulky paper concerning the formal basics of the General Theory of Relativity, wherein Einstein considered his equations of gravitation. — Editor's comment. D.R.

[†]Einstein A. Die Feldgleichungen der Gravitation. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1915, S. 844–847. — Editor's comment. D.R.

so that the determinant equals 1^*

$$x^1 = \frac{r^3}{2}\,, \quad x^2 = -\cos\vartheta\,, \quad x^3 = \varphi\,, \quad x^4 = t\,. \qquad (7)$$

The line-element should have the same form

$$ds^2 = f_4\,(dx^4)^2 - f_1\,(dx^1)^2 - f_2\,\frac{(dx^2)^2}{1-(x^2)^2} - f_3\,(dx^3)^2\left[1-(x^2)^2\right], \;\; (8)$$

so that we have

$$g_{11} = -f_1\,, \quad g_{22} = -\frac{f_2}{1-(x^2)^2}\,, \quad g_{33} = -f_2\left[1-(x^2)^2\right], \quad g_{44} = f_4\,,$$

while the other $g_{\mu\nu}$ are zero. These f are functions dependent only on x^1.

In the space outside this sphere, the solutions (10), (11), (12) were found†

$$f_4 = 1 - \alpha\left(3x^1 + \rho\right)^{-\frac{1}{3}}, \quad f_2 = \left(3x^1 + \rho\right)^{\frac{2}{3}}, \quad f_1(f_2)^2 f_4 = 1\,, \quad (9)$$

where α and ρ are two arbitrary constants, which should be determined on the basis of the mass and the radius of the sphere.

We are going to construct the field equations for the internal space of this sphere with use of the formula (8) for the line-element, then solve these equations. Concerning the right sides, we obtain

$$T_{11} = T_1^1\, g_{11} = -p f_1\,, \qquad T_{22} = T_2^2\, g_{22} = -\frac{p f_2}{1-(x^2)^2}\,,$$

$$T_{33} = T_3^3\, g_{33} = -p f_2\left[1-(x^2)^2\right], \qquad T_{44} = T_4^4\, g_{44} = \rho_0 f_4\,,$$

$$G_{11} = \frac{\varkappa f_1}{2}\,(p - \rho_0)\,, \qquad G_{22} = \frac{\varkappa f_2}{2}\,\frac{1}{1-(x^2)^2}\,(p-\rho_0)\,,$$

$$G_{33} = \frac{\varkappa f_2}{2}\left[1-(x^2)^2\right](p - \rho_0), \qquad G_{44} = -\frac{\varkappa f_4}{2}\,(\rho_0 + 3p)\,.$$

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*In the original — "polar coordinates". The same formulation was used in his first paper of 1916. Obviously Schwarzschild means the three-dimensional spherical coordinates, whose origin meets the centre of the sphere of incompressible liquid. — Editor's comment. D.R.

†Here Schwarzschild refers to the formulae (10), (11), and (12) obtained in his first paper: Über das Gravitationsfeld eines Massenpunktes nach der Einstein-schen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

We can assume that the components $\Gamma^{\alpha}_{\mu\nu}$ of the gravitational field expressed through these functions f, and also the left sides of the field equations are independent of the point-mass (see §4). Limiting our task again by considering the equator ($x^2 = 0$), we obtain the following system of equations.

First, these are three field equations

$$a) \quad -\frac{1}{2}\frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x^1}\right) + \frac{1}{4}\frac{1}{(f_1)^2}\left(\frac{\partial f_1}{\partial x^1}\right)^2 + \frac{1}{2}\frac{1}{(f_2)^2}\left(\frac{\partial f_2}{\partial x^1}\right)^2 +$$

$$+ \frac{1}{4}\frac{1}{(f_4)^2}\left(\frac{\partial f_4}{\partial x^1}\right)^2 = -\frac{\varkappa}{2}\,f_1\,(\rho_0 - p)\,,$$

$$b) \quad +\frac{1}{2}\frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_2}{\partial x^1}\right) - \frac{1}{2}\frac{1}{f_1 f_2}\left(\frac{\partial f_2}{\partial x^1}\right)^2 = -\frac{\varkappa}{2}\,f_2\,(\rho_0 - p)\,,$$

$$c) \quad -\frac{1}{2}\frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_4}{\partial x^1}\right) + \frac{1}{2}\frac{1}{f_1 f_4}\left(\frac{\partial f_4}{\partial x^1}\right)^2 = -\frac{\varkappa}{2}\,f_4\,(\rho_0 + 3p)\,.$$

We should add to these the determinant equation

$$d) \quad f_1(f_2)^2 f_4 = 1\,.$$

The equilibrium conditions provide just one equation

$$e) \quad -\frac{\partial p}{\partial x^1} = -\frac{p}{2}\left[\frac{1}{f_1}\frac{\partial f_1}{\partial x^1} + \frac{2}{f_2}\frac{\partial f_2}{\partial x^1}\right] + \frac{\rho_0}{2}\frac{1}{f_4}\frac{\partial f_4}{\partial x^1}\,.$$

Proceeding from the common consideration of Einstein's equations, it follows that the aforementioned 5 equations with respect to 4 variables f_1, f_2, f_4, p are consistent with each other.

We should find solutions of these 5 equations, which would be free of singularity inside the sphere. There on the surface of the sphere $p = 0$ should be true, the functions f in the neighbourhood of their derivatives should be continuous, and be transferred into the quantities (9) which are true outside the sphere.

We will omit the index 1 in x^1, for simplicity.

§4. The equation $e)$, due to the determinant equation, transforms into

$$-\frac{\partial p}{\partial x} = \frac{\rho_0 + p}{2}\frac{1}{f_4}\frac{\partial f_4}{\partial x}\,.$$

It can be easy integrated, and gives

$$(\rho_0 + p)\sqrt{f_4} = const = \gamma\,. \tag{10}$$

The field equations $a)$, $b)$, $c)$, after multiplication by the factors -2, $+2\frac{f_1}{f_2}$, $-2\frac{f_1}{f_4}$, transform into

$$a')\quad \frac{\partial}{\partial x}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x}\right) = \frac{1}{2\,(f_1)^2}\left(\frac{\partial f_1}{\partial x}\right)^2 + \frac{1}{(f_2)^2}\left(\frac{\partial f_2}{\partial x}\right)^2 +$$
$$+ \frac{1}{2\,(f_4)^2}\left(\frac{\partial f_4}{\partial x}\right)^2 + \varkappa f_1\,(\rho_0 - p)\,,$$

$$b')\quad \frac{\partial}{\partial x}\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x}\right) = 2\,\frac{f_1}{f_2} + \frac{1}{f_1 f_2}\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x} - \varkappa f_1\,(\rho_0 - p)\,,$$

$$c')\quad \frac{\partial}{\partial x}\left(\frac{1}{f_4}\frac{\partial f_4}{\partial x}\right) = \frac{1}{f_1 f_4}\frac{\partial f_1}{\partial x}\frac{\partial f_4}{\partial x} + \varkappa f_1\,(\rho_0 + 3p)\,.$$

Forming the combinations $a' + 2b' + c'$ and $a' + c'$, and using the determinant equation, we obtain, finally,

$$0 = 4\,\frac{f_1}{f_2} - \frac{1}{(f_2)^2}\left(\frac{\partial f_2}{\partial x}\right)^2 - \frac{2}{f_2 f_4}\frac{\partial f_2}{\partial x}\frac{\partial f_4}{\partial x} + 4\varkappa f_1 p\,, \qquad (11)$$

$$0 = 2\,\frac{\partial}{\partial x}\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x}\right) + \frac{3}{(f_2)^2}\left(\frac{\partial f_2}{\partial x}\right)^2 + 2\varkappa f_1\,(\rho_0 + p)\,. \qquad (12)$$

Now we introduce new variables, which are desirable due to the fact that, according to the results obtained for the point-mass, such variables behave simply outside the sphere as they are independent of the terms of these equations which contain ρ and p. So the equations, being expressed with the new variables, should have a simple form as well.

The new variables are

$$f_2 = \eta^{\frac{2}{3}}\,, \qquad f_4 = \zeta\,\eta^{-\frac{1}{3}}\,, \qquad f_1 = \frac{1}{\zeta\,\eta}\,. \qquad (13)$$

Then, according to (9) outside the sphere,

$$\eta = 3x + \rho\,, \qquad \zeta = \eta^{\frac{1}{3}} - \alpha\,, \qquad (14)$$

$$\frac{\partial \eta}{\partial x} = 3\,, \qquad \frac{\partial \zeta}{\partial x} = \eta^{-\frac{2}{3}}\,. \qquad (15)$$

We introduce these new variables and, at the same time, remove $\rho_0 + p$ with $\gamma f_4^{-\frac{1}{2}}$ according to (10). As a result the equations (11) and (12) transform into

$$\frac{\partial \eta}{\partial x}\frac{\partial \zeta}{\partial x} = 3\eta^{-\frac{2}{3}} + 3\varkappa\gamma\zeta^{-\frac{1}{2}}\eta^{\frac{1}{6}} - 3\varkappa\rho_0\,, \qquad (16)$$

$$2\zeta\,\frac{\partial^2\eta}{\partial x^2} = -\,3\,\varkappa\gamma\,\zeta^{-\frac{1}{2}}\,\eta^{\frac{1}{6}}\,. \tag{17}$$

Summation of these two equations gives

$$2\zeta\,\frac{\partial^2\eta}{\partial x^2} + \frac{\partial\eta}{\partial x}\,\frac{\partial\zeta}{\partial x} = 3\eta^{-\frac{2}{3}} - 3\,\varkappa\rho_0\,.$$

The integrating multiplier of this equation is $\frac{\partial\eta}{\partial x}$. We obtain, after integration,

$$\zeta\left(\frac{\partial\eta}{\partial x}\right)^2 = 9\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\,\eta + 9\lambda\,, \tag{18}$$

where λ is the constant of integration.

Raising it to a power of $\frac{3}{2}$ gives

$$\zeta^{\frac{3}{2}}\left(\frac{\partial\eta}{\partial x}\right)^3 = \left(9\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\,\eta + 9\lambda\right)^{\frac{3}{2}}\,.$$

Dividing (17) by this equation, we obtain that ζ vanishes so that the following differential equations with respect to η is obtained

$$\frac{2\frac{\partial^2\eta}{\partial x^2}}{\left(\frac{\partial\eta}{\partial x}\right)^3} = -\frac{3\,\varkappa\gamma\eta^{\frac{1}{6}}}{\left(9\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\,\eta + \lambda\right)^{\frac{3}{2}}}\,.$$

Again, $\frac{\partial\eta}{\partial x}$ is the integrating multiplier here. We obtain, after integration,

$$\frac{2}{\left(\frac{\partial\eta}{\partial x}\right)} = 3\,\varkappa\gamma\int\frac{\eta^{\frac{1}{6}}d\eta}{\left(9\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\,\eta + \lambda\right)^{\frac{3}{2}}} \tag{19}$$

and, because

$$\frac{2}{\frac{\delta\eta}{\delta x}} = \frac{2\,\delta x}{\delta\eta}\,,$$

iterated integration gives

$$x = \frac{\varkappa\gamma}{18}\int d\eta\int\frac{\eta^{\frac{1}{6}}d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^{\frac{3}{2}}}\,. \tag{20}$$

It follows from here that x is a function of η and, vice versa, that η is a function of x. Besides, ζ, due to (18), (19), and also (13), is a function of f. Thus our problem has came back to quadratures.

§5. Now we should find the constants of integration so that the internal region of the sphere would be free of singularity, and also the continuous transfer from the values of the functions f and their derivatives inside the sphere to the respective values outside it would be allowed in the surface.

There in the surface of the sphere $r = r_a$, $x = x_a$, $\eta = \eta_a$, etc. The continuity of η and ζ can be satisfied in any case through the respective choice of the constants α and ρ. If also, according to it, the derivatives remain continuous, and, due to (15), $\left(\frac{d\eta}{dx}\right)_a = 3$ and $\left(\frac{d\zeta}{dx}\right)_a = \eta_a^{-\frac{2}{3}}$, the equations (16) and (18) should be

$$\gamma = \rho_0 \zeta_a^{\frac{1}{2}} \eta_a^{-\frac{1}{6}}, \qquad \zeta_a = \eta_a^{\frac{1}{3}} - \frac{\varkappa \rho_0}{3}\,\eta_a + \lambda\,. \tag{21}$$

It follows from here that

$$\zeta_a \eta_a^{-\frac{1}{3}} = (f_4)_a = 1 - \frac{\varkappa \rho_0}{3}\,\eta_a^{\frac{2}{3}} + \lambda \eta_a^{-\frac{1}{3}}.$$

Thus we have

$$\gamma = \rho_0 \sqrt{(f_4)_a}\,. \tag{22}$$

Comparing it to (10), we see that it satisfies the condition $p = 0$ in the surface. The requirement $\left(\frac{d\eta}{dx}\right)_a = 3$ leads to the following determination of the limits of integration in (19)

$$\frac{3\,dx}{d\eta} = 1 - \frac{\varkappa \gamma}{6} \int\limits_{\eta}^{\eta_a} \frac{\eta^{\frac{1}{6}}\,d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa \rho_0}{3}\,\eta + \lambda\right)^{\frac{3}{2}}} \tag{23}$$

so that, with taking (20) into account, we arrive at the determination of the limits of integration

$$3\,(x - x_a) = \eta - \eta_a + \frac{\varkappa \gamma}{6} \int\limits_{\eta}^{\eta_a} d\eta \int\limits_{\eta}^{\eta_a} \frac{\eta^{\frac{1}{6}}\,d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa \rho_0}{3}\,\eta + \lambda\right)^{\frac{3}{2}}}\,. \tag{24}$$

The surface conditions are satisfied completely. The constants η_a and λ are still undetermined; we will determine the constants through the continuity conditions at the origin of the coordinates.

First, we should require that $\eta = 0$ at $x = 0$. If this condition were wrong, f_2 would take a finite numerical value at the origin of the coordinates, so the change of the angle $d\varphi = dx^3$ at the origin of the coordinates (that does not mean a real motion) would give a meaning to the line-element. Thus, as follows from (24), the following condition con-

nects x_a and η_a

$$3x_a = \eta_a - \frac{\varkappa\gamma}{6} \int\limits_0^{\eta_a} d\eta \int\limits_\eta^{\eta_a} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^{\frac{3}{2}}} \,. \tag{25}$$

Finally, λ is determined by the condition, according to which the pressure inside the sphere should be finite and positive, as follows from (10), and also f_4 should be finite and nonzero. Proceeding from (13), (18) and (23), we have

$$f_4 = \zeta\eta^{-\frac{1}{3}} = \left(1 - \frac{\varkappa\rho_0}{3}\eta^{\frac{2}{3}} + \lambda\eta^{-\frac{1}{3}}\right) \times$$
$$\times \left[1 - \frac{\varkappa\gamma}{6} \int\limits_\eta^{\eta_0} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^{\frac{3}{2}}}\right]^2 . \tag{26}$$

First, it is supposed here that $\lambda \gtrless 0$. Then, for very small numerical values of η we obtain

$$f_4 = \frac{\lambda}{\eta^{\frac{1}{3}}} \left[K + \frac{\varkappa\gamma}{7} \frac{\eta^{\frac{7}{6}}}{\lambda^{\frac{3}{2}}}\right]^2 ,$$

where

$$K = 1 - \frac{\varkappa\gamma}{6} \int\limits_0^{\eta_0} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\eta + \lambda\right)^{\frac{3}{2}}} \,. \tag{27}$$

At the middle point $(\eta = 0)$ f_4 is also infinite, with an exception under the condition $K = 0$ where f_4 vanishes at $\eta = 0$. There is no such case where there could be a finite and nonzero value of f_4 at $\eta = 0$. We see from here that the assumption $\lambda \gtrless 0$ does not lead to physically useful solutions. Hence, we should assume $\lambda = 0$.

§6. Now the condition $\lambda = 0$ constitutes all the constants of integration. If we introduce a new variable χ instead η as follows

$$\sin\chi = \sqrt{\frac{\varkappa\rho_0}{3}}\, \eta^{\frac{1}{3}}, \quad \text{where} \quad \sin\chi_a = \sqrt{\frac{\varkappa\rho_0}{3}}\, \eta_a^{\frac{1}{3}}, \tag{28}$$

the equations (13), (26), (10), (24), (25) after elementary algebra take the following form

$$f_2 = \frac{3}{\varkappa\rho_0}\sin^2\chi\,, \quad f_4 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2, \quad f_1(f_2)^2 f_4 = 1\,, \tag{29}$$

$$\rho_0 + p = \rho_0 \, \frac{2\cos\chi_a}{3\cos\chi_a - \cos\chi} \, , \tag{30}$$

$$3x = r^3 = \left(\frac{\varkappa\rho_0}{3}\right)^{-\frac{3}{2}} \left[\frac{9}{4}\cos\chi_a\left(\chi - \frac{1}{2}\sin 2\chi\right) - \frac{1}{2}\sin^3\chi\right]. \tag{31}$$

The constant χ_a is determined, through the density ρ_0 and the radius r_a of the sphere, by the ratio

$$\left(\frac{\varkappa\rho_0}{3}\right)^{\frac{3}{2}} r_a^3 = \frac{9}{4}\cos\chi_a\left(\chi_a - \frac{1}{2}\sin 2\chi_a\right) - \frac{1}{2}\sin^3\chi_a \, . \tag{32}$$

The constants α and ρ, in the case of the solution attributed to the external region, follow from (14) as

$$\rho = \eta_a - 3x_a \, , \qquad \alpha = \eta^{\frac{1}{3}} - \zeta_a$$

and take the form

$$\rho = \left(\frac{\varkappa\rho_0}{3}\right)^{-\frac{3}{2}} \left[\frac{3}{2}\sin^3\chi_a - \frac{9}{4}\cos\chi_a\left(\chi_a - \frac{1}{2}\sin 2\chi_a\right)\right], \tag{33}$$

$$\alpha = \left(\frac{\varkappa\rho_0}{3}\right)^{-\frac{1}{2}} \sin^3\chi_a \, . \tag{34}$$

If using the variables χ, ϑ, φ instead of x^1, x^2, x^3, the line-element in the region inside the sphere takes the simple form

$$ds^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 dt^2 - $$
$$- \frac{3}{\varkappa\rho_0}\left[d\chi^2 + \sin^2\chi \, d\vartheta^2 + \sin^2\chi \, \sin^2\vartheta \, d\varphi^2\right]. \tag{35}$$

Outside the sphere the line-element is still has the same form as that for a point-mass

$$\left. ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2\left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right) \right\}$$

where
$$R^3 = r^3 + \rho \qquad\qquad\qquad\qquad\qquad\quad \tag{36}$$

Here ρ is determined according to (33), while it was $\rho = \alpha^3$ in the case of a point-mass.

§7. A few following notes should be given on the complete solution of our problem, presented in the previous Paragraph.

1. The spatial element $(dt\!=\!0)$ inside the sphere is

$$- ds^2 = \frac{3}{\varkappa \rho_0} \left[d\chi^2 + \sin^2\!\chi \, d\vartheta^2 + \sin^2\!\chi \, \sin^2\!\vartheta \, d\varphi^2 \right].$$

This is the line-element of the so-called non-Euclidean geometry of a spherical space. The spherical space geometry holds also in the internal region of our sphere. The curvature radius of such a spherical space is $\sqrt{\frac{3}{\varkappa \rho_0}}$. Our sphere has formed not all of the spherical space, but only a region in it; this is because χ cannot grow up to $\frac{\pi}{2}$, but grows up only to the boundary limit χ_a. Concerning the Sun the curvature radius of the spherical space, which determine the geometry of the interior of the Sun, would be equal to about 500 radii of the Sun (see equations 39 and 42).

This is a very interesting sequel to Einstein's theory, which manifests the fact that this theory is demanded for the geometry of a spherical space as the reality inside a gravitating sphere (this geometry had the power of a purely theoretical consideration before that).

Inside the sphere the "naturally measurable" quantities of length are

$$\sqrt{\frac{3}{\varkappa \rho_0}} \, d\chi \,, \qquad \sqrt{\frac{3}{\varkappa \rho_0}} \, \sin\chi \, d\vartheta \,, \qquad \sqrt{\frac{3}{\varkappa \rho_0}} \, \sin\chi \, \sin\vartheta \, d\varphi \,. \qquad (37)$$

The radius of the sphere, "measured from within" to the surface, is

$$P_i = \sqrt{\frac{3}{\varkappa \rho_0}} \, \chi_a \,. \qquad (38)$$

The circumference of the sphere, measured along the meridian (or any other great circle) then divided by 2π, should be referred as the "measured-from-outside" radius P_a. It is[*]

$$P_a = \sqrt{\frac{3}{\varkappa \rho_0}} \, \sin\chi_a \,. \qquad (39)$$

According to the formula (36) describing the line-element outside the sphere, this formula for P_a is obviously identical to $R_a = \left(r_a^3 + \rho\right)^{\frac{1}{3}}$ the variable R takes in the surface of the sphere.

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[*]Schwarzschild denoted by i ("innen gemessene") the radius "measured from within", while a (that means "außen gemessene") was used for the radius "measured from outside" due to the original pronunciation of these terms in German. — Editor's comment. D.R.

The following simple relations were obtained for α from (34) through the radius P_a

$$\frac{\alpha}{P_\alpha} = \sin^2\chi_a\,, \qquad \alpha = \frac{\varkappa\rho_0}{3}\,P_a^3\,. \tag{40}$$

Then the volume of our sphere is

$$V = \left(\sqrt{\frac{3}{\varkappa\rho_0}}\,\right)^3 \int\limits_0^{\chi_a} d\chi\,\sin^2\chi \int\limits_0^{\pi} d\vartheta\,\sin\vartheta \int\limits_0^{2\pi} d\varphi =$$

$$= 2\pi\left(\sqrt{\frac{3}{\varkappa\rho_0}}\,\right)^3 \left(\chi_a - \frac{1}{2}\,\sin 2\chi_a\right).$$

Proceeding from here, the mass M of our sphere is

$$M = \rho_0 V = \frac{3}{4\,k^2}\,\sqrt{\frac{3}{\varkappa\rho_0}}\,\left(\chi_a - \frac{1}{2}\,\sin 2\chi_a\right), \tag{41}$$

where $\varkappa = 8\pi k^2$.

2. The following notes are related to the equations of motion of a point of infinitely small mass, located outside our sphere. These equations have the same form as those for a point-mass (see equations 15–17 for that[*]).

At large distances the point moves according to Newton's law, where $\frac{\alpha}{2\,k^2}$ plays a rôle of the attracting mass. Therefore we can refer to $\frac{\alpha}{2\,k^2}$ as the "gravitational mass" of our sphere.

If such a point moves from the rest state at infinity up to the surface of the sphere, the "naturally measurable" velocity of fall of this point we obtain is

$$v_a = \frac{1}{\sqrt{1 - \frac{\alpha}{R}}}\,\frac{dR}{ds} = \sqrt{\frac{\alpha}{R_a}}\,.$$

Then, according to (40),

$$v_a = \sin\chi_a\,. \tag{42}$$

Concerning the Sun, the velocity of the fall is about $\frac{1}{500}$ of the velocity of light. As easy to see in the case of the small numerical values of χ_a and χ (which is $\chi < \chi_a$) following from this velocity, all our equations

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[*]Here Schwarzschild refers to the equations obtained by him in his first paper: Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

(up to the Einsteinian effects of the second order) transform into to the effects of Newton's theory.

3. For the ratio of the gravitational mass $\frac{\alpha}{2\,k^2}$ to the mass of matter M we obtain

$$\frac{\alpha}{2k^2M} = \frac{2}{3}\,\frac{\sin^3\chi_a}{\chi_a - \frac{1}{2}\sin 2\chi_a}\,.\tag{43}$$

With the growing velocity of the fall $v_a = (\sin\chi_a)$ the growing concentration of the mass lowers the ratio of the gravitational mass to the mass of matter. This fact explains that, for instance, at a constant mass and growing density the body approaches the lesser radius than earlier due to the drainage of energy (the lowering of temperature due to radiation).

4. The velocity of light inside our sphere becomes

$$v = \frac{2}{3\cos\chi_a - \cos\chi}\,,\tag{44}$$

and it grows up from the value $\frac{1}{\cos\chi_a}$ in the surface to the value $\frac{2}{3\cos\chi_a - 1}$ at the central point. The value of the density $\rho_0 + p$ grows, according to (10) and (30), proportional to the velocity of light.

At the centre of the sphere ($\chi = 0$) the velocity of light and the density become infinity. Once $\cos\chi_a = \frac{1}{3}$ the velocity of fall reaches $\sqrt{\frac{8}{9}}$ of the (naturally measurable) velocity of light. This value sets the upper limit of the concentration; a sphere of incompressible liquid cannot be denser than this. If we like to apply our equations to the values $\cos\chi < \frac{1}{3}$, we obtain the break just out of the centre of the sphere. At the same time it is possible to find solutions of this problem on the greater values of χ_a continuous at least out of the centre of the sphere, if we move to the case where $\lambda \gtrless 0$ and the condition $K = 0$ (see equation 27) is true. On the path to these solutions, which are however nonsense in physics due to that fact that they give infinite density at the centre of the sphere, we can move to the boundary case where a mass is concentrated in a point, then find, again, the relation $\rho = \alpha^3$ which, according to the earlier study*, is true for a point-mass. We also note that it is possible to talk about only one point-mass in so far as we use the variable r, which in the opposite case (amazingly) does not play a rôle for the geometry and motion in the gravitational field. For

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*Schwarzschild K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

an external observer, as follows from (40), a sphere of the gravitational mass $\frac{\alpha}{2\,k^2}$ cannot have a radius measured from outside whose numerical value is less than

$$P_a = \alpha\,.$$

Concerning a sphere of incompressible liquid such a border should be $\frac{9}{8}\,\alpha$. (In the case of the Sun it should be $3\,\mathrm{km}$, while for a mass of 1 gramme it should be $1.5\times10^{-28}\,\mathrm{cm}$.)

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