The Velocity of Light in Uniformly Moving Frame

A Dissertation. Stanford University, 1958

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Abstract: This is a dissertation submitted to the Department of Physics and the Committee on Graduate Study of Stanford University in partial fulfillment of the requirements for the degree of Doctor of Philosophy, September 1958. Approved by Sidney D. Drell, thesis advisor, Leonard I. Schiff, Department Chair, and Albert H. Bowker, Dean of the Graduate Division. Note: an abstract was prepared separately and published by University Microfilms, Ann Arbor, Michigan 1959-01456, USA. Briefly, the thesis shows classically and quantum mechanically that a linear transformation can be used in special relativity that keeps simultaneity and the out-and-back speed of light invariant, while the one-way velocity of light varies, depending on the frame's velocity relative to an inertial rest frame.

I certify that I have read this thesis and that in iny opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Sidney D. Drell

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Approved for the University Committee on Graduate Study.

Albert H. Bowker, Dean of the Graduate Division

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Introduction

According to the ideas of special relativity the transformation connecting two uniformly moving frames must be such as to leave the metric tensor an invariant, and given by the diagonal tensor $\eta_{\mu\nu} = (1, -1, -1, -1)$, thus preserving the same value for the speed of light in all uniformly moving frames. On the other hand, from an intuitive standpoint, such a result is quite paradoxical, since one would expect that for two frames in relative motion with speed v, the velocity of light ought to differ in the two frames by quantities of the first order in $\frac{v}{c}$. The absence of such an effect cannot be explained by Lorentz contraction of rods and the slowing down of clocks, since these are second order effects. The answer to the paradox, of course, lies in the fact that in special relativity, one deals with clocks which have been synchronized in a certain manner involving quantities of the first order in $\frac{v}{c}$ so that a cancellation with the above expected effect can occur. Thus the original objection is removed — provided one agrees that this method of synchronization does not contain any assumptions about the propagation of light which involve a *petitio principii*.

It is the purpose of this paper to examine this method of synchronization from the broader mathematical viewpoint of general relativity which, based as it is on general covariance, enables one to envisage more general transformations connecting uniformly moving frames. Indeed we shall consider transformations in which clocks are synchronized with "absolute signals", that is, signals travelling with infinite or arbitrarily large velocity. In our discussion we have not enquired into the dynamics of such signals. For the purpose here, such signals serve merely as a kinematic method for formulating in the framework of general covariance certain types of experiments which are unthinkable in the more restricted framework of special relativity. In the concluding chapter some of the possibilities and difficulties associated with such signals are briefly examined.

Using these signals, one arrives at the view of an absolute rest frame (or ether frame) in which the velocity of light is the same in all directions; but for observers in motion relative to this frame with speed v, the velocity of light is not the same in all directions and differs in different directions to first order by amounts of $\frac{v}{c}$, in agreement with one's intuitive ideas. With absolute signals, it is possible to measure this speed v, and hence to linearly order all frames according to the magnitude of this quantity. On the other hand, measurements made with light signals do not make it possible to measure v. In the Ap-

pendix the present status of the absolute frame and Mach's principle in general relativity is reviewed in connection with effects observed in rotating frames.

Some of the basic physical ideas underlying the discussion here are contained in the work of H. E. Ives [1], wherein the view is expressed that the "out" and "back" velocities of light are in general different in uniformly moving frames, and the Lorentz transformation is recast to take this difference into account. The approach in this paper makes it possible to circumvent the unnecessarily cumbersome algebra of his formulation. Recently, in a comprehensive review of the foundations of special relativity Grünbaum [2] has criticized the viewpoint of Ives as being logically inadequate. However, since Grünbaum also observes that other synchronization procedures than the usual one are logically possible, and since an alternative synchronization procedure in general leads to an asymmetry between out and back velocities, the approach given here is mathematically equally valid from either the standpoint of Ives or Grünbaum. Some valuable general remarks on the problems of the one-way velocity of light are to be found in Bridgman [3].

As was already remarked, the mathematical technique that is employed is based on general covariance which permits one to write equations independently of the coordinate system, in contrast with special relativity, where one is restricted to coordinate systems connected by Lorentz transformations. However, while covariance makes it possible to formulate equations independently of the coordinate system, the results obtained by measurement would of course depend on these coordinates if they had direct physical significance in terms of measuring rods and clocks. For example, if one could construct rods and clocks that did not exhibit the Lorentz contraction and time dilatation, one could use these (non-physical) rods and clocks to define a Galilean coordinate system, or set of coordinate systems, in which the velocity of light would not be independent of the motion of the frame. The mathematical framework of general relativity is broad enough to handle measurements made in these arbitrary coordinate systems.

As was pointed out originally by Kretschmann [4], (see also Bridgman [5], Fock [6]) there is therefore a difference between the notion of "relativity" as it is employed in general relativity where it means, from the standpoint of general covariance, a removal of restriction on coordinate systems, and the notion as it is employed in special relativity where it entails a restriction on coordinate systems. As a consequence, from the standpoint of general covariance alone, there is no necessity for two uniformly moving frames to be connected by a Lorentz transformation.

However, the "relativity" of general relativity is to be found really in another equally important assumption [7], namely: in a sufficiently small region of a frame, the propagation of light as measured by (rigid) rods and clocks is such that it is locally describable by the line element of special relativity and more generally, the laws of special relativity hold locally — to a first approximation when such a line element cannot be introduced in the large. In the case of uniformly moving frames, where it is possible to introduce the special relativity line element in the large, it is this assumption which then leads to the Lorentz transformation connecting two such frames. It will be shown in what follows that this latter assumption of general relativity is unnecessarily restrictive on the basis of what is experimentally measured, and can be broadened to permit the use of a line element in which there is an asymmetry in the velocity of propagation of light.

Chapter 1. The Absolute Lorentz Transformation

In the absence of gravitational sources, the field equations of general relativity reduce simply to

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0\,, \qquad \mu,\nu = 0,1,2,3. \tag{1.1}$$

The solutions to (1.1) with $g_{\mu\nu}$ constant are called "Cartesian frames". It is in such frames that we shall work. Since $g_{\mu\nu}$ is by definition a symmetric tensor, the coefficients of the quadratic form $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$, there are ten constants at our disposal, and with no other assumptions, a tenfold infinity of such Cartesian frames. However, because of the symmetry of the $g_{\mu\nu}$ it can always be reduced by real linear transformations to a diagonal matrix with diagonal values given by ± 1, or 0. The case with zero we exclude, since we are interested in working with the full 1+3 dimensionality of the time and space coordinates. By further demanding that the spatial coordinates of the reduced form satisfy the Pythagorean law, the signature of the quadratic form becomes $\pm 1, \pm(1,1,1)$. In order that $ds^2 = 0$, have real solutions corresponding to displacements along the light cone, we finally arrive at the two signatures, $\pm 1, \mp(1,1,1)$, one "time-like", the other, "space-like". For such frames, the determinant of the metric tensor g satisfies the relation $g < 0$. If we adopt the convention that ds^2 should in the limit of small velocities $\frac{dx^i}{dx^0} \approx 0$ (where $i = 1,2,3$) reduce to $(dx^0)^2$, we finally arrive at the canonical time-like metric tensor $\eta_{\mu\nu}$ of special relativity.

However such frames are still too general, for consider a frame which

originally had the line element

$$ds^2 = -(d\bar{x}^0)^2 + (d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2 \tag{1.2}$$

by making the transformation, $\bar{x}^0 \to x^1$, $\bar{x}^1 \to x^0$, $\bar{x}^2 \to x^2$, $\bar{x}^3 \to x^3$ it can be brought into the canonical form. Nevertheless the spatial part of the line element originally does not satisfy the Pythagorean law. Frames for which this requirement is not satisfied (to within a spatial coordinate transformation) can be shown in some cases to be moving with velocities greater than that of light. For example, the transformation of [8],

$$\left.\begin{aligned} \bar{x}^1 &= \frac{x^1 - vx^0}{\sqrt{v^2-1}}, \qquad \bar{x}^2 = x^2 \\ \bar{x}^0 &= \frac{vx^1 - x^0}{\sqrt{v^2-1}}, \qquad \bar{x}^3 = x^3 \end{aligned}\right\} \tag{1.3}$$

also transforms (1.2) into the canonical form. Since it is not our purpose to consider phenomena in such frames here, it is necessary to restrict the metric tensor $g_{\mu\nu}$ in the following way. Solving for the time Δx^0 for a light signal to propagate through a distance Δx^i one has, setting $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = 0$,

$$\Delta x^0 = -\frac{g_{0i}}{g_{00}}\,\Delta x^i \pm \frac{1}{|g_{00}|}\sqrt{(g_{0i}g_{0j} - g_{ij}g_{00})\,\Delta x^i \Delta x^j}\ . \tag{1.4}$$

The average out-and-back time for a light signal to propagate is therefore, choosing the positive root in order to make the time delay positive,

$$\frac{1}{2}\left(\Delta x^0_{out} + \Delta x^0_{back}\right) = \frac{1}{|g_{00}|}\sqrt{\gamma_{ij}\Delta x^i \Delta x^j}\ , \tag{1.5}$$

with $\gamma_{ij} \equiv g_{0i}g_{0j} - g_{ij}g_{00}$. Now unless γ_{ij} is positive definite, there will be directions corresponding to the choice of the Δx^i for which the delay either vanishes or becomes imaginary. Such a situation occurs, for example, in a frame moving faster than light. Moreover, in such a frame, a light signal emitted say from the origin cannot be reflected back to the origin since it cannot overtake the frame. Or again, consider the line element,

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 + (dx^3)^2, \tag{1.6}$$

for which γ_{ij} is not positive definite: light cannot propagate in the cones opening above and below the (x^1, x^2) plane along the x^3 axis. Since, as remarked previously we wish to remain in frames in which light

propagates in the customary manner, freely in all directions and with non-zero average delay, and for such frames to provide an alternative description to that given by special relativity, we therefore impose the requirement,

$$\gamma_{ij}: \text{ positive definite}$$

$$\gamma_{11} > 0\,, \qquad \begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} > 0\,, \qquad \begin{vmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{vmatrix} > 0\,. \tag{1.7}$$

This requirement, needless to say, imposes a restriction on allowed coordinate transformations, for example, the transformation, $x^0 \to x^1$, $x^1 \to x^0$, $x^2 \to x^2$, $x^3 \to x^3$, is excluded. In order to arrive at the time-like definition of ds^2, it is necessary to further impose the restriction $g_{00} > 0$; a relation which will then be preserved under all real transformations which leave γ_{ij} positive definite.

By the above assumptions and restrictions we therefore arrive at a multiplicity of frames in which light propagates in the usual manner and such that by a real linear transformation the metric tensor may be brought into the canonical form. Such frames we shall call "Lorentz-reducible" frames. Because these frames are all related to one another by linear transformations, they are easily seen to be in uniform translation (or at rest) with respect to one another. Thus for two such frames

$$dx^\mu = b^\mu_\nu\, dx'^\nu, \tag{1.8}$$

where the b^μ_ν do not depend on the coordinates, then

$$\frac{dx^i}{dx^0} = \frac{b^i_0 + \dfrac{b^i_j\, dx'^j}{dx'^0}}{b^0_0 + \dfrac{b^0_j\, dx'^j}{dx'^0}}\,, \qquad i,j = 1,2,3. \tag{1.9}$$

Hence, if a point in the primed frame is at rest $\frac{dx'^i}{dx'^0} = 0$, its velocity in the unprimed frame is constant and given by

$$\frac{dx^i}{dx^0} = \frac{b^i_0}{b^0_0}\,. \tag{1.10}$$

The above result holds for more general frames than Lorentz-reducible ones (since all that is required is a linear transformation connecting the two frames), so that actually we are dealing with a subset of uniformly translating frames, namely, ones travelling less than the speed of light.

Now it is customary to impose a further restriction on Lorentz-reducible frames, that of special relativity, so that we exclude frames with $g_{\mu\nu}$ not given by $\eta_{\mu\nu}$. For example, we exclude Galilean frames. This exclusion is not demanded by anything in the structure of general covariance, or anything we have done in the above derivation. It is imposed by the hypothesis that all uniformly translating frames are in every sense equivalent, and consequently there should be nothing in the metric tensor which would imply a difference in the propagation of light signals in one frame as distinguished from another.

But if we do not impose this relativity requirement, a variety of other expressions are obtained for the line element, depending upon one's choice of coordinate system. It is our purpose to investigate to what extent some of these alternative line elements are physically permissible, in the sense that they do not violate experimental evidence, taking into account the manner in which the experiments are performed.

Consider a frame with the following expression for the line element,

$$ds^2 = g'_{\mu\nu}\, dx'^{\mu} dx'^{\nu} = dt'^2 - 2v dx' dt' - \left(1 - v^2\right) dx'^2 - dy'^2 - dz'^2, \quad (1.11)$$

where we introduce units such that $c = 1$, also $x'^0 = t'$, $x'^1 = x'$, $x'^2 = y'$, $x'^3 = z'$, and v is a parameter. From the customary standpoint, one would say that this line element represents an improper choice of coordinate system and that one should perform a further coordinate transformation to put the metric tensor in canonical, diagonal form and the observer in a special relativistically admissible coordinate system. But there is more than one way to diagonalize (1.11), each with a different physical significance.

Thus one method to diagonalize (1.11) is to make the coordinate transformation (provided $v < 1$),

$$\left.\begin{aligned} x' &= \gamma(x - vt)\,, & y' &= y \\ t' &= \frac{1}{\gamma}\, t\,, & z' &= z \end{aligned}\right\} \quad (1.12)$$

with $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$, (1.12) has the inverse

$$\left.\begin{aligned} x &= \frac{1}{\gamma}\, x' + \gamma v t', & y &= y' \\ t &= \gamma t', & z &= z' \end{aligned}\right\}. \quad (1.13)$$

What meaning are we to assign to the transformation (1.12)? We interpret the meaning as follows: 1) the frame with coordinates (t', x',

y', z'), which we shall call S', is in uniform translation with speed v in the x-direction with respect to the frame with coordinates (t, x, y, z), which we shall call S; 2) rods in S' are contracted with respect to those in S by the factor $\frac{1}{\gamma}$, and clocks in S' indicate less elapsed time than those in S, again by the factor $\frac{1}{\gamma}$.

Thus the transformation is in some respects similar to the Lorentz transformation, but clearly the clocks have been synchronized in a different manner, since the time t' in S' only depends on the time t in S, and not on the spatial coordinates.

Moreover, we note that unlike the case with the Lorentz transformation, a measurement of a rod at rest in S, by an observer in S', leads to the conclusion that the rod in S has *expanded* relative to a rod at rest in S', similarly such an observer would say that a clock in S is going *faster* than a clock in S'. One does not have the paradoxical situation of special relativity that both observers say each other's rods have shrunk, or each other's clocks are moving more slowly, rather, one has an absolute relationship. If we regard S as the fundamental frame, then it is the rods in S' which have contracted, so that conversely the rods in S appear expanded with respect to the contracted rods in S', and similarly for clocks. Consider a third frame, S'' in motion with respect to S, and with speed w; clearly, we can state whether S'' is moving faster or slower than S' with respect to S simply by comparing the rates of clocks in S'' and S', since

$$t'' = t' \sqrt{\frac{1 - w^2}{1 - v^2}} \ . \tag{1.14}$$

In other words, all uniformly moving frames S', S'', etc., may be linearly ordered with respect to S in terms of a parameter v, the speed of the moving frame relative to S, and this ordering is absolute in the sense that observers in the two frames S', S'' by comparing the relative rates of their clocks can assert which is moving faster than the other relative to the frame S — without referring to the frame S — a situation which is not possible in special relativity. Because of this absolute property, we shall refer to (1.12) as the *Absolute Lorentz Transformation* (A.L.T.), and S as the absolute frame.

So far we have not shown that the A.L.T. is actually physically allowable, in the sense that it doesn't violate experimental evidence. In the following chapters we shall show that when measurements are made in the customary manner this is indeed the case.

Let us now observe that instead of diagonalizing the quadratic form by the A.L.T., one might also have chosen to diagonalize it by the

transformation,

$$\left.\begin{aligned} t_L &= t' - vx', \qquad y_L = y' \\ x_L &= x', \qquad\quad\ z_L = z' \end{aligned}\right\}. \tag{1.15}$$

A point at rest in the primed frame is at rest in the frame S_L and conversely. Thus there is a different physical significance to the two transformations: In the one case (A.L.T), the diagonalized frame is in motion relative to the undiagonalized frame and in the second case, the two frames are at rest relative to one another but there has been a resynchronization of clocks.

Now we note that the transformation above connecting S_L with S', when multiplied by the A.L.T., connecting S' with S, is a Lorentz transformation. Thus since the frame S' may all be ordered with respect to S according to the parameter v, and since to each of these frames S' there is a corresponding Lorentz frame S_L at rest relative to S', it follows the Lorentz frames themselves may be ordered with respect to S. On the other hand, it is clear that unless the observer in S_L has some way of factoring out of the Lorentz transformation the above synchronization of clocks so as to make measurements in S', the ordering with respect to v is lost and one is back to the situation of special relativity.

Chapter 2. Factorization of the Lorentz Transformation

The results of diagonalization obtained above may be stated more elegantly in the following way. Define the three unimodular transformations (we are using "unimodular" in the sense that the determinant is unity),

$$\left.\begin{aligned} O_1 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad O_2 \equiv \begin{pmatrix} \gamma^{-1} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ O_3 \equiv \begin{pmatrix} 1 & -v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}\right\} \tag{2.1}$$

and the column vectors $X = (t, x, y, z)$, $X' = (t', x', y', z')$, $X_L = (t_L, x_L, y_L, z_L)$, then the A.L.T. may be written

$$X' = O_2 O_1 X \tag{2.2}$$

and the Lorentz transformation

$$X_{L} = O_3 O_2 O_1 X. \tag{2.3}$$

Thus we have a factorization of the Lorentz transformation into three sub-transformations O_1, O_2, O_3. Such a factorization is meaningless in special relativity (since only transformations which leave the diagonal form invariant are permitted), whereas it is not in general relativity because there is the freedom of considering arbitrary coordinate transformations. Thus, reading from right to left, the transformations say,

$O_1 X$: Make a Galilean transformation from the frame S to a frame moving with velocity v in the x-direction with respect to S;

$O_2 O_1 X$: In the new frame, shrink the rods (that are oriented along the x-axis) and slow down the clocks — renormalization of length and time;

$O_3 O_2 O_1 X$: Without changing the state of motion of the frame, resynchronize the clocks.

In addition, because the determinant of each of the transformations is unity, they preserve the four dimensional volume element $dx\,dy\,dz\,dt$ for each of the intermediate steps. Further, since O_1, O_2, O_3 do not commute among one another, the order in which they are performed is significant. For example, if O_2 is performed before O_1, one will pick a frame which does not have velocity v with respect to S, but a velocity $v(1-v^2)$. It is interesting to note that O_1 and O_3 generate subgroups in themselves, since $O_1^2(v) = O_1(2v)$, $O_1^{-1}(v) = O_1(-v)$, $O_1^0(v) = O_1(0) = 1$, the identity, and similarly for O_3, but O_2 does not have this property.

Let us now observe that in the original diagonalization of the line element in S', we might have proceeded by first performing the operation O_2^{-1} which would have brought us without changing the state of motion into the Galilean frame with line element

$$ds^2 = \left(1 - v^2\right) dt_g^2 - 2v dt_g dx_g - dx_g^2 - dy_g^2 - dz_g^2\,, \tag{2.4}$$

with

$$\left.\begin{aligned} t_g &= \gamma t', \qquad & y_g &= y' \\ x_g &= \frac{1}{\gamma} x', \qquad & z_g &= z' \end{aligned}\right\} \tag{2.5}$$

and then proceeded from the Galilean frame S_g to the rest frame S. Note that in the Galilean frame, the velocity of light in the principal

directions has the "classical" values,

$$\frac{dx_g}{dt_g} = \pm\,(1 \mp v)\,, \quad \frac{dy_g}{dt_g} = \pm\sqrt{1-v^2}\,, \quad \frac{dz_g}{dt_g} = \pm\sqrt{1-v^2}\,, \qquad (2.6)$$

while in the Lorentz frame S_L *at rest* with respect to S_g, the velocity has the values, ± 1. The values in S' we shall discuss in detail in the subsequent chapters.

Chapter 3. Velocity of Light and Synchronization of Clocks under the Absolute Lorentz Transformation

The physical picture presented by A.L.T., then, is that of clocks and rods which have experienced a change in rate and length due to their motion relative to the absolute frame (or ether). (This was the picture used in the era preceding special relativity.) But unlike the situation with the usual Lorentz transformation, we have not further synchronized the clocks in the moving frame by demanding that the velocity of light be the same in all direction as in the absolute frame. Rather, the clocks have been synchronized in the following way: all clocks in both the frame S' and S have been initially synchronized from one clock by a signal travelling with infinite velocity in all directions; upon being synchronized, the clocks keep time at their "natural" rate, the natural rate in the moving frame S' being slower than the natural rate in the rest frame S. This is the physical meaning of the transformation, $t' = \sqrt{1-v^2}\,t$. It is not our purpose here to enquire as to how one might generate such signals, for example, by using the frames previously mentioned which were travelling with $v > 1$. In a later chapter we shall examine the question as to whether such signals violate any fundamental ideas of causality. Consider now, measurements of the velocity of light made by observers in S'. The relative velocity in S' of a point travelling with constant velocity in the x'-direction is given by, upon using the A.L.T.,

$$\frac{dx'}{dt'} = \frac{\dfrac{dx}{dt} - v}{1 - v^2}\,. \qquad (3.1)$$

So that since the velocity of light in S is 1, one obtains in the positive and negative directions,

$$\frac{dx'}{dt'} = \frac{1}{1+v}\,, \; -\frac{1}{1-v}\,. \qquad (3.2)$$

A result which one obtains directly in the primed frame by setting $ds^2 = g'_{\mu\nu}\,dx'^{\mu}dx'^{\nu} = 0$, and solving for the roots. For the transverse

direction one finds

$$\frac{dy'}{dt'} = \pm 1\,, \qquad \frac{dz'}{dt'} = \pm 1\,. \tag{3.3}$$

One might feel that such results violate experience; we shall see that this is not the case because of the way in which measurements are made. Thus consider a measurement of the velocity of light along the x'-axis. One sends a light signal from the origin in S', to a point located at positive distance $\Delta x'$ from the origin and back again. The time required is

$$\left.\begin{aligned} \Delta t'_{out} &= (1+v)\,\Delta x' \\ \Delta t'_{back} &= (1-v)\,\Delta x' \end{aligned}\right\}. \tag{3.4}$$

And hence the average time, which is used in obtaining the velocity of light is

$$\frac{1}{2}\,(\Delta t'_{out} + \Delta t'_{back}) = 1 \cdot \Delta x', \tag{3.5}$$

so that one obtains the same value as in the unprimed frame, or the Lorentz frame. We see that there is an exact cancellation that comes about due to the fact that the reciprocal of the velocity or "slowness" of the light signal is a linear function of the velocity of the primed frame. For an arbitrary direction, corresponding to displacements, $\Delta x'$, $\Delta y'$, $\Delta z'$, we have, setting $ds^2 = 0$,

$$\Delta t' = v\,\Delta x' + \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}\,. \tag{3.6}$$

On the outward and return paths, $\Delta x'$ changes sign, hence

$$\frac{1}{2}\,(\Delta t'_{out} + \Delta t'_{back}) = \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}\,. \tag{3.7}$$

Thus the same average value of the out and back times is obtained in S' as would be obtained by Lorentz observers. We note further that it is the reciprocal of the average slowness which is obtained in a typical out-and-back determination of the "velocity of light".

In sending a light signal to a point $\Delta x'$ from the origin, we see that the delay consists of two parts: $\Delta x'$ and $v\Delta x'$, that is, the delay one uses in special relativity and an additional part associated with the fact that one has synchronized the clocks with absolute signals. Since the extra delay is constant for a given frame, depending only on the location of the clock and the speed v of the frame, one can introduce a new time $t_{\scriptscriptstyle L}$, given by

$$t_{\scriptscriptstyle L} = t' - vx'. \tag{3.8}$$

So that the delay in sending a light signal becomes

$$\Delta t_{_L} = \Delta t' - v\Delta x' = (1+v)\,\Delta x' - v\Delta x' = 1\cdot\Delta x', \qquad (3.9)$$

the delay assigned in special relativity for the process. Indeed, on introducing the expressions for the A.L.T., (3.8) reduces to

$$t_{_L} = \gamma\,(t - vx)\,, \qquad (3.10)$$

or the well known relativistic transformation for time. The transformation (3.8) is our previously described transformation O_3. It is interesting to note historically that the above transformation was first given by Lorentz [9] using Galilean coordinates, under the title, "local time", in order to eliminate first order effects, so that his original transformation was O_3O_1. After he discovered O_2, he still gave the transformation in the form (3.8), instead of the relativistic form. We see therefore that the possibility of introducing the "local time" arises as a consequence of the arbitrariness of the synchronization of separated clocks when there are no absolute signals present. However, one might wonder whether by considering two similar clocks, synchronized at the origin A, and then slowly moving one of the clocks to B, $\Delta x'$ from the origin, and then measuring the velocity of light, one could not perhaps determine $v\Delta x'$, and hence v. This is not possible for the following reason: In terms of a clock located at the origin in the unprimed frame, initially coincident with that of the primed frame, the time of the two clocks at the origin in the primed frame is given by

$$t' = \sqrt{1-v^2}\;t\,. \qquad (3.11)$$

Then, on slowly moving one of the clocks in the primed frame to the point B, one has a change in rate of the clock given by

$$\delta t' = -\frac{v\,\delta v}{\sqrt{1-v^2}}\,t\;. \qquad (3.12)$$

On the other hand, the time t required to move the clock through a distance Δx in the absolute frame is

$$(v+\delta v)\,t = \Delta x = \sqrt{1-v^2}\;\Delta x' + vt\,, \qquad (3.13)$$

or

$$t = \frac{\sqrt{1-v^2}}{\delta v}\,\Delta x', \qquad (3.14)$$

and hence,

$$\delta t' = -\, v \Delta x', \tag{3.15}$$

so that there is an exact cancellation in the limit $\delta v = 0$. We shall use this result later in "deriving" the absolute Lorentz transformation.

In the above we assumed $\delta v \to 0$; for clocks moving with finite velocities, we encounter the following: Since the rate of the clock varies with the velocity with which we move the clock, it is also necessary to know this velocity in order to correct for this change in rate. But how can we measure the velocity of the clock? In order to measure the velocity we need to know the time it left the origin t'_A which we can measure and the time t'_B which it arrived at the point B, which we cannot measure. We can of course send a light signal back to the origin when the clock arrives. But how long did it take the light signal to go from B back to the origin? This is precisely what we were looking for originally! Thus we arrive at the following remarkable and somewhat astonishing result:

> Unless one can synchronize separated clocks absolutely, it is impossible to determine the one-way velocity of an object, since velocity is defined non-locally and one has no way of determining the time of arrival in terms of the time of departure.

Einstein [10], in formulating special relativity attempted to circumvent this difficulty in the following way:

> "We have not defined a common "time" for A and B, for the latter cannot be defined at all unless we establish *by definition* that the "time" required "by light to travel from A to B equals the "time" it requires to travel from B to A".

Such a definition assumes more than is warranted by experiment, since only the out-and-back propagation time of light is measured, or, if measured one-way, the motion of clocks is involved. Eddington [11] considered in detail this problem of the one-way velocity of light and attempted to actually give a "formal proof" that the out and back velocities must be the same. Thus he says:

> "If $v(\theta)$ is the velocity of light in the direction θ, the experimental result is
>
> $$\frac{1}{v(\theta)} + \frac{1}{v(\theta+\pi)} = const = C$$
>
> $$\frac{1}{v'(\theta)} + \frac{1}{v'(\theta+\pi)} = const = C'$$
>
> (v, v' refer to S and S' respectively — our note) for all values of θ. The constancy has been established to about 1 part in 10^{10}.

It is exceedingly unlikely that the first equation would hold unless

$$v(\theta) = v(\theta + \pi) = const$$

and it is fairly obvious that the existence of the second equation excludes the possibility altogether".

We shall not attempt to discuss his "proof", but merely point out that these "unlikely" results are precisely what the line element associated with the A.L.T. yields. Thus as we have shown, $\Delta t' = v\,\Delta x' + + \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}$, hence introducing $d\sigma' = dx'^2 + dy'^2 + dz'^2$, where $d\sigma'$ is the spatial distance; one may write the above as

$$\frac{\Delta t'}{\Delta \sigma'} = v\cos\theta' + 1\,, \tag{3.16}$$

where $\cos\theta' = \frac{\Delta x'}{\Delta \sigma'}$, and we see that the slowness in the direction θ', and $\theta' + \pi$, satisfy

$$\frac{\Delta t'}{\Delta \sigma'}(\theta') + \frac{\Delta t'}{\Delta \sigma'}(\theta' + \pi) = 2\,, \tag{3.17}$$

for all θ', and this result is independent of the velocity v of the frame relative to S. The difference in slowness is given by

$$\frac{\Delta t'}{\Delta \sigma'}(\theta') - \frac{\Delta t'}{\Delta \sigma'}(\theta' + \pi) = 2v\cos\theta', \tag{3.18}$$

which, together with (3.17) summarizes our previous results expressed in terms of the principal directions.

Although for convenience in the above discussion we have chosen v to lie along x, this is clearly not necessary. Thus if the velocity of S' with respect to absolute frame S has components v_i, the line element in S' becomes

$$ds^2 = g'_{\mu\nu}\,dx'^{\mu}dx'^{\nu} = dt'^2 - 2v_i\,dx'^i dt' - dx'^i dx'^j + v_i v_j\,dx'^i dx'^j. \tag{3.19}$$

upon replacing the local time $dt_{_L}$ by $dt' - v_i dx'^i$ in the Lorentz line element for the corresponding Lorentz frame. Setting $ds^2 = 0$, the time $\Delta t'$, for light to traverse $\Delta x'^i$ is

$$\Delta t' = v_i\,\Delta x'^i + \Delta\sigma', \tag{3.20}$$

which may be written in the form (3.16) and the above results hold *pari passu*. To avoid confusion it should be noted that the notation "v_i" is not meant in a covariant sense, but as a simplified way of writing

quantities which are, mathematically, components of the various coordinate transformations relating S_L, S', S. Thus since S is connected with S' via $dx^\mu = \bar{a}^\mu_\nu dx'^\nu$, and since the v_i are defined as the velocity of S', one has $v_i = \frac{dx^i}{dx^0} = \frac{\bar{a}^i_0}{\bar{a}^0_0}$, since $\bar{a}^0_i = 0$. Also writing $dx^\mu_L = l^\mu_\nu dx'^\nu$, one has $v_i = -l^0_i$. Finally, from (3.19) it follows $g'_{0i} = -v_i$ and we shall also see $g'^{0i} = -v_i$.

Chapter 4. Derivation and Generalization of the Absolute Lorentz Transformation

In the preceding, some of the consequences of the A.L.T. have been examined. Let us now reverse the procedure and undertake to see what postulates are necessary in the framework of general covariance to derive the transformation.

We assume that there exists an absolute (or ether) frame S, and in this frame the propagation of light is governed by (assuming that $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$)

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = 0\,, \tag{4.1}$$

so that the time for light to go from A to B is the same as the time from B to A, the "time" being measured by clocks at rest in the absolute frame and synchronized with absolute signals. Such an expression is taken to hold irrespective of the state of motion of the source of the light , an assumption wherein general relativity, special relativity, and the ether theories of light all agree.

We now consider a frame S', with coordinates t', x', y', z' in uniform motion with speed v in the positive x-direction as measured in S, and look for a linear transformation of the form,

$$\left.\begin{aligned} t' &= g_0(v)\,t\,, & y' &= g_2(v)\,y \\ x' &= g_1(v)(x - vt)\,, & z' &= g_3(v)\,z \end{aligned}\right\}. \tag{4.2}$$

The physical interpretation of this transformation is that the rods and clocks in S' have changed their length and rate with respect to those in S, but that the synchronization of clocks in S' with those in S has been without delay. Under the above transformation, the line element becomes

$$ds^2 = \frac{\left(1 - v^2\right)}{g_0^2}\left(dt'\right)^2 - \frac{2v}{g_0\, g_1}\, dt' dx' - \\ - \frac{1}{g_1^2}\left(dx'\right)^2 - \frac{1}{g_2^2}\left(dy'\right)^2 - \frac{1}{g_3^2}\left(dz'\right)^2. \tag{4.3}$$

If we demand that the average slowness (or equivalently, the out-and-back velocity) be the same in all directions — as experiments so far have indicated — we arrive at

$$\frac{g_0}{g_1}\,\gamma^2 = n(v)\,, \qquad \frac{g_0}{g_2}\,\gamma = n(v)\,, \qquad \frac{g_0}{g_3}\,\gamma = n(v)\,, \tag{4.4}$$

where $n(v)$ represents the average slowness in S' and is a positive quantity. It should be remarked that nothing in the above derivation requires that $n(v)$ be unity.

Let us now require that for a slowly moved clock, the shift in setting on being moved from A to B be just such as to compensate the extra delay experienced by light in travelling from A to B, so that a one-way measurement of the velocity of light will give $n(v)$. For a displacement in the positive x' direction by amount $\Delta x'$, this extra delay is given by

$$\left[\frac{g_0}{g_1}\,\frac{1}{1-v} - n(v)\right]\Delta x'. \tag{4.5}$$

Then by the same argument as in the previous Chapter we are led to the differential equation,

$$\frac{g_0}{g_1}\frac{1}{1-v} - n(v) + \frac{1}{g_1}\frac{dg_0}{dv} = 0\,, \tag{4.6}$$

which becomes, upon substitution from (4.4),

$$\frac{1}{g_0}\frac{dg_0}{dv} = -\frac{v}{1-v^2} \tag{4.7}$$

and hence since $g_0(0) = 1$, because the clock rates are the same when S' is at rest relative to S,

$$g_0(v) = \sqrt{1-v^2}\ . \tag{4.8}$$

The same result would have been obtained had the clock been moved in the negative x' direction. For motion in the transverse direction, the change in setting is zero, since $\delta g_0(v) = -\frac{1}{\sqrt{1-v^2}}\,v_y\,\delta v_y = 0$, since $v_y = 0$. It is therefore necessary and sufficient that $g_0(v) = \sqrt{1-v^2}$ in order that a clock slowly moved in any direction, yield $n(v)$ for a one-way determination of the velocity of light. Using this value for $g_0(v)$ one finds,

$$\left.\begin{aligned} g_1(v) &= \gamma\, n(v)^{-1} \\ g_2(v) &= g_3(v) = n(v)^{-1} \end{aligned}\right\}. \tag{4.9}$$

Thus we are led to a transformation of the form

$$\left.\begin{aligned} t' &= \frac{1}{\gamma}\, t\,, & y' &= n(v)^{-1} y \\ x' &= \gamma\, n(v)^{-1}\,(x - vt)\,, & z' &= n(v)^{-1} z \end{aligned}\right\}, \tag{4.10}$$

which we shall refer to as the generalized A.L.T. Under this transformation, the line element becomes,

$$\begin{aligned} ds^2 = dt'^2 - 2v\, dx' dt'\, n(v) - \\ - n(v)^2 \left(1 - v^2\right) dx'^2 - n(v)^2 dy'^2 - n(v)^2 dz'^2. \end{aligned} \tag{4.11}$$

If one now assumes that in S', the average slowness must be the same as in S, one has $n(v) = 1$, and the A.L.T. is derived. More generally, if one obtains the same contraction of rods independently of whether the frame was moved in the positive or negative x direction, corresponding to setting $v \to -v$, $n(v) = n(-v)$. Further, for $v = 0$, since the frames coincide, $n(0) = 1$.

One can proceed to define a local time $t_{_L}$ for the generalized transformation (4.10) in the same way as for the A.L.T. Thus set,

$$t_{_L} = t' - n(v)\, v x' \tag{4.12}$$

so that,

$$\Delta t_{_L} = \Delta t' - n(v)\, v\, \Delta x' \tag{4.13}$$

and since the slowness of light from (4.10) is, for the positive and negative x' directions,

$$\frac{\Delta t'}{\Delta x'} = n(v)\,(1 + v)\,,\ -n(v)\,(1 - v)\,, \tag{4.14}$$

it follows

$$\Delta t_{_L} = n(v)\, \Delta x' \tag{4.15}$$

which is the delay desired. One therefore arrives at the transformation connecting $t_{_L}$ with the absolute frame,

$$t_{_L} = \gamma\,(t - vx) \tag{4.16}$$

so that the local time does not actually depend on $n(v)$. Using $t_{_L}$, the line element (4.11) becomes

$$ds^2 = dt_{_L}^2 - n(v)^2 \left(dx'^2 + dy'^2 + dz'^2\right). \tag{4.17}$$

It follows from the derivation of (4.10) and (4.17) that if $n(v)$ were not strictly independent of v, it still would *not* be possible to determine

v by non-absolute measurements of the velocity of light in S' alone. For example, a Michelson-Morley type of experiment in a uniformly moving frame can only lead one to conclude the relations (4.4) when combined with the assumption that the velocity of light is independent of the source.

On the other hand, since for the Earth, v varies as a function of time, an experiment with an interferometer having unequal arms, such as the one of Kennedy-Thorndike [12], would show a periodic shift in the fringe system from one time of the year to the next. Thus with unequal interferometer arms $\Delta x'$, $\Delta y'$, the difference of the average times in the two directions is

$$\Delta T' = n(v)\,(\Delta x' - \Delta y') \tag{4.18}$$

and hence,

$$\delta\Delta T' = 2\left(\frac{dn}{dv^2}\right) v\,\delta v\,(\Delta x' - \Delta y')\,. \tag{4.19}$$

The rotational and orbital motion of the Earth will give rise to periodicities in the term $v\,\delta v$ causing a displacement in the fringe system proportional to $\frac{dn}{dv^2}$. In the theory of their experiment, Kennedy and Thorndike did not consider the possibility that $n(v)$ was not unity and so regarded their measurements in terms of checking the time dilatation, $\Delta t' = \frac{1}{\gamma}\Delta t$ and interpreted their data correspondingly. Thus assuming only the Lorentz contraction and independence of the velocity of light on the source, they showed one is led to an expression of the form,

$$\Delta T = \gamma\,(\Delta x' - \Delta y') \approx \left(1 + \frac{1}{2}\,v^2\right)(\Delta x' - \Delta y') \tag{4.20}$$

which can be obtained from (4.18) by setting $n(v) = 1$ and $\Delta T' = \frac{1}{\gamma}\Delta T$. Hence as v varies,

$$\delta\Delta T = v\,\delta v\,(\Delta x' - \Delta y')\,. \tag{4.21}$$

However, if in fact $\Delta t' = \frac{1}{\gamma}\Delta t$, as experiments for example with meson lifetimes indicate, there is still an effect to be expected unless $\frac{dn}{dv^2} = 0$, as derived above. Interpreting their data from this standpoint, the values they quote for an absolute velocity are to be regarded as being the quantity $2v\frac{dn}{dv^2}$. They found from an analysis of the diurnal periodicities in the fringe system, $2v\frac{dn}{dv^2} = 24 \pm 19$ km/sec and for the annual periodicities $2v\frac{dn}{dv^2} = 15 \pm 4$ km/sec. They concluded that because these velocities were so small compared to the velocities of thousands of kilometers per second known to exist among the nebulae,

and since moreover the directions of the two velocities differed by 123°, their experiment was to be interpreted as yielding a null result. However the results could mean merely $\frac{dn}{dv^2}$ is small. Since in this paper we are primarily interested in showing that the A.L.T. and the associated line element contain the results of special relativity because of the way in which measurements are made, we shall assume that $\frac{dn}{dv^2}=0$. Under these circumstances, $n(v)=1$ and (4.10) reduces to the A.L.T.

Alternatively, in deriving the A.L.T. we could have proceeded in the following manner: after requiring that the average slowness be independent of direction (which led to equation 4.4), we could have further demanded that it also be independent of the absolute velocity of the frame. Under these circumstances, $n(v)=1$, and (4.4) becomes

$$g_1=\frac{g_0}{1-v^2}\,,\qquad g_2=g_3=\frac{g_0}{\sqrt{1-v^2}}\,. \tag{4.22}$$

The corresponding transformation is

$$\left.\begin{aligned} t'&=g_0\,t\,, & y'&=g_0\,\gamma\,y\\ x'&=g_0\,\gamma^2\,(x-vt)\,, & z'&=g_0\,\gamma\,z \end{aligned}\right\} \tag{4.23}$$

for which the line element takes the form,

$$ds^2=\frac{1}{(\gamma\,g_0)^2}\left(dt'^2-2v\,dt'dx'-\left(1-v^2\right)dx'^2-dy'^2-dz'^2\right). \tag{4.24}$$

For a line element of this form it is clear that no effect is to be expected in either the Michelson-Morley or Kennedy-Thorndike experiment. If one now makes the assumption that the one-way velocity as determined by slowly moved clocks is the same as that yielded by the out-and-back methods, then (4.6) reduces to

$$\frac{1}{g_0}\frac{dg_0}{dv}=-\frac{v}{1-v^2}\,, \tag{4.25}$$

and once again $g_0=\frac{1}{\gamma}$.

Thus we see, in summary, that the A.L.T. follows uniquely as a consequence of the following postulates:

1. There exists a frame S in which light propagates with a constant velocity, the same in all directions, independently of the motion of the source;
2. In a coordinate frame S', in uniform translation with respect to S, the out-and-back travel time for light is independent of direction, and the velocity of S' with respect to S;

3. The one-way velocity of light as measured with clocks that have been synchronized together and then slowly separated is the same as the value yielded by the out-and-back technique.

In addition, we have employed a synchronization procedure based on the hypothetical "absolute" or "instantaneous" signal, in contrast with the usual relativistic approach based on establishing, by definition, the equality of the out and back times for the propagation of a light signal.

A similar approach to the one above for obtaining the line element in the moving frame has been given by Robertson [13]. However, because he employs the relativistic synchronization procedure before using experiment to restrict the coefficients in the metric of the moving frame, his derivation leads to the ordinary Lorentz transformation and the metric $\eta_{\mu\nu}$, rather than the A.L.T.

Let us now observe that in the above derivation of the A.L.T., nowhere was the assumption made that ds is the proper time, but merely that $ds^2 = 0$ represents the propagation of light. But now setting $\frac{dx^i}{dt} = v_i$ in the line element viewed in the absolute frame, $ds^2 = \left(1 - v^2\right) dt^2$, so that for a clock at rest in S', since $dt'^2 = \left(1 - v^2\right) dt^2$, one has $ds^2 = dt'^2$. Thus the assumption about the property of slowly moved clocks which yielded $g_0 = \frac{1}{\gamma}$, is equivalent to demanding $ds^2 = dt'^2$ for a clock at rest in S', as indeed an examination of the line element (4.3) indicates.

Chapter 5. Velocity of Light in a Moving Refractive Medium and Further Applications Involving Relative Velocity

So far our discussions have pertained only to the velocity of light in the vacuum, and we have seen that in S' the relative velocity of light is different in different directions but unobservable with present techniques so that one cannot measure the velocity v of S'. The questions arise as to whether such a velocity v might be detectable by a) causing the light to pass through a refractive medium at rest in S' and determining whether the out-and-back time is a function of v, b) comparing the time it takes light to travel a distance $\Delta x'$ in the refractive medium to the time it takes to travel the same distance in the vacuum and seeing whether this time difference, varies with v. But we know by, experiment (at least to the approximation $n(v)$ is unity mentioned in the preceding chapter) that there are no effects of the kind a) and b). The problem is therefore to write down a line element for the propaga-

tion of light in a refractive medium at rest in S' which exhibits these properties.

Consider the line element,

$$ds^2 = dt'^2 - 2v\,dx'dt' - \left(n^2 - v^2\right) dx'^2 - n^2 dy'^2 - n^2 dz'^2, \qquad (5.1)$$

which reduces to the vacuum A.L.T. line element for $n=1$, where n is the index of refraction when the refractive medium is at rest in S, the absolute frame (the index of refraction n used here should not be confused with $n(v)$ used in the preceding chapter, although they are somewhat similar in character). The line element (5.1) has the property that under the local time transformation, $t_{_L} = t' - vx'$, it goes into the form

$$ds^2 = dt_{_L}^2 - n^2 \left(dx'^2 + dy'^2 + dz'^2\right), \qquad (5.2)$$

which, upon setting $ds^2 = 0$, yields the same slowness for light in all directions, $\Delta t_{_L} = n\Delta\sigma'$, where $\Delta\sigma'$ is as before $\sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}$ and thus (5.1) predicts the results of special relativity.

Setting $ds^2 = 0$ in (5.1) one finds

$$\Delta t' = v\Delta x' + n\Delta\sigma\,, \qquad (5.3)$$

and hence the average out-and-back time is

$$\frac{1}{2}\left(\Delta t'_{out} + \Delta t'_{back}\right) = n\Delta\sigma \qquad (5.4)$$

so that there are no effects of the kind mentioned in a). The slowness of light in the direction θ', with $\cos\theta' = \frac{\Delta x'}{\Delta\sigma'}$ is,

$$\frac{\Delta t'}{\Delta\sigma'} = n + v\cos\theta'. \qquad (5.5)$$

On the other hand, for a comparison stretch in the vacuum, as was shown previously, $\left(\frac{\Delta t'}{\Delta\sigma'}\right)_{VAC} = 1 + v\cos\theta'$. Hence the time difference per distance $\Delta\sigma'$ is

$$\frac{\Delta t'}{\Delta\sigma'} - \left(\frac{\Delta t'}{\Delta\sigma'}\right)_{VAC} = n - 1 \qquad (5.6)$$

and so there are no effects of the kind mentioned in b).

Let us now enquire as to what the velocity of light through the refractive medium at rest in S' appears to be as measured in the absolute frame. For simplicity we consider the light to be moving in the positive x' direction. Then using the formula for the relative velocity given by

the A.L.T., namely, $\frac{dx'}{dt'} = \frac{dx/dt - v}{1-v^2}$, we have

$$\frac{dx}{dt} = \left(1 - v^2\right) \frac{dx'}{dt'} + v \tag{5.7}$$

and hence since $\frac{dx'}{dt'} = \frac{1}{n+v}$ from (5.5), it follows

$$\frac{dx}{dt} = \frac{\frac{1}{n} + v}{1 + \frac{1}{n} v}\,, \tag{5.8}$$

so that we have the same result as that prescribed by special relativity, but without requiring the slowness of light in the refractive medium in S' to be n.

In the above, for simplicity, we considered only two frames S' and S, but suppose, as in the Fizeau experiment, the refractive medium is in motion relative to the terrestrial frame, which in turn is in motion relative to the absolute frame, what value do we obtain for the refractive index in the terrestrial frame as a function of the relative velocity of the refractive medium? Let the moving refractive medium have velocity v_2 in the positive x direction relative to the absolute frame, and the Earth frame a velocity v_1, then the relative velocity u_1, of the light in the refractive medium with respect to the Earth frame is

$$u_1 = \left(\frac{\frac{1}{n} + v_2}{1 + \frac{1}{n} v_2} - v_1 \right) \frac{1}{1 - v_1^2}\,, \tag{5.9}$$

and the slowness $\frac{1}{u_1}$. Now in order to measure this slowness one has to know, as remarked previously, the time $\Delta t'$ to traverse a distance $\Delta x'$ which one has no way of measuring without absolute signals. If we employ the special relativistic convention that light travels with unit speed in a comparison vacuum stretch $\Delta x'$, we are actually assigning a slowness $\frac{1}{u_1} - v_1$, to the light in the refractive medium, hence a velocity given by $\frac{u_1}{1-u_1 v_1}$. A simple calculation yields

$$\frac{u_1}{1 - u_1 v_1} = \frac{\frac{1}{n} + v_r}{1 + \frac{1}{n} v_r}\,, \quad \text{with} \quad v_r = \frac{v_2 - v_1}{1 - v_2 v_1}\,, \tag{5.10}$$

which is again the relativistic result. On the other hand, measurements made with absolute signals in the Earth frame would give the value u_1. Clearly if $v_1 = 0$, $\frac{u_1}{1-u_1 v_1}$ reduces to u_1 so that the relative velocity with respect to the absolute frame is the same for observers using the A.L.T. or special relativity, since under these circumstances both

observers agree that the slowness of light is unity. In order to avoid confusion with the various "relative velocities" that we encounter it will be convenient to use the following terminology:

$$\left.\begin{array}{ll} \text{"Galilean relative velocity":} & \dfrac{dx_g}{dt_g} = \dfrac{dx}{dt} - v \\ \text{"A.L.T. relative velocity":} & \dfrac{dx'}{dt'} = \dfrac{\frac{dx}{dt} - v}{1 - v^2} \\ \text{"relativistic relative velocity":} & \dfrac{dx_{\scriptscriptstyle L}}{dt_{\scriptscriptstyle L}} = \dfrac{\frac{dx}{dt} - v}{1 - \frac{dx}{dt}\, v} \end{array}\right\} \qquad (5.11)$$

and for the quantity dx/dt, the "velocity relative to the absolute frame" or simply, "absolute velocity".

Another example illustrative of calculating with the A.L.T. is the following: consider twin observers, initially at rest at the origin in the primed frame, and let them be moved with equal and opposite velocities in the positive and negative x' direction, by equal amounts $\Delta x'$; do they have the same age upon arriving at their respective destinations? In special relativity the answer is clearly, yes; however, one might wonder whether the same would be true in the conceptual framework presented here, since the twin that went in the positive x' direction will have a larger velocity relative to the absolute frame than the twin that went in the negative x' direction (except as discussed below) and hence the rate of ageing of the former is greater than the rate of ageing of the latter — and in an absolute sense. However, the key to the discrepancy in the two results lies in the phrase, "equal and opposite velocities". As discussed before, we have with current techniques no way of measuring their velocities; on the other hand, special relativity states that the two twins had equal and opposite velocities, if upon arrival, they each sent back light signals which arrived at the origin simultaneously. Let us calculate with this requirement in the framework of the A.L.T. and see what result is obtained.

If the twins have A.L.T. relative velocities u_-, u_+ to the left and right respectively, the total time elapsed after they have left the origin and the two light signals return is

$$\left(\frac{1}{u_-} + 1 + v\right)\Delta x' = \left(\frac{1}{u_+} + 1 - v\right)\Delta x', \qquad (5.12)$$

since the two signals are required to arrive simultaneously, and the slowness of light is $1+v$ in the positive direction and $1-v$ in the negative

direction. Hence, $\frac{1}{u_-} + v = \frac{1}{u_+} - v$. Now the twin that went to the right, required a time $\Delta t'$ in the primed frame given by $\Delta t' = \frac{1}{u_+}\Delta x'$, which in the absolute frame meant a time $\Delta t = \frac{1}{u_+}\Delta x'\gamma$. Hence a clock at rest with the twin indicated a time

$$\Delta t''_+ = \Delta t\sqrt{1-(v_2^+)^2} = \frac{1}{u_+}\,\Delta x'\,\gamma\sqrt{1-(v_2^+)^2}\ , \tag{5.13}$$

where v_2^+ is the absolute velocity of the twin that went to the right. From the relation, $u_+ = \frac{v_2^+ - v}{1-v^2}$, one has $v_2^+ = u_+(1-v^2) + v$ and inserting this in (5.13) and simplifying, there results,

$$\Delta t''_+ = \Delta x'\sqrt{\left(\frac{1}{u_+} - v\right)^2 - 1}\ , \tag{5.14}$$

and similarly for the twin that went to the left,

$$\Delta t''_- = \Delta x'\sqrt{\left(\frac{1}{u_-} + v\right)^2 - 1}\ . \tag{5.15}$$

Hence, since $\frac{1}{u_-} + v = \frac{1}{u_+} - v$, the two ages are the same. Moreover we note $\frac{1}{u_+} - v$, $\frac{1}{u_-} - v$ are nothing but the expressions for the reciprocal of the relativistic relative velocities, namely,

$$\frac{1}{u_+} - v = \frac{1-v^2}{v_2^+ - v} - v = \frac{1-v_2^+ v}{v_2^+ - v} \equiv \frac{1}{v_r^+}\,, \tag{5.16}$$

also, remembering $\frac{1}{u_-}$ is treated as a magnitude above,

$$\frac{1}{u_-} + v = \frac{1-v^2}{v - v_2^-} + v = \frac{1-v_2^- v}{v - v_2^-} \equiv \frac{1}{|v_r^-|}\,, \tag{5.17}$$

so that, as one would calculate relativistically,

$$\Delta t''_\pm = \frac{\Delta x'}{v_r^\pm}\sqrt{1-(v_r^\pm)^2}\ , \tag{5.18}$$

where we can omit the absolute magnitude sign of v_r^- treating $\Delta x'$ as negative for motion to the left.

From the above we see that since $\frac{1}{u_+} = \frac{1}{u_-} + 2v$, the twin that went to the right actually had less A.L.T. relative velocity than the twin that went to the left. So that an observer using absolute signals would not agree with a relativistic observer that the twins had arrived "simultaneously" at their (respective destinations. Had we therefore made

$u_+ = u_-$, their ages upon arrival would have been different, as may be seen from (5.14) and (5.15) and indeed, as was surmised initially, the twin that went to the right would have been younger when he arrived than-the twin that went to the left.

It should be noted that in the theory presented here there is no twin paradox of the kind in special relativity, since time can be measured in an absolute sense. For a "twin" moved to the right from the origin in S', the rate of ageing is *less* than a twin at the origin.

When the twin returns (provided of course he doesn't return so swiftly that his absolute velocity is greater than v) his rate of ageing is *greater* than the twin at the origin. But the total ageing for the journey is always less than that for the twin who remained at the origin. Thus the out-and-back time $\Delta\bar{t}'$ measured by a twin at the origin is,

$$\Delta\bar{t}' = \frac{1}{u_+}\,\Delta x' + \frac{1}{u_-}\,\Delta x' \tag{5.19}$$

and the total ageing of the twin that travelled out and back is,

$$\Delta\bar{t}'' = \Delta x' \sqrt{\left(\frac{1}{u_+} - v\right)^2 - 1} \; + \; \Delta x' \sqrt{\left(\frac{1}{u_-} + v\right)^2 - 1} \tag{5.20}$$

and using the relations, $\frac{1}{v_r^+} = \frac{1}{u_+} - v$, $\frac{1}{|v_r^-|} = \frac{1}{u_-} + v$, one has

$$\frac{\Delta\bar{t}'}{\Delta x'} = \frac{1}{v_r^+} + \frac{1}{|v_r^-|} \tag{5.21}$$

and

$$\frac{\Delta\bar{t}''}{\Delta x'} = \frac{1}{v_r^+}\sqrt{1 - (v_r^+)^2} + \frac{1}{|v_r^-|}\sqrt{1 - (v_r^-)^2} \tag{5.22}$$

and since all quantities are positive, one always has,

$$\frac{\Delta\bar{t}'}{\Delta x'} > \frac{\Delta\bar{t}''}{\Delta x'} \tag{5.23}$$

a result which is expected from simpler considerations using special relativity. On the other hand, the following result is meaningless in special relativity.

Let there be two pairs of identical clocks in S', one pair at A, call them a_1, and a_2, and the other pair at B, call them b_1, and b_2. And let B be at a positive distance $\Delta x'$ from A. Let both sets of clocks be synchronized with absolute signals at some time $t' = 0$ and then permitted to run at their natural rates. Now let a_2 be moved to the right to

B, and let b_2 be moved to the left to A. Then the above considerations show that while the time indicated by a_2 when it arrives at B will always be *less* than the time indicated by b_1, the time indicated by clock b_2 when it arrives at A will be greater than, equal to, or less than the time indicated by a_1, according to the following scheme,

$$\left.\begin{aligned} t_{b_2} > t_{a_1}\colon &\quad \left|\frac{-2v}{1-v^2}\right| > u_- > 0 \\ t_{b_2} = t_{a_1}\colon &\quad \left|\frac{-2v}{1-v^2}\right| = u_- \\ t_{b_2} < t_{a_1}\colon &\quad \left|\frac{-2v}{1-v^2}\right| < u_- \end{aligned}\right\}. \tag{5.24}$$

Since the elapsed time read by a_1 is $\frac{1}{u_-}\Delta x'$ and the time by b_2 is $\frac{1}{u_-}\Delta x'\gamma\sqrt{1-v_2^2}$, where v_2, is the absolute velocity of b_2 corresponding to u_-, $u_- = \left|\frac{(v_2-v)}{1-v^2}\right|$, and their ratio is $\gamma\sqrt{1-v_2^2}$ which by the above scheme may be adjusted to be equal to, or less than unity.

The relation between u^- and v_r^- which was found in the preceding by essentially a physical argument follows quite simply from the local time transformation: $t_{_L} = t' - vx'$, $x^i_{_L} = x'^i$, since this may be written, upon using $\Delta\sigma_{_L} = \Delta\sigma'$,

$$\frac{\Delta t_{_L}}{\Delta\sigma_{_L}} = \frac{\Delta t'}{\Delta\sigma'} - v\cos\theta', \tag{5.25}$$

and hence for $\theta' = 0$, $\frac{1}{v_r} = \frac{1}{u} - v$. If we take $\frac{\Delta t_{_L}}{\Delta\sigma_{_L}} = 1$, we obtain the expression, for the slowness of light found in (3.16), $\frac{\Delta t'}{\Delta\sigma'} = 1 + v\cos\theta'$.

Chapter 6. Measurements with Signals Travelling with Finite Velocities

As has been shown, it is possible to determine the asymmetries in the propagation of light in S' using absolute signals, but can one measure such asymmetries with signals travelling with merely finite velocities greater than that of light? Before determining the answer to this question, let us note it is possible to define operationally such superlight (or "supervidic") signals without any assumptions about the synchronization of separated clocks. Let there be two similar clocks in S', one at the origin A, and the other $\Delta x'$ from the origin at B. Let a light signal and the signal in question be sent out simultaneously from A, and let their respective times of arrival, t'_1 and t'_2 be measured at B. Then if

$t'_2 - t'_1 > 0$, the signal in question traveled more slowly than light, and if the time difference is negative, $t'_2 - t'_1 < 0$, the signal traveled faster than light and was a superlight signal. Note that there is nothing in the above definition about the magnitudes of the velocities, but rather a statement about their ordering as to magnitude.

Consider a set of such superlight signals, arriving at B at times t'_2, t'_3, etc., each travelling faster than the other, then

$$|t'_2 - t'_1| < |t'_3 - t'_1| < \cdots < |t'_m - t'_1| \,. \tag{6.1}$$

The upper bound of this sequence defines the absolute signal. Moreover there exists such a bound, since the absolute (and largest) delay for light is $(1+v)\,\Delta x'$ in the positive x'-direction, hence $|t'_m - t'_1| \leqslant$ $\leqslant (1+v)\,\Delta x'$, the equality sign holding for the absolute signal.

Let us now suppose we wish to determine the velocity of the frame S' (say the Earth frame) with respect to the frame S; can this be done with a superlight signaling apparatus? In order to make such a measurement one would do the following: Compare the difference in times of arrival of the light signal and the superlight signal from A to B with the difference in times of arrival from B to A. Thus

$$\left.\begin{aligned} \left(1 + v - \frac{1}{u_+}\right)\Delta x' &= \Delta t'_+ \\ \left(1 - v - \frac{1}{u_-}\right)\Delta x' &= \Delta t'_- \end{aligned}\right\}, \tag{6.2}$$

where u_+, u_- represent the magnitudes of the A.L.T. relative velocities of the superlight signals in the positive and negative directions. It will be seen one has two equations in three unknowns so that in general there is no solution. However for special cases there are solutions, the simplest situation being $\frac{1}{u_+}, \frac{1}{u_-} \ll v$ so that effectively we are dealing with absolute signals. Or again, if one discovers by experiment that the velocity of the super-light signal is independent of the velocity of the source (as is the case for light signals) and given by $w > 1$, then

$$u_+ = \frac{w - v}{1 - v^2}\,, \qquad u_- = \frac{|-w - v|}{1 - v^2}\,, \tag{6.3}$$

and one has two equations in two unknowns. For the time differences one finds

$$\Delta t'_+ - \Delta t'_- = 2v\left(\frac{w^2 - 1}{w^2 - v^2}\right)\Delta x', \tag{6.4}$$

and for the sum,

$$\Delta t'_{+} + \Delta t'_{-} = 2\left(1 - w\,\frac{1-v^2}{w^2-v^2}\right)\Delta x', \tag{6.5}$$

from which one can determine w and v.

One can for such a signal, formally define a line element ds in the absolute frame given by

$$d\bar{s}^2 = dt^2 - \frac{1}{w^2}\left(dx^2 + dy^2 + dz^2\right), \tag{6.6}$$

which under the A.L.T. becomes in S',

$$d\bar{s}^2 = \frac{1}{w^2}\left[\frac{w^2-v^2}{1-v^2}(dt')^2 - 2v\,dx'dt' - \left(1-v^2\right)dx'^2 - dy'^2 - dz'^2\right], \tag{6.7}$$

and hence setting $d\bar{s}^2 = 0$, the time $\Delta t'$ for such signals to traverse $\Delta x'$, $\Delta y'$, $\Delta z'$ is

$$\Delta t' = \frac{v\left(1-v^2\right)\Delta x'}{w^2-v^2} \pm \\ \pm \frac{1-v^2}{|w^2-v^2|}\sqrt{w^2\left(\Delta x'\right)^2 + \frac{w^2-v^2}{1-v^2}\left(\Delta y'^2 + \Delta z'^2\right)}\,. \tag{6.8}$$

Although the above results were derived assuming $w > 1$, it is interesting to note that they also hold if $w < 1$, except that under these circumstances the slower-than-light signal cannot propagate in certain directions in the primed frame if $w < v$, namely those directions for which

$$w^2\left(\Delta x'\right)^2 + \frac{w^2-v^2}{1-v^2}\left[\left(\Delta y'\right)^2 + \left(\Delta z'\right)^2\right] < 0\,, \qquad \Delta t' < 0 \tag{6.9}$$

since in these directions the delay in sending such a signal is neither real nor positive. The signals are therefore confined to a cone opening in the negative x' direction. On the other hand when $w > v$, all directions are allowed.

Using these slower-than-light signals it would also be possible to detect the absolute motion of the Earth as with superlight signals for which w is constant, employing (6.4) and (6.5) if $w > v$, and if $w < v$, measuring the slope of the cone of preferred directions and $\Delta t'_{-}$ above — from which it is possible to obtain v and w by a simple calculation. It is interesting to note that if w is zero, the cone shrinks to a line. Physically, this "signal" consists in the identification of a point in the absolute

frame which then in S' moves rearward with velocity $\frac{-v}{1-v^2}$. Comparing the delay of such a "signal" with a light signal we find using (6.2) $\Delta t'_{-} = \left(\frac{1}{v} - 1\right) \Delta x'$ and hence v can be found in this case as well.

Thus we see that a sufficient condition for it to be possible to detect the absolute motion of a frame S' is that there be at least one other signal propagating with constant absolute velocity w in the vacuum, with $w \neq 1$.

Chapter 7. Dynamics of a Free Particle

§7.1. Energy-momentum relations in an A.L.T. frame

As was remarked in the Introduction, a fundamental distinction between special relativity and general relativity (from the standpoint of general covariance) is that "invariance" in the former implies a restriction, on coordinate transformations, whereas invariance in the latter is really a tautology. Given any contravariant vector V^μ, and its covariant vector $V_\mu = g_{\mu\nu} V^\nu$, the statement,

$$V_\mu V^\mu = \text{"an invariant"} \tag{7.1}$$

is true independently of what coordinate transformation is made; it is a tautology of general covariance. On the other hand, the statement,

$$V^0 V^0 - V^i V^i = \text{"an invariant"} \tag{7.2}$$

is in general not true except for certain transformations, the Lorentz transformations, so that it is a "conditional" invariance relation. This relation in special relativity leads to the result, $p^0_L p^0_L - p^i_L p^i_L = m^2$, where p^μ_L are the momenta in a Lorentz frame. Let us now seek to find the analogous conditional invariance relation when the A.L.T. is employed, and finally, for further comparison, the relation when the Galilean transformation is employed.

Let a particle of mass m, be moving with absolute velocities, $\dot{x}$, $\dot{y}$, $\dot{z}$, the equations of motion are obtained from the variational principle,

$$\delta \int m ds = 0\,, \qquad ds = \sqrt{\eta_{\mu\nu}\, dx^\mu dx^\nu}\ . \tag{7.3}$$

In the primed frame S', under the A.L.T., the variational principle becomes

$$\delta \int m \sqrt{g'_{\mu\nu}\, dx'^\mu dx'^\nu} = \\ = \delta \int m \sqrt{1 - 2v u_x - (1 - v^2)\, u_x^2 - u_y^2 - u_z^2}\ dt' = 0\,, \tag{7.4}$$

when t' is taken as parameter, and where $u_x = \frac{dx'}{dt'}$, $u_y = \frac{dy'}{dt'}$, $u_z = \frac{dz'}{dt'}$. The Lagrangian is therefore,

$$L = m\sqrt{1 - 2vu_x - (1-v^2)\,u_x^2 - u_y^2 - u_z^2}\ . \tag{7.5}$$

The covariant momenta are,

$$\left.\begin{aligned} p'_x &= \frac{\partial L}{\partial u_x} = -m\Gamma\left[v + \left(1-v^2\right)u_x\right], \qquad p'_y = -m\Gamma u_y \\ p'_0 &= -u_i\frac{\partial L}{\partial u_i} + L = m\Gamma\left(1 - u_x v\right), \qquad p'_z = -m\Gamma u_z \\ \Gamma &= \frac{m}{L} = \frac{1}{\sqrt{1 - 2vu_x - (1-v^2)\,u_x^2 - u_y^2 - u_z^2}} \end{aligned}\right\}. \tag{7.6}$$

Alternatively, the expression for p'_0 is more generally obtainable using $p'_0 = \frac{\partial L}{\partial u_0}$, where $u_0 = \frac{dt'}{ds}$, and after differentiation, setting $\frac{dt'}{ds} = 1$, if t' is taken as parameter. The contravariant metric tensor $g'^{\mu\nu}$ is obtained from inverting $g'_{\mu\nu}$ is given by

$$\|\,g'^{\mu\nu}\| = \begin{Vmatrix} 1-v^2 & -v & 0 & 0 \\ -v & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{7.7}$$

If the velocity of the frame S' were not along the x axis but in an arbitrary direction v_x, v_y, v_z, $g'^{\mu\nu}$ is obtained from inverting the tensor given in (3.19) and is

$$\|\,g'^{\mu\nu}\| = \begin{Vmatrix} 1-v^2 & -v_x & -v_y & -v_z \\ -v_x & -1 & 0 & 0 \\ -v_y & 0 & -1 & 0 \\ -v_z & 0 & 0 & -1 \end{Vmatrix}. \tag{7.8}$$

For simplicity we shall continue to restrict our discussion, to motion of S' along the x-axis.

The contravariant momenta are then obtained from (7.6) and (7.7),

$$\left.\begin{aligned} p'^x &= g'^{x0}p'_0 + g'^{xx}p'_x = m\Gamma u_x\,, \qquad p'^y = m\Gamma u_y \\ p'^0 &= g'^{0x}p'_x + g'^{00}p'_0 = m\Gamma\,, \qquad\quad p'^z = m\Gamma u_z \end{aligned}\right\}. \tag{7.9}$$

In order to obtain the momenta in the unprimed frame, it is the contravariant quantities above which are to be transformed via the A.L.T., hence as is the case with the coordinates

$$\left.\begin{aligned} p^x &= \frac{1}{\gamma}\, p'^x + \gamma\, v\, p'^0, \qquad & p^y &= p'^y \\ p^0 &= \gamma\, p'^0, & p^z &= p'^z \end{aligned}\right\}. \tag{7.10}$$

Using the transformation properties of the absolute relative velocities,

$$u_x = (\dot{x} - v)\,\gamma^2, \qquad u_y = \gamma\,\dot{y}\,, \qquad u_z = \gamma\,\dot{z}\,, \tag{7.11}$$

one can rewrite Γ as

$$\Gamma = \frac{\bar{\gamma}}{\gamma}\,, \qquad \bar{\gamma} \equiv \frac{1}{\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}}\,. \tag{7.12}$$

Substituting this expression for Γ, together with the expressions for the velocities (7.11) and the momenta (7.10) into the transformation yields

$$p^0 = m\bar{\gamma}\,, \quad p^x = m\bar{\gamma}\,\dot{x}\,, \quad p^y = m\bar{\gamma}\,\dot{y}\,, \quad p^z = m\bar{\gamma}\,\dot{z} \tag{7.13}$$

as would have been obtained using the ordinary Lorentz transformation, or as we shall now show, any transformation. The line element, $ds = \sqrt{\eta_{\mu\nu}\, dx^\mu dx^\nu} = \sqrt{\eta_{\mu\nu}\, \frac{dx^\mu}{d\tau}\, \frac{dx^\nu}{d\tau}}\, d\tau$, where τ is an arbitrary parameter, under the transformation, $dx^\mu = b^\mu_\nu\, dx'^\nu$, becomes

$$ds = \sqrt{\eta_{\mu\nu}\, b^\mu_\lambda b^\nu_\rho dx'^\lambda dx'^\rho} = \sqrt{\eta_{\mu\nu}\, b^\mu_\lambda b^\nu_\rho\, \frac{dx'^\lambda}{d\tau}\, \frac{dx'^\rho}{d\tau}}\; d\tau', \tag{7.14}$$

where τ' is another arbitrary parameter. The momenta (per unit mass) are

$$p'_\rho = \eta_{\mu\nu}\, b^\mu_\lambda b^\nu_\rho\, \Gamma\, \frac{dx'^\lambda}{d\tau'}\,, \qquad p'^\rho = \Gamma\, \frac{dx'^\rho}{d\tau'}\,, \qquad \Gamma \equiv \frac{d\tau'}{ds}\,. \tag{7.15}$$

The coordinate transformation may be written,

$$\frac{dx^\mu}{ds} = b^\mu_\nu\, \frac{dx'^\nu}{ds} = b^\mu_\nu\, \frac{dx'^\nu}{d\tau'}\, \frac{d\tau'}{ds} \tag{7.16}$$

and if in the unprimed frame $\frac{d\tau}{ds} \equiv \bar{\gamma}$,

$$\bar{\gamma}\, \frac{dx^\mu}{d\tau} = b^\mu_\nu\, \Gamma\, \frac{dx'^\nu}{d\tau'}\,, \tag{7.17}$$

which is the desired result when $d\tau = dx^0 \equiv dt$, $d\tau' = dx'^0 \equiv dt'$.

Let us now observe that in the primed frame, the following relationship holds:

$$(p'_0)^2 - (p'^x)^2 - (p'^y)^2 - (p'^z)^2 = m^2. \tag{7.18}$$

That is, the *covariant* energy and the contravariant spatial momenta satisfy the usual relativistic energy-momentum relationship for a free particle of mass m. Since this invariance relation, does not depend upon v, the absolute velocity of the frame, it is true in all uniformly moving frames for which the A.L.T. holds. It is the conditional invariance relation for which we were seeking that is the analog of (7.2). Moreover one finds after some manipulation that,

$$\left.\begin{aligned} p'^x &= m\gamma_r v_{rx}\,, & p'^y &= m\gamma_r v_{ry} \\ p'_0 &= m\gamma_r\,, & p'^z &= m\gamma_r v_{rz} \end{aligned}\right\} \tag{7.19}$$

with $\gamma_r \equiv \frac{1}{\sqrt{1-(v_{rx})^2-(v_{ry})^2-(v_{rz})^2}}$, and

$$\left.\begin{aligned} v_{rx} &= \frac{\dot{x}-v}{1-v\dot{x}} = \frac{1}{\frac{1}{u_x}-v} \\ v_{ry} &= \frac{\sqrt{1-v^2}}{1-v\dot{x}}\,\dot{y} = \frac{u_y}{1-u_x v} \\ v_{rz} &= \frac{\sqrt{1-v^2}}{1-v\dot{x}}\,\dot{z} = \frac{u_z}{1-u_x v} \end{aligned}\right\}. \tag{7.20}$$

Thus p'_0, p'^i are to be identified with their relativistic counterparts, p^0_L, p^i_L based on using the ordinary Lorentz transformation. This result may be made more transparent by noting that the transformation connecting p'_0, p'^i to the absolute frame is the Lorentz transformation. Thus, as may be shown,

$$p'_0 = p'^0 - v p'^x \tag{7.21}$$

(which follows most easily from recognizing $dt = dt' - v dx'$ and making the appropriate identification — alternatively, from suitably reshuffling the terms in the line element or more formally, using $p'_0 = g'_{0\mu} p'^\mu$), and hence substituting for p'^0 in the transformation (7.10), there results,

$$\left.\begin{aligned} p^x &= (p'^x + v p'_0)\,\gamma\,, & p^y &= p'^y \\ p^0 &= (p'_0 + v p'^x)\,\gamma\,, & p^z &= p'^z \end{aligned}\right\}. \tag{7.22}$$

The fact that the (p'_0, p'^i) are to be identified with what are called the energy and momentum in special relativity, provides us therefore

with a very simple way of transcribing the dynamical laws of one theory in terms of the other, and moreover, as we have been showing, when measurements are made in certain ways, they are the *same* laws.

Let us now enquire as to the physical meaning of the momenta complementary to the above, p'^0, p'_i. On using the relations, $\Gamma = \frac{\bar{\gamma}}{\gamma}$, $\dot{x} = = (1-v^2)\,u_x + v$ etc., we find,

$$\left.\begin{aligned} p'_x &= -m\,\dot{x}\,\frac{\bar{\gamma}}{\gamma}\,, \qquad & p'_y &= -m\,\dot{y}\,\bar{\gamma} \\ p'^0 &= m\,\frac{\bar{\gamma}}{\gamma}\,, & p'_x &= -m\,\dot{z}\,\bar{\gamma} \end{aligned}\right\} \tag{7.23}$$

so that apart from the time dilatation factor $\frac{1}{\gamma}$, these quantities are nothing but the *covariant* momenta p_μ as measured in the absolute frame. They are therefore "unobservables" unless measurements are made with absolute signals or the equivalent. Note the square of these momenta satisfy,

$$\gamma^2\left[\left(p'^0\right)^2 - \left(p'_x\right)^2\right] - \left(p'_y\right)^2 - \left(p'_z\right)^2 = m^2, \tag{7.24}$$

which contains explicit reference to the absolute velocity of the frame and is therefore not an invariance relation.

These complementary momenta do not have the same kind of reflection properties that are possessed by the $(p'_0,\, p'^i)$. Thus consider a particle moving in S' along the x' axis, upon colliding elastically with a sufficiently heavy object, the quantities p'_0, p'^x satisfy (the subscripts i and f denoting initial and final states)

$$\left(p'_0\right)_f = \left(p'_0\right)_i\,, \qquad \left(p'^x\right)_f = -\left(p'^x\right)_i \tag{7.25}$$

the same as for the Lorentz observer. Whereas, for the quantities p'^0, p'_x one has since $p'_x = -p'^x - v\,p'_0$, $p'^0 = p'_0 + v\,p'^x$,

$$\left.\begin{aligned} \left(p'^0\right)_f &= \left(p'_0\right)_i - v\left(p'^x\right)_i \neq \left(p'^0\right)_i = \left(p'_0\right)_i + v\left(p'^x\right)_i \\ \left(p'_x\right)_f &= \left(p'^x\right)_i - v\left(p'_0\right)_i \neq -\left(p'_x\right)_i = \left(p'^x\right)_i + v\left(p'_0\right)_i \end{aligned}\right\}. \tag{7.26}$$

This result is the analogue of the effect that was discussed in Chapter 5, where we saw that a Lorentz observer says two objects are moving with equal speeds in opposite directions if $|v_r^-| = v_r^+$ whereas the A.L.T. observer using u_+, u_- finds the two objects are in fact travelling in general with different speeds. Despite this lack of symmetry, both species of momenta are conserved in a collision process. For indeed, if one is

conserved, so is the other, since they are linearly dependent. Thus if $\sum_i p'^x = \sum_f p'^x$, $\sum_i p'_0 = \sum_f p'_0$, then

$$\left.\begin{aligned} &\sum_i p'_x - \sum_f p'_x = -\Big(\sum_i p'^x - \sum_f p'^x\Big) - v\Big(\sum_i p'_0 - \sum_f p'_0\Big) = 0 \\ &\sum_i p'^0 - \sum_f p'^0 = \sum_i p'_0 - \sum_f p'_0 + v\Big(\sum_i p'^x - \sum_f p'^x\Big) = 0 \end{aligned}\right\}, \quad (7.27)$$

where $\sum_i$, $\sum_f$ represent the summation over the momenta of the particles in the initial and final states.

§7.2. Energy-momentum relations in a Galilean frame

It is of interest to see what the preceding method yields when applied to a Galilean frame. The line element $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ becomes under the Galilean transformation, $t_g = t$, $x_g = x - vt$, $y_g = y$, $z_g = z$, as given earlier in (2.5),

$$ds^2 = \left(1 - v^2\right) dt_g^2 - 2v\, dt_g dx_g - dx_g^2 - dy_g^2 - dz_g^2\,,$$

the contravariant metric tensor for the Galilean frame being the covariant metric tensor for the A.L.T. frame and conversely. The momenta are

$$\left.\begin{aligned} &p_{gx} = -m\Gamma_g\,(v + \dot{x}_g)\,, && p_{gy} = -m\Gamma_g\,\dot{y}_g \\ &p_{g0} = m\Gamma_g\left(1 - v\dot{x}_g - v^2\right), && p_{gz} = -m\Gamma_g\,\dot{z}_g \\ &p_g^x = m\Gamma_g\,\dot{x}_g\,, && p_g^y = m\Gamma_g\,\dot{y}_g \\ &p_g^0 = m\Gamma_g\,, && p_g^z = m\Gamma_g\,\dot{z}_g \\ &\Gamma_g = \frac{1}{\sqrt{(1 - v^2) - 2v\dot{x}_g - \dot{x}_g^2 - \dot{y}_g^2 - \dot{z}_g^2}} \end{aligned}\right\}. \quad (7.28)$$

Relating the contravariant momenta via the Galilean transformation to their values in the absolute frame by

$$\left.\begin{aligned} &p^x = p_g^x + v\,p_g^0\,, && p^y = p_g^y \\ &p^0 = p_g^0\,, && p^z = p_g^z \end{aligned}\right\} \quad (7.29)$$

one also obtains the expressions for the energy and momenta given by (7.13) as our general arguments showed must be the case.

However, unlike the situation with the A.L.T., these Galilean quantities can not be identified with appropriate relativistic counterparts in the Lorentz frame. Indeed, on using $\dot{x} = \dot{x}_g + v$, etc., and observing, $\Gamma_g = \frac{1}{\sqrt{1-(\dot{x}_g+v)^2-\dot{y}_g^2-\dot{z}_g^2}} = \bar{\gamma}$, we have the following identification,

$$\left.\begin{aligned} p_{gx} &= -m\bar{\gamma}\dot{x} = p_x\,, & p_{gy} &= -m\bar{\gamma}\dot{y} = p_y \\ p_g^0 &= m\bar{\gamma} = p_0\,, & p_{gz} &= -m\bar{\gamma}\dot{z} = p_z \end{aligned}\right\} \tag{7.30}$$

so that these quantities are in fact, the covariant absolute momenta. Thus the Galilean observer states the same invariance relation as the absolute observer but with a change in notation, i.e., $(p_g^0)^2 - (p_{gx})^2 - -(p_{gy})^2 - (p_{gz})^2 = m^2$.

Consider now the complementary quantities p_{g0}, p_g^i: we note Γ_g can also be written

$$\Gamma_g = \frac{\sqrt{1-v^2}\,\gamma_r}{1-v\dot{x}} \tag{7.31}$$

and using the expressions for v_{rx}, v_{ry}, v_{rz} we find,

$$\left.\begin{aligned} p_g^x &= \frac{m\gamma_r v_{rx}}{\gamma}\,, & p_g^y &= m\gamma_r v_{ry} \\ p_{g0} &= \frac{m\gamma_r}{\gamma}\,, & p_g^z &= m\gamma_r v_{rz} \end{aligned}\right\} \tag{7.32}$$

so that these quantities are almost, but not quite, the Lorentz momenta. They satisfy,

$$\gamma^2\left[\left(p_{g0}\right)^2 - \left(p_g^x\right)^2\right] - \left(p_g^y\right)^2 - \left(p_g^z\right)^2 = m^2 \tag{7.33}$$

which is not a conditional invariant for arbitrary Galilean observers, depending as it does on v. It is analogous to the expression (7.24).

Chapter 8. Transformation of Maxwell's Equations and Further Applications

§8.1. Transformation of Maxwell's equations

In the absolute frame S, Maxwell's equations may be written

$$\left.\begin{aligned} &\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0 \\ &\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu \end{aligned}\right\}. \tag{8.1}$$

Under a transformation to S', given by

$$dx'^{\mu} = a^{\mu}_{\nu}\, dx^{\nu}, \quad dx^{\nu} = \bar{a}^{\nu}_{\mu}\, dx'^{\mu}, \quad a^{\mu}_{\nu}\, \bar{a}^{\nu}_{\rho} = \delta^{\mu}_{\rho}\,, \tag{8.2}$$

where the a^{μ}_{ν}, $\bar{a}^{\nu}_{\mu}$ are the coefficients of the A.L.T. and its reciprocals, the above equations take the same tensor form,

$$\left.\begin{aligned} &\frac{\partial F'_{\mu\nu}}{\partial x'^{\lambda}} + \frac{\partial F'_{\lambda\mu}}{\partial x'^{\nu}} + \frac{\partial F'_{\nu\lambda}}{\partial x'^{\mu}} = 0 \\ &\frac{\partial F'^{\mu\nu}}{\partial x'^{\nu}} = j'^{\mu} \end{aligned}\right\}. \tag{8.3}$$

The above results of course are not conditional on using the A.L.T.; they are true for any transformation (for non-linear transformations with $F^{\mu\nu}$ and j^{μ} replaced by tensor densities). As remarked in the preceding chapter the above result is simply a tautology of covariance. On introducing the vector potential A'_{μ} defined by $F'_{\mu\nu} = \frac{\partial A'_{\mu}}{\partial x'^{\nu}} - \frac{\partial A'_{\nu}}{\partial x'^{\mu}}$, the second Maxwell equation becomes

$$g'^{\lambda\nu} \frac{\partial^2 A'^{\mu}}{\partial x'^{\lambda}\, \partial x'^{\nu}} = j'^{\mu} \tag{8.4}$$

with the imposition of the gauge condition, $\frac{\partial A'^{\mu}}{\partial x'^{\nu}} = 0$. Since $g'^{\lambda\nu}$ is an explicit function of the absolute velocity of the frame S', it is in this form of the Maxwell's equations that the fundamental difference between the A.L.T. and the Lorentz transformation manifests itself.

An invariant element of charge at rest in the frame S' is given by $\delta e = = j'^{0}\sqrt{-g'}\, dx'dy'dz'$ where g' is the determinant of the metric tensor g', but since A.L.T. is unimodular $-g' = -\eta = 1$ (where η is the determinant of $\eta_{\mu\nu}$). Hence $j'^{0}dx'dy'dz'$ is an invariant of the transformation , and one may write

$$\delta e = j'^{0}dx'dy'dz' = j^{0}\, dxdydz \tag{8.5}$$

which is the same law as for the Lorentz observer. However, for charges in motion in S', the quantity j'^{0} is not what a Lorentz observer would associate with $j^{0}_{L}(= j_{L0})$, as we shall see it is j'_{0}, which for charges at rest in S' is given by $j'_{0} = g'_{0\nu} j'^{\nu} = j'^{0}$. The transformation laws for the A'^{μ}, j'^{μ} are, as before, for the momenta,

$$\left.\begin{aligned} &A^{x} = \frac{1}{\gamma}\, A'^{x} + \gamma v A'^{0}, \qquad A^{y} = A'^{y} \\ &A^{0} = \gamma A'^{0}, \qquad\qquad\quad A^{z} = A'^{z} \end{aligned}\right\}, \tag{8.6}$$

$$\left.\begin{aligned} j^x &= \frac{1}{\gamma}\, j'^x + \gamma v j'^0, \qquad & j^y &= j'^y \\ j^0 &= \gamma\, j'^0, & j^z &= j'^z \end{aligned}\right\}. \tag{8.7}$$

As before in dealing with the momenta we may write

$$\left.\begin{aligned} A'_0 &= g'_{0\mu} A'^{\mu} = A'^0 - v A'^x \\ j'_0 &= g'_{0\mu} j'^{\mu} = j'^0 - v j'^x \end{aligned}\right\}. \tag{8.8}$$

And hence the transformation for the potential and current using these mixed quantities becomes that of the Lorentz quantities:

$$\left.\begin{aligned} A^x &= (A'^x + v A'_0)\,\gamma\,, \qquad & A^y &= A'^y \\ A^0 &= (A'_0 + v A'^x)\,\gamma\,, & A^z &= A'^z \end{aligned}\right\} \tag{8.9}$$

and similarly for the currents. Thus $(A'_0,\, A'^i)$, $(j'_0,\, j'^i)$ are to be identified with the Lorentz quantities $(A^0_{\scriptscriptstyle L},\, A^i_{\scriptscriptstyle L})$, $(j^0_{\scriptscriptstyle L},\, j^i_{\scriptscriptstyle L})$, whereas for example $(j'^0, j'_x)\gamma$ are $(j_0,\, j_x)$, the latter being the quantities measured in the absolute frame — as was the case for the momenta, and is indeed true for all vectors.

Let us now relate the electromagnetic field quantities $F'_{\mu\nu}$, $F'^{\mu\nu}$ to their values in the absolute frame. One has, $F_{\mu\nu} = a^{\rho}_{\mu} a^{\lambda}_{\nu} F'_{\rho\lambda}$, $F^{\mu\nu} = \bar{a}^{\mu}_{\rho} \bar{a}^{\nu}_{\lambda} F'^{\rho\lambda}$, which reduce to

$$\left.\begin{aligned} F_{0x} &= F'_{0x}\,, & F^{0x} &= F'^{0x} \\ F_{0y} &= \frac{1}{\gamma}\, F'_{0y} - \gamma v F'_{xy}\,, & F^{0y} &= \gamma F'^{0y} \\ F_{0z} &= \frac{1}{\gamma}\, F'_{0z} - \gamma v F'_{xz}\,, & F^{0z} &= \gamma F'^{0z} \end{aligned}\right\}, \tag{8.10}$$

$$\left.\begin{aligned} F_{yz} &= F'_{yz}\,, & F^{yz} &= F'^{yz} \\ F_{xy} &= \gamma F'_{xy}\,, & F^{xy} &= \frac{1}{\gamma}\, F'^{xy} + \gamma v F'^{0y} \\ F_{zx} &= \gamma F'_{zx}\,, & F^{zx} &= \frac{1}{\gamma}\, F'^{zx} + \gamma v F'^{z0} \end{aligned}\right\}. \tag{8.11}$$

In order to compare the quantities $F'_{\mu\nu}$, $F'^{\mu\nu}$ with the values obtained by a Lorentz observer $F_{{\scriptscriptstyle L}\mu\nu}$, $F^{\mu\nu}_{\scriptscriptstyle L}$, we use the transformation O_3 relating Lorentz coordinates to the primed coordinates,

$$dx^{\mu}_{\scriptscriptstyle L} = \ell^{\mu}_{\nu}\, dx'^{\nu}, \quad dx'^{\mu} = \bar{\ell}^{\mu}_{\nu}\, dx^{\nu}_{\scriptscriptstyle L}, \quad \ell^{\mu}_{\nu}\, \bar{\ell}^{\nu}_{\lambda} = \delta^{\mu}_{\lambda}\,. \tag{8.12}$$

as given in Chapter 1. So that $F_{L\mu\nu} = \bar{\ell}^{\rho}_{\mu}\bar{\ell}^{\lambda}_{\nu}F'_{\rho\lambda}$, $F_L^{\mu\nu} = \ell^{\mu}_{\rho}\ell^{\nu}_{\lambda}F'^{\rho\lambda}$ one obtains,

$$\left.\begin{aligned} F_{L0x} &= F'_{0x}\,, & F_L^{0x} &= F'^{0x} \\ F_{L0y} &= F'_{0y}\,, & F_L^{0y} &= F'^{0y} - vF'^{xy} \\ F_{L0z} &= F'_{0z}\,, & F_L^{0z} &= F'^{0z} - vF'^{xz} \end{aligned}\right\}, \tag{8.13}$$

$$\left.\begin{aligned} F_{Lyz} &= F'_{yz}\,, & F_L^{yz} &= F'^{yz} \\ F_{Lxy} &= F'_{xy} + vF'_{0y}\,, & F_L^{xy} &= F'^{xy} \\ F_{Lzx} &= F'_{zx} + vF'_{z0}\,, & F_L^{zx} &= F'^{zx} \end{aligned}\right\}. \tag{8.14}$$

Thus the electromagnetic field quantities F'_{0i}, F'^{ij} are what correspond to the electric and magnetic fields as measured by a Lorentz observer at rest with respect to S'. Using the above expressions one can rewrite the transformations from the primed frame to the unprimed frame in the form,

$$\left.\begin{aligned} F_{0x} &= F'_{0x}\,, & F^{yz} &= F'^{yz} \\ F_{0y} &= \left(F'_{0y} - vF'^{xy}\right)\gamma\,, & F^{xy} &= \left(F'^{xy} - vF'_{0y}\right)\gamma \\ F_{0z} &= \left(F'_{0z} - vF'^{xz}\right)\gamma\,, & F^{zx} &= \left(F'^{zx} - vF'_{0z}\right)\gamma \end{aligned}\right\} \tag{8.15}$$

thereby exhibiting explicitly the Lorentz-like behaviour of $(F'_{0i},\ F'^{ij})$. Denoting these quantities then by $\mathcal{E}', \mathcal{H}'$, it follows that $\mathcal{E}'^2 - \mathcal{H}'^2$, $\mathcal{E}' \cdot \mathcal{H}'$ are the conditional invariants under the A.L.T. On the other hand, the quantities $(F'^{0i},\ F'_{ij})$ do not have this property, as may be inferred from the manner in which they are connected with the absolute frame as given above, such a product would contain explicit references to the absolute velocity of the frame.

§8.2. Equations of motion of a charged particle

Let us now consider the equations of motion of a particle interacting with the electromagnetic field, as observed in S'. We shall see that they may be written in a form identical to that seen by a Lorentz observer and for the *same* quantities but with a different label.

The variational principle in the absolute frame is

$$\delta \int m ds + eA_{\mu}\dot{x}^{\mu} ds = 0\,, \tag{8.16}$$

(where $\dot{x}^{\mu} = \frac{dx^{\mu}}{ds}$) and under the A.L.T., or indeed any transformation,

becomes

$$\delta \int m ds + e A'_\mu \dot{x}'^\mu ds = 0\,, \tag{8.17}$$

so that the equations of motion, written in both contra-and covariant form, are

$$\left.\begin{aligned} \frac{d\dot{x}'^\nu}{ds} &= \frac{e}{m}\,\dot{x}'_\mu F'^{\mu\nu} \\ \frac{d\dot{x}'_\nu}{ds} &= \frac{e}{m}\,\dot{x}'^\mu F'_{\mu\nu} \end{aligned}\right\}. \tag{8.18}$$

Consider the equation for the development of the energy,

$$\frac{d\dot{x}'_0}{ds} = \frac{e}{m}\,\dot{x}'^i F'_{i0}\,. \tag{8.19}$$

As we saw $\dot{x}'_0$ has the same value as the Lorentz quantity, $\dot{x}_{L0}$, similarly $\dot{x}'^i = \dot{x}^i_{\scriptscriptstyle L}$, $F'_{i0} = F_{Li0}$ hence, this equation may be written

$$\frac{d\dot{x}_{L0}}{ds} = \frac{e}{m}\,\dot{x}_{\scriptscriptstyle L} F_{Li0} \tag{8.20}$$

and is therefore identical to the corresponding equation as seen by the Lorentz observer. Consider now the equations

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\,\dot{x}'_\mu F'^{\mu i}; \tag{8.21}$$

they may be written

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\,\left(\dot{x}'_0 F'^{0i} + \dot{x}'_j F'^{ji}\right); \tag{8.22}$$

but as we saw in (8.13), $F^{0y}_{\scriptscriptstyle L} = F'^{0y} - v F'^{xy}$ which generalized for motion of the frame S, with velocity v_x, v_y, v_z, becomes $F^{0i}_{\scriptscriptstyle L} = F'^{0i} - v_j F'^{ji}$, but $F^{0i}_{\scriptscriptstyle L} = -F_{L0i} = -F'_{0i}$, hence

$$F'^{0i} = -F'_{0i} + v_j F'^{ji}; \tag{8.23}$$

(which may also be derived using $F'_{0i} = g'_{0\mu} g'_{iv} F'^{\mu\nu}$) so that the above equation may be written

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\left(-\dot{x}'_0 F'_{0i} + \left(\dot{x}'_0 v_j + \dot{x}'_j\right) F'^{ji}\right) \tag{8.24}$$

but $\dot{x}'^j = g'^{j\mu}\dot{x}'_\mu = -v\dot{x}'_0 - \dot{x}'_j$, hence

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\left(-\dot{x}_0 F'_{0i} - \dot{x}'^j F'^{ji}\right), \tag{8.25}$$

and now noting $F'_{0i} = -F_L^{0i}$, $\dot{x}'^j = \dot{x}'^j_L = -\dot{x}_{Lj}$, the equation is identical to

$$\frac{d\dot{x}'^i_L}{ds} = \frac{e}{m}\,\dot{x}_{L\mu}F_L^{\mu i} \tag{8.26}$$

and our original observation is proved.

The importance of this result is that it means provided an observer in S' makes measurements of velocity using light signals or slowly moving clocks, he always arrives at the same equations of motion as the Lorentz observer at rest with respect to S'; on the other hand, if he makes observations using absolute signals, he can arrive at a second set of equations of motion, namely those given by

$$\left.\begin{aligned}\frac{d\dot{x}'_i}{ds} &= \frac{e}{m}\,\dot{x}'^{\mu}F'_{\mu i}\\ \frac{d\dot{x}'^0}{ds} &= \frac{e}{m}\,\dot{x}'_{\mu}F'^{\mu 0}\end{aligned}\right\} \tag{8.27}$$

which do *not* reduce to the equations of motion as seen by the Lorentz observer. With current techniques, these equations are "unobservables", since they involve knowledge of the absolute velocity of the frame.

Finally, we note that in the presence of an electromagnetic field the conditional invariance relation on the momentum under the A.L.T. (7.18) becomes

$$\left(p'_0 - eA'_0\right)^2 - \left(p'^i - eA'^i\right)^2 = m^2. \tag{8.28}$$

§8.3. Unobservability of a correction to the wave number under the A.L.T.

In the discussions given in previous chapters it was shown that there were no effects to be expected due to the asymmetric propagation of light in S' because of the way in which measurements are made. This was done using the line element and observing the cancellation in the out-and-back slowness. It is also possible to give an analogous discussion from a wave standpoint working with the D'Alembertian equation in S'. One has

$$\left[\left(1-v^2\right)\frac{\partial^2}{\partial t'^2} - 2v\frac{\partial^2}{\partial x'\partial t'} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}\right]A'^i = 0\,, \tag{8.29}$$

and one looks for plane wave solutions of the form, $\exp \pm i\left(k'_\mu x'^\mu\right)$, the k'_μ satisfying

$$\left(1-v^2\right)k'^2_0 - 2vk'_0k'_x - k'^2_x - k'^2_y - k'^2_z = 0\,. \tag{8.30}$$

This expression may be diagonalized by using $k'_x = -v\,k'_0 - k'^x$, so that (upon introducing $k'^y = -k'_y$, $k'^z = -k'_z$), the above reduces to the usual conditional invariance relation,

$$k'^2_0 - (k'^x)^2 - (k'^y)^2 - (k'^z)^2 = 0\,. \tag{8.31}$$

And hence for the phase we may write

$$k'_0\,t' - (v k'_0 + k'^x)\,x' - k'^y\,y' - k'^z\,z'. \tag{8.32}$$

By introducing the local time $t_{\scriptscriptstyle L} = t' - vx'$, the above expression becomes the usual relativistic one,

$$k'_0\,t_{\scriptscriptstyle L} - k'^x\,x' - k'^y\,y' - k'^z\,z' = k^0_{\scriptscriptstyle L}\,t_{\scriptscriptstyle L} - k^i_{\scriptscriptstyle L}\,x^i_{\scriptscriptstyle L} \tag{8.33}$$

with the previously noted identity between $(k'_0,\, k'^i)$, and the corresponding Lorentz quantities $(k^0_{\scriptscriptstyle L},\, k^i_{\scriptscriptstyle L})$ in the case of particle dynamics.

It is now our purpose to show that even though one has an expression for the phase given by (8.32), that in any typical measurement involving this phase, the quantity vk'_0x' cancels out, so that effectively one is dealing with the Lorentz expression (8.33). The argument is a trivial extension of ones given previously.

Thus consider a typical interference experiment involving two beams of light. They will each propagate from an initial point $P'_1(x'_1, y'_1, z'_1)$, where they were initially in phase, via two different paths, to a final point $P'_2(x'_2, y'_2, z'_2)$ where a phase comparison is to be made. Then we have for their respective phases along the two paths

$$\left.\begin{aligned} \text{Path 1:}\quad & \int\limits_{P'_1}^{P'_2} (v k'_0 + k'^x)\,dx' + k'^y\,dy' + k'^z\,dz' \\ \text{Path 2:}\quad & \int\limits_{P'_1}^{P'_2} (v k'_0 + k'^x)\,dx' + k'^y\,dy' + k'^z\,dz' \end{aligned}\right\}. \tag{8.34}$$

And we see that in the phase difference, the term of interest,

$$\int\limits_{P'_1}^{P'_2} v k'_0\,dx' - \int\limits_{P'_1}^{P'_2} v k'_0\,dx' = v k'_0 \oint dx' \tag{8.35}$$

reduces to a line integral around a closed contour and hence vanishes, leaving the customary relativistic expression for the phase difference.

In the above we have assumed the light paths to be in vacuum, however the interposition of refractive media causes no difficulty. Taking the inverse of the metric tensor given in (5.1), the D'Alembertian equation on these paths becomes

$$\left[\left(1-\frac{v^2}{n^2}\right)\frac{\partial^2}{\partial t'^2}-\frac{2v}{n^2}\frac{\partial^2}{\partial t'\partial x'}-\frac{1}{n^2}\frac{\partial^2}{\partial x'^i\partial x'^i}\right]A'^i=0. \tag{8.36}$$

Proceeding as before, the wave numbers satisfy

$$\left(1-\frac{v^2}{n^2}\right)k_0'^2-\frac{2v}{n^2}\,k_0'k_x'-\frac{1}{n^2}\,(k_i')^2=0\,. \tag{8.37}$$

Using $k'^\mu=g'^{\mu\nu}k'_\nu$, one has $k'_x=-vk'_0-n^2k'^x$, $k'_y=-n^2k'^y$, $k'_z=$ $=-n^2k'^z$, and the above may be written, $k_0'^2-n^2k'^ik'^i=0$. Hence nk'^i, rather than k'^i represents the wave number in vacuum. Denoting nk'^i by $\underline{k}'^i$, they satisfy $k_0'^2-\underline{k}'^i\underline{k}'^i=0$, and the phase may be written

$$k_0't'-\left(vk_0'+n\underline{k}'^x\right)x'-n\underline{k}'^yy'-n\underline{k}'^zz', \tag{8.38}$$

and as before, $\int vk_0'dx'$ vanishes along a closed path leaving the usual expression. This result may be also derived noting that for a Lorentz observer, the phase in a refractive medium is, $k_L^0t_L-n\underline{k}_L^ix_L^i$, where $\underline{k}_L^i$ are the vacuum wave numbers, hence setting $t_L=t'-vx'$ and making the appropriate correspondences, the result follows.

It is interesting to note that what a Lorentz observer describes to be a plane wave propagating perpendicular to the x' axis (say in the y' direction) with $k_L^x=k'^x=0$, an A.L.T. observer using absolute signals describes as propagating in a direction tilted with respect to the y' axis and with wave number vk_0' along the x' axis. This follows immediately from the expression for the phase, but it is interesting to give a physical reason for this result. Now a Lorentz observer would declare the relative phase of two portions of a wave front crossing the x' axis to be the same if, as they crossed say at $\frac{-\Delta x'}{2}$, $\frac{\Delta x'}{2}$, they each triggered a device which sent light signals to the origin, one from the left, one from the right, which arrived simultaneously. But as we have seen, the slowness of the light signal propagating from the right to the origin is $(1-v)$, and slowness from the left to the origin is $(1+v)$, hence two signals arriving simultaneously correspond to a time difference of $(1-v)\frac{\Delta x'}{2}-(1+v)\frac{\Delta x'}{2}$, and a phase difference of $k_0'\left[(1-v)\frac{\Delta x'}{2}-(1+v)\frac{\Delta x'}{2}\right]=-vk_0'\Delta x'$, as indicated above. Thus for

an A.L.T. observer it is necessary for $-k'_x = k'^x + v k'_0$ to vanish to have transverse propagation; one then has, $k'^x = -v k'_0$ also using (8.30) $k'^y = \frac{1}{\gamma} k'_0$, so that a relativistic observer would declare the wave is propagating with direction $\frac{k^x_L}{k^y_L} = \frac{k'^x}{k'^y} = -v\gamma$.

§8.4. Transformation of energy-momentum and angular momentum tensors

Let us now consider the transformation properties of the energy-momentum tensor $T^{\mu\nu}$ which for the electromagnetic field is given by $T^{\mu\nu} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}F_{\lambda\rho}F^{\lambda\rho}g^{\mu\nu}$. However, in the following, for generality, we shall consider $T^{\mu\nu}$ to be an arbitrary energy-momentum tensor that in the absolute frame has the properties of being symmetric and satisfying the conservation law $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0$. Then since these are tensor properties they also hold in the A.L.T. frame,

$$\left.\begin{aligned} T'^{\lambda\mu} &= a^{\lambda}_{\nu} a^{\mu}_{\rho} T^{\nu\rho} = a^{\mu}_{\rho} a^{\lambda}_{\nu} T^{\rho\nu} = T'^{\mu\lambda} \\ \frac{\partial T'^{\lambda\mu}}{\partial x'^{\mu}} &= \bar{a}^{\nu}_{\mu} a^{\lambda}_{\rho} a^{\mu}_{\varsigma} \frac{\partial T^{\rho\varsigma}}{\partial x^{\nu}} = a^{\lambda}_{\rho} \frac{\partial T^{\rho\nu}}{\partial x^{\nu}} = 0 \end{aligned}\right\}. \tag{8.39}$$

It follows from these two properties that angular momentum will be conserved in the A.L.T. frame. Define the generalized angular momentum density $\mathfrak{M}'^{\mu\lambda\nu}$ about the origin to be

$$\mathfrak{M}'^{\mu\lambda\nu} \equiv x'^{\mu} T'^{\lambda\nu} - x'^{\lambda} T'^{\mu\nu}, \tag{8.40}$$

then

$$\frac{\partial \mathfrak{M}'^{\mu\lambda\nu}}{\partial x'^{\nu}} = T'^{\lambda\mu} - T'^{\mu\lambda} = 0\,. \tag{8.41}$$

Since this derivation does not depend on the coordinate system (although we are here only working in Cartesian frames), this conservation law is a typical example of a result which follows from general covariance rather than one which follows from working in the restricted (Lorentz) coordinate frames of special relativity.

The tensors $T'_{\mu\nu}$, $T'^{\mu\nu}$, T'^{μ}_{ν} are related to the tensors in the corresponding Lorentz frame by

$$\left.\begin{aligned} T_{L\mu\nu} &= \bar{\ell}^{\rho}_{\mu} \bar{\ell}^{\lambda}_{\nu} T'_{\rho\lambda} \\ T^{\mu\nu}_{L} &= \ell^{\mu}_{\rho} \ell^{\nu}_{\lambda} T'^{\rho\lambda} \\ T^{\nu}_{L\mu} &= \bar{\ell}^{\lambda}_{\mu} \ell^{\nu}_{\rho} T'^{\rho}_{\lambda} \end{aligned}\right\}. \tag{8.42}$$

Obtaining ℓ^{μ}_{ν}, $\bar{\ell}^{\mu}_{\nu}$, from $dt_{L} = dt' - v_j dx'^j$, $dx^j_{L} = dx'^j$, one has

$$\left.\begin{aligned}
&T_{L00} = T'_{00}\\
&T_{L0i} = T'_{0i} + v_i T'_{00}\\
&T_{Lij} = T'_{ij} + v_i T'_{0j} + v_j T'_{0i} + v_i v_j T'_{00}\\
&T^{00}_{L} = T'_{00} - 2v_i T'^{i0} + v_i v_j T'^{ij}\\
&T^{0i}_{L} = T'^{0i} - v_j T'^{ji}\\
&T^{ij}_{L} = T'^{ij}\\
&T^{0}_{L0} = T'^{0}_{0} - v_i T'^{i}_{0}\\
&T^{i}_{L0} = T'^{i}_{0}\\
&T^{0}_{Li} = T'^{0}_{i} + v_i T'^{0}_{0} - v_j T'^{j}_{i} - v_i v_j T'^{j}_{0}\\
&T^{i}_{Lj} = T'^{i}_{j} + v_j T'^{i}_{0}
\end{aligned}\right\}. \tag{8.43}$$

Because of the general relations $T'^{\mu\nu} = g'^{\mu\lambda} g'^{\nu\rho} T'_{\lambda\rho}$, $T'^{\mu}_{\nu} = g'^{\mu\lambda} T'_{\lambda\nu}$ there are actually only ten linearly independent components to the stress tensor. As can be seen from the above, the ten components which are to be identified with the quantities measured by the Lorentz observer are T'_{00}, T'^{i}_{0}, T'^{ij}; the other twenty-six components $T'_{\mu j}$, $T'^{0\mu}$, T'^{0}_{μ} ,T'^{i}_{j}, except for special cases of symmetry, are unobserveables unless measurements are made with absolute signals. It will be noted that while $T'_{\mu\nu}$, $T'^{\mu\nu}$ are both symmetric tensors T'^{μ}_{ν} does not have this property. In the Lorentz frame one has $T^{0}_{Li} = -T^{i}_{L0}$, $T^{i}_{Lj} = T^{j}_{Li}$ while for T'^{μ}_{ν}, from above

$$T'^{i}_{j} - T'^{j}_{i} = v_i T'^{j}_{0} - v_j T'^{i}_{0}\,, \tag{8.44}$$

and using $T'^{i}_{0} = g'^{i\lambda} T'_{\lambda 0}$, $T'^{0}_{i} = g'^{0\lambda} T'_{\lambda i}$,

$$\left.\begin{aligned}
&T'^{i}_{0} = -v_i T'_{00} - T'_{i0}\\
&T'^{0}_{i} = \left(1 - v^2\right) T'_{0i} - v_j T'_{ji}
\end{aligned}\right\}. \tag{8.45}$$

In the limiting case $v_i = 0$, the relations that hold on the mixed tensor in the absolute frame and the Lorentz frame follow.

The momentum of the field as measured by the Lorentz observer is $P^{\mu}_{L} = \int T^{\mu}_{L0}\, d^3 x_{L}$, and since $d^3 x_{L} = d^3 x'$, $T^{j}_{L0} = T'^{j}_{0}$, $T^{0}_{L0} = T'_{00}$, this corresponds to the A.L.T. quantities

$$P'_{0} = \int T'_{00}\, d^3 x', \qquad P'^{i} = \int T'^{i}_{0}\, d^3 x' \tag{8.46}$$

in analogy to the correspondence found in the particle case. Consider now, for simplicity, the field to be sufficiently localized (that of a particle) so that the total angular momentum about the origin may be written

$$M'^{ij} = x'^i p'^j - x'^j p'^i; \tag{8.47}$$

because of the above correspondence, this is the same as that obtained by the Lorentz observer, that is,

$$M'^{ij} = M_L^{ij}\,. \tag{8.48}$$

On the other hand, employing the relations $p'^i = -p'_i - v_i p_0$, $x'^i = -x'_i - v_i x'_0$, one obtains

$$M'^{ij} = M'_{ij} - x'_0\left(v_i p'^j - v_j p'^i\right) - p'_0\left(x'^i v_j - x'^j v_i\right) \tag{8.49}$$

whereas for the Lorentz observer $M_L^{ij} = M_{Lij}$.

Thus, in summary, we have shown by explicit calculation that in an A.L.T. frame there always exists a set of tensors which are identical (apart from label) to a corresponding set obtained in the Lorentz frame. Consequently, it is always possible for the A.L.T. observer to write his equations in the same form employing the same quantities as the Lorentz observer. On the other hand, the A.L.T. observer finds there are additional quantities involving the absolute motion of the frame. The fact that the A.L.T. observer finds there are these additional quantities is an expression of the possibility he has for another method of measurement (based on absolute synchronization) not available to the Lorentz observer, the results of which will not in general yield the same value as for the Lorentz observer. This was seen most clearly in connection with the one-way velocity of light. On the other hand, as we also saw in this connection, when the A.L.T. observer performs his measurements in the same way as the Lorentz observer, he obtains the same results — indeed this formed the basis for the derivation of the A.L.T.

Chapter 9. Kinematic Implications of Superlight Signals for Relativistic Causality

The basic idea of causal propagation is that a disturbance from a measurement propagates only forward in time. In order to violate causal propagation it would be necessary to have "signals" that propagate backwards in time. As we shall see, however, this condition is not sufficient due to the different ways in which various observers define time. For brevity, signals that propagate backwards in time will be called

"acausal", independently of whether or not they actually lead to a violation of causality.

It will be shown that when there are faster-than-light signals present, they can appear to an infinite class of Lorentz observers as exhibiting this acausal propagation, although for an A.L.T., observer this is not the case. However it will also be shown that if two measurements do not interfere for an A.L.T. observer, they do not interfere for a Lorentz observer, so that the latter cannot use such acausal signals to influence events before they occurred. In the language of general relativity one would say this acausal propagation is due to an improper choice of coordinate system, namely: the use of the Lorentz local time. Indeed one can always in a given frame make any signal propagating forwards in time propagate backwards in time by redefining time, say $T = t - qx$, and choosing q sufficiently large; this will be discussed in detail anon.

Let us now temporarily confine our attention to the absolute frame so that observers using the ordinary Lorentz transformation, the A.L.T. and the Galilean transformation all agree that the velocity of light is the same in all directions and of magnitude unity. Consider two measurements being made on the x axis (for convenience) one at the point x_0 at time t_0, and the other at $x_1 > x_0$ and at time $t_1 > t_0$. Then in order for a light signal emitted at (x_0, t_0) to be unable to interfere with the measurement at (x_1, t_1), it is necessary that

$$t_1 - t_0 < x_1 - x_0 \tag{9.1}$$

or more generally, $t_1 - t_0 < |\vec{x}_1 - \vec{x}_0|$, that is, the interval must be space-like, to guarantee non-interference of an earlier measurement with a later measurement. This is the "relativistic causality" assumption. But actually it contains two distinct assumptions, namely:

1. There are no signals that travel faster than light forward in time;
2. There are no signals that travel backward in time.

For, if merely 1 were satisfied, a signal violating 2 could leave the later measurement at (x_1, t_1) and arrive in time to interfere with the earlier measurement at (x_0, t_0).

Let us now suppose that 1 is violated and there are indeed superlight signals available, of constant velocity v_s in the absolute frame S, but that there are no acausal signals present. Under these circumstances, the assumption that the interval be space-like to guarantee non-interference is no longer sufficient. A Lorentz observer, an A.L.T. observer and a Galilean observer at rest in S would all agree that the

above requirement is to be replaced by

$$t_1 - t_0 < \frac{1}{v_s}\,(x_1 - x_0)\,, \tag{9.2}$$

and more generally,

$$t_1 - t_0 < \frac{1}{v_s}\,|\vec{x}_1 - \vec{x}_0|\,. \tag{9.3}$$

An interesting situation now arises in the limit $v_s \to \infty$, since the condition for non-interference reduces to the requirement $t_1 - t_0 < 0$, that is the measurement at (x_1) was earlier in time than the measurement at (x_0). This is in contradiction with our initial assumption that $t_1 > t_0$. Indeed, if we allow t_1 to be earlier than t_0, the former measurement could then have influenced the latter measurement, and hence the requirement (9.2) would no longer be applicable in guaranteeing non-interference. In other words we are actually working with a set of inequalities

$$0 < t_1 - t_0 < \frac{1}{v_s}\,(x_1 - x_0)\,. \tag{9.4}$$

The first inequality, to guarantee the measurement at t_1 does not interfere with the one at t_0 (based on assumption 2), and the second inequality to prevent interference of the earlier measurement with the later measurement by putting it outside the superlight cone. Clearly the limiting case $v_s \to \infty$ does not belong to the set.

On the other hand, if instead of the above we employ

$$0 \leqslant t_1 - t_0 \leqslant \frac{1}{v_s}\,(x_1 - x_0)\,, \tag{9.5}$$

we do not arrive at a self-contradictory requirement in the limit. Hence, if we wish to include the absolute signal as a limiting case of causal propagation, the conditions for non-interference of a measurement at t_1 with one at t_0 is that the former measurement be later than or simultaneous with the latter measurement, $t_1 \geqslant t_0$, rather than simply $t_1 > t_0$ as we have been working with above. Since the inclusion of the equality sign may seem paradoxical, it is necessary to include a stipulation that the "effect" or disturbance occur after the arrival of the initiating causal impulse. For example, in a classical case, the effect or disturbance might be a pointer displacement, the cause, a force producing a unit acceleration commencing at time $t = 0$. Then the displacement is $d = \frac{1}{2}t^2$ and there is no displacement until $t > 0$. For simplicity in what follows, v_s will be taken to be finite although arbitrarily large.

With the above qualification on v_s, let us now return to the requirement (9.2) and ask the question, does this requirement guarantee that two measurements which did not interfere in the absolute frame will not interfere for the several observers in the moving frame? Moreover, if a signal propagates causally in the absolute frame, will it propagate causally in the moving frame? We shall consider three cases: Galilean observer, A.L.T. observer, and Lorentz observer.

Case I: Galilean observer. The transformation notation is as before, $x_g = x - vt$, $t_g = t$. The elapsed time $\Delta t(v_s)$ for the signal in the unprimed frame to travel the distance $x_1 - x_0$ is $\frac{x_1 - x_0}{v_s}$; hence the Galilean observer describes the signal as having occupied the same (positive) interval of time

$$\Delta t_g(v_s) = \Delta t(v_s) = \frac{x_1 - x_0}{v_s}\,. \tag{9.6}$$

The time interval between measurements is

$$\Delta t_g(1,0) = \Delta t(1,0) = t_1 - t_0\,. \tag{9.7}$$

Hence by our original assumptions $t_1 - t_0 < \frac{x_1 - x_0}{v_s}$, it follows

$$\Delta t_g(1,0) - \Delta t_g(v_s) < 0\,, \tag{9.8}$$

and the measurements do not interfere; also, the signal is clearly causal, since it traverses the distance, $\Delta x_g(v_s) = (x_1 - x_0) - \frac{v\,(x_1 - x_0)}{v_s} = (x_1 - x_0) \times$ $\times \left(1 - \frac{v}{v_s}\right)$, in a positive interval of time.

Case II: The A.L.T. observer. The above argument is basically unchanged. One has, for the signal,

$$\Delta t'(v_s) = \frac{x_1 - x_0}{v_s\,\gamma} \tag{9.9}$$

and for the time interval between measurements,

$$\Delta t'(1,0) = \frac{t_1 - t_0}{\gamma} \tag{9.10}$$

and hence,

$$\Delta t'(1,0) - \Delta t'(v_s) < 0 \tag{9.11}$$

and the measurements do not interfere. Moreover the signal is causal since the time interval is positive to traverse the distance $\Delta x'(v_s) =$ $= \Delta x_g \gamma$.

Case III: The Lorentz observer. Proceeding as above, the time intervals are

$$\Delta t_{L}(v_s) = \left[\frac{x_1 - x_0}{v_s} - v\,(x_1 - x_0)\right]\gamma \tag{9.12}$$

$$\Delta t_{L}(1,0) = \left[(t_1 - t_0) - v\,(x_1 - x_0)\right]\gamma \tag{9.13}$$

and once again, $\Delta t_{L}(1,0) - \Delta t_{L}(v_s) < 0$.

It will be noted however, that for an infinite class of Lorentz observers it is possible to choose v such that $vv_s > 1$, and the time interval $\Delta t_{L}(v_s)$ becomes negative. Hence the above class of Lorentz observers declare the superlight signal to have propagated acausally. However if $vv_s > 1$, since $\frac{x_1 - x_0}{t_1 - t_0} > v_s$, $\frac{v(x_1 - x_0)}{t_1 - t_0} > vv_s$, $\Delta t_{L}(1,0)$ is also negative, and the Lorentz observer asserts that the measurement at x_1 took place *earlier* than the measurement at x_0. Moreover since $\Delta t_{L}(1,0) < \Delta t_{L}(v_s)$, he would then say the superlight acausal signal did not propagate backwards in time sufficiently fast to influence the earlier measurement. Thus, insofar as genuinely violating causality is concerned, this "acausal" propagation of the superlight signal is spurious, and arises only because of the method of synchronization employed by the Lorentz observer.

To complete the above discussion, it is necessary to show that when v is chosen such that the later event is mapped onto an earlier event, so that $\Delta t_{L}(1,0)$ is negative, a superlight signal that was emitted from the event at $(x_1,\, t_1)$ cannot arrive before the event at $(x_0,\, t_0)$ as seen in the new Lorentz frame. Denoting by $\Delta t_{L}(v_s)$ the interval for the signal to propagate from right to left, one has

$$\Delta t_{L}(v_s) = \left[\frac{x_1 - x_0}{v_s} + v\,(x_1 - x_0)\right]\gamma \tag{9.14}$$

so that $\Delta t_{L}(v_s)$ is positive. Hence it is necessary to show $\Delta t_{L}(v_s) - \big(-\Delta t_{L}(1,0)\big) > 0$. Using the expression for $\Delta t_{L}(1,0)$ given in (9.13) one obtains

$$\Delta t_{L}(v_s) - \big(-\Delta t_{L}(1,0)\big) = \frac{x_1 - x_0}{v_s} + (t_1 - t_0) > 0 \tag{9.15}$$

and hence the signal arrives later.

Tolman [14] has given a discussion of this problem, but upon showing that a superlight signal can propagate backwards in time for a class of Lorentz observers, he inferred that such signals violate causality. As we have just seen this conclusion is invalid, since it is not sufficient to show

merely that such signals propagate backwards in time in the new Lorentz frame, one must also show that the signals interfere with measurements that, in the original frame, occurred before their arrival — which is not the case.

Let us now re-derive the above results working in the moving frame S' alone, thus giving us a more general proof that if two measurements do not interfere for an A.L.T. observer, they do not interfere for a Lorentz observer, although the latter will in general have to admit of acausal propagation to describe the propagation of causal superlight signals.

In the primed frame S', the A.L.T. observer describes the two measurements as having occurred at $(x'_0,\, t'_0)$ and $(x'_1,\, t'_1)$, with $t'_1 > t'_0 > 0$, since the latter measurement is by assumption later. A superlight signal leaves the earlier event and propagates to the point x'_1, with A.L.T. relative velocity u, moreover since the measurements did not interfere

$$t'_1 - t'_0 < \frac{x'_1 - x'_0}{u}\,. \tag{9.16}$$

The time interval for the signal to propagate is,

$$\Delta t'(u) = \frac{x'_1 - x'_0}{u}\,. \tag{9.17}$$

The local time interval for the propagation is,

$$\Delta t_{\scriptscriptstyle L}(u) = \Delta t' - v\Delta x' = \frac{x'_1 - x'_0}{u} - v\,(x'_1 - x'_0) \tag{9.18}$$

and the local time interval between measurements is

$$\Delta t_{\scriptscriptstyle L}(1,0) = (t'_1 - t'_0) - v\,(x'_1 - x'_0)\,. \tag{9.19}$$

Hence,

$$\Delta t_{\scriptscriptstyle L}(1,0) - \Delta t_{\scriptscriptstyle L}(u) = (t'_1 - t'_0) - \frac{x'_1 - x'_0}{u} < 0 \tag{9.20}$$

and the measurements did not interfere. It will be noted that the acausality condition is now $vu > 1$, but since $u = \frac{v_s - v}{1 - v^2}$, this is equivalent to $\frac{v_s v - 1}{1 - v^2} + 1 > 1$, and hence $vv_s > 1$, as before.

Alternatively, the above discussion might be carried out using the expressions for relative velocity in the moving frame. The Galilean relative velocity and the A.L.T. relative velocity both transform a superlight signal that was causal in the absolute frame into a causal signal in the moving frame. But the denominator of the relativistic relative velocity, $v_r = \frac{v_s - v}{1 - vv_s}$ changes sign for $vv_s > 1$, which does not mean the signal propagated in a reversed direction in positive time, but as we saw, the

signal propagated in a positive direction backwards in time, an interesting example of the ambiguity in velocity. For $vv_s = 1$, the Lorentz observer says the signal propagated with infinite velocity. Consider now the relation between v_r and the A.L.T. relative velocity u developed in preceding chapters. The Lorentz observer has corrected (or phased) his clocks so as to make the slowness of light unity by effectively subtracting $v\Delta x'$, and hence declares a signal of slowness $\frac{1}{u}$ to be of slowness $\frac{1}{v_r} = \frac{1}{u} - v$. For $vv_s < 1$, $\frac{1}{u} > v$, and the relativistic slowness $\frac{1}{v_r}$ is positive, for $v_s = \frac{1}{v}$, u also equals $\frac{1}{v}$ (interestingly enough), and $\frac{1}{v_r}$ vanishes, becoming negative for $v_s > \frac{1}{v}$, and hence $\frac{1}{u} < v$. Thus v_r has the character of a phase velocity in these regions, which can be made to run forward or backward in time by appropriate relative synchronization of clocks corresponding to the choice of Lorentz frame.

Chapter 10. The A.L.T. Line Element under Improper Transformations

As we have seen, given an A.L.T. frame S' with the general line element

$$ds^2 = dt'^2 - 2v_i\,dt'dx'^i - dx'^i dx'^i + v_i v_j\,dx'^i dx'^j$$

it is possible to pass, via the local time transformation, to the corresponding Lorentz frame S_L with line element $ds^2 = dt_L^2 - dx_L^i dx_L^i$. Now the metric tensor $\eta_{\mu\nu}$ of the Lorentz line element is invariant under the improper transformations T: $(t_L \to -t_L,\ x_L^i \to x_L^i)$, P: $(x_L^i \to -x_L^i,\ t_L \to t_L)$ and the problem is to study the behaviour of the A.L.T. line element under similar transformations.

Denoting the improper coordinate transformations by

$$\left.\begin{aligned} T' &: \left(t' \to -t',\ x'^i \to x'^i\right) \\ P' &: \left(x'^i \to -x'^i,\ t' \to t'\right) \end{aligned}\right\}, \tag{10.1}$$

one has that under either T' or P', the line element becomes

$$ds^2 = dt'^2 + 2v_i\,dt'dx'^i - dx'^i dx'^i + v_i v_j\,dx'^i dx'^j \tag{10.2}$$

so that in the reflected coordinate system $g'_{0i} \to -g'_{0i}$, the other components of the metric tensor remaining unchanged. Thus unlike the situation with the Lorentz observer, the metric tensor is not invariant under the improper transformations, the new line element being that for an A.L.T. frame translating with absolute velocity $-v_i$, without reflection of time or space. Let us examine how this lack of invariance would show up in a classical experiment performed in S'.

Consider first an experiment designed to check invariance under T'. An observer in S' sends a light signal from the origin in the direction θ' through a distance $\Delta\sigma'$ to a point A and measures the delay to be $\Delta t' = (1 + v\cos\theta')\,\Delta\sigma'$. Now in reversed time, the signal returned from A back to the origin in the direction $\theta'+\pi$. The expression for slowness in reversed time obtained from (10.2) is $\Delta t' = (1 - v\cos\theta')\,\Delta\sigma'$ and hence for $\theta' \to \theta' + \pi$, the delay is the same as on the outward journey. On the other hand, invariance under time reversal in a given frame means that the motion observed in the time reversed frame is a possible state of motion in the original frame before time reversal. Hence a light signal sent from A to the origin in the original frame should also exhibit the same delay as it did on its outward path, which is of course not the case, the delay being $\Delta t' = (1 + v\cos(\theta' + \pi))\,\Delta\sigma' = (1 - v\cos\theta')\,\Delta\sigma'$. Thus T' is not an invariance operation in the given frame S'.

The physical reason for this lack of invariance is clear: in reversed time not only did the light signal return from A to the origin but the frame itself also reversed its direction of absolute motion, whereas in the above experiment only the direction of motion of the light signal was reversed, not the frame. Thus in order to preserve overall invariance with respect to time reversal it is necessary to go outside the given frame and include the frame travelling with absolute velocity $-v_i$, the only exception being the absolute frame for which $v_i = 0$.

Similarly the parity transformation P': $(x'^i \to -x'^i,\ t' \to t')$ is not an invariance operation in the given frame: a clock slowly moved in the direction θ' does not read the same as a clock moved in the opposite direction $\theta' + \pi$. Once again, to obtain overall invariance with respect to parity one must include the frame travelling in the opposite direction with absolute velocity $-v_i$.

On the other hand, strong reversal, $T'P'$, does represent an invariant transformation, since the two operations have the effect of cancelling the asymmetry produced by the absolute motion so that the metric tensor is left unchanged. Thus unlike the Lorentz line element, for which T, P, TP represent invariant improper coordinate transformations, the A.L.T. line element possesses only one, $T'P'$. However the possibility of performing the local time transformation $t_{_L} = t' - v_i x'^i$ has the effect of restoring the full symmetry of the absolute frame to one in uniform motion.

The lack of invariance under P' in a given A.L.T. frame is of course of an entirely different character than the parity violations observed in the weak interactions. In the former case v_i is a polar vector and the correlated asymmetry in the propagation of light is likewise polar,

whereas in the weak interactions one has a correlation between an axial vector and a polar vector. To obtain the latter kind of correlation on the basis of the metrical structure of the line element it would be necessary that g'_{0i} be an axial vector, so that light propagated with different slownesses, for example, parallel and anti-parallel to the "direction" of an axial vector.

Chapter 11. Invariance under the Local Time Transformation

Although it has been shown that for the usual classical-mechanical and electromagnetic type of experiments, the absolute velocity v cancels out in a typical measurement, so that utilization of the A.L.T. does not lead to any contradictions, the question arises as to what are some of the effects to be expected at the quantum level. It will be shown that in a given Lorentz frame, upon resynchronizing the clocks absolutely, by means of the local time transformation, so as to transform to the A.L.T. frame, the Schrödinger state function undergoes a unitary transformation so that the measureables of the two observers are the same. (The method of proof, however, will lead to a result of somewhat greater generality which will be discussed below.) That such a unitary transformation should exist follows on general principles from the fact that as was shown in Chapter 7, the energy-momentum relation for an A.L.T. observer satisfies the conditional in variance relation $p'^2_0 - p'^j p'^j = m^2$, and as was shown in Chapter 8, the equations of motion for an A.L.T. observer can be written in a form involving the same quantities in the same way as for a Lorentz observer.

Consider first the Schrödinger representation in a given Lorentz frame. In this representation, the state vector satisfies the equation, for units in which $\hbar = 1$,

$$i \frac{\partial}{\partial t_{_L}} \Psi(t_{_L}) = H \Psi(t_{_L}) \,, \tag{11.1}$$

where the Hamiltonian H is a time independent Hermitian operator whose relativistic transformation properties will discussed below. Choose a representation for the Ψ's in which H is diagonal and the Ψ's are the energy eigenstates of H, then

$$i \frac{\partial}{\partial t_{_L}} \Psi_{_E}(t_{_L}) = E \Psi_{_E}(t_{_L}) \,. \tag{11.2}$$

Consider now a transformation from the Lorentz frame to the A.L.T. frame, employing $t_{_L} = t' - v_j x'^j$, $x^j_{_L} = x'^j$, then

$$\frac{\partial}{\partial t_{_L}} \to \frac{\partial}{\partial t'} \,, \qquad \Psi_{_E}(t_{_L}) \to \Psi_{_E}(t' - v_j x'^j) \,. \tag{11.3}$$

But since the Ψ_E are energy eigenstates, their time dependence is of the form $\exp(-iEt_L)$ and hence under the above time transformation,

$$\Psi_E(t_L) = U\Psi_E(t'), \qquad U \equiv \exp\left(iEv_j x'^j\right). \tag{11.4}$$

Hence the Schrödinger equation becomes

$$i\frac{\partial}{\partial t'}\Psi_E(t') = U^{-1}EU\Psi_E(t') = E\Psi_E(t') \tag{11.5}$$

and the energy levels and eigenfunctions are the same for the A.L.T. observer as for the Lorentz observer.

It is interesting to note that the proof of (11.5) did not rely on the relativistic properties of H, all that was required was that H be a time independent operator and that there exist stationary solutions of the form: $\Psi_E(t_L) = \exp(-iEt_L)\,\Phi_E$, with $\frac{\partial\Phi_E}{\partial t_L} = 0$, $H\Phi_E = E\Phi_E$. Invariance under the local time transformation is therefore an extremely fundamental property of Schrödinger-type equations which deserves to be further exploited. On the other hand, when we are not dealing with a Lorentz invariant system, upon performing the local time transformation, we are not of course transforming to an A.L.T. frame, since the concepts of Lorentz frame and A.L.T. frame are no longer defined; we are then simply transforming from a frame with coordinates labelled x_L^μ to one with coordinates labelled x'^μ.

So far we have confined our remarks to systems in an eigenstate of energy with H diagonal, when this is not the case, the generalization of U is the displacement operator, which may be formally represented by

$$U = \exp\left(-v_j x'^j \frac{\partial}{\partial t'}\right). \tag{11.6}$$

We arrive at such an operator by looking for a generalization of (11.4), namely: an operator for which the following holds

$$\Psi\left(t' - v_j x'^j\right) = U\Psi(t') \tag{11.7}$$

even when Ψ is not eigenstate of energy. Although a great many interesting mathematical questions occur in connection with such an operator, its use here will be justified by showing for the space with which we are working U is unitary. This is done rather simply by first noting that our above analysis has actually given us the unitary "eigenvalues" of U, that is,

$$U\Psi_E(t') = \mathrm{e}^{iEv_j x'^j}\,\Psi_E(t'). \tag{11.8}$$

Now in order to establish unitarity, we must show that U preserves the "lengths" of vectors: $\|U\Psi\| = \|\Psi\|$. But since U clearly leaves invariant the lengths of the orthogonal base vectors Ψ_E, which by completeness span the space, U is unitary and $U^\dagger = U^{-1} = \exp\left(v_j\, x'^j \frac{\partial}{\partial t'}\right)$.

Thus we have in general that under the above assumptions, under the local time transformation,

$$i\,\frac{\partial}{\partial t'}\,\Psi(t') = U^{-1} H U \Psi(t') = H\,\Psi(t')\,. \tag{11.9}$$

We may see more explicitly how U acts by expressing H in the form $H\left(\frac{\partial}{\partial x_L^j},\, x_L^j\right)$ and noting that under the local time transformation H becomes $H\left(\frac{\partial}{\partial x'^j} + v_j \frac{\partial}{\partial t'},\, x'^j\right)$ but since $\left(\frac{\partial}{\partial x'^j} + v_j \frac{\partial}{\partial t'}\right)^n U\Psi = U\left(\frac{\partial}{\partial x'^j}\right)^n \Psi$, we have, assuming we can expand H in a power series, $H\left(\frac{\partial}{\partial x'^j} + v_j \frac{\partial}{\partial t'},\, x'^j\right) U$ $= U\, H\left(\frac{\partial}{\partial x'^j},\, x'^j\right)$ and the result (11.9) follows.

As an illustration of the above in the relativistic case, we consider the Dirac equation $\left(-i\gamma^\mu \frac{\partial}{\partial x_L^\mu} + m\right)\Psi(x_L^\mu) = 0$, since the invariance of the Klein-Gordon equation is immediate. Transforming to the A.L.T. frame under the local time transformation, the Dirac equation becomes

$$\left[-i\left(\gamma^0 + \gamma^j v_j\right)\frac{\partial}{\partial t'} - i\,\gamma^j \frac{\partial}{\partial x'^j} + m\right]\Psi\left(t' - v_j x'^j,\, x'^j\right) = 0\,. \tag{11.10}$$

It will be noted that if we define $\gamma'^0 \equiv \gamma^0 + \gamma^j v_j$, $\gamma'^j \equiv \gamma^j$ they satisfy

$$\left.\begin{aligned} &\gamma'^\mu \gamma'^\nu + \gamma'^\nu \gamma'^\mu = 2\,g'^{\mu\nu} \\ &g'^{00} = \left(1 - v^2\right), \quad g'^{j0} = -v_j\,, \quad g'^{jk} = -\delta_{jk} \end{aligned}\right\} \tag{11.11}$$

as would be the case if we had formulated the equation in the A.L.T. frame directly. Proceeding as above, we set $\Psi\left(t' - v_j x'^j,\ x'^j\right) =$ $= U\,\Psi(t', x'^j)$, hence since

$$-\,i\,\gamma^j \frac{\partial}{\partial x'^j}\, U\Psi\left(t', x'^j\right) = U\left(-\,i\,\gamma^j \frac{\partial}{\partial x'^j} + i\,v_j\,\gamma^j \frac{\partial}{\partial t'}\right)\Psi\left(t', x'^j\right), \tag{11.12}$$

we obtain finally $\left(-i\gamma^\mu \frac{\partial}{\partial x'^\mu} + m\right)\Psi(x'^\mu) = 0$.

Let us now observe in connection with the above that after making the unitary transformation, the spatial momenta obtained from $i\frac{\partial}{\partial x'^j}$ are *not* p'_j but $-p'^j$ and are therefore the Lorentz observer's covariant momenta p_{Lj}. Before performing the unitary transformation on Ψ for a plane wave state Ψ_E one has

$$p'_j\,\Psi_E = i\,\frac{\partial}{\partial x'^j}\,\Psi_E = -\left(p'^j + v_j E\right)\Psi_E\,. \tag{11.13}$$

The unitary transformation is therefore a method for eliminating the "unobservable" components of the A.L.T. covariant spatial momenta, i.e., $-v_j E$. Alternatively stated, from the standpoint of quantum mechanics, the reason that they are unobservable is that they can be eliminated by a unitary transformation. Thus a breakdown of some one or more of the assumptions we have employed here would be necessary to make v an observable.

The possibility of introducing an absolute time and ether velocity into quantum field theory has been discussed by Dirac [15] in connection with a reformulation of electrodynamics.

Chapter 12. Conclusions

The preceding analysis shows that the experimental results of special relativity may be obtained without imposing the usual requirement that the line element be the same in all uniformly moving frames. Rather, we may employ the A.L.T. line element which leads to an asymmetry in the propagation of light, depending on the absolute velocity of the frame. As we have seen in the various examples presented, this absolute velocity always cancels when measurements are performed in the usual manner. Under these circumstances, from the standpoint of mathematical simplicity, it is advantageous to further introduce the local time transformation, since the results do not depend on the absolute synchronization of separated clocks. Thus the final diagonalization of the line element in the moving frame appears as a convenient but unnecessary step.

On the other hand, one might legitimately raise the question: if this velocity relative to an absolute frame were to always cancel out, would it and the absolute frame have any physical significance? Certainly it would be unsatisfactory to introduce these concepts, together with others employed here such as instantaneous synchronization, superlight signals, etc., in order to justify certain intuitive ideas about the propagation of light, and then show that they play no role in physical phenomena.

At the present time this unsatisfactory situation does seem to exist insofar as uniformly moving frames are concerned; but if we consider phenomena in rotating frames, as is discussed in the Appendix, the situation is somewhat different. As is pointed out there, general relativity does not entail Mach's principle, without which, inertia must be regarded as being relative to space rather than the "fixed stars", and hence, rotation as absolute, rather than merely relative to these fixed stars, indeed, throughout the preceding discussion, we have worked with

solutions to $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$, that is, a space in which the effect of other matter is vanishingly small. Nevertheless, this did not prevent us from assigning inertia to a particle, and propagation properties to light, which would not have been possible if Mach's principle were contained in the theory. Under these circumstances, one has no choice but to regard the $g_{\mu\nu}$ as representing a description of space-time itself, as manifested in the propagation of light, the behaviour of rods and clocks, and the inertial properties of bodies.

Once this view of the $g_{\mu\nu}$ is accepted, the objections raised above are considerably lessened but not eliminated, since one still has the problem of how uniform motion relative to space is to be measured and the absolute velocity thereby determined. As we have seen, this determination could be made if, in the simplest case, there were signals propagating with arbitrarily large velocities. The well known arguments for excluding such signals are based on the following: Since the energy of a particle increases according to $m\gamma$, it would require infinite energy even to achieve to the speed of light, while beyond the speed of light the energy would become imaginary — both of which are physically untenable requirements. However, while the first objection is certainly valid in classical mechanics, where, to produce a particle travelling faster than light, one would first have to accelerate it through the speed of light, the situation is somewhat different in quantum field theory. For in this case, one can conceive of the possibility of creating, via a collision process, particles (e.g. a pair) that are already in the faster-than-light region. Thus the infinite energy at the speed of light would then divide the spectrum of particles with non-zero mass into two classes: those travelling with $v < 1$, and those with $v > 1$.

It is therefore the second objection, the imaginary energy for $v > 1$, that represents the serious problem. From a classical standpoint, this imaginary energy can be avoided by transferring to the space-like branch of the energy momentum relations as seen in the absolute frame S. Thus we formally define the line element for a particle moving $v > 1$ to be

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2, \tag{12.1}$$

where the coordinates of the particle are measured in S. Then ds^2 is real for $v > 1$, and the energy and momentum satisfy $p^2 - E^2 = m^2$ and are also real. Also setting $ds^2 = 0$, yields, necessarily, the same propagation properties for light in S as the time-like definition. (It should be noted that frequently in the literature (12.1) appears in connection with the usual relativistic theory; however, since $-ds^2$ is employed for the particle variational principle, $\delta \int m\sqrt{-ds^2} = 0$, one is really working

with the time-like branch. Similarly in quantum field theory although $g_{\mu\nu} = -\eta_{\mu\nu}$ is sometimes used, this is compensated for by employing $\pm im$ where one would have had m working with the time-like metric.) By employing the transformation (1.3), one can transform to the moving frame bringing the line element into the diagonal form

$$ds^2 = d\bar{t}^{\,2} - d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2, \tag{12.2}$$

or, employing the analogue of the A.L.T., into the non-diagonal form

$$ds^2 = dt'^2 + 2v\,dx'dt' + \left(v^2 - 1\right) dx'^2 + dy'^2 + dz'^2. \tag{12.3}$$

In either case, light cannot propagate freely in all directions as must be the case for a frame travelling with $v > 1$. However, whether one can build a consistent extension to field theory that includes this space-like branch is an open question.

At present, therefore, we can only conclude that the above objections to particles travelling with $v > 1$ do not as yet suffice to exclude such states and further study is necessary. The principal objection, then, that can be raised is that such states have not been experimentally observed — in agreement with the fundamental viewpoint of special relativity. However, this objection can be turned around and used in the construction of a faster-than-light field theory which makes such states difficult to observe. Thus the fact that they have not been observed could be taken to imply some combination of the following: the coupling is weak; the lifetime is short; the threshold for production is high; the particles are neutral. (Charged particles with $v > 1$ would exhibit Cherenkov-like radiation, since space-like energy-momentum relations permit a spontaneous "wake" radiation. Such states need not on this account be eliminated since the threshold might be high, the lifetime short.) Thus the rather extensive possibilities that exist in quantum field theory simply do not permit any firm conclusions about the non-existence of such states to be drawn from present experimental data.

Let us now turn to a question of a different nature: the significance of the transformation O_2. As we have seen, it is this transformation in conjunction with O_1, the Galilean transformation, which gives rise to an extension of the relativity of Newtonian mechanics to include fields propagating with the speed of light. Were it not for O_2, it would not be possible to eliminate the absolute velocity from the line element even after performing a local time transformation. For example, the local time transformation, $dt_\ell = dt_g - dx_g \frac{v}{1-v^2}$, diagonalizes the line element in the Galilean frame but it does not eliminate v; an observer could still detect his motion through space by a Michelson-Morley type of ex-

periment. Thus, while we have made it clear in the derivation of the A.L.T. that O_2 must arise in conjunction with O_1, to make v unobservable in the usual experiments, we have not in turn indicated why this should be the case from the standpoint of some more fundamental dynamical principle. It would perhaps be more satisfactory from the standpoint of logical economy of postulates if it were possible to show that just as O_3 is generated from O_2 and O_1, that O_2 in turn follows from O_1, and as a consequence of the nature of the equations with which we are dealing. It should be stressed that this problem does not exist for the Lorentz observer for whom it is impossible in principle to ever determine v and hence O_2 by measurements made in his frame. But for the A.L.T. observer such measurements are in principle possible, and therefore the contractions and dilatations are something to be explained in the sense that they are for him an observable function of v. On the other hand, as was pointed out when he makes measurements in the same way as the Lorentz observer, v is no longer determinable, and as was proved in Chapter 11, to the extent the usual quantum mechanical principles hold, it is in this manner that he will make measurements. Moreover this rather general invariance that was found under the local time transformation, even when the concepts of Lorentz observer and A.L.T. observer no longer apply, indicate O_3 is in every respect as fundamental to the problem as O_2 and O_1, and so one might rather regard O_2 as being generated as a consequence of O_3 and O_1.

In conclusion then, there are two theoretical problems to be solved to complete the point of view developed here:

1. An extension of quantum field theory to include states with $v > 1$ (or alternatively, a demonstration that there can exist another signal propagating with constant velocity $\neq 1$ in the absolute frame);
2. A further simplification of the postulates underlying the A.L.T.

On the other hand, to lend support to the opposite view that motion relative to the absolute frame has no physical significance, it would be logically necessary to:

a) develop a modification or extension of general relativity that fully incorporates Mach's principle;

b) demonstrate that any extension of field theory along the lines of 1 above would lead to a contradiction with known phenomena.

Appendix. Mach's Principle and the Concept of an Absolute Frame

As the development in the preceding chapters indicates, it is possible to construct a mathematical formalism yielding the same experimental results as special relativity, but in which uniform motion is referred to an absolute frame. Since the treatment was premised on the assumption of absolute signals, for which there is as yet no experimental evidence, the question arises as to whether there are any known physical phenomena which would lend support to the notion of an absolute frame. To find such phenomena it is necessary to go outside the framework of uniformly translating systems and consider other types of motion such as, for example, rotation. Because of the non-inertial character of a rotating frame, an observer located in such a frame can determine he is in rotation, without reference to the "fixed stars", by a variety of mechanical and optical experiments: Foucault pendulum, precessing gyro, the rotating-interferometer of Sagnac [16], the Michelson-Gale experiment [17].

The latter experiment, which may be regarded as the optical analogue of the Foucault pendulum, determines the angular velocity of the Earth by sending two beams of light around a large rectangle (in the actual experiment 2010×1113 feet) in opposite directions, whereupon the beams are made to interfere and a fringe displacement is measured relative to a fringe system produced by sending the beams around a smaller rectangle as a reference. The shift can be calculated very simply from a classical picture in which one takes the Earth as rotating relative to an absolute frame in which the velocity of light is c, so that the velocity of light is different in opposite directions relative to the terrestrial path. Alternatively, one may calculate the effect from the standpoint of general relativity employing the line element

$$ds^2 = \left(1 - \frac{\Omega^2 r^2}{c^2}\right) c^2 dt^2 - 2r^2 \Omega \, d\phi dt - dr^2 - r^2 d\phi - dz^2, \qquad \text{(A.1)}$$

obtained from the usual expression for the line element in cylindrical coordinates by substituting $\phi \to \phi + \Omega t$, and leaving the other coordinates unchanged. In both cases one finds [18] to first order in Ω (which represents the limits of experimental accuracy) that the time difference is $\frac{1}{c^2} 4A\Omega$ for the two beams of light to traverse a figure of area A.

In neither method of calculation is there any reference to the other matter of the universe, the result being a consequence of purely kinematical considerations which, to first order in Ω, do not even involve

relativity. Of particular interest is the appearance in the above line element of the cross term $2r^2\Omega\, d\phi dt$, which has the consequence that the time it takes light to go a distance $r\Delta\phi$ is different in opposite directions $\pm\Delta\phi$. The effect of this cross term is therefore entirely analogous to that of the cross term $2v\, dx' dt'$ we encountered in the A.L.T. line element which also gave rise to a difference in the velocity of propagation in different directions. Without absolute signals, however, as we saw, this cross term cancels out in a typical interference measurement, because effectively all that is measured is the average slowness of light which does not involve v. But in a rotating frame one has the opportunity, due to the symmetry involved, to measure the difference of the slowness in opposite directions around the light path and hence obtain the effect of the cross term. Thus, using the A.L.T. expression for the slowness (with $c=1$) $\frac{\Delta t'}{\Delta\sigma} = 1 + v\cos\theta'$, we have the following expression for the time difference for the two light beams traversing a closed circuit in opposite directions $(+,-)$,

$$\int\limits_{(+)} \Delta t' - \int\limits_{(-)} \Delta t' = \int\limits_{(+)} v\cos\theta'\Delta\sigma' - \int\limits_{(-)} v\cos\theta'\Delta\sigma' =$$

$$= \int (\nabla\times\vec{v})\, d\vec{A}' + \int (\nabla\times\vec{v})\, d\vec{A}' = 4\bar{\Omega}A' \qquad \text{(A.2)}$$

where $\bar{\Omega}$ is the average normal component of $\frac{1}{2}\nabla\times\vec{v}$ over the area. This derivation is of course not rigorous since the expression for the slowness was derived assuming uniform motion; however, one can always consider a series of uniformly moving frames oriented along the light path as having instantaneously the same value of $\vec{v}$ as the point the light is traversing. Since the contractions and dilatations are second order, the use of Stokes' theorem to first order is justified. Thus from the standpoint of the A.L.T., the effect observed in rotation is simply the measurement of the curl of the absolute velocity appearing in the set of A.L.T. line elements instantaneously defining the light path in the rotating frame, and hence Ω is to be regarded as the angular velocity of the terrestrial frame relative to the absolute frame.

Needless to say such an interpretation is inadmissible if one holds to the relativistic viewpoint that motion of a material body has meaning only with respect to other material bodies or reference frames. Under these circumstances it is logically necessary for the relativist to interpret the apparent absolute character of the effects observed in rotating frames (or more generally, non-inertial frames) from the standpoint of Mach's idea [19], as formulated into a principle by Einstein [20]. Ac-

cording to this principle, bodies do not have inertia relative to space but relative to the totality of matter in the universe which not only influences the inertia of a body but somehow produces it. This totality of matter, of which the "fixed stars" constitute the visible and presumably principal component, then determines via an averaging process the fundamental inertial frame (to within an inertial motion: uniform translation, free-fall in a local gravitational field) relative to which rotations and other apparently "absolute" motions are to be referred.

From the standpoint of the relativity of motion and the elimination of non-observable frames, Mach's idea is very attractive; however, it has never been successfully incorporated into a dynamical scheme. Thus in general relativity, as we have seen, a possible solution to the field equations in the absence of sources is $g_{\mu\nu} = \eta_{\mu\nu}$, or $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, and hence the usual Euler-Lagrange equations follow, indicating a single particle can have inertia without other masses being present. The field equations therefore admit of solutions in which inertia is relative to space. In order to satisfy Mach's principle, however, it would be necessary that the field equations have no solutions admitting of inertia in the absence of other matter. It was with the idea of securing this result that Einstein introduced the modification of the field equations involving the famous cosmological constant, an attempt which was later abandoned since, among other reasons, the equations still possessed a solution admitting of inertia in the absence of other matter, de Sitter's "empty" universe [21]. The rotating shell model of Thirring [22], which is sometimes taken as suggesting that general relativity contains Mach's principle, suffers from the difficulties that the "shell" would have to be travelling faster than the speed of light even for the nearest stars, let alone distances as great as the fixed stars, and hence Thirring's solution does not apply. Also the mass of the shell is introduced *ad hoc* into the equations, whereas according to Mach's principle the mass of the shell must arise as a consequence of the interaction.

Thus the situation still remains that when one calculates the effects observed in rotating bodies using general relativity, one is not making a calculation whose physical interpretation is Machian, but rather Newtonian (in the sense of an absolute frame). The view that general relativity in its present form does not entail Mach's principle has been expressed by many authors including Beck [23], Bondi [24] and at the 1955 Jubilee of Relativity at Bern by Heckmann and Robertson and by Pauli [25]. A recent interesting attempt to construct an alternative theory to general relativity by R. H. Dicke [26] can be easily shown to admit a line element of the form $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ even

in the absence of matter and therefore likewise does not entail Mach's principle.

On the basis of experimental evidence the simplest assumption that summarizes the facts of rotation seems to be:

> There is a universal frame, embracing the "fixed stars", relative to which it is possible to determine that a material body is in rotation by mechanical and electromagnetic measurements made on the body without reference to the stars. (This frame we shall call the "absolute frame" or simply the "ether".)

The absolute frame as described and defined above is essentially what Newton referred to as "absolute space", except that we have now attributed to it, on the basis of experiment, electromagnetic properties as well as mechanical ones. In so doing, it has been tacitly assumed that the angular velocity one determines by the mechanical experiment (Foucault pendulum) agrees with that determined by the optical experiment (Michelson-Gale experiment), an assumption in accordance with the experimental facts but not *a priori* necessary. The assumption of "universality" is needed to correlate these determinations of Ω with those determined from the observations of the fixed stars. The apparent rotation of the fixed stars is then due simply to the fact that the Earth is rotating in the ether and the stars travelling with velocities less than c relative to the ether, and at these distances $\frac{c}{R\Omega} \ll 1$. Their influence on the events (e.g. precession of Foucault pendulum) observed in the Earth frame is, in the absence of Mach's principle, presumably very small and would appear only through their influence on the metric structure of the absolute frame, say via the field equations of general relativity. In addition to the fixed stars, as Eddington [27] points out, more locally gravitating bodies can produce effects simulating a rotation of the coordinate system, however such effects are quite small (a few seconds per century for the Moon's orbit) and do not entail Mach's principle*. However because of these effects, one cannot regard the absolute frame as a rigid structure existing independently of matter as in the Newtonian theory or Lorentz's theory of the ether, but rather as in general relativity, a (space-time) structure capable of being influenced and perturbed by the distribution of matter. Once an absolute frame is admitted on the basis of providing a simple explanation for the rotation experiments,

*Eddington is not talking about the Thirring model, but about the modification of the Moon's orbit based on the analysis due to de Sitter. See pp. 95–99 of the reference, not just p. 99. See also the detailed analysis in Ciufolini & Wheeler's book *Gravitation and Inertia*, 1995, pp. 133–134. — Note by the Author, 2009.

there is no reason for rejecting it on the basis of the Michelson-Morley experiment, etc., since as was shown, the A.L.T. is capable of providing the same results as special relativity without requiring the complete equivalence of uniformly moving frames — the requirement that was primarily responsible for discarding the absolute frame.

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P.S. Note by the Author, 2009:

Since the writing of the thesis fifty-one years ago, with subsequent study, I found that there is an intriguing ambiguity in the Minkowski metric: for suitable choice of the coordinates, in addition to it being seen as an "anti-Machian" metric, it can also be seen as the limiting case of a static, spherical universe with an infinite radius, and an *infinite* amount of mass at *zero* density, together with a *vanishing* cosmological term, and hence Machian. See the following references:

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Tangherlini F. R. Schwarzschild field in n dimensions and the dimensionality of space problem. *Nuovo Cimento*, 1963, vol. 27, 636–651.

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