Gravitational Waves and Gravitational Inertial Waves According to the General Theory of Relativity

Larissa Borissova

Abstract: This research concerns gravitational waves and gravitational inertial waves, considered as waves of the curvature of space (space-time). It was produced using the mathematical apparatus of chronometric invariants, which, being the projections of the four-dimensional quantities onto the line of time and the spatial section of an observer, are physically observable quantities. The wave functions (d'Alembertian) of the chronometrically invariant (physically observable) projections X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor are deduced. The conditions of the non-stationarity of the wave functions are taken into focus. It is shown that, even in the absence of the deformation of space ($D_{ik}=0$), the non-stationarity of the wave functions is possible. Four such cases were found, depending on the gravitational inertial force F_i and the rotation of space A_{ik}: 1) $F_i=0$, $A_{ik}=0$; 2) $F_i=0$, $A_{ik}\neq 0$; 3) $F_i\neq 0$, $A_{ik}=0$; 4) $F_i\neq 0$, $A_{ik}\neq 0$. It is shown that in the first case, where $F_i=0$ and $A_{ik}=0$, in emptiness, space is flat. If one of the quantities F_i and A_{ik} differs from zero, the metric remains stationary in emptiness and in the medium. If both F_i and A_{ik} are nonzero, the metric can be non-stationary in both emptiness and the medium, if the field F_i is vortical. The main conclusion is that it is not necessary that only the deformation of space be a source of gravitational waves and gravitational inertial waves. The waves can exist even in non-deforming spaces, if the gravitational inertial force F_i and the rotation of space A_{ik} differ from zero, and the field F_i is vortical.

Contents:

Preface. This is my BSc diploma study, which I produced during 1968 at the Sternberg Astronomical Institute of the Moscow State University, where I was a student in the years 1962–1969. I presented the study on January 27, 1969*. Six years later, on March 04, 1975, the Faculty of Theoretical Physics of the Patrice Lumumba University in Moscow considered this study (with minor changes) as a PhD thesis, and bestowed upon me a PhD degree†.

This study met much interest from the side of the local scientific community working in General Relativity and gravitation. This popularity, however, was very unfortunate to me: the person who had been formally nominated as my supervisor (despite the fact that I produced this research by my own solitary strength), had included my study, without any permission from my side, let alone a sense of dignity, as a substantial part of his book surveying the gravitational wave problem‡. This was a standard of poor behaviour in the formerly USSR, where a young researcher (especially among women, who are a significant minority among the scientists), even a highly potential one, was often treated with a deliberate dose of tyranny and neglect.

After four decades, I have decided to publish my first study in its original form, in accordance with my unpublished draft of 1968. This is because I think that the main research results (manifested in the resume outlined above) may still be of interest to the scientists working on the theory of gravitational waves.

September 16, 2010 *Larissa Borissova*

§1. Introduction. The gravitational wave problem remains unsolved until this day, in both the theoretical and experimental parts of it. The theoretical foundations for gravitational waves have arrive from the General Theory of Relativity. It is commonly accepted that the experimental registration of gravitational waves in the future will be one more direct verification of Einstein's field equations in particular, and Einstein's theory in general. Just after Albert Einstein introduced the General Theory of Relativity, Arthur Eddington considered a linearized

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†Borissova L. Gravitational Waves and Gravitational Inertial Waves. PhD Thesis. Patrice Lumumba University, Moscow, 1975.

‡Zakharov V. D. Gravitational Waves in Einstein's Theory of Gravitation. Translated by R. N. Sen, Halsted Press — John Wiley & Sons, Jerusalem — New York, 1973 (originally published in Russian by Nauka Publishers, Moscow, 1972).

form of Einstein's equations. He had found that the linearized equations have a non-stationary solution in emptiness. The discovered functions depend on the argument $ct + x^1$. Therefore, the non-stationary solution was interpreted as an elliptically polarized plane wave of the gravitational field (in other words, a *gravitational wave*) travelling in the direction x^1. Subsequently, Eddington suggested that the waves should transfer gravitational radiation, which was already predicted by Einstein. Commencing in the 1920's, this kind of solutions has been commonly assumed as a basis of the theory of gravitational waves. This is because the cosmic bodies which could theoretically be the sources of gravitational radiation are located very distant from the observer, thus the arriving gravitational wave can be assumed to be weak and plane.

Meanwhile, I am convinced that we should not limit ourselves to the single (simplest) metric of weak plane gravitational waves (I will refer to it as the *Einstein-Eddington metric*). We should consider the gravitational wave problem, including the Einstein-Eddington metric, from different viewpoints.

Apart from the Einstein-Eddington theory, outlined above, there are numerous other research directions, in which another determination has been applied to gravitational waves, thus introducing not only weak gravitational waves as in the Einstein-Eddington theory, but also strong gravitational waves, including also gravitational inertial waves.

Many problems can be met in this way. From a formal point of view, weak gravitational waves should serve as an approximation to strong gravitational waves. However, the problem concerning which definition should be applied to strong gravitational waves remains open until this day. Besides that, there is another serious problem: we still have not exact solution of the problem of the gravitational field energy. In other words, we still have not real energy-momentum tensor of the gravitational field in the theory, but only particular solutions of the problem: this is the energy-momentum pseudo-tensor of the field, in its different versions suggested by Einstein, Møller and Mitskievich, Stanyukovich, and others.

As follows from that has been said above, we still have not final clarity in the theoretical part of the gravitational wave problem. On the other hand, it is obvious that there are many non-stationary processes such as supernova explosions, binary star systems, and others, which, according to Einstein's theory, should produce gravitational radiation, thus filling space with gravitational waves travelling in all directions. In other words, the existence of gravitational waves is out of doubt. Hence, we should continue research in the theory of gravitational waves in looking for new approaches which could give a better chance for understanding the nature of the phenomenon. It is possible that one of the new approaches will give the energy-momentum tensor of the gravitational field, thus resolving the problem of the gravitational field energy, including the wave energy of the field.

Generally speaking, all theoretical studies of gravitational waves can be split into three main groups:

1) The first group consists of studies, which give a generally covariant definition for gravitational waves; the presence of such waves in space does not depend on the frame of reference of the observer. These are studies produced by Pirani [1], Lichnérowicz [2–4], Bel [5–8], Debever [9–11], Hély [12, 13], Trautman [14], Bondi [15], and others. I refer to it as the *generally covariant approach* to the gravitational wave problem.

2) The second group consists of studies, which give a chronometrically invariant definition for gravitational waves. This definition is invariant with respect to the transformations of time along the three-dimensional spatial section of the observer, and is based on the mathematical apparatus of chronometric invariants (physically observable quantities) introduced by Zelmanov [16,17]. Due to the common consideration of the fields of gravitation and rotation, which is specific to the mathematical apparatus, this definition is common to both gravitational waves (derived from masses) and gravitational inertial waves (derived from the fields of rotation) which thus are considered as two manifestations of the same phenomenon. These studies were started by Zelmanov himself (his results were surveyed by his student, Zakharov, in the publication of 1966 [18]), then continued in my early studies, and also in the present paper. I refer to it as the *chronometrically invariant approach*.

3) The third group joins studies around the search for such solutions of Einstein's equations, which, proceeding from physical considerations, could describe gravitational radiation. These are studies produced by Bondi [19], Einstein and Rosen [20, 21], Peres [22, 23], Takeno [24–26], Petrov [27], Kompaneetz [28], Robinson and Trautman [29, 30], and others. I refer to it as the *physical approach*.

Most criteria for gravitational waves were introduced proceeding from the properties of the Riemann-Christoffel curvature tensor. Therefore, it is commonly assumed that they are travelling waves of the curvature of space (space-time).

Besides that, the theory of gravitational waves is directly linked to the classification of spaces introduced by Alexei Petrov [27], which is known as *Petrov's classification.* This is a classification according to the algebraic structure of the Riemann-Christoffel curvature tensor. According to the classification, three main kinds of spaces (gravitational fields) exist. Petrov referred to them as *Einstein spaces*:

Einstein spaces of kind I. Fields of gravitation of kind I are derived from island-like distributions of masses. An example of such a field is that of a spherical mass, and is described by the Schwarzschild mass-point metric. Spaces containing such fields approach a flat space at an infinite distance from the gravitating island;

Einstein spaces of kinds II and III. Spaces filled with gravitational fields of kinds II and III cannot asymptotically approach a flat space even in the case where they are empty. Such spaces are curved themselves, independently of the presence of gravitating matter. They satisfy most of the invariant definitions given to gravitational waves [18, 29–32].

As is known (see Problem 1 to §102 *Gravitational Waves* in [33], and also page 41 herein), the metric of weak plane gravitational waves is one of the sub-kind N of kind II according to Petrov's classification.

Note that we mean herein the Riemannian (four-dimensional) curvature, whose formula contains the acceleration, rotation, and deformation of the observer's reference space. However, most analysis of the wave solutions to Einstein's equations has been limited to the idea that gravitational waves have a purely *deformational origin*, i.e. are waves of the deformation of space.

Thus, considering only all aforementioned physical factors of gravitational waves, we can arrive at understanding the true origin of the phenomenon. This is the main task of this study. We will do so by employing the mathematical methods of chronometric invariants.

§2. The gravitational wave problem according to the classical theory of differential equations. So, there are three main approaches to the gravitational wave problem according to the General Theory of Relativity: 1) the generally covariant criteria for gravitational waves, whose existence does not depend on our choice of the reference frame; 2) the chronometrically invariant approach, which gives definitions for both gravitational waves and gravitational inertial waves, determined in the real frame of reference connected to a real observer;

3) gravitational waves, defined on the basis of physical considerations. Before focusing on the approaches, I will consider the gravitational wave problem from the viewpoint of the classical theory of differential equations.

An exact theory of gravitational waves became possible after de Donder [34] and Lanczos [35] who proved that Einstein's equations are a system of partial differential equations of the hyperbolic kind. The classical theory of differential equations characterizes a wave by a *Hadamard break* [36] in the solutions of the wave equations in a hypersurface S along the wave front (named after Jacques Salomon Hadamard). The hypersurface wherein the field functions have a break is known as the wave front surface, and is a characteristic hypersurface of the field equations. Therefore, looking for characterstics of the hypersurface is one of the main tasks of the theory. The gravitational wave problem as a particular problem of the solutions to Einstein's equations is also linked to Cauchy's problem formulated for the system of quasi-linear partial differential equations of the hyperbolic kind. Solving this problem containing initial data depends on not only the class of smooth functions, but also on the initial shape of the hypersurface. Because Hadamard break plays a very important rôle in the further development of the generally covariant theory of gravitational waves, it is reasonable to say more on the topic.

At first, consider a scalar function ψ as an example. Let the function ψ be continuous in each of the neighbourhoods 1 and 2, obtained due to the hypersurface S which splits the given region of space (space-time). Let also the function ψ approach to the boundary numerical values ψ_1^0 and ψ_2^0, once x^α approaches to a point $P_0(x_0^\alpha)$ of the respective neighbourhoods 1 and 2 of the hypersurface. Given these assumptions, a break of the function ψ in the hypersurface S is the following function of the point P_0

$$[\psi]\,(P_0) = \psi_1^0 - \psi_2^0\,. \tag{2.1}$$

Let the function ψ be continuous everywhere near S, but several of their first partial derivatives $\frac{\partial\psi}{\partial x^\alpha}$ have finite breaks in S

$$[\psi] = 0\,, \qquad \left[\frac{\partial\psi}{\partial x^\alpha}\right] \neq 0\,. \tag{2.2}$$

Let the hypersurface S be determined by the equation $\varphi\,(x^\alpha = 0)$. The normal vector $\frac{\partial\varphi}{\partial x^\alpha}$ of the hypersurface S is characterized by the relation

$$\frac{\partial\varphi}{\partial x^\alpha}\, dx^\alpha = 0\,, \qquad \alpha = 0,1,2,3, \tag{2.3}$$

if the increment dx^α lies in the hypersurface S. Hadamard [36] showed that, in this case, the break of the first derivative of the function is proportional to the derivative itself

$$\left[\frac{\partial\psi}{\partial x^\alpha}\right] = \chi\,\frac{\partial\varphi}{\partial x^\alpha}\,, \tag{2.4}$$

where χ is a coefficient of proportion. If the first derivatives are continuous, it is possible to show that the break of the second derivatives is expressed with the formula

$$\left[\frac{\partial^2\psi}{\partial x^\alpha\,\partial x^\beta}\right] = \chi\,\frac{\partial^2\varphi}{\partial x^\alpha\,\partial x^\beta}\,. \tag{2.5}$$

We are mostly interested in Cauchy's problem for the tensorial function $g_{\alpha\beta}$ obtained from Einstein's equations. In the case of Einstein's equations which determine an empty field of gravitation

$$R_{\alpha\beta} = 0\,, \tag{2.6}$$

where $R_{\alpha\beta}$ is Ricci's tensor, Cauchy's problem is formulated as follows:

Cauchy's problem. Consider an initial hypersurface S described by the equation

$$\varphi\,(x^\alpha) = 0\,. \tag{2.7}$$

Let functions $g_{\alpha\beta}\,(x^\sigma)$ and their first derivatives $\frac{\partial g_{\alpha\beta}(x^\sigma)}{\partial x^\rho}$ are present on the hypersurface. It is required to find these functions outside the hypersurface S given that they and their first derivatives meet the respective functions on the hypersurface S, and that all functions $g_{\alpha\beta}$ satisfy Einstein's equations in emptiness.

A Hadamard break of a tensorial function $g_{\mu\nu}$ is determined according to the equation

$$\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha\,\partial x^\beta}\right] = a_{\mu\nu}\,\frac{\partial^2\varphi}{\partial x^\alpha\,\partial x^\beta}\,, \tag{2.8}$$

where $a_{\mu\nu}$ are *coefficients of the breaking* [14, 37]. The studies [14, 37] manifest that, given all second derivatives of the function $g_{\mu\nu}$, only those by x^0 can experience some breaking in the hypersurface S

$$\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^0\,\partial x^0}\right] = a_{\mu\nu}\,. \tag{2.9}$$

Concerning Einstein's equations, this problem seems more particular. As is known, Einstein's equations do not contain the second derivatives of $g_{0\alpha}$ with respect to $x^0 = ct$. It is important to know that, given all second derivatives of $g_{\mu\nu}$ which are included into Einstein's equations, only the second derivatives of the three-dimensional components g_{ij} by x^0 (where $i, j = 1, 2, 3$) can experience a break in the hyperspace S. André Lichnérowicz [38] had showed that Einstein's equations in emptiness, considered under the following condition $g^{00} \neq 0$, can have a solution which has not a Hadamard break in S. This coincides with the case where the second derivatives of g_{ij} by x^0 are unambiguously determined in common with the Cauchy initial data. If, however, $g^{00} = 0$ in the neighbourhood of S, the derivatives and, hence, the respective components R_{0i0j} of the Riemann-Christoffel curvature tensor cannot be unambiguously determined by the Cauchy initial data and Einstein's equations, thus the second derivatives of g_{ij} with respect to x^0 experience a Hadamard breaking in the hypersurface S. This is known as a *Hadamard weak break of the 1st kind.*

The condition $g^{00} = 0$, which determines the Hadamard break of the Riemann-Christoffel curvature tensor in the initial hypersurface, can be re-formulated in the generally covariant form

$$g^{\alpha\beta}\,\frac{\partial^2\varphi}{\partial x^\alpha\,\partial x^\beta} = 0\,, \tag{2.10}$$

which is the same as the eikonal equation (equation of the wave phase) known in geometrical optics. This is a necessary and sufficient condition of the isotropy of the hypersurface S. Hence, the break of the Riemann-Christoffel curvature tensor, which is the condition of that the gravitational field in an empty space is a wave, is possible only if the initial hypersurface is isotropic.

Lichnérowicz [39] had proven the following theorem (I refer to it as *Lichnérowicz' theorem*):

Lichnérowicz' theorem. A Hadamard break of the curvature tensor $R_{\alpha\beta\gamma\delta}$ in an empty space is possible only in the characteristic hypersurface S of Einstein's equations in emptiness, which is determined by the eikonal equation.

A *characteristic hypersurface* is thus such that satisfies the eikonal equation. An enveloping arc of the hyperplanes, which are tangential to all hypersurfaces which are conceived at the given point, is known as a *characteristic cone* [40].

Because the characteristic hypersurface of Einstein's equations in emptiness is isotropic (the interval of length is zero therein), the characteristic cone of Einstein's equations meets the light cone in an empty space [38]. *Bicharacteristics* of Einstein's equations, known also as *rays*, meet the lines of the current of a vectorial field l^α, which is orthogonal to the characteristic hypersurface S

$$l^\alpha = g^{\alpha\beta}\,\frac{\partial\varphi}{\partial x^\beta}\,, \tag{2.11}$$

and are characterized by the equation

$$\frac{dx^\alpha}{d\tau} = g^{\alpha\beta}\,\frac{\partial\varphi}{\partial x^\beta}\,, \tag{2.12}$$

where τ is a nonzero parameter taken along the ray. Lichnérowicz [38] also showed that the functions of x^α are geodesics of a Riemannian space, whose metric is $g_{\alpha\beta}$.

The theory of partial differential equations says that the bicharacteristics (rays) belong to the charactersic hypersurface, hence the lines oriented tangentially to them are elements of the characteristic cone, which, in this case, meets the light cone [38]. The following conclusion follows herefrom:

> The travelling rays of gravitational waves are isotropic geodesic lines, as well as the travelling rays of light.

Proceeding from this analogy, Lichnérowicz [38] considered Cauchy's problem for Maxwell's equations in a Riemannian space V_4. He had proved the following theorem (I refer to it as *Lichnérowicz' theorem on characteristic manifolds*):

Theorem on characteristic manifolds. The characteristic manifolds of Einstein's equations and Maxwell's equations meet each other in a Riemannian space V_4, and are determined by the solution of the eikonal equation of these fields.

Analysis of this theorem, while taking into account that has been said on the rays of the travel of gravitational waves, necessarily leads to the obvious conclusion:

> The bicharacteristics of Einstein's equations (the *rays of gravitational waves*) coincide with the bicharacteristics of Maxwell's equations (the *rays of electromagnetic waves*). Thus, proceeding from the classical theory of differential equations, gravitational waves and electromagnetic waves travel at the velocity of light, along the same isotropic geodesics.

In brief, the main results obtained due to the classical theory of differential equations are such that the characteristic manifold of Einstein's equations is a hypersurface, wherein the Riemann-Christoffel curvature tensor has a Hadamard break. Therefore, this hypersurface is the front of a gravitational wave. The bicharacteristics of Einstein's equations are trajectories of an isotropic vector, which is orthogonal to the wave front, thus this is a wave vector. Because the characteristics of the characteristic manifold are generally covariant quantities, the hypersurface of Einstein's equations can be considered as an invariantly determined front of a gravitational wave, while the bicharacteristics of Einstein's equations — as invariantly determined rays. The front of an electromagnetic wave in a Riemannian space V_4 is determined by the characteristic hypersurface of Maxwell's equations. According to Lichnérowicz' theorem on the characteristic manifolds, the front of an electromagnetic wave coincides with the front of a gravitational wave, while the electromagnetic rays (bicharacteristics of Maxwell's equations) coincide with the gravitational rays (bicharacteristics of Einstein's equations).

Despite having a general method determining gravitational waves as kinds of Einstein's equations in emptiness, or as kinds of the Einstein-Maxwell equations in a space filled with both gravitational and electromagnetic fields, we cannot obtain exact solutions of the system of Einstein's equations (or the Einstein-Maxwell equations), because we meet the following difficulties:

1) Einstein's equations have a complicate non-linear structure. They have not universal boundary conditions;

2) We have not an universal form of d'Alembert's operator, which could be explicitly expressed from Einstein's equations. The core of this problem is that the unknown variables of Einstein's equations are the components of the fundamental metric tensor $g_{\alpha\beta}$, which conserves in the generally covariant meaning: it satisfies the generally covariant conservation law, thus $\nabla_\sigma g_{\alpha\beta}=0$ (here ∇_σ is the symbol of absolute differentiation). Therefore, the generally covariant d'Alembertian of the fundamental metric tensor is zero: $\Box g_{\alpha\beta} \equiv g^{\rho\sigma}\nabla_\rho\nabla_\sigma g_{\alpha\beta} \equiv 0$.

Einstein's theory interprets gravitational fields as distortions of space (space-time). Therefore, it is a naturally valid idea to connect gravitational waves to the properties of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$. The four-dimensional pseudo-Riemannian space, which is the basic space-time of General Relativity, is characterized by the curvature tensor: if the tensor is zero in a region, gravitational fields are absent therein. The Riemann-Christoffel curvature tensor is not a direct part of Einstein's equations. Only its contacted forms, namely — Ricci's tensor and scalar, form the basis of the equations. Therefore, other methods should be applied in order to study its structure. In particular, we can impose some conditions (criteria) on the tensor, which could allow to consider the curvature field as a gravitational wave. In this direction, an emergent goal in the theory of gravitational waves was included due to studies of the algebraic properties of the Riemann-Christoffel curvature tensor produced by Petrov [27]. His classification of the curvature tensor according to its algebraic structure allowed him to determine several kinds of the solutions of Einstein's equations as gravitational wave fields.

We will consider the invariant criteria for gravitational waves, and also Petrov's results related to the algebraic structure of the curvature tensor, in the next paragraphs §3 and §4.

§3. Generally covariant criteria for gravitational waves and their link to Petrov's classification. As was mentioned in the end of §1, most analysis of the wave solutions to Einstein's equations was limited by an idea that they are only due to the factor of the deformation of space, thus gravitational waves are waves of the deformation of space. Here the next question arises. How well is this statement justified? General covariant criteria for the wave solutions to Einstein's equations will be our task in this paragraph.

Einstein's equations (gravitational field equations) have the form

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,, \tag{3.1}$$

where $R_{\alpha\beta} = R^{\sigma\cdots}_{\cdot\alpha\sigma\beta}$ is Ricci's tensor, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, $\varkappa = \frac{8\pi G}{c^2}$ is Einstein's gravitational constant, G is Gauss' constant of gravitation, λ is the cosmological term.

When studying gravitational waves, most scientists assume $\lambda = 0$ thus considering a particular case of Einstein's equations, which is

$$R_{\alpha\beta} = \kappa\, g_{\alpha\beta}\,. \tag{3.2}$$

This is a case of spaces known, after Petrov [27], as *Einstein spaces.* They can be either empty ($\kappa = 0$) or filled with homogeneously distributed matter (in this case, $R_{\alpha\beta} \sim \varkappa T_{\alpha\beta}$). If $\kappa = 0$ in an Einstein space, there is not distributed matter. If there is not islands of mass as well, such an empty space can also be curved: in this case, it is related to kinds II and III according to Petrov's classification (see page 29).

As was mentioned in §2, according to the classical theory of differential equations, gravitational wave fields are determined by solutions of Einstein's equations, taken with the initial conditions of a characteristic hypersurface of the equations. A gravitational wave is a Hadamard break in the initial characteristic hypersurface of the equations; this surface is the front of a gravitational wave. Let us re-write the formula of a Hadamard break of a tensorial function $g_{\mu\nu}$ in a Riemannian space, (2.8), as

$$\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\right] = a_{\mu\nu}\, l_\alpha l_\beta\,, \qquad l_\alpha \equiv \frac{\partial\varphi}{\partial x^\alpha}\,. \tag{3.3}$$

According to Lichnérowicz [2–4], who followed with Hadamard's studies, a Hadamard break of the second derivatives of $g_{\mu\nu}$ can be in a characteristic hypersurface of Einstein's equation only due to a Hadamard break in the field of the Riemann-Christoffel curvature tensor, i.e. due to $[R_{\alpha\beta\gamma\delta}]$, which satisfies the equations (see [4])

$$l_\lambda \left[R_{\mu\alpha\beta\nu}\right] + l_\alpha \left[R_{\mu\beta\lambda\nu}\right] + l_\beta \left[R_{\mu\lambda\alpha\nu}\right] = 0\,. \tag{3.4}$$

Proceeding from this condition realized in a characteristic hypersurface of Einstein's equations, and also because the break $[R_{\alpha\beta\gamma\delta}]$, located at the front of a gravitational wave, is proportional to the curvature tensor $R_{\alpha\beta\gamma\delta}$ itself (see §2 herein for detail), Lichnérowicz was able to formulate his generally covariant criterion for gravitational waves [2–4]:

Lichnérowicz' criterion. The Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta} \neq 0$ determines the state of "pure gravitational radiation", only if there is a vector l^α, which is orthogonal to the characteristic surface of Einstein's equations, is isotropic ($l_\alpha l^\alpha = 0$), and satisfies the equations

$$\left.\begin{array}{l} l^\mu R_{\mu\alpha\beta\nu} = 0 \\ l_\lambda R_{\mu\alpha\beta\nu} + l_\alpha R_{\mu\beta\lambda\nu} + l_\beta R_{\mu\lambda\alpha\nu} = 0 \end{array}\right\}. \tag{3.5}$$

If $R_{\alpha\beta} = 0$ (in an empty space, which is free of distributed matter of any kind), the equations determine the field of "pure gravitational radiation".

There is also another generally covariant criterion for gravitational waves, formulated by Zelmanov* [18]. It is indirectly connected to Lichnérowicz' criterion. Zelmanov's criterion proceeds from the d'Alembert generally covariant operator

$$\Box^\sigma_\sigma \equiv g^{\rho\sigma}\, \nabla_\rho \nabla_\sigma\,, \tag{3.6}$$

and is formulated as follows:

*This criterion was introduced by Abraham Zelmanov in the early 1960's, and was presented to a close circle of his associates. It was first published in 1966, in the survey on the gravitational wave problem [18] authored by Zakharov, who was a student of Zelmanov. Zakharov referred to Zelmanov in the publication.

Zelmanov's criterion. A space satisfies the state of gravitational radiation, only if the Riemann-Christoffel curvature tensor a) does not conserve ($\nabla_\sigma R_{\mu\alpha\beta\nu} \neq 0$), and b) satisfies the generally covariant condition

$$\Box^\sigma_\sigma R_{\mu\alpha\beta\nu} = 0\,. \tag{3.7}$$

Any empty space, satisfying Zelmanov's criterion, satisfies Lichnérowicz' criterion as well. And vice versa: any empty space, which satisfies Lichnérowicz' criterion (excluding constant curvature spaces, where $\nabla_\sigma R_{\mu\alpha\beta\nu} = 0$), also satisfies Zelmanov's criterion.

There are also numerous other generally covariant criteria for gravitational waves, introduced by Bel, Pirani, Debever, Maldybaeva and others. Each of the criteria has its own advantages and drawbacks, therefore none of the criteria can be considered as the final solution of the gravitational wave problem. Therefore, it would be a good idea to consider those characteristics of gravitational wave fields, which are common to most of the criteria. Such an integrating factor is Petrov's classification according to the algebraic structure of the Riemann-Christoffel curvature tensor [27]. This is a classification of spaces, which satisfy the particular Einstein equations (3.2) and are known as Einstein spaces. Thus, gravitational fields, which satisfy (3.2), can also be classified in this way.

As is known, the Riemann-Christoffel curvature tensor satisfies the following identities

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}\,, \qquad R_{\alpha[\beta\gamma\delta]} = 0\,. \tag{3.8}$$

Because of these identities, the curvature tensor is related to tensors of a special family, known as *bitensors*. They satisfy two conditions:

1) Their covariant and contravariant valencies are even;
2) Both covariant and contravariant indices of the tensors are split into pairs, and inside each pair the tensor $R_{\alpha\beta\gamma\delta}$ is antisymmetric.

A set of tensor fields located in an n-dimensional Riemannian space is known as a *bivector set*, and its representation at a point is known as a *local bivector set*. Every antisymmetric pair of indices $\alpha\beta$ is denoted by a common index a, and the number of common indices is $N = \frac{n(n-1)}{2}$. It is obvious that if $n = 4$ we have $N = 6$. Hence a bitensor $R_{\alpha\beta\gamma\delta} \rightarrow R_{ab}$, located in a four-dimensional Riemannian space, maps itself into a six-dimensional bivector space. It can be metrized by introducing the specific metric tensor

$$g_{ab} \rightarrow g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}\,. \tag{3.9}$$

The metric tensor g_{ab} (where $a, b = 1, 2, \dots N$) is symmetric and non-degenerate. If the metric is given for the sign-alternating $g_{\alpha\beta}$, it is sign-alternating as well, having a respective signature. So, for Minkowski's signature $(+---)$ of $g_{\alpha\beta}$, the signature of g_{ab} is $(+++---)$.

Mapping the curvature tensor $R_{\alpha\beta\gamma\delta}$ onto the metric bivector space V_N, we obtain a symmetric tensor R_{ab} (where $a, b = 1, 2, \dots N$) which can be associated with a lambda-matrix

$$\| R_{ab} - \Lambda g_{ab} \| . \tag{3.10}$$

Solving the classical problem of linear algebra (reducing a lambda-matrix to its canonical form along a real distance), we can find a classification for V_n under a given n. A specific kind of spaces, which are Einstein spaces we are considering, is set up by a characteristic of the respective lambda-matrix. This kind remains unchanged in the area, where this characteristic remains unchanged.

Bases of the elementary divisors of the lambda-matrix for any V_n have an ordinary geometric meaning as *stationary curvatures*. Naturally, the Riemannian curvature K of V_n in a two-dimensional direction is determined by an ordinary (single-sheet) bivector $V^{\alpha\beta} = \underset{(1)}{V^{\alpha}} \underset{(2)}{V^{\beta}}$ as

$$K = \frac{R_{\alpha\beta\gamma\delta} V^{\alpha\beta} V^{\gamma\delta}}{g_{\alpha\beta\gamma\delta} V^{\alpha\beta} V^{\gamma\delta}} . \tag{3.11}$$

If $V^{\alpha\beta}$ is non-ordinary, the invariant K is known as the *bivector curvature in the direction of the vector*. Mapping K onto the bivector space, we obtain

$$K = \frac{R_{ab} V^a V^b}{g_{ab} V^a V^b} , \qquad a, b = 1, 2, \dots N. \tag{3.12}$$

The ultimate numerical values of K are known as *stationary curvatures at a given point*, while the vectors V^a corresponding to them are known as *stationary non-simple bivectors*. In this case

$$V^{\alpha\beta} = \underset{(1)}{V^{\alpha}} \underset{(2)}{V^{\beta}} , \tag{3.13}$$

so the stationary curvature is the same as the Riemannian curvature in the given two-dimensional direction.

Finding the ultimate numerical values of K is the same as finding those vectors V^a, where K takes the ultimate numerical values. This is the same as finding *undoubtedly stationary directions*. The necessary and sufficient condition of a stationary state of V^a is

$$\frac{\partial}{\partial V^a} K = 0 . \tag{3.14}$$

The problem of finding the stationary curvatures for Einstein spaces had been solved by Petrov [27]. If the space metric is sign-alternating, the stationary curvatures are complex as well as the stationary bivectors relating to them in the space V_n. For Einstein spaces of four dimensions with Minkowski's signature, Petrov had formulated a theorem:

Petrov's theorem. Given an ortho-frame $g_{\alpha\beta} = \{+1, -1, -1, -1\}$, there is a symmetric paired matrix $\|R_{ab}\|$

$$\|R_{ab}\| = \left\| \begin{array}{c|c} M & N \\ \hline N & -M \end{array} \right\|, \tag{3.15}$$

where M and N are two symmetric square matrices of the 3rd order, whose components satisfy the relationships

$$m_{11} + m_{22} + m_{33} = -\kappa\,, \qquad n_{11} + n_{22} + n_{33} = 0\,. \tag{3.16}$$

After transformations, the lambda-matrix $\|R_{ab} - \Lambda g_{ab}\|$, where $g_{ab} = \{+1, +1, +1, -1, -1, -1\}$, takes the form

$$\|R_{ab} - \Lambda g_{ab}\| = \left\| \begin{array}{c|c} M + iN + \Lambda\varepsilon & 0 \\ \hline 0 & M - iN + \Lambda\varepsilon \end{array} \right\| \equiv$$

$$\equiv \left\| \begin{array}{cc} Q\,(\Lambda) & 0 \\ 0 & \bar{Q}\,(\Lambda) \end{array} \right\|, \tag{3.17}$$

where $Q\,(\Lambda)$ and $\bar{Q}\,(\Lambda)$ are three-dimensional matrices, whose elements are complex conjugates, and ε is the three-dimensional unit matrix.

The matrix $Q\,(\Lambda)$ can have only one of the following three kinds of characteristics: I) [111]; II) [21]; III) [3].

As a matter of fact that the initial lambda-matrix can have only one characteristic drawn from: I) $[111, \overline{111}]$; II) $[21, \overline{21}]$; III) $[3, 3]$.

The numbers in brackets means the multiplicity of roots of the characteristic equation $\det\|R_{ab} - \Lambda g_{ab}\| = 0$ (see Chapter 2 in [27]). Consider a 6×6 matrix g_{ab}. Construct the characteristic equation for it. This is a 6th order equation: it has 6 roots and, as Petrov showed, the ultimate number of different roots is 3 as for a 3×3 matrix (also several of these 3 pairs of roots can be complex conjugates). Obtain the roots, then compare the obtained pairs of solutions. If all 3 pairs of roots differ from each other, this is kind [111]. If two of them are the same, this is kind [21]. If all 3 pairs of roots are the same, this is kind [3].

The bar in the second half of a characteristic means an imaginary part of the complex conjugates. There is not bar in kind $[3, 3]$, because the elementary divisors are always real therein.

Taking a lambda-matrix of each of three possible kinds, Petrov [27] had deduced the canonical form of the matrix $\|R_{ab}\|$ in a non-holonomic ortho-frame

$$\left.\begin{array}{c}\underline{\text{Kind I}}\\[4pt] \|R_{ab}\| = \left\| \begin{array}{cc} M & N \\ N & -M \end{array} \right\|, \\[8pt] M = \left\| \begin{array}{ccc} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{array} \right\|, \quad N = \left\| \begin{array}{ccc} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{array} \right\| \end{array}\right\}, \qquad (3.18)$$

where $\sum_{i=1}^{3}\alpha_i = -\kappa$ and $\sum_{i=1}^{3}\beta_i = 0$ (so, here are 4 independent parameters, determining the space structure by an invariant form),

$$\left.\begin{array}{c}\underline{\text{Kind II}}\\[4pt] \|R_{ab}\| = \left\| \begin{array}{cc} M & N \\ N & -M \end{array} \right\|, \\[8pt] M = \left\| \begin{array}{ccc} \alpha_1 & 0 & 0 \\ 0 & \alpha_2+1 & 0 \\ 0 & 0 & \alpha_2-1 \end{array} \right\|, \quad N = \left\| \begin{array}{ccc} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 1 \\ 0 & 1 & \beta_2 \end{array} \right\| \end{array}\right\}, \qquad (3.19)$$

where $\alpha_1 + 2\alpha_2 = -\kappa$ and $\beta_1 + 2\beta_2 = 0$ (here are 2 independent parameters determining the space structure by an invariant form),

$$\left.\begin{array}{c}\underline{\text{Kind III}}\\[4pt] \|R_{ab}\| = \left\| \begin{array}{cc} M & N \\ N & -M \end{array} \right\|, \\[8pt] M = \left\| \begin{array}{ccc} -\frac{\kappa}{3} & 1 & 0 \\ 1 & -\frac{\kappa}{3} & 0 \\ 0 & 0 & -\frac{\kappa}{3} \end{array} \right\|, \quad N = \left\| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array} \right\| \end{array}\right\}, \qquad (3.20)$$

thus no independent parameters determining the space structure by an invariant form exist in this case.

Thus Petrov has successfully resolved the problem of reducing a lambda-matrix to its canonical form along a real path in a Riemannian space with a sign-alternating metric. Despite the fact that his solution is obtained only at a given point, the obtained classification is invariant because the results are applicable to any point in the space.

Stationary curvatures take the form

$$\Lambda_i = \alpha_i + i\beta_i \qquad (3.21)$$

in spaces of kind III, where they take real values ($\Lambda_1 = \Lambda_2 = \Lambda_3 = -\frac{\kappa}{3}$).

Numerical values of some stationary curvatures in spaces (gravitational fields) of kinds I and II can coincide with each other. If they are the same, we have sub-kinds of the spaces (fields). Kind I has 3 sub-kinds: I ($\Lambda_1 \neq \Lambda_2 \neq \Lambda_3$); D ($\Lambda_2 = \Lambda_3$); O ($\Lambda_1 = \Lambda_2 = \Lambda_3$). If the space is empty ($\kappa = 0$), the sub-kind O of kind I gives a flat space. Kind II has 2 sub-kinds: II ($\Lambda_1 \neq \Lambda_2$, $\Lambda_2 = \Lambda_3$) and N ($\Lambda_1 = \Lambda_2$). Kinds I and II are the basic kinds of Petrov's classifications.

In empty spaces (empty gravitational fields) the stationary curvatures are $\Lambda = 0$, so empty spaces (fields) are degenerate.

Studies of the algebraic structure of the Riemann-Christoffel curvature tensor for known solutions of Einstein's equations showed that most of the solutions are related to kind I. The curvature decreases with distance from a gravitating mass. In the extreme case, where the distance becomes infinite, the space approaches a flat space. As was shown in my early (unpublished) study, reported to Zelmanov when I was a student, the Schwarzschild mass-point solution, which represents a spherically symmetric gravitational field derived from an island of mass located in emptiness, is classified as the sub-kind D of kind I.

General covariant criteria for gravitational waves are linked to the algebraic structure of the curvature tensor, and thus should be associated with the aforementioned types of Einstein spaces. Most gravitational wave solutions of Einstein's equations are attributed to the sub-kind N of kind I. Several gravitational wave solutions are attributed to kinds II and III. Note, spaces of kinds II and III cannot approach a flat space, because the components of the curvature tensor matrix $\|R_{ab}\|$ contain $+1$ and -1. This makes the approach of the curvature tensor to zero impossible, and thus excludes approaching a flat space at infinity. Therefore, gravitational waves (waves of the curvature) are present everywhere in spaces of kinds II and III. Pirani [1] holds that gravitational waves are solutions to the gravitational field equations in spaces of the sub-kind N of kind II, or of kind III. The following solutions are classified to the sub-kind N of kind II: Peres' solution [22, 23] which describes plane gravitational waves

$$ds^2 = (dx^0)^2 - 2\alpha(dx^0 + dx^3)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \qquad (3.22)$$

Takeno's solution [24–26]

$$\begin{aligned} ds^2 = (\gamma + \rho)\,(dx^0)^2 - 2\rho\, dx^0 dx^3 - \\ - \alpha(dx^1)^2 - 2\delta\, dx^1 dx^2 - \beta(dx^2)^2 + (\rho - \gamma)\,(dx^3)^2, \end{aligned} \qquad (3.23)$$

where $\alpha = \alpha(x^1 - x^0)$, while γ, ρ, β, δ are functions of $(x^3 - x^0)$, and also Petrov's solution [27], which was represented by Bondi, Pirani, and Robertson in another coordinate system [15] as

$$ds^2 = (dx^0)^2 - (dx^1)^2 + \alpha(dx^2)^2 + 2\beta\, dx^2 dx^3 + \gamma(dx^3)^2, \qquad (3.24)$$

where α, β, γ are functions of $(x^1 + x^0)$.

A detailed survey of the relations between the generally covariant criteria for gravitational waves and Petrov's classification was presented in the publication [18]. Among the other issues, two following theorems were discussed therein:

Theorem. In order that a space satisfies the state of "pure gravitational radiation" (in the Lichnérowicz sense), it is a necessary and sufficient condition that the space is an Einstein space of the sub-kind N of kind II according to Petrov's classification, thus characterized by zero curvature matrix $\|R_{ab}\|$ in the bivector space.

Theorem. An Einstein space satisfying Zelmanov's criterion can only be an empty space ($\varkappa = 0$) of the sub-kind N of kind II. And vice versa, any empty space V_4 of the sub-kind N satisfies Zelmanov's criterion as well. This is true excluding symmetric spaces* of this kind; symmetric spaces of this kind have the metric

$$ds^2 = 2\, dx^0 dx^1 - \mathrm{sh}^2 dx^0\, (dx^2)^2 - \sin^2 dx^0\, (dx^3)^2. \qquad (3.25)$$

Proceeding from the theorems, we immediately arrive at a relation between Zelmanov's criterion for gravitational wave fields in emptiness and Lichnérowicz' criterion for "pure gravitational radiation":

Theorem. Any empty space V_4, satisfying Zelmanov's criterion for gravitational wave fields located in empty spaces, also satisfies Lichnérowicz' criterion for "pure gravitational radiation". And vice versa, any empty space V_n, satisfying Lichnérowicz' criterion (excluding the case of symmetric spaces), satisfies Zelmanov's criterion as well.

How are these criteria related to each other in a general case? This problem is still open for discussion.

In [18] it was shown that all known solutions to Einstein's equations in emptiness, which satisfy Zelmanov's criterion and Lichnérowicz' criterion, can be obtained as particular cases of a generalized metric whose space permits a vector field l^α, which conserves in the space and thus

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*A space is referred to as *symmetric*, if its curvature tensor $R_{\alpha\beta\gamma\delta}$ conserves and thus satisfies the conservation condition $\nabla_\sigma R_{\alpha\beta\gamma\delta} = 0$.

satisfies the conservation law

$$\nabla_\sigma l^\sigma = 0\,. \tag{3.26}$$

It is obvious that this condition leads to Lichnérowicz' condition (3.5), hence this empty space is classified as the sub-kind N of kind II, and, also, the vector l^α playing the rôle of a gravitational wave vector, is unique and isotropic $l_\alpha l^\alpha = 0$. According to Eisenhart's theorem [41], a Riemannian space V_4 containing a unique isotropic vector l^α (in other words, an absolute parallel vector field), has the metric

$$\begin{aligned} ds^2 = \varepsilon\,(dx^0)^2 + 2\,dx^0 dx^1 + 2\varphi\, dx^0 dx^2 + \\ + 2\psi\, dx^0 dx^3 + \alpha(dx^2)^2 + 2\gamma\, dx^2 dx^3 + \beta(dx^3)^2, \end{aligned} \tag{3.27}$$

where ε, φ, ψ, α, β, γ are functions of x^0, x^2, x^3, and $l^\alpha = \delta^\alpha_1$. This metric satisfies the particular form (3.2) of Einstein's equations. So this is an exact solution to Einstein's equations in emptiness or vacuum, and satisfies Zelmanov's criterion and Lichnérowicz' criterion for gravitational waves. This solution generalizes those solutions suggested by Takeno, Peres, Bondi, Petrov and others, which satisfy the Zelmanov and Lichnérowicz criteria.

The metric (3.27), taken under additional conditions suggested by Bondi [18], satisfies Einstein's equations in their general form (3.1) in the case where $\lambda = 0$ and the energy-momentum tensor $T_{\alpha\beta}$ describes an isotropic electromagnetic field. Given an isotropic electromagnetic field, Maxwell's tensor $F_{\mu\nu}$ of the field satisfies the conditions

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F_{\mu\nu}F^{*\mu\nu} = 0\,, \tag{3.28}$$

where $F^{*\mu\nu} = \frac{1}{2}\,\eta^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is a pseudotensor dual to Maxwell's tensor, while $\eta^{\mu\nu\rho\sigma}$ is the completely antisymmetric discriminant tensor (it makes pseudotensors out of tensors). Direct substitution shows that this metric satisfies the following requirements: the Riner-Wheeler condition discussed by Peres [42]

$$R = 0\,, \qquad R_{\alpha\rho}R^{\rho\beta} = \frac{1}{4}\,\delta^\beta_\alpha\,(R_{\rho\sigma}R^{\rho\sigma}) = 0\,, \tag{3.29}$$

where $\delta^\alpha_\beta = g^\alpha_\beta$, and the Nordtvedt-Pagels condition [43]

$$\eta_{\mu\varepsilon\gamma\sigma}\left(R^{\delta\gamma,\sigma}R^{\varepsilon\tau} - R^{\delta\varepsilon,\sigma}R^{\gamma\tau}\right) = 0\,, \tag{3.30}$$

where $R^{\delta\gamma,\sigma} = g^{\sigma\mu}\nabla_\mu R^{\delta\gamma}$.

We have an interest in isotropic electromagnetic fields because an observer, who accompanies such a field, should be moving at the velocity of light [1,4]. Hence, isotropic electromagnetic fields can be interpreted as fields of electromagnetic radiation without sources. On the other hand, according to Eisenhart's theorem [41], a Riemannian space V_4 having the metric (3.27) permits an absolute parallel vector field l^α. Therefore, we conclude that the vector l^α considered in this case satisfies Lichnérowicz' criterion for "pure gravitational radiation".

Thus the metric (3.27), satisfying the conditions

$$\left.\begin{aligned} & R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\, T_{\alpha\beta} \\ & T_{\alpha\beta} = \frac{1}{4}\, F_{\rho\sigma} F^{\rho\sigma} g_{\alpha\beta} - F_{\alpha\sigma} F_{\beta\cdot}^{\cdot\sigma} \\ & F_{\alpha\beta} F^{\alpha\beta} = 0\,, \qquad F_{\alpha\beta} F^{*\alpha\beta} = 0 \end{aligned}\right\} \tag{3.31}$$

and taken under the additional condition suggested by Bondi [18]

$$R_{2323} = R_{0232} = R_{0323} = 0\,, \tag{3.32}$$

is an exact solution to Einstein's equations, which describes both gravitational waves and electromagnetic waves without sources. This solution does not satisfy Zelmanov's criterion in a general case, but satisfies it in particular cases where $T_{\alpha\beta} \neq 0$, and also $R_{\alpha\beta} \neq 0$.

A recursion curvature space is a Riemannian space, which has a curvature satisfying the relationship

$$\nabla_\sigma R_{\alpha\beta\gamma\delta} = l_\sigma R_{\alpha\beta\gamma\delta}\,. \tag{3.33}$$

Due to Bianchi's identity, such a space satisfies

$$l_\sigma R_{\alpha\beta\gamma\delta} + l_\alpha R_{\beta\sigma\gamma\delta} + l_\beta R_{\sigma\alpha\gamma\delta} = 0\,. \tag{3.34}$$

A common classification for recursion curvature spaces had been suggested by Walker [44]. His classification was then applied to the four-dimensional pseudo-Riemannian space (the basic space-time of General Relativity). Concerning the class of prime recursion spaces*, we are particularly interested in two metrics, which are

$$ds^2 = \psi\,(x^0, x^2)(dx^0)^2 + 2\,dx^0 dx^1 - (dx^2)^2 - (dx^3)^2, \tag{3.35}$$

$$ds^2 = 2\,dx^0 dx^1 + \psi\,(x^1, x^2)(dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{3.36}$$

*A recursion curvature space is prime or simple, if it contains $n-2$ parallel vector fields (isotropic and non-isotropic). Here n is the dimension of the space.

where $\psi > 0$. In these metrics, only one component of Ricci's tensor is nonzero. It is $R_{00} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^0 \partial x^0}$ in (3.35), and $R_{11} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^1 \partial x^1}$ in (3.36). Einstein spaces with these metrics can only be empty ($\kappa = 0$) and flat ($R_{\alpha\beta\gamma\delta} = 0$). This can be proved by checking that both metrics satisfy the Riner-Wheeler condition (3.29) and the Nordtvedt-Pagels condition (3.30), which determine isotropic electromagnetic fields.

Both metrics (3.35) and (3.36) are interesting due to their physical meaning: in such a space, the space curvature is due to an isotropic electromagnetic field. Moreover, if we remove this field from the space, the space becomes flat.

There are also numerous other metrics which are exact solutions to the Einstein-Maxwell equations, related to the class of isotropic electromagnetic fields. However no one of them satisfies Zelmanov's criterion and Lichnérowicz' criterion.

Minkowski's signature permits only two metrics for non-simple recursion curvature spaces. These are the metric

$$\left.\begin{aligned} ds^2 &= \psi\,(x^0, x^2, x^3)(dx^0)^2 + 2\,dx^0 dx^1 + \\ &\quad + K_{22}(dx^2)^2 + 2K_{23}\,dx^2 dx^3 + K_{33}(dx^3)^2 \\ K_{22} &< 0\,, \qquad K_{22}K_{33} - K_{23}^2 < 0 \end{aligned}\right\}, \tag{3.37}$$

where $\psi = \chi_1(x_0)\big(a_{22}(x^2)^2 + 2a_{23}x^2x^3 + a_{33}(x^3)^2\big) + \chi_2(x^0)x^2 + \chi_3(x^0)x^3$, and the metric

$$\left.\begin{aligned} ds^2 &= 2\,dx^0 dx^1 + \psi\,(x^1, x^2, x^3)(dx^1)^2 + \\ &\quad + K_{22}(dx^2)^2 + 2K_{23}\,dx^2 dx^3 + K_{33}(dx^3)^2 \\ K_{22} &< 0\,, \qquad K_{22}K_{33} - K_{23}^2 < 0 \end{aligned}\right\}, \tag{3.38}$$

where $\psi = \chi_1(x_1)\big(a_{22}(x^2)^2 + 2a_{23}x^2x^3 + a_{33}(x^3)^2\big) + \chi_2(x^1)x^2 + \chi_3(x^1)x^3$. Here a_{ij} and K_{ij} $(i, j = 2, 3)$ are constants.

The metrics (3.37) and (3.38) satisfy the Einstein space condition $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ (3.2) only if $\kappa = 0$ that leads to the relationship

$$K_{33}\,a_{22} + K_{22}\,a_{33} - 2K_{23}\,a_{23} = 0\,. \tag{3.39}$$

The metrics (3.37) and (3.38) are related to the sub-kind N of kind II according to Petrov's classification. It is interesting that the metric (3.38) is stationary and, at the same time, describes "pure gravitational radiation" (in the Lichnérowicz sense).

In a general case, where $R_{\alpha\beta} \neq \kappa g_{\alpha\beta}$, the metrics (3.37) and (3.38) satisfy the Riner-Wheeler condition (3.29) and the Nordtvedt-Pagels condition (3.30). Therefore these metrics are exact solutions to the Einstein-Maxwell equations, which describe both gravitational waves and electromagnetic waves without sources. In this general case both metrics satisfy Zelmanov's criterion and Lichnérowicz' criterion.

§4. The chronometrically invariant criterion for gravitational waves and its link to Petrov's classification. All that has been detailed above represents a generally covariant approach to the gravitational wave problem: the presence of such waves in space does not depend on the frame of reference of the observer. There is also another approach to the gravitational wave problem. It determines not only gravitational waves (they are derived from masses), but also gravitational inertial waves (derived from the fields of rotation), both in a frame of reference connected to a real observer. This approach is due to Zelmanov's mathematical apparatus of chronometric invariants [16, 17], which are physically observable quantities in the basic space (space-time) of General Relativity.

In all experimental tests of the General Theory of Relativity, the most important fact is that any real observer, who processes the measurements, rests with respect to his laboratory reference frame and all physical standards located in it. In other words, he is located in a reference frame which accompanies his physical standards (the body of reference). Zelmanov [16, 17] showed that quantities measured by the observer in the accompanying reference frame possess the property of *chronometric invariance*: they are invariant along the three-dimensional section determined by the observer's reference frame (along his three-dimensional space). Keeping this fact in mind, Zelmanov formulated a *chronometrically invariant criterion* for gravitational waves. This criterion is invariant only for the transformations of that reference frame, which rests with respect to the observer and his laboratory references. Following this way, in contrast to the generally covariant approach, we can match our theoretical conclusions and the results obtained from real physical experiments.

Zelmanov showed that the property of chronometric invariance means invariance with respect to the transformations

$$\left.\begin{aligned} &\widetilde{x}^0 = \widetilde{x}^0(x^0, x^1, x^2, x^3) \\ &\widetilde{x}^i = \widetilde{x}^i(x^1, x^2, x^3)\,, \qquad \frac{\partial \widetilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\}, \tag{4.1}$$

then he proved that chronometrically invariant quantities are the respective projections of four-dimensional (generally covariant) quantities onto the line of time and the spatial section of the observer. He had developed a versatile mathematical apparatus, which allows one to derive the chronometrically invariant projections from any generally covariant quantities (and equations) and is known as the *theory of chronometric invariants*. The core of the theory and necessary details were presented by him in the publications [16, 17].

In the framework of the theory, a chronometrically invariant d'Alembert operator was introduced as

$$^*\Box = h^{ik\,*}\nabla_i{}^*\nabla_k - \frac{1}{a^2}\frac{^*\partial^2}{\partial t^2}\,, \tag{4.2}$$

where $h^{ik} = -g^{ik}$ is the chr.inv.-metric tensor presented in its contravariant form (its contravariant, covariant, and mixed forms differ, see below), $^*\nabla_i$ is the symbol of chr.inv.-differentiation (a chr.inv.-analogue to the symbol ∇_σ of generally covariant differentiation), a is the linear velocity at which the attraction of gravity spreads, $\frac{^*\partial}{\partial t}$ is the chr.inv.-differential operator with respect to time.

This is Zelmanov's chronometrically invariant criterion for gravitational waves and gravitational inertial waves:

Zelmanov's chr.inv.-criterion. If the metric of a space possesses wave properties, the chr.inv.-quantities f, characterizing the local reference space of an observer, such as the gravitational inertial force F_i, the angular velocity of the rotation of the space A_{ik}, the deformation tensor D_{ik}, the spatial curvature tensor C_{iklj} (also the scalar quantities, derived from them), and the chr.inv.-projections X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor must satisfy the chr.inv.-d'Alembert equation

$$^*\Box\, f = A\,, \tag{4.3}$$

where A is an arbitrary function of the four-dimensional coordinates, and contains only first derivatives of the chr.inv.-quantities represented by f.

Zelmanov's chr.inv.-criterion is true for the generalized gravitational wave metric (3.27) in the case where the gravitational inertial force F^i is a wave function. At the same time, the generally covariant criteria for gravitational waves are derived from a limitation imposed on the Riemann-Christoffel curvature tensor in order that it be a wave function. Therefore, it would be interesting to study the chr.inv.-components of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ [16]

$$X^{ik} = -\,c^2\,\frac{R_{0\cdot0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \qquad Y^{ijk} = -\,c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \qquad Z^{ijkl} = c^2 R^{ijkl} \tag{4.4}$$

in the case, where they are wave functions as well.

What is common among Zelmanov's generally covariant criterion (3.7) and his chr.inv.-criterion (4.3)? The answer to the question arrives from Zelmanov's generally covariant criterion, $\Box^{\sigma}_{\sigma} R_{\mu\alpha\beta\nu} = 0$ (3.7), rewritten in chr.inv.-form

$$^*\Box\, X^{ij} = A^{ij}_{(1)}\,, \qquad ^*\Box\, Y^{ijk} = A^{ijk}_{(2)}\,, \qquad ^*\Box\, Z^{iklj} = A^{iklj}_{(3)}\,, \tag{4.5}$$

where $A^{ij}_{(1)}$, $A^{ijk}_{(2)}$, $A^{iklj}_{(3)}$ are chr.inv.-tensors, which contain only first derivatives of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}. From these formulae we arrive at an obvious conclusion, which is:

> Spaces, which satisfy Zelmanov's generally covariant criterion, also satisfy Zelmanov's chr.inv.-criterion. Therefore, the chr.inv.-components of the Riemann-Christoffel curvature tensor play a rôle of wave functions in gravitational wave fields.

Looking at the formula (4.2) of the chr.inv.-d'Alembert operator, together with Zelmanov's chr.inv.-criterion, we see two necessary conditions for *physically observable* gravitational waves:

1) The chr.inv.-quantities f are non-stationary, i.e. $\frac{^*\partial f}{\partial t} \neq 0$;
2) The field of each quantity f is inhomogeneous, i.e. $^*\nabla_i\, f_k \neq 0$.

The wave functions X_{ij}, Y_{ijk}, Z_{iklj} satisfy the requirements only if the observable mechanical characteristics of the observer's reference space (the chr.inv.-quantities F_i, A_{ik}, D_{ik}) and its observable geometric characteristic (the chr.inv.-curvature C_{iklj}) also satisfy them.

When Zelmanov began to construct his cosmological theory of an inhomogeneous anisotropic universe [16], he introduced conditions of the inhomogeneity of a finite region of space. The conditions of inhomogeneity are formulated, in the framework of the chronometrically invariant formalism, as follows [16, 17]

$$^*\nabla_i\, F_k \neq 0\,, \quad ^*\nabla_j\, A_{ik} \neq 0\,, \quad ^*\nabla_j\, D_{ik} \neq 0\,, \quad ^*\nabla_j\, C_{ik} \neq 0\,. \tag{4.6}$$

It is obvious that the wave functions X^{ij}, Y^{ijk}, Z^{iklj}, being taken under these conditions, shall be inhomogeneous as well.

Considering the chr.inv.-formulae of the gravitational inertial force F_i and the angular velocity of the rotation of space A_{ik} [16, 17]

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad \mathrm{w} = c^2\,(1 - \sqrt{g_{00}}\,)\,, \tag{4.7}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\,(F_i\, v_k - F_k\, v_i)\,, \tag{4.8}$$

we see that non-stationary states of a gravitational inertial force field are due to the non-stationarity of its gravitational potential w or the linear velocity v_i of the rotation of space, determined as

$$v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \quad v^i = -c\,g^{0i}\sqrt{g_{00}}\,, \quad v_i = h_{ik}\,v^k, \quad v^2 = h_{ik}\,v^i v^k. \qquad (4.9)$$

Two fundamental chronometrically invariant identities

$$\left.\begin{aligned} &\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0 \\ &\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0 \end{aligned}\right\} \qquad (4.10)$$

introduced by Zelmanov (I refer to them as *Zelmanov's identities*), linking F_i and A_{ik}, lead us to the conclusion that the source of non-stationary states of v_i is the vortical nature of the gravitational inertial force F_i (the vorticity means ${}^*\nabla_k F_i - {}^*\nabla_i F_k \neq 0$).

The cause of non-stationary states of the deformation D_{ik} of space, which is determined in chr.inv.-form as [16, 17]

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \quad D = h^{ik} D_{ik} = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \qquad (4.11)$$

where $h = \det\|h_{ik}\|$, is the non-stationarity of the physically observable metric tensor h_{ik}, determined by Zelmanov [16, 17] as

$$h_{ik} = -\,g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}} = -\,g_{ik} + \frac{1}{c^2}\,v_i v_k\,, \quad h^{ik} = -\,g^{ik}, \quad h^i_k = \delta^i_k\,. \qquad (4.12)$$

The non-stationarity of the chr.inv.-metric tensor h_{ik} is also the cause of non-stationary states of the chr.inv.-curvature

$$\begin{aligned} C_{lkij} = H_{lkij} - \frac{1}{c^2}\big(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + \\ + A_{kl}D_{ij} + A_{li}D_{jk}\big) \end{aligned} \qquad (4.13)$$

and the chr.inv.-quantities $C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im}C_{kimj}$ and $C = C^j_j = h^{lj}C_{lj}$ derived from it (the chr.inv.-scalar C is the *three-dimensional observable curvature*). They are determined through the Schouten chr.inv.-tensor $H_{lki\cdot}^{\cdots j}$ and the Christoffel chr.inv.-symbols Δ^k_{ij}

$$H_{lki\cdot}^{\cdots j} = \frac{{}^*\partial \Delta^j_{kl}}{\partial x^i} - \frac{{}^*\partial \Delta^j_{il}}{\partial x^k} + \Delta^m_{kl}\Delta^j_{im} - \Delta^m_{il}\Delta^j_{km}\,, \qquad (4.14)$$

$$\Delta^k_{ij} = h^{km}\Delta_{ij,m} = \frac{1}{2}\,h^{km}\left(\frac{{}^*\partial h_{im}}{\partial x^j} + \frac{{}^*\partial h_{jm}}{\partial x^i} - \frac{{}^*\partial h_{ij}}{\partial x^m}\right), \qquad (4.15)$$

which are Zelmanov's remakes [16,17] of Schouten's tensor and Christoffel's symbols according to the chronometrically invariant formalism.

Here $\frac{^*\partial}{\partial t}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+\frac{1}{c^2}v_i\frac{^*\partial}{\partial t}$ are the chr.inv.-differential operators with respect to time and the spatial coordinates.

Zelmanov [16] had obtained how the chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ are expressed through the (observable) chr.inv.-characteristics of space. These formulae, having indices lowered by the chr.inv.-metric tensor h_{ik}, are

$$X_{ij}=\frac{^*\partial D_{ij}}{\partial t}-\left(D_i^l+A_{i\cdot}^{\cdot l}\right)\left(D_{jl}+A_{jl}\right)+\frac{1}{2}\left(^*\nabla_i F_j+{}^*\nabla_j F_i\right)-\frac{1}{c^2}F_iF_j\,, \tag{4.16}$$

$$Y_{ijk}={}^*\nabla_i\left(D_{jk}+A_{jk}\right)-{}^*\nabla_j\left(D_{ik}+A_{ik}\right)+\frac{2}{c^2}A_{ij}F_k\,, \tag{4.17}$$

$$Z_{iklj}=D_{ik}D_{lj}-D_{il}D_{kj}+A_{ik}A_{lj}-A_{il}A_{kj}+2A_{ij}A_{kl}-c^2C_{iklj}\,. \tag{4.18}$$

We see from here that non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} are due to the non-stationarity of the chr.inv.-characteristics of space (F_i, A_{ik}, D_{ik}, C_{iklj}), thus — the non-stationarity of the components of the fundamental metric tensor $g_{\alpha\beta}$, namely

$$g_{00}=\left(1-\frac{\mathrm{w}}{c^2}\right)^2,\quad g_{0i}=-\frac{1}{c}\,v_i\left(1-\frac{\mathrm{w}}{c^2}\right),\quad g_{ik}=-h_{ik}+\frac{1}{c^2}\,v_iv_k\,. \tag{4.19}$$

We consider each of these cases here, mindful of the need to find theoretical grounds for the gravitational wave problem:

1) Non-stationary states of the time component g_{00} derive from the time variation of the gravitational potential w;
2) Non-stationary states of the mixed components g_{0i} derive from the non-stationarity of the rotation of space or the gravitational potential w (or from both these factors);
3) Non-stationary states of the spatial components g_{ik} derive from the aforementioned two factors as well.

The metric of weak plane gravitational waves has the form

$$ds^2=c^2dt^2-(dx^1)^2-(1+a)(dx^2)^2+2b\,dx^2dx^3-(1-a)(dx^3)^2, \tag{4.20}$$

where $a=a(ct+x^1)$ and $b=b(ct+x^1)$ if the wave travels in the direction x^1, and they are small values.

As seen, in this metric there is not a gravitational potential ($\mathrm{w}=0$) as soon as there is not rotation of space ($v_i=0$). For this reason we arrive at a very important conclusion:

> Weak plane gravitational waves are derived from sources other than gravitational fields of masses.

An analogous situation arises in relativistic cosmology, where, until this day, the main rôle is played by the theory of a homogeneous isotropic universe. This theory is based on the metric of a homogeneous isotropic space (see Chapter 1 in [16], for detail)

$$\left.\begin{aligned} & ds^2 = c^2dt^2 - R^2\frac{(dx^1)^2+(dx^2)^2+(dx^3)^2}{\left[1+\frac{k}{4}\left[(dx^1)^2+(dx^2)^2+(dx^3)^2\right]\right]^2} \\ & R = R\,(t)\,, \quad k = 0, \pm 1 \end{aligned}\right\}. \qquad (4.21)$$

When one substitutes this metric into Einstein's equations, one obtains a spectrum of solutions, which are known as homogeneous isotropic models, or the *Friedmann cosmological models* [16].

Taking our previous conclusion on the origin of weak plane gravitational waves into account, we come to another important conclusion:

> No gravitational wave fields derived from masses can exist in any Friedmann universe. Moreover, any Friedmann universe is free of gravitational inertial waves derived from the fields of rotation.

Currently there is not indubitable observational data supporting the absolute rotation of the Universe. This problem is under considerable discussion among astronomers and physicists over decades, and remains open. Rotations of bulky space bodies like planets, stars, and galaxies are beyond any doubt. But these rotations do not result from the absolute rotation of the whole Universe, including the absolute rotation of its common gravitational field.

Looking back at the question of whether or not gravitational waves and gravitational inertial waves exist, or whether or not non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} exist, we conclude that non-stationary states of the wave functions are derived from:

1) The case, where the field of the acting gravitational inertial force F_i is vortical (the non-stationarity of g_{00} and g_{0i});
2) Non-stationary states of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$.

In the first case, the effect of gravitational inertial waves or gravitational inertial waves manifests itself as non-stationary corrections to the clock of the observer. In the second case, the proper time of the observer flows unchanged, while gravitational waves or gravitational inertial waves are presented as waves of only the deformation of space.

My task herein is to construct basics of the chronometrically invariant theory of gravitational waves and gravitational inertial waves.

It is possible to show that the chr.inv.-components of the Riemann-Christoffel curvature tensor, which are the wave functions X^{ij}, Y^{ijk}, Z^{iklj}, possess the properties

$$X_{ij} = X_{ji}\,, \quad X^k_k = -\kappa c^2, \quad Y_{[ijk]} = 0\,, \quad Y_{ijk} = -Y_{ikj}\,. \tag{4.22}$$

Equations (4.4) being taken in an ortho-frame (where $g_{00}=1$, $g_{0i}=0$, and $g_{ik}=\delta_{ik}$, thus there is not difference between the covariant and contravariant components of a tensor) take the form

$$X_{ij} = -c^2 R_{0i0j}\,, \quad Y_{ijk} = -c R_{0ijk}\,, \quad Z_{iklj} = c^2 R_{iklj}\,. \tag{4.23}$$

Once we re-write the Einstein space condition $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ (3.2) in an ortho-frame, we take the formulae (4.23) into account. Then, introducing three-dimensional matrices x and y such that

$$x \equiv \|\,x_{ik}\| = -\frac{1}{c^2}\,\|\,X_{ik}\|\,, \qquad y \equiv \|\,y_{ik}\| = -\frac{1}{2c}\,\|\,\varepsilon_{imn} Y_{k\cdot\cdot}^{\cdot mn}\|\,, \tag{4.24}$$

where ε_{imn} is the three-dimensional completely antisymmetric discriminant chr.inv.-tensor, we compose a six-dimensional matrix $\|R_{ab}\|$

$$\|\,R_{ab}\| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|, \qquad a, b = 1, 2, \dots 6\,, \tag{4.25}$$

which satisfies the conditions

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad y_{11} + y_{22} + y_{33} = 0\,. \tag{4.26}$$

Now, let us compose a lambda-matrix

$$\|\,R_{ab} - \Lambda g_{ab}\| = \left\| \begin{matrix} x + \Lambda\varepsilon & y \\ y & -x - \Lambda\varepsilon \end{matrix} \right\|, \tag{4.27}$$

where ε is the three-dimensional unit matrix. After elementary transformations, we reduce this lambda-matrix to the form

$$\left\| \begin{matrix} x+iy+\Lambda\varepsilon & 0 \\ 0 & -x-iy-\Lambda\varepsilon \end{matrix} \right\| = \left\| \begin{matrix} \bar{Q}\,(\Lambda) & 0 \\ 0 & \bar{Q}\,(\Lambda) \end{matrix} \right\|. \tag{4.28}$$

As is known according to Petrov [27], the initial lambda-matrix can have only one of characteristics drawn from three kinds: I) $[111, \overline{111}]$; II) $[21, \overline{21}]$; III) $[3, 3]$. Using, according to Petrov, the canonical form of the matrix $\|R_{ab}\|$ in a non-holonomic ortho-frame for each of these three kinds of the curvature tensor, we express the matrix $\|R_{ab}\|$ through the chr.inv.-tensors X_{ij} and Y_{ijk}. We obtain, for kind I,

$$\left.\begin{array}{c} \underline{\text{Kind I}} \\ \|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \\ x = \begin{Vmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{Vmatrix}, \quad y = \begin{Vmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{Vmatrix} \end{array}\right\}, \tag{4.29}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \quad y_{11} + y_{22} + y_{33} = 0\,. \tag{4.30}$$

Using (4.24) we also express the stationary curvatures Λ_i (3.21) $(i = 1, 2, 3)$ through X_{ij} and Y_{ijk}

$$\left.\begin{aligned} \Lambda_1 &= -\frac{1}{c^2}\,X_{11} + \frac{i}{c}\,Y_{123} \\ \Lambda_2 &= -\frac{1}{c^2}\,X_{22} + \frac{i}{c}\,Y_{231} \\ \Lambda_3 &= -\frac{1}{c^2}\,X_{33} + \frac{i}{c}\,Y_{312} \end{aligned}\right\}. \tag{4.31}$$

Hence the chr.inv.-quantities X_{ik} consist of the real parts of the stationary curvatures Λ_i (the term α_i in 3.21), while the chr.inv.-quantities Y_{ijk} consist the imaginary parts (the term $i\beta_i$ in formula 3.21). In spaces of the sub-kind D $(\Lambda_2 = \Lambda_3)$ we have: $X_{22} = X_{33}$, $Y_{231} = Y_{312}$. In spaces of the sub-kind O $(\Lambda_1 = \Lambda_2 = \Lambda_3)$ we have: $X_{11} = X_{22} = X_{33} = -\frac{\kappa c^2}{3}$, $Y_{123} = Y_{231} = Y_{312} = 0$. Hence Einstein spaces of the sub-kind O have only real curvatures, while being empty they are flat.

For kind II we have

$$\left.\begin{array}{c} \underline{\text{Kind II}} \\ \|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \\ x = \begin{Vmatrix} x_{11} & 0 & 0 \\ 0 & x_{22}+1 & 0 \\ 0 & 0 & x_{22}-1 \end{Vmatrix}, \quad y = \begin{Vmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 1 \\ 0 & 1 & y_{22} \end{Vmatrix} \end{array}\right\}, \tag{4.32}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \quad x_{22} - x_{33} = 2\,, \quad y_{11} + 2y_{22} = 0\,. \tag{4.33}$$

The stationary curvatures in this case are

$$\left.\begin{aligned} \Lambda_1 &= -\frac{1}{c^2}\,X_{11} + \frac{i}{c}\,Y_{123} \\ \Lambda_2 &= -\frac{1}{c^2}\,X_{22} - 1 + \frac{i}{c}\,Y_{231} \\ \Lambda_3 &= -\frac{1}{c^2}\,X_{33} + 1 + \frac{i}{c}\,Y_{312} \end{aligned}\right\}. \tag{4.34}$$

From these results we conclude that the stationary curvatures Λ_2 and Λ_3 can never become zero in this case, so Einstein spaces (gravitational fields) of kind II are curved in any case. They cannot approach a flat space.

In spaces of kind II ($\Lambda_1 = \Lambda_2 = 0$; if this is the sub-kind N of kind II, there is also $\kappa = 0$), in an ortho-frame, we have

$$X_{11} = X_{22} - \kappa c^2 = X_{33} + \kappa c^2, \qquad Y_{123} = Y_{231} = Y_{312} = 0\,, \tag{4.35}$$

so the stationary curvatures take real numerical values. In an empty space of this kind, the matrices x and y are degenerate (determinants of these matrices are zero). For this reason spaces of the sub-kind N of kind II are *degenerate*. Thus, I refer to gravitational fields which fill spaces of the sub-kind N of kind II as *degenerate gravitational fields*. In emptiness ($\kappa = 0$) several elements of the matrices x and y take the numerical values $+1$ and -1 thereby making an ultimate transition to a flat space impossible.

For kind III we have

$$\left.\begin{gathered} \underline{\text{Kind III}} \\ \|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \\ x = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad y = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix} \end{gathered}\right\}. \tag{4.36}$$

Here the stationary curvatures are zero and both the matrices x and y are degenerate. Einstein spaces of kind III can only be empty ($\kappa = 0$), but, at the same time, they can never be flat.

These are the basics of the chronometrically invariant theory of gravitational waves and gravitational inertial waves, which I have introduced in this paragraph for the case of empty Einstein spaces. Numerous important conclusions follow from the theory.

The conclusions are related to the (observable) chr.inv.-components X^{ik}, Y^{ijk}, Z^{ijkm} of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$, which are wave functions in a wave gravitational field. Further, I will refer to the chr.inv.-components according to their physical meaning:

1) X^{ik}, as a projection onto the line of time, manifests the variation of the curvature tensor with time at the same location. This is the *stationary observable component* of the curvature tensor;
2) Y^{ijk} is a mixed (space-time) projection. It manifests a shift of the time variation of the curvature tensor with the variation of the three-dimensional (spatial) coordinates. This is the *dynamical observable component* of the curvature tensor. This is a "truly gravitational wave component", which, being nonzero ($Y^{ijk} \neq 0$), manifests the presence of gravitational waves or gravitational inertial waves travelling in space;
3) Z^{ijkm}, which is a purely spatial projection, is an "instant three-dimensional shot" (or "section") of the curvature tensor. This is the *distributive observable component*.

Proceeding from the equations deduced for the canonical form of the matrix $\| R_{ab} \|$, obtained in the framework of the chr.inv.-theory, we conclude:

> The dynamical observable component Y^{ijk} of the curvature tensor can be zero ($Y^{ijk} = 0$) only in spaces of kind I (the stationary curvatures take real values in this case). Moreover, $Y^{ijk} = 0$ in all known metrics of kind I. Gravitational fields of spaces of kind I are derived from islands of mass located in emptiness. Thus, gravitational waves and gravitational inertial waves cannot derive from islands of mass located in an empty space (at least, in the framework of all known metrics of kind I).

In particular, this means that search for gravitational radiation, targeting rotating cosmic bodies in emptiness as its source, cannot be a proper experimental test to the General Theory of Relativity.

According to most of the gravitational wave criteria, the presence of gravitational waves is linked to spaces of the sub-kind N of kind II, and kind III, where the matrix y_{ik} has components equal to $+1$ or -1. Moreover, in the fields of the sub-kind N of kind II, and kind III, the numerical values $+1$ or -1 are attributed also to components of the matrix x_{ik}. This implies that:

> Spaces, which contain gravitational fields satisfying the gravitational wave criteria (these are spaces of the sub-kind N of kind II,

and kind III), are curved independently of whether or not they are empty ($R_{\alpha\beta}=0$) or filled with distributed matter ($R_{\alpha\beta}=\kappa g_{\alpha\beta}$). In any case, in these spaces gravitational radiation is derived from the "interaction" between the stationary observable component X^{ij} and the dynamical observable component Y^{ijk} of the curvature tensor, which are nonzero therein.

Petrov's classification of spaces (gravitational fields) applied here to the gravitational wave problem is valid only to Einstein spaces. Solving this problem for spaces of a general kind, where $R_{\alpha\beta}\neq\kappa g_{\alpha\beta}$, is a highly complicate task due to some mathematical difficulties. Namely, when having an arbitrary distribution of matter in a space, the matrix $\|R_{ab}\|$, taken in a non-holonomic ortho-frame, is not symmetrically doubled; on the contrary, the matrix takes the form

$$\|R_{ab}\| = \left\| \begin{array}{cc} x & y \\ y' & z \end{array} \right\|, \tag{4.37}$$

where the three-dimensional matrices x, y, z are constructed on the following elements, respectively*

$$\left.\begin{aligned} x_{ik} &= -\frac{1}{c^2}\,X_{ik} \\ z_{ik} &= \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq} \\ y_{ik} &= \frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn} \end{aligned}\right\}, \tag{4.38}$$

and y' means transposition. It is obvious that reducing this matrix to its canonical form will meet severe mathematical difficulties, thus becoming a highly complicate task.

Nevertheless Petrov's classification, which has successfully been applied here to the chr.inv.-theory of gravitational waves and gravitational inertial waves, allows us to conclude:

The stationary observable component X^{ij} and the dynamical observable component Y^{ijk} of the curvature tensor are different in their physical origin†. Space metrics can exist even in a case, where

*In ortho-frames there is not difference between the covariant and contravariant components of a tensor [27]. Therefore, we can replace $z_{ik}=\frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq}$ and $y_{ik}=\frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}$ with $z_{ik}=\frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z_{mnpq}$ and $y_{ik}=\frac{1}{2c}\,\varepsilon_{imn}Y_{kmn}$ in (4.38). This can also be applied to the equations of formula (4.24).

†We do not discuss the spatial observable component Z^{iklj}, because, in an ortho-frame, the matrices x and z are connected by the ratio $x=-z$. Therefore, the components X^{ik} and Z^{iklj} are connected to each other in this case.

$Y^{ijk} = 0$ but $X^{ij} \neq 0$ and $Z^{iklj} \neq 0$ (these are spaces of kind I). However, among all known solutions of Einstein's equations, there is not a metric for which $Y^{ijk} \neq 0$ but $X^{ij} = 0$ and $Z^{iklj} = 0$. Therefore, in gravitational wave fields and gravitational inertial wave fields, $Y^{ijk} \neq 0$ and $X^{ij} \neq 0$ (and $Z^{iklj} \neq 0$ as well: see the footnote on page 56) everywhere and always.

§5. Physical conditions of the existence of gravitational waves in non-empty spaces. In §4, I suggested a chr.inv.-theory of gravitational waves and gravitational inertial waves for empty Einstein spaces. Now, I extend the theory to non-empty Einstein spaces.

As was shown in §4, in the framework of the chr.inv.-theory of gravitational waves and gravitational inertial waves, the necessary condition of the existence of the waves are the inhomogeneity and non-stationarity of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}, which are the observable components of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$. The conditions of homogeneity in the presence of distributed matter (medium) are formulated, in the framework of the chronometrically invariant formalism [16], as follows

$$\left.\begin{array}{l} {}^*\nabla_i F_k = 0\,, \quad {}^*\nabla_j A_{ik} = 0\,, \quad {}^*\nabla_j D_{ik} = 0\,, \quad {}^*\nabla_j C_{ik} = 0 \\ \dfrac{{}^*\partial\rho}{\partial x^i} = 0\,, \quad {}^*\nabla_j J_i = 0\,, \quad {}^*\nabla_j U_{ik} = 0 \end{array}\right\}, \tag{5.1}$$

where ρ, $J_i = h_{ik}J^k$, and $U_{ik} = h_{im}h_{kn}U^{mn}$ are the observable density of matter, the observable density of momentum, and the observable stress tensor, which are the respective chr.inv.-projections

$$\rho = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{cT_0^i}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik} \tag{5.2}$$

of the energy-momentum tensor $T_{\alpha\beta}$ of the matter (from which we can also obtain $U = h^{ik}U_{ik}$).

Once the conditions of inhomogeneity (5.1) are satisfied, the wave functions represented by f in Zelmanov's chr.inv.-criterion for gravitational waves and gravitational inertial waves (4.3) are homogeneous as well, thus the d'Alembertian (4.3) becomes trivial.

Now, let us study the conditions of the non-stationarity of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} in the presence of a distributed matter. To do it, we should express them through the chr.inv.-characteristics of the matter. We will use Einstein's equations and also the conservation law of the energy-momentum tensor, written in chr.inv.-form. In [16], Zelmanov considered Einstein's generally covariant equations (3.1) in the general case, where any kind of distributed matter is presented: the formula for the energy-momentum tensor $T_{\alpha\beta}$ is not detailed there. According to Zelmanov, they have chr.inv.-projections as follows (I refer to them as the *Einstein chr.inv.-equations*)

$$\frac{^*\partial D}{\partial t}+D_{jl}D^{jl}+A_{jl}A^{lj}+{}^*\nabla_j F^j-\frac{1}{c^2}F_jF^j=-\frac{\varkappa}{2}\left(\rho c^2+U\right)+\lambda c^2, \quad (5.3)$$

$$^*\nabla_j\left(h^{ij}D-D^{ij}-A^{ij}\right)+\frac{2}{c^2}F_jA^{ij}=\varkappa J^i, \quad (5.4)$$

$$\frac{^*\partial D_{ik}}{\partial t}-\left(D_{ij}+A_{ij}\right)\left(D_k^j+A_{k\cdot}^{\cdot j}\right)+DD_{ik}+3A_{ij}A_{k\cdot}^{\cdot j}+$$
$$+\frac{1}{2}\left({}^*\nabla_iF_k+{}^*\nabla_kF_i\right)-\frac{1}{c^2}F_iF_k-c^2C_{ik}=$$
$$=\frac{\varkappa}{2}\left(\rho c^2h_{ik}+2U_{ik}-Uh_{ik}\right)+\lambda c^2h_{ik}\,, \quad (5.5)$$

where $^*\nabla_i$ is the symbol of chr.inv.-differentiation (a chr.inv.-analogue to the symbol ∇_σ of generally covariant differentiation). He also considered the general covariant conservation law equation

$$\nabla_\sigma T^{\alpha\sigma}=0 \quad (5.6)$$

of the energy-momentum tensor (also in the general case of arbitrary matter). It has the following chr.inv.-projections [16]

$$\frac{^*\partial\rho}{\partial t}+D\rho+\frac{1}{c^2}D_{ij}U^{ij}+{}^*\nabla_iJ^i-\frac{2}{c^2}F_iJ^i=0\,, \quad (5.7)$$

$$\frac{^*\partial J^k}{\partial t}+DJ^k+2\left(D_i^k+A_{i\cdot}^{\cdot k}\right)J^i+{}^*\nabla_iU^{ik}-\frac{2}{c^2}F_iU^{ik}-\rho F^k=0\,. \quad (5.8)$$

We begin the study from the simplest case, where all kinematic characteristics of a non-empty space are zero. In this case, the reference frame of the observer (his local space of reference) falls freely, is free of rotation, and does not deform. In other words,

$$F_i=0\,,\qquad A_{ik}=0\,,\qquad D_{ik}=0\,, \quad (5.9)$$

thus the chr.inv.-components of the curvature tensor (the wave functions) take the form

$$X^{ik}=0\,,\qquad Y^{ijk}=0\,,\qquad Z^{iklj}=-c^2C^{iklj}. \quad (5.10)$$

It is easy to see that, in this case, the solely nonzero component Z^{iklj}

of the curvature tensor is stationary. Therefore, gravitational waves and gravitational inertial waves are impossible in this case.

Construct the metric of a respective space (space-time) for this case. The conditions $F_i = 0$ and $A_{ik} = 0$ mean, respectively, that $g_{00} = 1$ and $g_{0i} = 0$ in the space. The fact that the space does not deform ($D_{ik} = 0$) points to the stationarity of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$. According to Cotton [45], in this case the three-dimensional metric can be reduced to diagonal form. Therefore, a space which satisfies the physical conditions (5.9) is a reducible space, whose metric takes the form

$$ds^2 = c^2 dt^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2, \tag{5.11}$$

where the components g_{ii} do not depend on time.

Thus, we arrive at the following obvious conclusion:

> In non-empty spaces, whose all kinematic characteristics are zero, gravitational waves and gravitational inertial waves are impossible due to the stationarity of all the chr.inv.-components of the curvature tensor (the wave functions of space).

Consider another kind of non-empty spaces, which do not contain fields of acceleration (the gravitational potential is homogeneously distributed therein), do not deform, but rotate. A typical instance of such spaces are those described by Gödel's metric [46], where

$$F_i = 0\,, \qquad D_{ik} = 0\,, \qquad A_{ik} \neq 0\,. \tag{5.12}$$

The first condition of these, $F_i = 0$, according to the chronometrically invariant formalism, means

$$g_{00} = 1\,, \qquad \frac{{}^*\partial g_{0i}}{\partial t} = 0\,, \tag{5.13}$$

therefore the rotation of a Gödel space is stationary. Because the chr.inv.-metric tensor has the form $h_{ik} = -\,g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}$, we see that g_{ik} does not depend on time in this case. This means, being applied to the wave functions X^{ik}, Y^{ijk}, Z^{iklj}, that

$$\frac{{}^*\partial X^{ik}}{\partial t} = 0\,, \qquad \frac{{}^*\partial Y^{ijk}}{\partial t} = 0\,, \qquad \frac{{}^*\partial Z^{iklj}}{\partial t} = 0\,, \tag{5.14}$$

i.e. Gödel's metric is completely stationary. Hence,

> In non-empty spaces, which do not deform but rotate with a constant linear velocity, gravitational waves and gravitational inertial waves are impossible.

Therefore, in searching for gravitational radiation, where binary stars are targeted as its source, we should focus onto only those binaries, whose rotation is non-stationary. In particular, if a satellite-star decelerates (due to some reasons) when orbiting the main star, the binary system should emit gravitational radiation.

Now, consider that case of non-empty spaces, where spaces do not deform, do not rotate, but contain fields of acceleration (the gravitational inertial force is nonzero therein). In this case,

$$F_i \neq 0\,, \qquad A_{ik} = 0\,, \qquad D_{ik} = 0\,. \tag{5.15}$$

The condition $A_{ik}=0$ means $g_{0i}=0$. The condition $D_{ik}=0$, as was explained above, means that the observable metric h_{ik} of the space is stationary, hence g_{ik} does not depend on time: in this case, according to Cotton [45], the three-dimensional metric can be transformed to diagonal form. Finally, the metric of such a space takes the form

$$ds^2 = g_{00}\,(ct,\, x^1,\, x^2,\, x^3)\,c^2dt^2 + g_{ii}\,(x^1,\, x^2,\, x^3)(dx^i)^2, \tag{5.16}$$

so the chr.inv.-components of the curvature tensor take the form

$$\left.\begin{aligned} X_{ik} &= \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}\,F_iF_k \\ Y^{ijk} &= 0\,, \qquad Z_{iklj} = -\,c^2 C_{iklj} \end{aligned}\right\}. \tag{5.17}$$

Due to the absence of the rotation and deformation, the wave function Z^{iklj} is stationary. So, only the non-stationarity of the wave function X^{ik} can be supposed. Using the Einstein chr.inv.-equations while taking the physical conditions (5.15) into account, we express X^{ik} (5.17) through the chr.inv.-characteristics of the distributed matter

$$X_{ik} = c^2C_{ik} + \frac{\varkappa}{2}\left(\rho c^2h_{ik} + 2U_{ik} - Uh_{ik}\right) + \lambda c^2h_{ik}\,. \tag{5.18}$$

This, however, does not matter in this case. Anyhow, due to the fact that the dynamical observable component Y^{ijk} of the curvature tensor is zero in such spaces, we immediately arrive at the folowing conclusion:

> In non-empty spaces, which contain fields of the gravitational inertial force, but are free of rotation and deformation, gravitational waves and gravitational inertial waves are impossible.

Now, the last case of non-empty spaces remains under focus. In this case, the space does not deform, but rotates and contains the field of the gravitational inertial force

$$F_i \neq 0\,, \qquad A_{ik} \neq 0\,, \qquad D_{ik} = 0\,. \tag{5.19}$$

Running ahead of the obtained result, I announce that this is the most interesting case of non-deforming non-empty Einstein spaces, because it permits gravitational radiation.

The wave functions in this case take the form

$$X^{ik} = 3A^{i\cdot}_{\cdot j}A^{kj} - c^2C^{ik} + \frac{\varkappa}{2}\left(\rho c^2 h^{ik} + 2U^{ik} - Uh^{ik}\right) + \lambda h^{ik}, \quad (5.20)$$

$$Y^{ijk} = {}^*\nabla^j A^{ik} - {}^*\nabla^i A^{jk} + \frac{2}{c^2}\,A^{ji}F^k, \quad (5.21)$$

$$Z^{iklj} = A^{ik}A^{lj} - A^{il}A^{kj} + 2\,A^{ij}A^{kl} - c^2C^{iklj}. \quad (5.22)$$

Analyzing the formulae, we apply Zelmanov's 1st identity (4.10), which links the non-stationarity of A_{ik} to the vortex of F_i. We take into account that $\frac{{}^*\partial A_{ik}}{\partial t} = h_{im}h_{kn}\frac{{}^*\partial A^{mn}}{\partial t}$ in non-deforming spaces. We obtain that: 1) the non-stationarity of X^{ik} can be due to the vortex of the gravitational inertial force F_i, the non-stationarity of the factors of the observable three-dimensional curvature C^{ik}, the observable components of the energy-momentum tensor, and the cosmological term, or due to all these factors; 2) the non-stationarity of Y^{ijk} can only be due to the common presence of the vortex of the field F_i and the non-stationarity of the force F_i; 3) the non-stationarity of Z^{iklj} can be due to the vortex of the field F_i or the non-stationarity of the observable three-dimensional curvature C^{iklj}, or due to both these factors.

As was explained in §4, page 55, the dynamical observable component Y^{ijk} of the Riemann-Christoffel curvature tensor is a "truly gravitational wave component", which manifests the presence of gravitational waves or gravitational inertial waves travelling in space. The fact that $Y^{ijk} \neq 0$ in spaces of this kind means that gravitational waves and gravitational inertial waves are possible therein.

Because $Y^{ijk} \neq 0$ (5.21) in the case, we obtain $J^i \neq 0$ from the Einstein chr.inv.-vectorial equation (5.4), and $\frac{{}^*\partial\rho}{\partial t} \neq 0$ due to the chr.inv.-scalar conservation equation (5.7).

The first result, $J^i \neq 0$, implies the presence of a flow of energy-momentum of the medium that fills the space. In other word, the observer (and his frame of reference) does not accompany the medium, but moves with respect to it. As was already shown in §2, the travelling rays of gravitational radiation in emptiness are isotropic geodesics (the rays of the light's travel). Hence, gravitational wave fields and gravitational inertial wave fields are non-isotropic in spaces of this kind: the waves travel at another velocity than light, depending on the properties of the medium.

The second result, $\frac{{}^*\partial\rho}{\partial t} \neq 0$, means that the density of the medium does not remain stationary, but changes with time according to the transit of gravitational waves and gravitational inertial waves. In a barotropic medium, as we know, $p = p\,(\rho)$ is true. Therefore, if a space of this kind is filled with a barotropic medium, gravitational waves and gravitational inertial waves travelling therein are linked to the non-stationarity of the pressure. If a space of this kind is filled with a baroclinе medium (it is characterized by the condition $p = p\,(\rho, T)$, where T is the absolute temperature of the medium), gravitational waves and gravitational inertial waves are linked to the non-stationarity of the pressure and temperature.

Thus, concerning non-empty spaces characterized by the physical conditions (5.19), we conclude:

> Non-empty spaces, which do not deform, but rotate and contain fields of the gravitational inertial force, gravitational waves and gravitational inertial waves are possible. In a barotropic medium, the waves are linked to the non-stationarity of the pressure, while in a barocline medium they are linked to the non-stationary of the pressure and temperature. The waves travel with a velocity different than that of light, depending on the properties of the medium that fills the space.

An important note should be said in the end. When we considered the physical conditions of the existence of gravitational waves in non-empty spaces, we meant that the spaces do not deform ($D_{ik} = 0$). This has been the main assumption and task of this study. As a matter of fact, gravitational waves and gravitational inertial waves can exist in deforming spaces as waves of the space deformation. Therefore, all that has been obtained in this paragraph is related only to non-deforming spaces. The main result obtained herein is:

> It is not necessary that only the deformation of space is the source of gravitational waves and gravitational inertial waves. The waves can exist even in non-deforming spaces, if the gravitational inertial force F_i and the rotation of space A_{ik} differ from zero, and the field F_i is vortical (that means the non-stationarity of A_{ik}).

§6. Chronometrically invariant representation of Petrov's classification for non-empty spaces. In §4, I suggested a chr.inv.-theory of gravitational waves and gravitational inertial waves in empty Einstein spaces. The geometrical structure of Einstein spaces of all three kinds was presented in terms of chronometric invariants. This study was extended to non-empty Einstein spaces in §5: physical conditions of the existence of gravitational radiation in medium were discussed. Now, I shall obtain chr.inv.-components of Weyl's conformal curvature tensor, and find their connexion with the chr.inv.-components of the Riemann-Christoffel curvature tensor. The main task of this deduction is understanding the rôle of matter in forming gravitational radiation in non-empty non-Einstein spaces.

Petrov's classification of spaces where $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ (Einstein spaces) was a resolvable mathematical problem, because the matrix $\|R_{ab}\|$ of the Riemann-Christoffel curvature tensor in an ortho-frame of a six-dimensional Riemannian space is symmetrically doubled due to Einstein's equations. In the case where a space is filled with distributed matter of an arbitrary kind, Einstein's equations manifest that

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y' & z \end{matrix} \right\|, \tag{6.1}$$

where y' is a matrix transposed to the matrix y. This fact makes classification of the curvature tensor in non-empty spaces a very difficult task (see page 56). Therefore, Petrov [27] suggested another solution to this problem. He had constructed a special curvature tensor

$$P_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - S_{\alpha\beta\gamma\delta} + \sigma \left(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma} \right), \tag{6.2}$$

which satisfies all algebraic properties of the Riemann-Christoffel tensor in non-empty spaces, while the additional tensor $S_{\alpha\beta\gamma\delta}$, which takes the energy-momentum tensor (distributed matter) into account, possesses all the properties as well, i.e.

$$S_{\alpha\beta\gamma\delta} = \frac{\varkappa}{2} \left(g_{\alpha\beta}\, T_{\delta\gamma} - g_{\alpha\gamma}\, T_{\beta\delta} + g_{\beta\gamma}\, T_{\alpha\delta} - g_{\beta\delta}\, T_{\alpha\gamma} \right). \tag{6.3}$$

After contraction of the tensor by indices β and δ, and taking Einstein's equations into account, we obtain

$$P_{\alpha\gamma} = (R + 3\sigma)\, g_{\alpha\gamma}\,, \tag{6.4}$$

where σ is a scalar. Once distribution of matter (the energy-momentum tensor $T_{\alpha\beta}$) has been determined, the curvature of space can be found with a precision to within the scalar σ. However the physical meaning of the scalar is still unclear. Therefore, in order to introduce an algebraic classification of non-empty spaces, Weyl's conformal curvature tensor

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2} \left(R_{\alpha\gamma}\, g_{\beta\delta} + R_{\beta\delta}\, g_{\alpha\gamma} - R_{\alpha\delta}\, g_{\beta\gamma} - R_{\beta\gamma}\, g_{\alpha\delta} \right) + \\ + \frac{R}{6} \left(g_{\beta\gamma}\, g_{\alpha\delta} - g_{\beta\delta}\, g_{\alpha\gamma} \right) \tag{6.5}$$

should be applied. This tensor also possesses all the algebraical properties of the Riemann-Christoffel curvature tensor. Also, contracting it by indices β and δ, we obtain

$$C_{\alpha\gamma} = 0\,. \tag{6.6}$$

All these mean that applying Weyl's contracted tensor $C_{\alpha\beta}$ to non-empty spaces, we arrive at an analogy to Ricci's tensor $R_{\alpha\beta}$. Therefore, classification of non-empty non-Einstein spaces according to the algebraic properties of Weyl's conformal curvature tensor $C_{\alpha\beta\gamma\delta}$ should be analogous to Petrov's classification of Einstein spaces. The difference is only that the matrices $\widetilde{x}$ and $\widetilde{y}$ should be used in Weyl's tensor, instead the matrices x and y of the Riemann-Christoffel tensor.

Here we suggest an algebraic classification of Weyl's conformal curvature tensor in terms of chronometric invariants. First, we define the (observable) chr.inv.-components of Weyl's tensor

$$\widetilde{X}^{ik} = -\,c^2\,\frac{C_{0\cdot0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \qquad \widetilde{Y}^{ijk} = -\,c\,\frac{C_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \qquad \widetilde{Z}^{iklj} = c^2 C^{ijkl}, \tag{6.7}$$

which are formulated in analogy to those of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ (4.4) as well as those of any 4th rank tensor of the antisymmetric kind as these tensors. The chr.inv.-components (6.7) possess the following properties

$$\widetilde{X}_{ik} = \widetilde{X}_{ki}\,, \quad \widetilde{X}_k^k = 0\,, \quad \widetilde{Y}_{[ijk]} = 0\,, \quad \widetilde{Y}_{ijk} = -\,Y_{ikj}\,, \tag{6.8}$$

where Y_{ikj} is that of $R_{\alpha\beta\gamma\delta}$ (4.4). In an ortho-frame, we have

$$\widetilde{X}_{ik} = -\,c^2\,C_{0i0k}\,, \quad \widetilde{Y}_{ijk} = -\,c\,C_{oijk}\,, \quad \widetilde{Z}_{iklj} = c^2\,C_{iklj}\,. \tag{6.9}$$

Now, we express the chr.inv.-components of Weyl's tensor through the (observable) chr.inv.-characteristics of the distributed matter that fills the space. To do it, we apply the Einstein chr.inv.-equations (they were presented in §5). In an ortho-frame, we obtain

$$C_{0i0k} = -\frac{1}{c^2}\,X_{ik} - \frac{\varkappa}{2c^2}\,U_{ik} + \frac{\varkappa\rho}{6}\,h_{ik} + \frac{\varkappa c^2}{3}\,U h_{ik}\,, \tag{6.10}$$

$$C_{i0jk} = \frac{1}{c}\,Y_{ijk} - \frac{\varkappa}{2c}\left(h_{ik}J_i - h_{ij}J_k\right), \tag{6.11}$$

$$C_{iklj} = \frac{1}{c^2}\,Z_{iklj} - \frac{\varkappa}{2c^2}\left(h_{ij}U_{kl} - h_{il}U_{kj} + h_{kl}U_{ij} - h_{kj}U_{il}\right) - \\ - \frac{\varkappa}{3}\left(\rho - \frac{U}{c^2}\right)\left(h_{ik}h_{jl} - h_{il}h_{jk}\right). \tag{6.12}$$

In analogy to (4.24), we introduce three-dimensional matrices

$$\left.\begin{aligned} \widetilde{x} &= \|\,\widetilde{x}_{ik}\,\| = -\frac{1}{c^2}\left\|\widetilde{X}_{ik}\right\| \\ \widetilde{y} &= \|\,\widetilde{y}_{ik}\,\| = \frac{1}{2c}\left\|\varepsilon_{imn}\widetilde{Y}_{k\cdot\cdot}^{\cdot mn}\right\| \\ \widetilde{z} &= \|\,\widetilde{z}_{ik}\,\| = \frac{1}{4c^2}\left\|\varepsilon_{imn}\,\varepsilon_{kpq}\,\widetilde{Z}^{mnpq}\right\| \end{aligned}\right\}. \qquad (6.13)$$

It is possible to show, from the Einstein equations $C_{\alpha\beta}=0$ written in an ortho-frame, in analogy to Petrov [27] who did it for Einstein's original equations $R_{\alpha\beta}=0$, that $\widetilde{x}_{ik}=-\,\widetilde{z}_{ik}$. Therefore, we compose a six-dimensional matrix $\|C_{ab}\|$ from Weyl's conformal curvature tensor $C_{\alpha\beta\gamma\delta}$. We obtain a symmetrically paired matrix

$$\|C_{ab}\| = \left\|\begin{matrix} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{matrix}\right\| \qquad (6.14)$$

whose elements are connected by the relations

$$\widetilde{x}_{11}+\widetilde{x}_{22}+\widetilde{x}_{33}=0\,, \qquad \widetilde{y}_{11}+\widetilde{y}_{22}+\widetilde{y}_{33}=0\,, \qquad (6.15)$$

and, as is possible to show, the diagonal components of the matrix $\widetilde{y}$ meet the respective diagonal components of the matrix y.

$$\left.\begin{aligned} \widetilde{y}_{11} &= \frac{1}{c}\,\widetilde{Y}_{123} = \frac{1}{c}\,Y_{123} = y_{11} \\ \widetilde{y}_{22} &= \frac{1}{c}\,\widetilde{Y}_{231} = \frac{1}{c}\,Y_{321} = y_{22} \\ \widetilde{y}_{33} &= \frac{1}{c}\,\widetilde{Y}_{312} = \frac{1}{c}\,Y_{312} = y_{33} \end{aligned}\right\}. \qquad (6.16)$$

Composing a lambda-matrix $\|\,C_{ab}-\Lambda\,g_{ab}\,\|$ then reducing it to the canonical form in analogy to Petrov, who did it for the lambda-matrix $\|\,R_{ab}-\Lambda\,g_{ab}\,\|$, we obtain three kinds of non-empty non-Einstein spaces, which are characterized according to Weyl's tensor.

After transformations, we obtain the lambda-matrix $\|\,C_{ab}-\Lambda\,g_{ab}\,\|$ in the form

$$\|C_{ab}-\Lambda\,g_{ab}\| = \left\|\begin{array}{c|c} \widetilde{x}+i\widetilde{y}+\Lambda\varepsilon & 0 \\ \hline 0 & \widetilde{x}-i\widetilde{y}+\Lambda\varepsilon \end{array}\right\| \equiv$$

$$\equiv \left\|\begin{matrix} Q(\Lambda) & 0 \\ 0 & \bar{Q}\,(\Lambda) \end{matrix}\right\|. \qquad (6.17)$$

In analogy to Petrov's classification of the matrix $\|R_{ab}\|$, we obtain, in an ortho-frame, three respective kinds of the matrix $\|C_{ab}\|$

$$\left.\begin{array}{c} \underline{\text{Kind I}} \\ \|C_{ab}\| = \left\| \begin{array}{cc} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{array} \right\|, \\ \widetilde{x} = \left\| \begin{array}{ccc} \widetilde{x}_{11} & 0 & 0 \\ 0 & \widetilde{x}_{22} & 0 \\ 0 & 0 & \widetilde{x}_{33} \end{array} \right\|, \qquad \widetilde{y} = \left\| \begin{array}{ccc} \widetilde{y}_{11} & 0 & 0 \\ 0 & \widetilde{y}_{22} & 0 \\ 0 & 0 & \widetilde{y}_{33} \end{array} \right\| \end{array} \right\}, \tag{6.18}$$

where $\widetilde{x}_{11} + \widetilde{x}_{22} + \widetilde{x}_{33} = 0$, $\widetilde{y}_{11} + \widetilde{y}_{22} + \widetilde{y}_{33} = 0$ (so in this case there are 4 independent parameters, determining the space structure by an invariant form),

$$\left.\begin{array}{c} \underline{\text{Kind II}} \\ \|C_{ab}\| = \left\| \begin{array}{cc} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{array} \right\|, \\ \widetilde{x} = \left\| \begin{array}{ccc} \widetilde{x}_{11} & 0 & 0 \\ 0 & \widetilde{x}_{22}+1 & 0 \\ 0 & 0 & \widetilde{x}_{22}-1 \end{array} \right\|, \quad \widetilde{y} = \left\| \begin{array}{ccc} \widetilde{y}_{11} & 0 & 0 \\ 0 & \widetilde{y}_{22} & 1 \\ 0 & 1 & \widetilde{y}_{22} \end{array} \right\| \end{array} \right\}, \tag{6.19}$$

where $\widetilde{x}_{11} + \widetilde{x}_{22} + \widetilde{x}_{33} = 0$, $\widetilde{x}_{22} - \widetilde{x}_{33} = 2$, $\widetilde{y}_{11} + 2\widetilde{y}_{22} = 0$ (so in this case there are 2 independent parameters determining the space structure by an invariant form),

$$\left.\begin{array}{c} \underline{\text{Kind III}} \\ \|C_{ab}\| = \left\| \begin{array}{cc} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{array} \right\|, \\ \widetilde{x} = \left\| \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\|, \qquad \widetilde{y} = \left\| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array} \right\| \end{array} \right\}. \tag{6.20}$$

As was shown in §3, the diagonal components of the matrices x and y represent, respectively, the real and imaginary parts of stationary curvatures $\Lambda_i = \alpha_i + i\beta_i$ $(i = 1, 2, 3)$ of the Riemann-Christoffel curvature tensor. Accordingly, we obtain stationary curvatures of Weyl's conformal curvature tensor. They are

$$\widetilde{\Lambda}_1 = \widetilde{x}_{11} + i\widetilde{y}_{11}\,, \quad \widetilde{\Lambda}_2 = \widetilde{x}_{22} + i\widetilde{y}_{22}\,, \quad \widetilde{\Lambda}_3 = \widetilde{x}_{33} + i\widetilde{y}_{33}\,. \tag{6.21}$$

As was mentioned above, the diagonal components of the matrix $\widetilde{y}$ coincide with the respective diagonal components of the matrix y.

Now, we write the formulae of the stationary curvatures while taking into account the obtained formulae of the components of Weyl's tensor $C_{\alpha\beta\gamma\delta}$, expressed through the chr.inv.-properties of the medium that fills the space. We obtain, for all three kinds of non-empty non-Einstein spaces, respectively

$$\left.\begin{aligned} &\underline{\text{Kind } \widetilde{\text{I}}} \\ \widetilde{\Lambda}_1 &= -\frac{1}{c^2}\,X_{11} - \frac{\varkappa}{2c^2}\,U_{11} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{123} \\ \widetilde{\Lambda}_1 &= -\frac{1}{c^2}\,X_{22} - \frac{\varkappa}{2c^2}\,U_{22} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{231} \\ \widetilde{\Lambda}_1 &= -\frac{1}{c^2}\,X_{33} - \frac{\varkappa}{2c^2}\,U_{33} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{312} \end{aligned}\right\}, \tag{6.22}$$

$$\left.\begin{aligned} &\underline{\text{Sub-kind } \widetilde{\text{D}} \text{ of kind } \widetilde{\text{I}} \;\; (\widetilde{\Lambda}_2 = \widetilde{\Lambda}_3)} \\ &X_{22} - X_{33} = \frac{\varkappa}{2}\,(U_{33} - U_{22}) \\ &Y_{231} = Y_{312} \end{aligned}\right\}, \tag{6.23}$$

$$\left.\begin{aligned} &\underline{\text{Sub-kind } \widetilde{\text{O}} \text{ of kind } \widetilde{\text{I}} \;\; (\widetilde{\Lambda}_1 = \widetilde{\Lambda}_2 = \widetilde{\Lambda}_3)} \\ &X_{11} + \frac{\varkappa}{2}\,U_{11} = X_{22} + \frac{\varkappa}{2}\,U_{22} = X_{33} + \frac{\varkappa}{2}\,U_{33} \\ &Y_{123} = Y_{231} = Y_{312} = 0 \end{aligned}\right\}, \tag{6.24}$$

$$\left.\begin{aligned} &\underline{\text{Kind } \widetilde{\text{II}}} \\ \widetilde{\Lambda}_1 &= -\frac{1}{c^2}\,X_{11} - \frac{\varkappa}{2c^2}\,U_{11} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{123} \\ \widetilde{\Lambda}_2 &= -\frac{1}{c^2}\,X_{22} - 1 - \frac{\varkappa}{2c^2}\,U_{22} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{231} = \\ &= -\frac{1}{c^2}\,X_{33} + 1 - \frac{\varkappa}{2c^2}\,U_{22} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{231} \end{aligned}\right\}, \tag{6.25}$$

$$\left.\begin{aligned} &\underline{\text{Sub-kind } \widetilde{\text{N}} \text{ of kind } \widetilde{\text{II}} \;\; (\widetilde{\Lambda}_1 = \widetilde{\Lambda}_2)} \\ &X_{11} + \frac{\varkappa}{2}\,U_{11} = X_{22} + \frac{\varkappa}{2}\,U_{22} - c^2 = X_{33} + \frac{\varkappa}{2}\,U_{33} + c^2 \\ &Y_{123} = Y_{231} = Y_{312} = 0 \end{aligned}\right\}, \tag{6.26}$$

$$\left.\begin{array}{c}\underline{\text{Kind }\widetilde{\text{III}}}\\ X_{11}+\frac{\varkappa}{2}\,U_{11}=X_{22}+\frac{\varkappa}{2}\,U_{22}=X_{33}+\frac{\varkappa}{2}\,U_{33}=0\\ Y_{123}=Y_{231}=Y_{312}=0\end{array}\right\}. \tag{6.27}$$

As seen, in spaces of kind $\widetilde{\text{III}}$ all stationary curvatures are zero. However the aforementioned canonical representation of the matrix $\|C_{ab}\|$ of Weyl's tensor in an ortho-frame manifests that both matrices $\widetilde{x}$ and $\widetilde{y}$ are nonzero in any case, and this fact does not depend on the kind of matter that fills the space.

Finally, our consideration of the canonical forms of Weyl's conformal curvature tensor, and its stationary curvatures for non-empty non-Einstein spaces of all three kinds leads to the following conclusion:

> The presence of distributed matter (medium) in a non-Einstein space changes only the real parts of the stationary curvatures. The impossibility of gravitational waves and gravitational inertial waves, which is the condition $\widetilde{Y}_{ijk}=0$ (equality to zero of the dynamical observable component of Weyl's tensor), can only be realized in the non-empty spaces (gravitational field) of kind $\widetilde{\text{I}}$, where the stationary curvatures take real values. In non-empty non-Einstein spaces of the kinds other than kind $\widetilde{\text{I}}$, gravitational waves and gravitational inertial waves are possible.

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