Behaviour of the EGR Persistent Vacuum Field Following the Lichnérowicz Matching Conditions

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Abstract: Recently, the author has proposed an extension of the General Theory of Relativity — the EGR theory, which allows for a persistent gravity-like field to exist as a homogeneous energy density background. In this paper, we demonstrate the continuity of this field with respect to the gravitational field of a massive body. To achieve this goal, we make use of the Lichnérowicz conjecture which formulates the conditions required to match a hyperbolic 4-metric characterized by a material-energy tensor, with a similar type of vacuum-solution metric. This is herein applied to a spherically symmetric class of the general relativistic solutions compatible with the Schwarzschild exterior metric. The EGR covariant derivatives of the metric are then only radial and time-dependent functions: the radial persistent field tensor component vanishes on a hypersurface separating the vacuum from the matter state. As a consequence, when this hypersurface is narrowed down to the size of a particle, it follows a non-Riemannian geodesic describing the trajectory of the particle whose mass slightly increased: this effect can be interpreted as the bare mass carrying its subsequent gravitational field.

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Introduction. The problem of matching two Riemannian hyperbolic metrics in the sense of Lichnérowicz [1] can be stated as follows:

> Given a metric solution corresponding to a "normal material tensor" [2], we look for a hypersurface S where some "junction" conditions must be fulfilled to match a similar vacuum metric, so that some degrees of smoothness are not lost when approaching it form either side of the hypersurface.

This mathematical procedure is derived from the evolution of Einstein's equations, which necessarily involves the Cauchy problem.

In order to have an appropriate simple picture of the situation, we begin by regarding one of the "material metric" corresponding to a massive source, as generating a four-dimensional space-time "world tube". It is thus convenient to visualize the tube walls as a hypersurface S. Lichnérowicz admissible coordinates [3] can be introduced from either side of the hypersurface. Within the tube, the space metric satisfies the "material" Einstein equations. Outside the tube, the metric satisfies the source-free Einstein equations. The admissible coordinate conditions imply $G^4_a = 0$ for the Einstein tensor G_{ab} along with the time component $u^4 = 0$ of the unit vector u^a on the dividing hypersurface S. In this case, Lichnérowicz proved that the hypersurface S is generated by a congruence of time-like geodesics, since S is tangent to those lines and is thus itself time-like.

Let us imagine that the material tensor represents a massive particle, if the section of the tube is narrowing down to a particle size. In this case, we easily verify that such a particle would follow a time-like geodesic which is imposed by the field of the exterior metric.

Earlier on, guided by the equivalence principle whereby inertia is not locally distinguishable from gravitation, Einstein extended the special relativistic law of motion for a test particle to a gravitational field geodesic. On the other hand, the fundamental consequence of the matching conditions (which was later acknowledged by Einstein himself) results in the following: the geodesic principle is no longer a *postulate*, but a straightforward *consequence* of Einstein's equations.

In this paper, we will be primarily concerned with the discontinuity which the EGR persistent field [4] might undergo when switching from the source-free metric to the material (matter-filled) metric. To answer this question we are going to follow Lichnérowicz' program applied to a spherically symmetric class of Einstein's field solutions that are to match the Schwarzschild exterior metric. A particular importance is the assumption of a homogeneously distributed EGR field, which in this case results in an *extended Schwarzschild exterior solution* fully compatible with the standard Schwarzschild exterior solution (Riemannian geometry). Through this derivation, we are eventually led to reconsider the matter under the form of a modified density.

Chapter 1. The Cauchy Problem in General Relativity

§1.1. Problem statement. As is well known, Einstein's equations are non-linear. The gravitational fields corresponded to the equations, even when singled out, define the space-time over which they propagate. As a result, the solution of the equations can be found to be unique up to a diffeomorphism, and hence one is forced to introduce a fixed background or hypersurface S onto which a set of initial data are given. From these Cauchy data, it is thus possible to predict and study the further evolution of Einstein's equations in the neighbourhood of S.

The Cauchy problem in General Relativity was pioneered by Darmois and Lichnérowicz [5], then extensively studied in [6]. We restrict this topic to local considerations of the problem. For a full treatment of the global aspect of the Cauchy problem in General Relativity, see for instance Choquet-Bruhat and Geroch [7, 8], and others [9–12]. From a strict mathematical point of view, the Cauchy problem can be formulated as follows [13]:

> Let S be a given three-dimensional manifold and a set of n initial data on it. We look for a four-dimensional Lorentzian manifold (M, g) and an embedding $f : \mathrm{S} \to \mathrm{M}$ such that the metric $\mathrm{g} = g_{ab}\, dx^a \otimes dx^b$ satisfies Einstein's equations and the initial conditions on $f(\mathrm{S})$, and that $f(\mathrm{S})$ constitutes a Cauchy hypersurface for the manifold (M, g).

§1.2. The exterior situation. Following Lichnérowicz, we assume that components of the metric tensor g_{ab} (as well as their first derivatives) should be smooth and continuous on a given hypersurface S. In the neighbourhood of S of any event, the potentials g_{ab} satisfy the source-free Einstein equations

$$G_{ab} = 0\,, \qquad a, b = 1, 2, 3, 4, \tag{1.1}$$

where the right-hand side can include the cosmological term.

We consider a space-like hypersurface $x^4 = 0$: therein g_{ab} and their first derivatives $\partial_4\, g_{ab}$ are thus defined as the set of n initial data.

From the contracted Bianchi identities

$$G^{a4}_{,4} = -\,G^{a\sigma}_{,\sigma} - \left\{{}^{a}_{bc}\right\} G^{cb} - \left\{{}^{b}_{bc}\right\} G^{ac} \tag{1.2}$$

we see that the right hand side contains at most two differentiations with respect to time and so it must be the case for the left-hand side. Therefore,

$$G^{a4} = 0 \tag{1.3}$$

contains only first derivatives of the metric tensor with respect to time.

The second-order derivative $\partial_{44}\, g_{ab}$ cannot be determined by the field equations. Hence, no information can be extracted about the time evolution from the four equations (1.3).

These equations are regarded as the *constraint Einstein equations* for the set of n initial data, i.e. for g_{ab} and $\partial_4\, g_{ab}$. If they are satisfied by the initial data, there exists a solution of the Cauchy problem for the field equations $G_{ab}=0$ in the neighbourhood of S.

So, we are left with 6 dynamical field equations

$$G_{\alpha\beta} = 0\,, \qquad \alpha, \beta = 1, 2, 3. \tag{1.4}$$

For the second-order derivatives $\partial_{44}\, g_{ab}$ (they are 10), we have a four-fold ambiguity which can be removed by imposing four conditions (known as the *harmonicity conditions*) on the metric tensor g_{ab}.

Explicitly, these conditions are

$$F^b = \frac{\partial \mathcal{G}^{ab}}{\partial x^a} = 0\,, \tag{1.5}$$

where $\mathcal{G}^{ab} = \sqrt{-g}\, g^{ab}$ is the metric tensor density. With this choice of harmonic coordinates, the Einstein tensor G_{ab} can be written as

$$G^{ab} = \left(G^{ab}\right)_{\rm harm} + A^{ab} \tag{1.6}$$

with

$$\left.\begin{aligned} A^{ab} &= \frac{1}{2}\left(g^{ac}\,\partial_c F^b + g^{bc}\,\partial_c F^a\right) \\ \left(G^{ab}\right)_{\rm harm} &= \frac{g^{ik}}{2\sqrt{-g}}\,\frac{\partial^2 \mathcal{G}^{ab}}{\partial x^i \partial x^k} + H^{ab} \end{aligned}\right\}, \tag{1.7}$$

where H_{ab} depends on the potentials and their first derivatives.

Hence, we can solve the *reduced Einstein equations*

$$\left(G^{ab}\right)_{\rm harm} = 0\,. \tag{1.8}$$

The solutions of the initial problem should satisfy the constraint equations (1.3) at any later time.

Consider the conservation equations

$$\nabla_a G^a_b = 0\,, \tag{1.9}$$

that is

$$\nabla_4 G_b^4 + \nabla_\alpha G_b^\alpha = 0\,. \tag{1.10}$$

The constraints (1.3) are imposed so that G^{ab} vanishes everywhere along with $G^{\alpha\beta}$. It can also be shown that, taking into account (1.10) on S where (1.3) is satisfied, the constraint equations are also satisfied in the neighbourhood of S.

Therefore the equations (1.3) propagate, and the Einstein equations are said to be in *involution* (not evolution), in the sense of Cartan.

§1.3. Interior situation. Here the problem is somewhat more complex. Put it simply, the field equations are a part of the system

$$\left.\begin{aligned} G^{ab} &= \varkappa T^{ab} \\ \nabla_a T^{ab} &= 0 \end{aligned}\right\}, \tag{1.11}$$

which is also in involution in the sense of Cartan. On the hypersurface S ($x^4 = const$), we choose initial data satisfying the four conditions

$$G_a^4 = \varkappa T_a^4 \tag{1.12}$$

for $x^4 = 0$. Inspection shows that the Cauchy problem has a solution in the neighbourhood of S, provided that the data are sufficiently differentiable in the case of a massive tensor.

Chapter 2. Application to Spherically Symmetric Metrics

§2.1. The general solution. We begin by redefining a spherically symmetric Lorentzian manifold (M, g) as a manifold admitting the group SO(3) as an isometric group, in such a way that the group orbits are two-dimensional space-like surfaces.

The group orbits are necessarily surfaces of constant positive curvature. Thus, it is always possible to introduce coordinates such that the metric has the regular form

$$ds^2 = e^{2a(T,R)}dT^2 - e^{2b(T,R)}dR^2 + e^{2c(T,R)}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.1}$$

According to the EGR theory, 1) we keep the spherical symmetry and maintain the normalization of R so that a circle has the circumference $2\pi R$; 2) we make the legitimate assumption that the EGR covariant metric tensor variations only apply to T and R.

The general form of the EGR metric

$$(ds^2)_{\text{EGR}} = ds^2 + dJ \tag{2.2}$$

has been postulated [4], where the linear form $dJ = f(J_a)\,dx^a$ depends on the covariant derivative of the metric tensor

$$\mathrm{D}_a\, g_{bc} = \frac{1}{3}\left(J_c\, g_{ab} + J_b\, g_{ac} - J_a\, g_{bc}\right). \tag{2.3}$$

With our second assumption, we write the spherically symmetric EGR metric as

$$(ds^2)_{\mathrm{EGR}} = ds^2 + (J_T\, dT - J_R\, dR)\,. \tag{2.4}$$

A quick comparison with (2.1) readily leads to $dT\,(e^{2a(T,R)}\,dT + J_T)$, which we write in the form $dT^2\,(e^{2\mathcal{A}})$. In the same way, $dR^2\,(e^{-2\mathcal{B}})$. Finally, we have the modified coefficients

$$\left.\begin{aligned} \mathcal{A} &= a + \text{correction}\,(R,T) \\ \mathcal{B} &= b + \text{correction}\,(R,T) \\ \mathcal{C} &= c + \text{correction}\,(R,T) \end{aligned}\right\}, \tag{2.5}$$

thus we write the EGR spherical metric in the form

$$(ds^2)_{\mathrm{EGR}} = e^{2\mathcal{A}(R,T)}dT^2 - e^{-2\mathcal{B}(R,T)}dR^2 - - e^{2\mathcal{C}(R,T)}\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big). \tag{2.6}$$

Using Cartan's calculus, we will be able to obtain formulae for the EGR Ricci tensor and the EGR Einstein tensor.

First, we re-write the metric (2.6) with the Pfaffian forms

$$(ds^2)_{\mathrm{EGR}} = (\omega^4)^2 - (\omega^a)^2, \tag{2.7}$$

where the local basis Pfaffian forms are given by

$$\omega^4 = e^{\mathcal{A}dT}dT, \quad \omega^1 = e^{\mathcal{B}}dR, \quad \omega^2 = e^{\mathcal{C}}d\theta, \quad \omega^3 = e^{\mathcal{C}}\sin\theta\, d\varphi. \tag{2.8}$$

Now, we need the *connection forms*, which will be obtained from the first Cartan structure equations

$$d\omega = -\,\omega^a_b \wedge \omega^b. \tag{2.9}$$

Determining first the exterior derivatives

$$\left.\begin{aligned} d\omega^4 &= \mathcal{A}' e^{-\mathcal{B}}\omega^1\wedge\omega^4 \\ d\omega^1 &= \dot{\mathcal{B}}\, e^{-\mathcal{A}}\omega^4\wedge\omega^1 \\ d\omega^2 &= \dot{\mathcal{C}}\, e^{-\mathcal{A}}\omega^4\wedge\omega^2 + \dot{\mathcal{C}}\, e^{-\mathcal{B}}\omega^1\wedge\omega^2 \\ d\omega^3 &= \dot{\mathcal{C}}\, e^{-\mathcal{A}}\omega^4\wedge\omega^3 + \dot{\mathcal{C}}\, e^{-\mathcal{B}}\omega^1\wedge\omega^3 + \frac{1}{R}\cot\theta\left(\omega^2\wedge\omega^3\right) \end{aligned}\right\}, \tag{2.10}$$

where $\mathcal{A}' = \frac{\partial \mathcal{A}}{\partial R}$ and $\dot{\mathcal{B}} = \frac{\partial \mathcal{B}}{\partial T}$, then using of (2.9), we find

$$\left.\begin{aligned} \omega_1^4 &= \omega_4^1 = \mathcal{A}' e^{-\mathcal{B}} \omega^4 + \dot{\mathcal{B}} e^{-\mathcal{A}}\, \omega^1 \\ \omega_2^4 &= \omega_4^2 = \dot{\mathcal{C}}\, e^{-\mathcal{A}}\, \omega^2 \\ \omega_3^4 &= \omega_4^3 = e^{-\mathcal{A}}\, \dot{\mathcal{C}}\, \omega^3 \\ \omega_1^2 &= -\,\omega_2^1 = \mathcal{C}' \\ \omega_1^3 &= -\,\omega_3^1 = \mathcal{C}'\, e^{-\mathcal{B}}\, \omega^3 \\ \omega_2^3 &= \omega_3^2 = -\frac{1}{R}\, \cot\theta\, \omega^3 \end{aligned}\right\} . \tag{2.11}$$

The ansatz (2.11) satisfies

$$\omega_{ab} + \omega_{ba} = 0\,, \tag{2.12}$$

since the basis ω_a is chosen to be orthonormal.

From the second structure equation

$$\Omega_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c\,, \tag{2.13}$$

we obtain the EGR curvature forms

$$\left.\begin{aligned} \Omega_1^4 &= E\,\omega^4 \wedge \omega^1 \\ \Omega_2^4 &= F\,\omega^4 \wedge \omega^2 + H\omega^1 \wedge \omega^2 \\ \Omega_3^4 &= F\,\omega^4 \wedge \omega^3 + H\omega^1 \wedge \omega^3 \\ \Omega_2^1 &= I\,\omega^1 \wedge \omega^2 - H\omega^4 \wedge \omega^2 \\ \Omega_3^1 &= I\,\omega^1 \wedge \omega^3 - H\omega^4 \wedge \omega^3 \\ \Omega_3^2 &= D\,\omega^2 \wedge \omega^3 \end{aligned}\right\} , \tag{2.14}$$

where we use the short denotations

$$\left.\begin{aligned} E &= e^{-2\mathcal{A}}\big(\ddot{\mathcal{B}} + \dot{\mathcal{B}}^2 - \dot{\mathcal{B}}\dot{\mathcal{A}}\big) - e^{-2\mathcal{B}}\big(\mathcal{A}'' + \mathcal{A}'^2 - \mathcal{A}'\mathcal{B}'\big) \\ F &= e^{-(\mathcal{A}+\mathcal{B})}\big(\dot{\mathcal{C}}' + \dot{\mathcal{C}}\mathcal{C}' - \dot{\mathcal{C}}\mathcal{A}' - \dot{\mathcal{B}}\mathcal{C}'\big) \\ H &= e^{-2\mathcal{A}}\big(\ddot{\mathcal{C}} + \dot{\mathcal{C}}^2 - \dot{\mathcal{C}}\dot{\mathcal{A}}\big) - e^{-2\mathcal{B}}\mathcal{A}'\mathcal{C}' \\ D &= e^{-2\mathcal{A}}\dot{\mathcal{C}}^2 - e^{-\mathcal{B}}\mathcal{C}'^2 + e^{-2\mathcal{C}} \\ I &= e^{-2\mathcal{A}}\dot{\mathcal{C}}\dot{\mathcal{B}} - e^{-2\mathcal{B}}\big(\mathcal{C}'' + \mathcal{C}'^2 - \mathcal{C}'\mathcal{B}'\big) \end{aligned}\right\} . \tag{2.15}$$

Hence, we can infer the needed diagonal components of the EGR Ricci tensor $(R_{ab})_{\rm EGR}$. We obtain

$$\left.\begin{aligned}(R_{44})_{\rm EGR} &= -E-2F\\ (R_{11})_{\rm EGR} &= E+2I\\ (R_{22})_{\rm EGR} &= (R_{33})_{\rm EGR} = E+D+I\end{aligned}\right\}, \tag{2.16}$$

while the curvature scalar is given by

$$(R)_{\rm EGR} = -\,2\,(E+I) - 4\,(F+I)\,. \tag{2.17}$$

We can now calculate the useful components of the EGR Einstein tensor [4], which is defined as follows

$$(G_{ab})_{\rm EGR} = (R_{ab})_{\rm EGR} - \frac{1}{2}\left[g_{ab}\,(R)_{\rm EGR} - \frac{2}{3}\,J_{ab}\right]. \tag{2.18}$$

In our particular case, the diagonal components reduce this tensor to the Riemannian form

$$G_{ab} = R_{ab} - \frac{1}{2}\,g_{ab}\,R\,, \tag{2.19}$$

so that the diagonal components of the EGR Einstein tensor are

$$\left.\begin{aligned}(G_4^4)_{\rm EGR} &= (G_{44})_{\rm EGR} = D+2I\\ (G_1^1)_{\rm EGR} &= (G_{11})_{\rm EGR} = 2F+D\\ (G_{22})_{\rm EGR} &= (G_{33})_{\rm EGR} = E+I+F\end{aligned}\right\}. \tag{2.20}$$

According to the EGR theory [4], for the *interior metric*, and assuming for the EGR unit 4-velocity that $(u_a u^a)_{\rm EGR} = 1$, these components are associated with the material tensor and the persistent field

$$(G_4^4)_{\rm EGR} = \varkappa\left[\,T_4^4 + (t_4^4)_{\rm EGR}\right] = \varkappa\left[\,\rho + (t_4^4)_{\rm EGR}\right] = \varkappa\rho^*, \tag{2.21}$$

$$(G_1^1)_{\rm EGR} = \varkappa\,(t_1^1)_{\rm EGR}\,, \tag{2.22}$$

and $(G_{22})_{\rm EGR} = \varkappa\,(t_{22})_{\rm EGR}$, $(G_{33})_{\rm EGR} = \varkappa\,(t_{33})_{\rm EGR}$, thus

$$(t_{22})_{\rm EGR} = (t_{33})_{\rm EGR}\,, \tag{2.23}$$

where ρ^* stands for the modified material density, which was already introduced by the EGR theory.

Our main task will be to show that the external (radial) component of the EGR persistent field is vanishing at the contact of the spherical mass (source of the field), thus ensuring the continuity with the modified massive density. To be more specific, we expect to see that the vacuum EGR persistent field will actually vanish when approaching asymptotically a region around the bare mass. Such a global region represents the modified massive quantity, i.e. the bare mass carries its subsequent gravitational field.

§2.2. The Schwarzschild metric (classical solution). We first consider the classical Schwarzschild solution, which is obtained in the framework of Riemannian geometry. Then we represent it according to the EGR theory.

As is known, the Schwarzschild metric in the spherical coordinates has the form

$$ds^2 = e^{2a(r)}\, dt^2 - e^{2b(r)}\, dr^2 - r^2 \left(d\zeta^2 + \sin^2\zeta\, d\varphi^2\right). \tag{2.24}$$

We rewrite this linear element with the Pfaffian forms

$$ds^2 = (\theta^4)^2 - (\theta^a)^2, \tag{2.25}$$

where we have chosen

$$\theta^4 = e^a dt\,, \quad \theta^1 = e^b\, dr\,, \quad \theta^2 = r\, d\zeta\,, \quad \theta^3 = r \sin\zeta\, d\varphi\,. \tag{2.26}$$

Exterior differentiation of these results immediately in

$$\left.\begin{aligned} d\theta^4 &= a' e^a dr \wedge dt \\ d\theta^1 &= 0 \\ d\theta^2 &= dr \wedge d\zeta \\ d\theta^3 &= \sin\zeta\, dr \wedge d\varphi + r\cos\zeta\, d\zeta \wedge d\varphi \end{aligned}\right\}. \tag{2.27}$$

Comparison with the first structure equation leads to the following expressions for the connection forms

$$\left.\begin{aligned} \omega^4_1 &= \omega^1_4 = a' e^{-b}\, \theta^4 \\ \omega^2_1 &= \omega^1_2 = \frac{1}{r}\, e^{-b}\, \theta^2 \\ \omega^3_1 &= -\omega^1_3 = \frac{1}{r}\, e^{-b}\, \theta^3 \\ \omega^3_2 &= -\omega^2_3 = \frac{1}{r}\, (\cot\zeta)\, \theta^3 \\ \omega^4_2 &= \omega^2_4 = \omega^4_3 = \omega^3_4 = 0 \end{aligned}\right\}. \tag{2.28}$$

From the second structure equation, we obtain the curvature forms

$$\left.\begin{aligned}\Omega^4_1 &= e^{-2b}\big(a'b' - a'' - a'a'\big)\,\theta^4\wedge\theta^1\\ \Omega^4_2 &= -\frac{a'e^{-b}}{r}\left(\theta^4\wedge\theta^2\right)\\ \Omega^4_3 &= -\frac{a'e^{-2b}}{r}\left(\theta^4\wedge\theta^3\right)\\ \Omega^1_2 &= \frac{b'e^{-2b}}{r}\left(\theta^1\wedge\theta^2\right)\\ \Omega^1_3 &= \frac{b'e^{-2b}}{r}\left(\theta^1\wedge\theta^3\right)\\ \Omega^2_3 &= \frac{1-e^{-2b}}{r^2}\left(\theta^2\wedge\theta^3\right)\end{aligned}\right\}. \tag{2.29}$$

For the Einstein tensor, we obtain the useful mixed diagonal components

$$G^4_4 = \frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right), \tag{2.30}$$

$$G^1_1 = \frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right), \tag{2.31}$$

$$G^2_2 = G^3_3 = -e^{-2b}\left(a'^2 - a'b' + a'' + \frac{a'-b'}{r}\right). \tag{2.32}$$

The vacuum solutions are then given by

$$G^4_4 + G^1_1 = 0\,, \tag{2.33}$$

which imply that $a' + b'$, and hence $a + b = 0$, since a, b approach zero asymptotically such that the Schwarzschild metric becomes asymptotically flat, i.e. $a = -b$.

Integrating (2.30), we obtain

$$e^{2a} = e^{-2b} = 1 - \frac{m}{r}\,. \tag{2.34}$$

The constant m is determined as follows: at large distances we must have the Newtonian limit

$$g^{44} \approx 1 + 2U, \tag{2.35}$$

where $U = -\frac{\mathfrak{G}M}{r}$ is the classical gravitational potential, where M is the mass producing the field. Hence, $m = \mathfrak{G}M$ (we have assumed $c = 1$).

Once $\left(1-\frac{2m}{r}\right)$ has been substituted into the curvature forms (2.29), we find, for the curvature tensor components,

$$R_{4141}=-R_{2323}=2L\,, \quad R_{1212}=R_{1313}=R_{4242}=-R_{4343}=L\,, \tag{2.36}$$

where

$$L=\frac{m}{r^3}\,. \tag{2.37}$$

§2.3. The Schwarzschild metric (the EGR formulation). Following the same procedure as in §2.2, we write the extended Schwarzschild solution

$$(ds^2)_{\rm EGR}=e^{2A(r,t)}\,dt^2-e^{2B(r,t)}\,dr^2-r^2\big(d\zeta^2+\sin^2\zeta\,d\varphi^2\big)\,, \tag{2.38}$$

where the coefficients A and B are formulated as

$$A=a+\text{correction }(r,t)\,, \quad B=b+\text{correction }(r,t)\,. \tag{2.39}$$

In Riemannian geometry, the Schwarzschild metric is obtained as a vacuum solution. According to the EGR theory, there is not source-free solution: the field equations are characterized by a persistent field t_{ab}. Therefore, applying the "vacuum" formulae (2.30) and (2.31) of the classical Schwarzschild solution, we obtain

$$(G^4_4)_{\rm EGR}=\varkappa\,t^4_4\,, \quad (G^1_1)_{\rm EGR}=\varkappa\,t^1_1\,, \tag{2.40}$$

thus we have

$$(G^4_4)_{\rm EGR}+(G^1_1)_{\rm EGR}=\varkappa\left(t^4_4+t^1_1\right). \tag{2.41}$$

According to the EGR theory, the persistent field is assumed to be homogeneously distributed as a background energy density. Under the assumption of spherical symmetry we thus need only the components (2.40), so that we have

$$t^4_4=-\,t^1_1\,. \tag{2.42}$$

The obtained EGR formula (2.41) is similar to that according to Riemannian geometry (2.32). Therefore, we have the most important result which formulates as:

> According to the aforementioned (mixed) diagonal conditions, the classical Schwarzschild exterior solution is equivalent to the EGR Schwarzschild metric.

This circumstance enables us to set forth

$$e^{2A}=e^{-2B}=1-\frac{m^*}{r}\,, \tag{2.43}$$

where m^* is a *modified mass* we have introduced through the relation

$$m^* = m + \text{correction}. \tag{2.44}$$

We note here the very consistency with our previous result (2.21), where we have been able to determine a modified density ρ^*. We thus extrapolate (2.37) as

$$(L)_{\text{EGR}} = \frac{m^*}{r^3}\,. \tag{2.45}$$

The corresponding EGR curvature forms are

$$\left.\begin{aligned}
(\Omega^4_1)_{\text{EGR}} &= 2\,(L)_{\text{EGR}}\left(\theta^{4*}\wedge\theta^{1*}\right)\\
(\Omega^4_2)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{4*}\wedge\theta^{2*}\right)\\
(\Omega^4_3)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{4*}\wedge\theta^{3*}\right)\\
(\Omega^2_3)_{\text{EGR}} &= 2\,(L)_{\text{EGR}}\left(\theta^{2*}\wedge\theta^{3*}\right)\\
(\Omega^1_3)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{3*}\wedge\theta^{1*}\right)\\
(\Omega^1_2)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{1*}\wedge\theta^{2*}\right)
\end{aligned}\right\}, \tag{2.46}$$

where θ^{a*} are the *EGR Pfaffian forms* which are determined by the EGR coefficients of the metric (2.38).

Chapter 3. The Local Matching Conditions

§3.1. General definition. In a space-time manifold (M, g), where matter generates a world-tube limited by a hypersurface S, we are in the presence of an interior metric satisfying the massive field equations, and an exterior metric satisfying the source-free Einstein equations. From either side, g_{ab} are defined and smoothly and continuous in each open sub-domain. The purpose of the current work is to analyse the continuous properties required for the metrics when approaching and crossing S. To start with, we indicate the matching conditions as was first stated by Lichnérowicz:

> Given $x \in S$, there exists a frame of admissible coordinates whose domain includes x, and the potentials g_{ab} (related to this frame) as well as their first derivatives be continuous when crossing S.

Anticipating on the final proof result, Lichnérowicz also showed that the matching conditions requires for S to be a time-like hypersurface.

Let a Riemannian metric

$$ds_1^2 = g_{ab}\,dx^a dx^b \tag{3.1}$$

be defined on an open subset $O_1 \subset (M, g)$ (a four-dimensional manifold). Denote the respective metric tensor and Riemannian connection as $g_1(x)$ and $\{\ \}_1(x)$, with $x \in O_1$. Consider another metric

$$ds_2^2 = g_{c'd'}\, dx^{c'} dx^{d'} \tag{3.2}$$

defined on O_2 which is connected to (M, g), with $g_2(x')$ and $\{\ \}_2(x')$ (we mean here that $x' \in O_2$).

Provided that ds_1^2 and ds_2^2 are attributed to the same hyperbolic type (and having the same signature), they can be matched in the sense of Lichnérowicz, if there exists:

1) Functions
$$x^{e'} = x^{e'}(x^a)\,, \tag{3.3}$$
whose non-vanishing Jacobian $J_a^{c'} = \frac{\partial x^{c'}}{\partial x^a}$ satisfies $J_a^{c'} J_{c'}^b = \delta_a^b$;

2) A hypersurface S represented by a local equation $f(x^a) = 0$ on which holds
$$g_{ab} = J_a^{i'} J_b^{k'} g_{i'k'}\ , \tag{3.4}$$

$$\left\{ {}^{\,c}_{ab} \right\} = J_{d'}^c J_a^{i'} J_b^{k'} \left\{ {}^{\,d'}_{i'k'} \right\} + J_{d'}^c\, \partial_a J_b^{d'}\,. \tag{3.5}$$

§3.2. Application to a natural basis. In view of applying our next program for the matching conditions, it suffices to adopt the approximated Minkowskian forms of the metrics (3.1) and (3.2)

$$ds_1^2 = \eta_{ab}\, \omega^a \wedge \omega^b, \tag{3.6}$$

$$ds_2^2 = \eta_{e'l'}\, \theta^{e'} \wedge \theta^{l'}, \tag{3.7}$$

where the Pfaffian forms are defined by

$$\omega^a = A_b^a\, dx^b, \qquad \theta^{e'} = B_{l'}^{e'}\, dx^{l'}. \tag{3.8}$$

The change of variables (3.3) performed on $\theta^{e'}$ yields

$$\theta^{e'} = \mathcal{L}_a^{e'} \omega^a, \tag{3.9}$$

where

$$\mathcal{L}_a^{e'} = B_{l'}^{e'}\, \mathcal{L}_b^{l'}\, (A^{-1})_a^b\,. \tag{3.10}$$

It can be shown that the conditions (3.5) are equivalent to

$$\omega_b^a = \mathcal{L}_{e'}^a\, \theta_{l'}^{e'}\, \mathcal{L}_b^{l'} + \mathcal{L}_{k'}^a\, d\mathcal{L}_b^{k'}\,. \tag{3.11}$$

Since we consider a Lorentzian manifold (M, g) with $n=4$, the matrix $\mathcal{L}$ is an element of the special Lorentz group which is reduced into the boost

$$\mathcal{L}_a^{e'} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}\,. \tag{3.12}$$

The Lorentz invariance is also obtained by setting $\tanh\chi = v$, in which case we have for $\mathcal{L}$ the following form

$$\mathcal{L}_{e'}^{a} = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.13}$$

$$\mathcal{L}_{a}^{e'} = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.14}$$

where the parameter χ will be defined later on.

Remarkably, we check in passing that the components of the Riemann tensor in the classical Schwarzschild metric (2.24) are unchanged with respect to a radial co-moving reference frame ($u^\alpha = 0$, $u_4 u^4 = 1$)

$$\begin{aligned} R'_{4141} &= \mathcal{L}_{4'}^{a}\,\mathcal{L}_{1'}^{b}\,\mathcal{L}_{4'}^{c}\,\mathcal{L}_{1'}^{d}\,R_{abcd} = \\ &= R_{4141} = \left(\cosh^4\chi - 2\cosh^2\chi\,\sinh^2\chi + \sinh^4\chi\right) R_{4141}\,, \end{aligned} \tag{3.15}$$

since $\cosh^2\chi - \sinh^2\chi = 1$. In a similar manner, inspection shows the invariance of the other components in this particular frame of reference.

Let us now set

$$\mathcal{Q}_b^a = \omega_b^a - \mathcal{L}_{e'}^{a}\,\theta_{l'}^{e'}\,\mathcal{L}_b^{l'} - \mathcal{L}_{k'}^{a}\,d\mathcal{L}_b^{k'}\,. \tag{3.16}$$

According to (3.11), the form $\mathcal{Q}_b^a$ should be zero on the hypersurface S. Therefore the surface element $d\mathcal{Q}_b^a$ must satisfy

$$d\mathcal{Q}_b^a \wedge df = 0\,, \tag{3.17}$$

where df is the normal to S.

We are going to find the matching relation between the considered two metrics referred to the same basis ω^a.

To this effect, we first notice that the curvature form related to the metrics ds_1^2 and ds_2^2 is expressed, respectively, by

$$\Omega_b^a = \frac{1}{2}\, R_{\cdot bki}^{a\cdots}\, \omega^k \wedge \omega^i, \tag{3.18}$$

$$\Omega_{l'}^{e'} = \frac{1}{2}\, R_{\cdot l'c'd'}^{e'\cdots}\, \omega^{c'} \wedge \omega^{d'}. \tag{3.19}$$

Hence,

$$\mathcal{L}_{e'}^{a}\, \Omega_{l'}^{e'} \mathcal{L}_{b}^{l'} = \frac{1}{2}\, \mathcal{L}_{e'}^{a}\, R_{\cdot l'c'd'}^{e'\cdots}\, \mathcal{L}_{b}^{l'} \mathcal{L}_{k}^{c'} \mathcal{L}_{i}^{d'}\, \omega^k \wedge \omega^i \tag{3.20}$$

and $d\mathcal{Q}_b^a \wedge df$ is now written

$$\left(\Omega_b^a - \mathcal{L}_{e'}^{a}\, \Omega_{l'}^{e'} \mathcal{L}_{b}^{l'}\right) \wedge df = 0 \tag{3.21}$$

on the hypersurface S.

In particular, the latter equation can be written in terms of the Einstein tensor G_{ab} as

$$\left(G_b^a - \mathcal{L}_{e'}^{a}\, G_{l'}^{e'} \mathcal{L}_{b}^{l'}\right) \partial_a f = 0 \tag{3.22}$$

on the hypersurface S (this has been formulated, in another form than 3.22, by Lichnérowicz [5, p. 62]).

§3.3. Conditions for matching the EGR metrics. The curvature form is here given by

$$(\Omega_b^a)_{\text{EGR}} = \frac{1}{2}\, (R_{\cdot bki}^{a\cdots})_{\text{EGR}}\, \omega^k \wedge \omega^i, \tag{3.23}$$

where the EGR curvature tensor has the form

$$(R_{abki})_{\text{EGR}} = R_{abki} + B_{abki}\,, \tag{3.24}$$

where

$$B_{abki} = B_{abki}(J_{mn})\,, \quad J_{mn} = \partial_m J_n - \partial_n J_m\,, \tag{3.25}$$

and the Pfaffian forms ω_a are adapted accordingly.

We then denote the EGR Schwarzschild solutions as $(ds_1^2)_{\text{EGR}}$ and $(ds_2^2)_{\text{EGR}}$. The spherical symmetry suggests us to set

$$\theta = \zeta\,, \qquad \phi = \varphi\,, \qquad t = t\,(T)\,. \tag{3.26}$$

In order to investigate the possible consequences of matching the EGR metrics $(ds_1^2)_{\text{EGR}}$ and $(ds_2^2)_{\text{EGR}}$, we compute the exact components of $\mathcal{Q}^a_b$ with the help of (2.29) and (2.46), where the EGR Pfaffian forms of ds^2_2 are implicitly denoted by θ^{a*}. We eventually obtain

$$\left.\begin{aligned} \mathcal{Q}^4_1 &= (E-2L)\,\omega^4\wedge\omega^1 \\ \mathcal{Q}^4_2 &= (F+L)\,\omega^4\wedge\omega^2 \\ \mathcal{Q}^4_3 &= (F+L)\,\omega^4\wedge\omega^3 \\ \mathcal{Q}^2_3 &= (D-2L)\,\omega^2\wedge\omega^3 \\ \mathcal{Q}^3_1 &= (I+L)\,\omega^3\wedge\omega^1 \\ \mathcal{Q}^1_2 &= (I+L)\,\omega^1\wedge\omega^2 \end{aligned}\right\}. \tag{3.27}$$

A short inspection shows that fulfilling the condition (3.17), implies that we must set

$$df \propto \omega^1, \tag{3.28}$$

$$D-2L=0\,, \qquad F+L=0 \tag{3.29}$$

on S. That is the hypersurface S is *time-like* as it should be.

Indeed, had we chosen $df = d\omega^4$, we would then have been left with vanishing conditions involving the terms E and $I\neq 0$, whose coefficient $\mathcal{B}$ is time-dependent, and therefore contradicting the nature of the hypersurface S which would be space-like in this case.

From (3.28) and (3.29), we have

$$2F+D=0 \tag{3.30}$$

on S, which, taking (2.40) into account, yields the fundamental result

$$(t^1_1)_{\rm EGR}=0\,. \tag{3.31}$$

The radial component of the EGR persistent field tensor vanishes on the time-like hypersurface S.

Ultimately, as is easy to show, the EGR coefficients of (2.5) and (2.39) define the parameter χ of (3.13) and (3.14) so that

$$\sinh\chi = -\,\dot{C}\,e^{B+C-\mathcal{A}}, \quad \cosh\chi = C'\,e^{B+C-\mathcal{B}}. \tag{3.32}$$

Discussion and concluding remarks. Under the above symmetry assumptions, the radial component is only the "dynamical" component, which is of importance here.

Therefore, we clearly see that, provided that the hypersurface S strictly divides the exterior EGR Schwarzschild solution from the EGR spherically symmetric interior metric, there exists a physical continuity between the exterior EGR persistent field tensor and the interior modified material tensor.

In Riemannian geometry, an *interior* spherically symmetric class of solutions of Einstein's equations corresponds to a normal material-energy tensor, i.e., to that of the generic form

$$T_{ab} = \rho\, u_a u_b - \Pi_{ab} \tag{3.33}$$

compatible with the spherical symmetry.

It was shown [14, 15] that under the assumption of spherical symmetry ($u^\alpha = 0,\ u_k u^k = 1$), all such *interior* solutions can be matched with the Schwarzschild exterior solution, provided that the radial pressure component $\Pi^1_1 = p$ vanishes on the time-like hypersurface S. This purely theoretical result has not any physical grounds.

On the contrary, the EGR theory provides here a much better interpretation: the continuity of the EGR persistent field presents indeed a physical consistency with the Lichnérowicz conjecture imposed as metric-matching conditions, which is a direct consequence of the Cauchy problem.

Following this pattern applied to two spherically symmetric models, it has indeed been rigorously shown that the EGR persistent field which pre-exists in the EGR "no-mass" metric, vanishes on the *contact separation* S between another metric containing a material source.

Reverting to the aforementioned picture where the "S-tube" section is considered as narrowing down to a particle's size, we can extend this proof by stating that the resulting principle of geodesics, still holds in the EGR theory for a neutral particle.

The essential difference lies in that the time-like geodesic is derived from the non-Riemannian EGR connection. As a result, the material source behaves as if it was modified by the "absorbed EGR field" presented in the matter.

As a matter of fact that a body's mass is not affected by the absorption of the EGR persistent field, but rather, the mass is now considered together with its own gravitational field, which has so far implicitly been described by an energy-momentum pseudo-tensor.

The EGR theory allows for an explicit description of a massive particle accompanied by its gravitational field, thus forming a single dynamical entity. If one still adopts the Riemannian picture, the "bare" proper mass of the particle is seen as being subjected to the influence of an environmental hidden medium that causes this mass to "fluctuate", according to de Broglie's Double Solution Theory [16]. Now, we clearly see that the random fluctuations are the manifestation of the particle's gravitational field, which is linked to the surrounding EGR field.

In conclusion, it should be noted that of importance is a pertinent analysis about the diagonal Gauss coordinates adopted in the framework of the admissible Lichnérowicz coordinate conditions, and related to the matching conditions applied to the Schwarzschild metric [17].

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