Instanton Representation of Plebanski Gravity. Application to the Schwarzschild Metric

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Abstract: In this paper we apply the instanton representation method to the construction of spherically symmetric blackhole solutions. The instanton representation implies the existence of additional Type D solutions which are axially symmetric. We explicitly construct these solutions, and show that they are fully consistent with Birkhoff's theorem.

Contents:

§1. Introduction. In [1] a new formulation of General Relativity has been introduced, known as the instanton representation of Plebanski gravity. The basic variables are a SO(3,C) gauge connection A_μ^a and a 3×3 matrix Ψ_{ae} which takes its values in two copies of SO(3,C). The equations of motion of the instanton representation imply the Einstein

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equations when the initial value constraints of General Relativity are satisfied, and imply that the gauge curvature of A_μ^a is Hodge self-dual with respect to the same metric $g_{\mu\nu}$ solving these equations.* As a consistency condition on this formulation, one should require that the 3-metric h_{ij} determined using the constraint solutions and the 3-metric defined by the Hodge duality condition be equal to one another. In this way, which we will refer to as the instanton representation method, one has a new recipe for constructing General Relativity solutions.

The initial value constraint solutions of General Relativity can be classified according to the Petrov classification of spacetime, which depends on the multiplicity of eigenvalues and eigenvectors of Ψ_{ae} (see e.g. [2, 3]). The instanton representation is concerned with the cases where Ψ_{ae} has three linearly independent eigenvectors, such as for Petrov Types I, D and O where its equivalence with General Relativity is manifest. In the Petrov Type D case there are two distinct eigenvalues of Ψ_{ae}, which can be permuted in three different ways. In the Type O case there is only one distinct eigenvalue and permutation, whereas in the Petrov Type I case there are three distinct eigenvalues with six possible permutations. The instanton representation method implies that there should be a separate General Relativity solution associated with each permutation of eigenvalues of Ψ_{ae}.

In this paper we apply the instanton representation method to the construction of spherically symmetric General Relativity solutions. According to Birkhoff's theorem [4], any spherically symmetric vacuum solution of the Einstein field equations must be static and must agree with the Schwarzschild solution. The Schwarzschild metric is a Type D vacuum solution, which as we will show in the instanton representation corresponds to a particular permutation $\vec{\lambda}_{(1)}$ of eigenvalues solving the initial value constraints. There are two additional permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ of this same set of eigenvalues. The instanton representation implies that these latter permutations should also correspond to solutions, which leads to the following obvious question. Are the $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ solutions consistent with the Birkhoff theorem or do they lead to a contradiction? In other words, is the Hodge duality condition of the instanton representation subject to the initial value constraints consistent with the ansatz of spherical symmetry and time-independence for any metrics other than the Schwarzschild metric? In this paper we find that the $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ metrics are different from the Schwarzschild metric, yet in a sense which this paper will make precise are not in contradiction

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*The latter actually follows from the equations of motion, and does not have to be added in as a separate postulate.

with Birkhoff's theorem.

The organization of this paper is as follows. In §2 we present some basic background on the initial value constraints problem in terms of the instanton representation phase space variables. In §3 we specialize the constraints to Type D spacetimes for a diagonal Ψ_{ae} for simplicity. §4 puts in place the ingredients necessary to produce spherically symmetric solutions. This uses a particular ansatz for the spatial connection A_i^a of a certain form, which includes time-independence of its components. §5, §6 and §7 apply the aforementioned instanton representation method to the construction of the metrics for the eigenvalue permutations $\vec{\lambda}_{(1)}$, $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$. The $\vec{\lambda}_{(1)}$ permutation leads to the Schwarzschild metric, and the remaining permutations lead to metrics which do not meet the conditions under which Birkhoff's theorem holds. §8 provides a summary and a brief discussion of these results.

§2. The initial value constraints. The dynamical variables in the instanton representation of Plebanski gravity are a SO(3,C) gauge connection A_μ^a and a $3{\times}3$ complex matrix $\Psi_{ae} \in \mathrm{SO(3,C)} \otimes \mathrm{SO(3,C)}$.[*] The variables are subject to the following constraints on each three-dimensional spatial hypersurface Σ

$$\mathbf{w}_e\{\Psi_{ae}\} = 0\,, \qquad \epsilon_{dae}\Psi_{ae} = 0\,, \qquad \Lambda + \operatorname{tr}\Psi^{-1} = 0\,, \tag{1}$$

where Λ is the cosmological constant.[†] We require that $\det\|\Psi\|\neq 0$, which means that the eigenvalues λ_1, λ_2 and λ_3 of Ψ_{ae} must be nonvanishing. The first equation of (1) is defined as

$$\mathbf{w}_e\{\Psi_{ae}\} = \mathbf{v}_e\{\Psi_{ae}\} + C_{be}\big(f_{abf}\delta_{ge} + f_{ebg}\delta_{af}\big)\Psi_{fg} = 0\,, \tag{2}$$

where f_{abc} are the SO(3) structure constants, and we have defined the vector fields $\mathbf{v}_a$ and a magnetic helicity density matrix C_{ae} given by

$$\mathbf{v}_a = B_a^i\,\partial_i\,, \qquad C_{ae} = A_i^a B_e^i\,. \tag{3}$$

In (3) we have defined the magnetic field B_a^i, which we assume to have nonvanishing determinant $\det\|B\|\neq 0$, as

$$B_a^i = \epsilon^{ijk}\,\partial_j A_k^a + \frac{1}{2}\,\epsilon^{ijk}f_{abc}A_j^b A_k^c\,. \tag{4}$$

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[*] For index conventions we use lower case symbols from the beginning of the Latin alphabet $a,b,c,\ldots$ to denote internal SO(3,C) indices, and from the middle $i,j,k,\ldots$ for spatial indices. Spacetime indices are denoted by $\mu,\nu,\ldots$

[†] The constraints in (1) are respectively the Gauss' law, diffeomorphism and Hamiltonian constraints. These constraints were also written down by Capovilla, Dell and Jacobson in the context of the initial value problem of General Relativity [5].

These variables define a spacetime metric $g_{\mu\nu}$, written in $3+1$ form, as follows

$$ds^2 = -N^2 dt^2 + h_{ij}\,\omega^i \otimes \omega^j, \tag{5}$$

where h_{ij} is the spatial 3-metric with one forms $\omega^i = dx^i - N^i dt$, where $N^\mu = (N, N^i)$ are the lapse function and shift vector. The 3-metric h_{ij} can be constructed from the constraint solutions, and is given by

$$(h_{ij})_{\text{Constraints}} = (\det\|\Psi\|)\big(\Psi^{-1}\Psi^{-1}\big)^{ae}\big(B^{-1}\big)^a_i\big(B^{-1}\big)^e_j\,(\det\|B\|)\,, \tag{6}$$

where Ψ_{ae} and A^a_i are solutions to (1). The constraints (1) do not fix N^μ, and make use only of the spatial part of the connection A^a_μ.

From the four-dimensional curvature $F^a_{\mu\nu}$ and using $F^a_{0i} = \dot{A}^a_i - \mathrm{D}_i A^a_0$ for the temporal component one can construct a matrix c_{ij}, given by

$$c_{ij} = F^a_{0i}\big(B^{-1}\big)^a_j\,, \qquad c \equiv \det\|c_{(ij)}\|\,. \tag{7}$$

The separation of c_{ij} into symmetric and antisymmetric parts defines a 3-metric $(h_{ij})_{\text{Hodge}}$ and a shift vector N^i, given by

$$(h_{ij})_{\text{Hodge}} = -\frac{N^2}{c}\,c_{(ij)}\,, \qquad N^i = -\frac{1}{2}\,\epsilon^{ijk}c_{jk}\,. \tag{8}$$

Equation (8) arises from the Hodge duality condition implied by the instanton representation [1]. Equations (8) and (6) are 3-metrics constructed using two separate criteria, and as a consistency condition must be set equal to each other. This is the basic feature of the instanton representation method in constructing General Relativity solutions in practice, which enables one to also write (5) as

$$ds^2 = -N^2\bigg[dt^2 + \frac{1}{c}\,c_{(ij)}\Big(dx^i + \frac{1}{2}\,\epsilon^{imn}c_{mn}dt\Big)\Big(dx^j + \frac{1}{2}\,\epsilon^{jrs}c_{rs}dt\Big)\bigg]. \tag{9}$$

Since Ψ_{ae} is a nondegenerate complex matrix by supposition, then it is diagonalizable when there are three linearly independent eigenvectors [2]. This enables one to classify solutions according to the Petrov type of the self-dual Weyl tensor ψ_{ae}. The matrix ψ_{ae} is symmetric and traceless, and related to Ψ_{ae} in the following way

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\,\delta_{ae} + \psi_{ae}\,. \tag{10}$$

So for this paper we assume that Ψ_{ae} is invertible, which requires the existence of three linearly independent eigenvectors. Hence, the results of this paper are limited to Petrov Types I, D and O. For each such Ψ_{ae}, combined with a connection A^a_i solving the constraints (1), the Hodge

duality condition (8) should yield a metric solving the vacuum Einstein equations.

§3. Application to Petrov Type D spacetimes. For the purposes of this paper we will restrict attention to the case where $\Psi_{ae} = \mathrm{diag}\,(\Psi_{11}, \Psi_{22}, \Psi_{33})$ is diagonal. Then from equation (1) the diffeomorphism constraint is automatically satisfied since a diagonal matrix is already symmetric. We can then associate the elements of Ψ_{ae} with its eigenvalues, and the Hamiltonian constraint is given by

$$\Lambda + \frac{1}{\Psi_{11}} + \frac{1}{\Psi_{22}} + \frac{1}{\Psi_{33}} = 0 \,. \tag{11}$$

The Gauss' law constraint can be written as

$$\mathbf{v}_e\{\Psi_{ae}\} + C_{be}\big(f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}\big) = 0 \,. \tag{12}$$

Since restricting to diagonal Ψ_{ae}, we need only consider the terms of (12) with $e = a$ on the first term, $e = f$ on the second and $a = g$ on the third. This is due to the fact that a is a free index while the remaining are dummy indices. Then we get the following equations

$$\left. \begin{array}{l} \mathbf{v}_1\{\Psi_{11}\} + C_{23}(\Psi_{33} - \Psi_{11}) + C_{32}(\Psi_{11} - \Psi_{22}) = 0 \\[4pt] \mathbf{v}_2\{\Psi_{22}\} + C_{31}(\Psi_{11} - \Psi_{22}) + C_{13}(\Psi_{22} - \Psi_{33}) = 0 \\[4pt] \mathbf{v}_3\{\Psi_{33}\} + C_{12}(\Psi_{22} - \Psi_{33}) + C_{21}(\Psi_{33} - \Psi_{11}) = 0 \end{array} \right\} \,. \tag{13}$$

Equation (13) is a set of three differential equations which can be put into the operator-valued matrix form

$$\begin{pmatrix} \mathbf{v}_1 - C_{[23]} & -C_{32} & C_{23} \\ C_{31} & \mathbf{v}_2 - C_{[31]} & -C_{13} \\ -C_{21} & C_{12} & \mathbf{v}_3 - C_{[12]} \end{pmatrix} \begin{pmatrix} \Psi_{11} \\ \Psi_{22} \\ \Psi_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

where we have defined $C_{[ae]} = C_{ae} - C_{ea}$. Since we have already removed three degrees of freedom by choosing Ψ_{ae} to be diagonal, and Gauss' law is a set of three conditions, we would rather not overconstrain Ψ_{ae} any further. In other words, we will regard the Gauss' law constraint as a set of conditions fixing three elements of the connection A_i^a, with Ψ_{ae} constrained only by the Hamiltonian constraint (11). We will from now on make the identifications

$$\Psi_{11} = \varphi_1 \,, \qquad \Psi_{22} = \varphi_2 \,, \qquad \Psi_{33} = \varphi_3 \,, \tag{14}$$

defined as the eigenvalues of Ψ_{ae}. We will now specialize to the Petrov Type D case, where two of the eigenvalues are equal with no vanishing eigenvalues.

§3.1. The Hamiltonian constraint.

Denote the eigenvalues of Ψ_{ae} by $\lambda_f = (\varphi_1, \varphi, \varphi)$ and all permutations thereof. Then the Hamiltonian constraint (11) reduces to

$$\frac{1}{\varphi_1} + \frac{2}{\varphi} + \Lambda = 0. \tag{15}$$

Equation (15) yields the following relations which we will use later

$$\varphi_1 = -\left(\frac{\varphi}{\Lambda\varphi + 2}\right), \qquad \varphi_1 - \varphi = -\varphi\left(\frac{\Lambda\varphi + 3}{\Lambda\varphi + 2}\right). \tag{16}$$

The diagonalized self-dual Weyl curvature for a spacetime of Type D is of the form $\psi_{ae} = \text{diag}(-2\Psi, \Psi, \Psi)$ for some function Ψ. The corresponding CDJ matrix is given by adding to this a cosmological contribution as in (10), which in matrix form is given by

$$\Psi_{ae}^{-1} = \begin{pmatrix} -\frac{\Lambda}{3} - 2\Psi & 0 & 0 \\ 0 & -\frac{\Lambda}{3} + \Psi & 0 \\ 0 & 0 & -\frac{\Lambda}{3} + \Psi \end{pmatrix}.$$

One can then read off the value of φ in (15) as

$$\varphi = \frac{1}{-\frac{\Lambda}{3} + \Psi}, \qquad \Lambda\varphi + 2 = \left(\frac{\frac{\Lambda}{3} + 2\Psi}{-\frac{\Lambda}{3} + \Psi}\right), \qquad \Lambda\varphi + 3 = \frac{3\Psi}{-\frac{\Lambda}{3} + \Psi}. \tag{17}$$

From (17) the following quantities Φ and ψ can be constructed

$$\left.\begin{aligned} \Phi &= \frac{\varphi(\Lambda\varphi + 3)^2}{(\Lambda\varphi + 2)^3} = 9\left(\frac{1}{2\Psi^{1/3} + \frac{\Lambda}{3}\Psi^{-2/3}}\right)^3 \\ \psi &= \varphi^2(\Lambda\varphi + 3) = 3\left(\frac{1}{-\frac{\Lambda}{3}\Psi^{-1/3} + \Psi^{2/3}}\right)^3 \end{aligned}\right\}, \tag{18}$$

which will become useful later in this paper.

§3.2. The Gauss' law constraint.

Next, we must set up the Gauss' law constraint (13) for the Type D case. There are three distinct permutations of eigenvalues to consider

$$\vec{\lambda}_{(1)} = (\varphi_1, \varphi, \varphi), \quad \vec{\lambda}_{(2)} = (\varphi, \varphi_1, \varphi), \quad \vec{\lambda}_{(3)} = (\varphi, \varphi, \varphi_1), \tag{19}$$

which we will treat individually. The steps which follow will refer to $\vec{\lambda}_{(1)}$, with the remaining cases obtainable by cyclic permutation. The Gauss'

law constraint for permutation $\vec{\lambda}_{(1)}$ reduces to

$$
\begin{pmatrix}
\mathbf{v}_1 - C_{[23]} & -C_{32} & C_{23} \\
C_{31} & \mathbf{v}_2 - C_{[31]} & -C_{13} \\
-C_{21} & C_{12} & \mathbf{v}_3 - C_{[12]}
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi \\
\varphi
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
$$

which leads to the following equations

$$
\left.
\begin{aligned}
\mathbf{v}_1\{\varphi_1\} &= C_{[23]}(\varphi_1 - \varphi) \\
\mathbf{v}_2\{\varphi\} &= C_{31}(\varphi - \varphi_1) \\
\mathbf{v}_3\{\varphi\} &= C_{21}(\varphi_1 - \varphi)
\end{aligned}
\right\}. \tag{20}
$$

Using the results from (16), the first equation of (20) implies that

$$
-\mathbf{v}_1\left\{\left(\frac{\varphi}{\Lambda\varphi + 2}\right)\right\} = -C_{[23]}\,\varphi\left(\frac{\Lambda\varphi + 3}{\Lambda\varphi + 2}\right). \tag{21}
$$

Since the vector fields $\mathbf{v}_a$ are first-derivative operators, equation (21) can be written as

$$
\frac{1}{\varphi}\left(\frac{\Lambda\varphi + 2}{\Lambda\varphi + 3}\right)\mathbf{v}_1\left\{\left(\frac{\varphi}{\Lambda\varphi + 2}\right)\right\} = C_{[23]} =
$$
$$
= \frac{1}{\varphi}\left(\frac{\Lambda\varphi + 2}{\Lambda\varphi + 3}\right)\left[\frac{(\Lambda\varphi + 2)\,\mathbf{v}_1\{\varphi\} - \varphi\,\mathbf{v}_1\{\Lambda\varphi + 2\}}{(\Lambda\varphi + 2)^2}\right], \tag{22}
$$

where we have used the Leibniz rule. Equation (22) then simplifies to

$$
\frac{2\mathbf{v}_1\{\varphi\}}{\varphi\,(\Lambda\varphi + 2)(\Lambda\varphi + 3)} = \frac{1}{3}\,\mathbf{v}_1\{\ln\Phi\} = C_{[23]}, \tag{23}
$$

which gives

$$
\mathbf{v}_1\{\ln\Phi\} = 3C_{[23]} \tag{24}
$$

with Φ given by (18).

The second equation of (20) implies that

$$
\mathbf{v}_2\{\varphi\} = C_{31}\,\varphi\left(\frac{\Lambda\varphi + 3}{\Lambda\varphi + 2}\right). \tag{25}
$$

Using (16), equation (25) simplifies to

$$
\frac{1}{\varphi}\left(\frac{\Lambda\varphi + 2}{\Lambda\varphi + 3}\right)\mathbf{v}_2\{\varphi\} = \frac{1}{3}\,\mathbf{v}_2\{\varphi^2(\Lambda\varphi + 3)\} = C_{31}, \tag{26}
$$

which gives

$$
\mathbf{v}_2\{\ln\psi\} = 3C_{31}. \tag{27}
$$

The manipulations of the third equation of (20) are directly analogous to (26) and (27), which implies that

$$\mathbf{v}_3\{\varphi\} = -C_{21}\,\varphi\left(\frac{\Lambda\varphi+3}{\Lambda\varphi+2}\right) \;\longrightarrow\; \mathbf{v}_3\{\ln\psi\} = -3C_{21}. \qquad (28)$$

Hence the three equations for $\vec{\lambda}_{(1)}$ can be written as

$$\mathbf{v}_1\{\ln\Phi\} = 3C_{[23]}, \quad \mathbf{v}_2\{\ln\psi\} = 3C_{31}, \quad \mathbf{v}_3\{\ln\psi\} = -3C_{21}, \qquad (29)$$

where Φ and ψ are given by (18).

For the second permutation of eigenvalues $\vec{\lambda}_{(2)}$ we have $\vec{\varphi} = (\varphi, \varphi_1, \varphi)$, which leads to the Gauss' law equations

$$\mathbf{v}_2\{\ln\Phi\} = 3C_{[31]}, \quad \mathbf{v}_3\{\ln\psi\} = 3C_{12}, \quad \mathbf{v}_1\{\ln\psi\} = -3C_{32}. \qquad (30)$$

For the third permutation of eigenvalues $\vec{\lambda}_{(3)}$ we have $\vec{\varphi} = (\varphi, \varphi, \varphi_1)$, which leads to the Gauss' law equations

$$\mathbf{v}_3\{\ln\Phi\} = 3C_{[12]}, \quad \mathbf{v}_1\{\ln\psi\} = 3C_{23}, \quad \mathbf{v}_2\{\ln\psi\} = -3C_{13}. \qquad (31)$$

The implication of this is the following. If there exists a General Relativity solution for a particular eigenvalue permutation, say $\vec{\lambda}_{(1)}$, then there must exist solutions corresponding to the remaining permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$.

§4. The spherically symmetric case.

We are now ready to proceed with the instanton representation method. We must first choose a connection A_μ^a which will play dual roles. On the one hand A_μ^a will define a metric based on the Hodge duality condition, and on the other hand its spatial part A_i^a will in conjunction with Ψ_{ae} form a metric based on the solution to the Gauss' law and the Hamiltonian constraints. For the purposes of this paper we will choose a connection A_μ^a which is known to produce spherically symmetric blackhole solutions. This paragraph will show that the requirements on $(h_{ij})_{\text{Hodge}}$ and on $(h_{ij})_{\text{Constraints}}$ are in a sense complementary. Then in the subsequent paragraphs of this paper we will equate these two metrics, which, as we will see, imposes stringent conditions on the form of the final solution.

§4.1. Ingredients for the Hodge duality condition.

Let the connection A_μ^a be defined by the following one-forms

$$A^1 = i\frac{f'}{g}\,dt + \left(\cos\theta\right)d\phi, \quad A^2 = -\left(\frac{\sin\theta}{g}\right)d\phi, \quad A^3 = \frac{d\theta}{g}, \qquad (32)$$

where $f = f(r)$ and $g = g(r)$ are at this stage arbitrary functions of radial distance r and a prime denotes differentiation with respect to r. Equation (32) yields the curvature 2-forms $F^a = dA^a + \frac{1}{2} f^{abc} A^b \wedge A^c$, given by

$$\left. \begin{aligned} F^1 &= -\left(\frac{if'}{g}\right)' dt \wedge dr - \sin\theta \left(1 - \frac{1}{g^2}\right) d\theta \wedge d\phi \\ F^2 &= -\frac{g'}{g^2}\, \sin\theta \, d\phi \wedge dr - \frac{if'}{g^2}\, dt \wedge d\theta \\ F^3 &= -\frac{g'}{g^2}\, dr \wedge d\theta - \frac{if'}{g^2}\, \sin\theta \, dt \wedge d\phi \end{aligned} \right\} . \tag{33}$$

From this we can read off the nonvanishing components of the magnetic field B_a^i and the temporal component of the curvature F_{0i}^a, given by

$$\left. \begin{aligned} B_1^1 &= \sin\theta \left(1 - \frac{1}{g^2}\right), \quad B_2^2 = -\frac{g'}{g^2}\, \sin\theta\,, \quad B_3^3 = -\frac{g'}{g^2} \\ F_{01}^1 &= -\left(\frac{if'}{g}\right)', \quad F_{02}^2 = -\left(\frac{if'}{g^2}\right), \quad F_{03}^3 = -\left(\frac{if'}{g^2}\right) \sin\theta \end{aligned} \right\} . \tag{34}$$

Since (34) form diagonal matrices, then the antisymmetric part of $(B^{-1})_i^a F_{0j}^a$ is zero which according to (8) makes the shift vector N^i equal to zero. Then following suit with (7) we have

$$c_{ij} = F_{0i}^a (B^{-1})_j^a = -i \begin{pmatrix} \dfrac{(f'/g)'}{\sin\theta\left(1-\frac{1}{g^2}\right)} & 0 & 0 \\[2ex] 0 & (f'/g')\dfrac{1}{\sin\theta} & 0 \\[2ex] 0 & 0 & (f'/g')\sin\theta \end{pmatrix} .$$

The determinant of $c_{(ij)}$ is given by

$$c = \det \left\| (B^{-1})_0^a F_{0j}^a \right\| = i\, \frac{(f'/g)'\,(f'/g')^2}{\left(1 - \frac{1}{g^2}\right) \sin\theta} \,. \tag{35}$$

So Hodge duality for the chosen connection A_μ^a implies, using (8), that the 3-metric $(h_{ij})_{\mathrm{Hodge}}$ is given by

$$(h_{ij})_{\mathrm{Hodge}} = -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\[2ex] 0 & \dfrac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')} & 0 \\[2ex] 0 & 0 & \dfrac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')}\sin^2\theta \end{pmatrix} .$$

According to Birkhoff's theorem, any spherically symmetric solution for vacuum General Relativity must be given by the Schwarzschild solution. Hodge duality alone is insufficient to select this solution, since it presently allows for three free functions f, g and N. Let us determine the minimal set of additional conditions necessary to obtain the Schwarzschild solution. Spherical symmetry ($g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2\theta$) in conjunction with the choice $N = f$ leads to the condition

$$\frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')} = r^2, \tag{36}$$

which still contains one degree of freedom in the choice of g. For example let us further choose $g = \frac{1}{f}$. Then $g' = -\frac{f'}{f^2}$, which yields

$$\frac{1}{2}\frac{d^2 f^2}{dr^2} = \frac{1}{r^2}\left(f^2 - 1\right). \tag{37}$$

Defining $u = \ln r$, then this leads to the equation

$$\left(\frac{d^2}{du^2} - \frac{d}{du} - 2\right) f^2 = -2 \tag{38}$$

with solution $f^2 = 1 + k_1 e^{-u} + k_2 e^{2u}$ for arbitrary constants k_1 and k_2. This yields the solution

$$f^2 = 1 + k_1 r^{-1} + k_2 r^2. \tag{39}$$

Upon making the identification $k_1 \equiv -2GM$ and $k_2 \equiv -\frac{\Lambda}{3}$ one recognizes (39) as the solution for a Schwarzschild-de Sitter black hole.[*]

§4.2. Ingredients for the Gauss' law constraint. The conditions determining $(h_{ij})_{\text{Constraints}}$ are fixed by the spatial connection A_i^a and Ψ_{ae} solving the constraints (1). Note in (32) that A_i^a depends only on g and not on f. This means that only g can be fixed by the Gauss' law constraint, and that f must be fixed by equality of (8) with (6). We will now proceed to solve the Gauss' law constraint for our connection (32), with spatial part given in the matrix form

$$A_i^a = \begin{pmatrix} A_r^1 & A_\theta^1 & A_\phi^1 \\ A_r^2 & A_\theta^2 & A_\phi^2 \\ A_r^3 & A_\theta^3 & A_\phi^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & -\frac{\sin\theta}{g} \\ 0 & \frac{1}{g} & 0 \end{pmatrix},$$

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[*]We will show that the set of conditions leading to (39) arise precisely from the equality of (8) with (6), namely that the Hodge-duality metric solve the Einstein equations. Without this, the solution is not unique.

where $g = g(r)$ is an arbitrary function only of radial distance r from the origin. By this choice we have also made the choice of a coordinate system (r, θ, ϕ) to whose axes various quantities will be referred. From (4), one can construct the magnetic field B_a^i

$$
B_a^i = \begin{pmatrix} -\left(1 - \frac{1}{g^2}\right)\sin\theta & 0 & 0 \\ 0 & \sin\theta \,\frac{d}{dr}\,g^{-1} & 0 \\ 0 & 0 & \frac{d}{dr}\,g^{-1} \end{pmatrix},
$$

and the magnetic helicity density matrix C_{ae}, given by

$$
C_{ae} = A_i^a B_e^i = \frac{\partial}{\partial r} \begin{pmatrix} 0 & 0 & \frac{\cos\theta}{g} \\ 0 & 0 & \frac{\sin\theta}{2}\left(1 - \frac{1}{g^2}\right) \\ 0 & -\frac{\sin\theta}{2}\left(1 - \frac{1}{g^2}\right) & 0 \end{pmatrix}.
$$

The vector field $\mathbf{v}_a = B_a^i \partial_i$ can be read off from the magnetic field matrix

$$
\left.\begin{aligned} \mathbf{v}_1 &= -\sin\theta\left(1 - \frac{1}{g^2}\right)\frac{\partial}{\partial r} \\[2mm] \mathbf{v}_2 &= \frac{d}{dr}\left(\frac{1}{g}\right)\sin\theta\,\frac{\partial}{\partial\theta}, \qquad \mathbf{v}_3 = \frac{d}{dr}\left(\frac{1}{g}\right)\frac{\partial}{\partial\phi} \end{aligned}\right\}. \tag{40}
$$

These will constitute the differential operators in the Gauss' law constraint. The ingredients for (6) for the configuration chosen are

$$
(\det\|\Psi\|)\left(\Psi^{-1}\Psi^{-1}\right)^{ae} = -\begin{pmatrix} \frac{\frac{\Lambda}{3} + 2\Psi}{\left(-\frac{\Lambda}{3} + \Psi\right)^2} & 0 & 0 \\ 0 & \frac{1}{\frac{\Lambda}{3} + 2\Psi} & 0 \\ 0 & 0 & \frac{1}{\frac{\Lambda}{3} + 2\Psi} \end{pmatrix}
$$

for the part involving Ψ_{ae}, and

$$
\eta_{ij}^{ae} \sim (B^{-1})_i^a (B^{-1})_j^e (\det\|B\|) \longrightarrow
$$

$$
\longrightarrow -\begin{pmatrix} \frac{\left(\frac{d}{dr}\,g^{-1}\right)^2}{1 - \frac{1}{g^2}} & 0 & 0 \\ 0 & 1 - \frac{1}{g^2} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2}\right)\sin^2\theta \end{pmatrix}
$$

for the part involving the magnetic field B^i_a. We have, in an abuse of notation, anticipated the result of multiplying the matrices needed for (6) for this special case where the matrices are diagonal. We will be particularly interested in the $\Lambda = 0$ case, as it is the simplest case to test for the Hodge duality condition.* For $\Lambda = 0$ the 3-metric based on the initial value constraints (6) is given by

$$
(h_{ij})_{\Lambda=0} = \frac{1}{2\Psi}
\begin{pmatrix}
\dfrac{4\left(\frac{d}{dr}\, g^{-1}\right)^2}{1-\frac{1}{g^2}} & 0 & 0 \\[2ex]
0 & 1 - \frac{1}{g^2} & 0 \\[2ex]
0 & 0 & \left(1 - \frac{1}{g^2}\right)\sin^2\theta
\end{pmatrix}.
$$

We are now ready to apply the instanton representation method to the construction of solutions.

§5. First permutation of eigenvalues $\vec{\lambda}_{(1)}$.

We will now produce some of the known blackhole solutions corresponding to the eigenvalue permutation $\vec{\lambda}_{(1)}$. The first equation of (29) for the chosen connection reduces to

$$
\mathbf{v}_1\{\ln\Phi\} = 3C_{[23]} \longrightarrow -\sin\theta\left(1-\frac{1}{g^2}\right)\frac{\partial\ln\Phi}{\partial r} = 3\sin\theta\,\frac{\partial}{\partial r}\left(1-\frac{1}{g^2}\right) \quad (41)
$$

where we have used (40), which integrates to

$$
\Phi = c\,(\theta,\phi)\left(1 - \frac{1}{g^2}\right)^{-3}, \tag{42}
$$

where c at this stage is an arbitrary function of two variables not to be confused with the c in (7). The second equation of (29) is given by

$$
\mathbf{v}_2\{\ln\psi\} = 3C_{31} \longrightarrow \left(\frac{d}{dr}\,g^{-1}\right)\sin\theta\,\frac{\partial\ln\psi}{\partial\theta} = 0, \tag{43}
$$

which implies that $\psi = \psi(r,\phi)$. The third equation of (29) is given by

$$
\mathbf{v}_3\{\ln\psi\} = -3C_{21} \longrightarrow \left(\frac{d}{dr}\,g^{-1}\right)\frac{\partial\ln\psi}{\partial\phi} = 0. \tag{44}
$$

In conjunction with the results from (43), one has that $\psi = \psi(r)$ must be a function only of r. Note that this is consistent with Φ being solely a function of r as in (42), which requires that $c(\theta,\phi) = c$ be a num-

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*The $\Lambda \neq 0$ case will be relegated for future research.

erical constant. Continuing from (42) we have

$$\left(\frac{1}{2\Psi^{1/3} + \frac{\Lambda}{3}\,\Psi^{-2/3}}\right)^3 = c\left(1 - \frac{1}{g^2}\right)^{-3}, \tag{45}$$

which upon redefining the parameter c yields the solution

$$g^2 = \left(1 - \frac{2}{c}\,\Psi^{1/3} - \frac{\Lambda}{3c}\,\Psi^{-2/3}\right)^{-1}. \tag{46}$$

So knowing Ψ, which comes directly from the CDJ matrix for Petrov Type D, enables us to determine the connection A_i^a explicitly in this case.

We can now proceed to compute the 3-metric h_{ij} for the chosen configuration. We would rather like to express the metric directly in terms of Ψ, which is the fundamental degree of freedom for the given Petrov Type. Hence from (45) we have

$$1 - \frac{1}{g^2} = \frac{1}{c}\,\Psi^{-2/3}\left(2\Psi + \frac{\Lambda}{3}\right), \tag{47}$$

which yields

$$\frac{d}{dr}\,g^{-1} = -\frac{1}{3c}\,\Psi^{-5/3}\left(1 - \frac{2}{c}\,\Psi^{1/3} - \frac{\Lambda}{3c}\,\Psi^{-2/3}\right)^{-1/2}\left(-\frac{\Lambda}{3} + \Psi\right)\Psi', \tag{48}$$

where $\Psi' = \frac{d\Psi}{dr}$. Then the magnetic field part η_{ij}^{ae} of the metric can be written explicitly in terms of Ψ

$$\eta_{ij}^{ae} = -\frac{1}{c}\begin{pmatrix} \frac{1}{9}\frac{\Psi^{-8/3}(\Psi')^2}{1-\frac{2}{c}\Psi^{1/3}-\frac{\Lambda}{3c}\Psi^{-2/3}}\frac{(-\frac{\Lambda}{3}+\Psi)^2}{2\Psi+\frac{\Lambda}{3}} & 0 & 0 \\ 0 & \Psi^{-2/3}\left(2\Psi+\frac{\Lambda}{3}\right) & 0 \\ 0 & 0 & \Psi^{-2/3}\left(2\Psi+\frac{\Lambda}{3}\right)\sin^2\theta \end{pmatrix}.$$

Multiplying this matrix by $(\det\|\Psi\|)\left(\Psi^{-1}\Psi^{-1}\right)^{ae}$, we obtain the 3-metric

$$\left(h_{ij}\right)_{\vec{\lambda}_{(1)}} = \frac{1}{c}\begin{pmatrix} \frac{1}{9}\frac{\Psi^{-8/3}(\Psi')^2}{1-\frac{2}{c}\Psi^{1/3}-\frac{\Lambda}{3c}\Psi^{-2/3}} & 0 & 0 \\ 0 & \Psi^{-2/3} & 0 \\ 0 & 0 & \Psi^{-2/3}\sin^2\theta \end{pmatrix}.$$

This is a general solution for the permutation sequence $\vec{\lambda}_{(1)}$ for the chosen connection. As a doublecheck, let us eliminate the constant of

integration c via the rescaling $\Psi \to \Psi c^{-3/2}$. But the shift vector N^i and the lapse N have remained undetermined based purely on the initial value constraints. For $N^i = 0$, which is a result of the Hodge duality condition, this yields a spacetime metric of

$$ds^2 = -N^2 dt^2 + \frac{1}{9}\left(\frac{\Psi^{-8/3}(\Psi')^2}{1 - 2\Psi^{1/3}c^{-3/2} - \frac{\Lambda}{3}\Psi^{-2/3}}\right)dr^2 +$$
$$+ \Psi^{-2/3}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \qquad (49)$$

Already, it can be seen that (49) can lead to some known General Relativity solutions. Taking $\Psi = \frac{1}{r^3}$, $c = (GM)^{-2/3}$, $N^i = 0$ and $N^2 = 1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2$ where N is the lapse function, we obtain

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2GM}{r} - \frac{\Lambda}{3}r^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

which is the solution for a Euclidean Schwarzschild-de Sitter blackhole. For $\Lambda = 0$, $g_{\mu\nu}$ reduces to the Schwarzschild metric and for $G = 0$, it reduces to the de Sitter metric.* There clearly exist solutions corresponding to $\vec{\lambda}_{(1)}$, since it is known that the Einstein equations admit blackhole solutions. On the other hand, the instanton representation implies that there must be additional solutions corresponding to the remaining permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$. We must now check for consistency of these additional solutions, if they exist, with Birkhoff's theorem. Let us examine the different eigenvalue permutations in turn.

§5.1. Hodge duality condition for $\lambda_{(1)}$ for $\Lambda = 0$. Note that the lapse function N at the level of (5) is freely specifiable and not fixed by (6). To make progress we will need to impose the Hodge duality condition, namely the equality of (6) with (8). From the Gauss' law constraint we can read off from (47) in the $\Lambda = 0$ case that

$$\Psi = \frac{1}{8}\left(1 - \frac{1}{g^2}\right)^3. \qquad (50)$$

So upon implementation of the Hodge duality condition, then the

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*Setting $M = 0$ corresponds to a transition from Type D to Type O spacetime, where $\Psi = 0$.

3-metric must satisfy the condition $(h_{ij})_{\text{Constraints}} = (h_{ij})_{\text{Hodge}}$, or

$$(h_{ij})_{\Lambda=0} = - \begin{pmatrix} 16\left(\frac{d}{dr}g^{-1}\right)^2\left(1-\frac{1}{g^2}\right)^{-4} & 0 & 0 \\ 0 & 4\left(1-\frac{1}{g^2}\right)^{-2} & 0 \\ 0 & 0 & 4\left(1-\frac{1}{g^2}\right)^{-2}\sin^2\theta \end{pmatrix} =$$

$$= -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\ 0 & \dfrac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')} & 0 \\ 0 & 0 & \dfrac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')}\sin^2\theta \end{pmatrix}.$$

As a consistency condition on the radial component g_{rr} we must require that

$$N^2\left(\frac{g'}{f'}\right)^2 = 16\,\frac{g'}{g^2}\left(1-\frac{1}{g^2}\right)^{-4}, \tag{51}$$

and as a consistency condition on $g_{\theta\theta}$ we must require that

$$N^2\,\frac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')} = 4\left(1-\frac{1}{g^2}\right)^{-2}. \tag{52}$$

Equations (51) and (52) are a set of two equations in three unknowns N, g and f. Upon dividing equation (52) into (51), then N^2 drops out and we have the following relation between f and g

$$\frac{1}{f'}\left(\frac{f'}{g}\right)' = \frac{4}{g^2-1} \quad \longrightarrow \quad \frac{f''}{f'} = \frac{g'}{g} + \frac{4g}{g^2-1}. \tag{53}$$

Integration of (53) determines $f = f[g]$ explicitly in terms of g, and substitution of the result into (51) determines $N = N[g]$ via

$$\left. \begin{aligned} f &= k_2 + k_1 \int dr\, g \exp\left[4\int \frac{g\,dr}{g^2-1}\right] \\ N^2 &= \frac{16}{g'g^2}\left(1-\frac{1}{g^2}\right)^{-2}\left(k_2 + k_1\int dr\, g\exp\left[4\int \frac{g\,dr}{g^2-1}\right]\right)^2 \end{aligned} \right\}. \tag{54}$$

Recall that g is fixed by the Gauss' law constraint on the spatial hypersurface Σ, and that f and N have to do with the temporal part of the metric. The function g is apparently freely specifiable, and each g

determines f and N. So the Hodge duality condition determines the temporal parts of $g_{\mu\nu}$ from the spatial part.

There are an infinite number of solutions parametrized by the function g. But according to Birkhoff's theorem there should be only one static spherically symmetric vacuum solution, namely the Schwarzschild solution. The Hodge duality condition by itself is insufficient to select this solution. First, we must impose the spherically symmetric form $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2\theta$, which implies

$$4\left(1 - \frac{1}{g^2}\right)^{-2} = r^2 \longrightarrow g = \left(1 - \frac{2}{r}\right)^{-1/2}, \quad g' = -r^{-2}\left(1 - \frac{2}{r}\right)^{-3/2} \quad (55)$$

in units where $GM = 1$. Substitution of (55) into (51) and (52) yields

$$N^2 = r^4\left(1 - \frac{2}{r}\right)\phi^2, \quad N^2 = -\frac{1}{2} r^5\left(1 - \frac{2}{r}\right)\phi'\phi, \quad \phi = f'\left(1 - \frac{2}{r}\right)^{1/2}. \quad (56)$$

Equating the first and second equations of (56) leads to the condition that $\phi = r^{-2}$. Putting this into the third equation allows us to find f

$$f = \int dr \; r^{-2}\left(1 - \frac{2}{r}\right)^{-1/2} = -\left(1 - \frac{2}{r}\right) \longrightarrow N^2 = 1 - \frac{2}{r}, \quad (57)$$

as well as the lapse function N. Putting (57) back into (51) then determines g_{rr}, given by

$$g_{rr} = -\frac{1}{1 - \frac{2}{r}} \, . \quad (58)$$

The final result is that the condition of spherical symmetry $g_{\theta\theta} = r^2$ in addition to Hodge duality of the curvature of the chosen A_i^a fixes the lapse function N, which yields the spacetime line element

$$-ds^2 = \left(1 - \frac{2}{r}\right)dt^2 + \left(1 - \frac{2}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \quad (59)$$

The result is the Euclidean Schwarzschild metric, as predicted by Birkhoff's theorem.

§6. Second permutation of eigenvalues $\vec{\lambda}_{(2)}$. We have found spherically symmetric blackhole solutions using the first permutation $\vec{\lambda}_{(1)}$. According to the Birkhoff theorem there should be no additional spherically symmetric time-independent solutions. But we will nevertheless proceed with the construction of any solutions implied by the

instanton representation for the second permutation $\vec{\lambda}_{(2)}$. The Gauss' law constraint equations associated with $\vec{\lambda}_{(2)}$ are given by (30)

$$\mathbf{v}_2\{\ln\Phi\} = 3C_{[31]}, \quad \mathbf{v}_3\{\ln\psi\} = 3C_{12}, \quad \mathbf{v}_1\{\ln\psi\} = -3C_{32} \qquad (60)$$

with Φ and ψ given by (18). The first equation of (60) yields

$$\mathbf{v}_2\{\ln\Phi\} = 3C_{[31]} \;\longrightarrow\; \left(\frac{d}{dr}\,g^{-1}\right)\sin\theta\,\frac{\partial\ln\Phi}{\partial\theta} = 3\frac{\partial}{\partial r}\left(-\frac{\cos\theta}{g}\right) \quad (61)$$

which integrates to

$$\Phi = c\,(r,\phi)\,\sin^{-3}\theta\,, \qquad (62)$$

where c is at this stage an arbitrary function of two variables. The second equation of (60) yields

$$\mathbf{v}_3\{\ln\psi\} = 3C_{12} = 0 \;\longrightarrow\; \left(\frac{d}{dr}\,g^{-1}\right)\frac{\partial\ln\psi}{\partial\phi} = 0\,, \qquad (63)$$

which implies that $\psi = \psi(r,\theta)$. The third equation of (60) yields

$$\mathbf{v}_1\{\ln\psi\} = -3C_{32} \;\longrightarrow$$
$$\longrightarrow\; -\sin\theta\left(1 - \frac{1}{g^2}\right)\frac{\partial\ln\psi}{\partial r} = \frac{3}{2}\sin\theta\,\frac{\partial}{\partial r}\left(1 - \frac{1}{g^2}\right), \qquad (64)$$

which integrates to

$$\psi = k\,(\theta,\phi)\left(1 - \frac{1}{g^2}\right)^{-3/2}. \qquad (65)$$

For consistency of (65) with the results of (62) and (63), we must have that $c\,(r,\phi) = c\,(r)$ and $k\,(\theta,\phi) = k(\theta)$. Therefore ψ and Φ are given by

$$\left.\begin{array}{l} \psi = 3\left(-\dfrac{\Lambda}{3}\,\Psi^{-1/3} + \Psi^{2/3}\right)^{-3} = k\,(\theta)\left(1 - \dfrac{1}{g^2}\right)^{-3/2} \\[3mm] \Phi = 9\left(\dfrac{\Lambda}{3}\,\Psi^{-2/3} + 2\Psi^{1/3}\right)^{-3} = c\,(r)\,\sin^{-3}\theta \end{array}\right\}. \qquad (66)$$

Equations (66) yield the following two conditions which must be satisfied

$$\left.\begin{array}{l} -\dfrac{\Lambda}{3}\,\Psi^{-1/3} + \Psi^{2/3} = k\,(\theta)\sqrt{1 - \dfrac{1}{g^2}} \\[3mm] \dfrac{\Lambda}{3}\,\Psi^{-2/3} + 2\Psi^{1/3} = c\,(r)\sin\theta \end{array}\right\}. \qquad (67)$$

It appears not possible to satisfy both conditions in (67) unless $\Lambda = 0$. Setting $\Lambda = 0$, then we have the following consistency condition

$$
\left.
\begin{array}{l}
\left[c\left(r \right) \sin \theta \right]^2 = k\left(\theta \right) \sqrt{1 - \dfrac{1}{g^2}} \quad \longrightarrow \quad c(r) = \left(1 - \dfrac{1}{g^2} \right)^{1/4} \\[2mm]
k\left(\theta \right) = \sin^2 \theta
\end{array}
\right\} . \qquad (68)
$$

Substituting (68) back into (67), we obtain

$$
\Psi = \Psi\left(r, \theta \right) = \left(1 - \frac{1}{g^2} \right)^{3/4} \sin^3 \theta . \qquad (69)
$$

Using the magnetic field for the configuration chosen, which is the same as for the previous permutation $\vec{\lambda}_{(1)}$, then (6) yields a 3-metric

$$
(h_{ij})_{\vec{\lambda}_{(2)}} = -\frac{1}{2} \left(1 - \frac{1}{g^2} \right)^{-3/4} \sin^{-3} \theta \times
$$

$$
\times \begin{pmatrix}
\dfrac{4\left(\frac{d}{dr} g^{-1} \right)^2}{1 - \frac{1}{g^2}} & 0 & 0 \\[3mm]
0 & 1 - \dfrac{1}{g^2} & 0 \\[3mm]
0 & 0 & \left(1 - \dfrac{1}{g^2} \right) \sin^2 \theta
\end{pmatrix} .
$$

This particular permutation of eigenvalues is allowed only for $\Lambda = 0$.

§6.1. Hodge duality condition for $\lambda_{(2)}$ for $\Lambda = 0$. The initial value constraints imply the existence of a spatial 3-metric $(h_{ij})_{\vec{\lambda}_{(2)}}$. We must enforce the Hodge duality condition as a consistency condition, and examine the implications with respect to the Birkhoff theorem. From the Gauss' law constraint we can read off from (69) that

$$
\Psi = \left(1 - \frac{1}{g^2} \right)^{3/4} \sin^3 \theta . \qquad (70)
$$

So upon implementation of the Hodge duality condition, which requires equality of (6) with (8), the 3-metric must satisfy the condition

$$
(h_{ij})_{\Lambda = 0} = -\frac{1}{2} \sin^{-3} \theta \times
$$

$$
\times \begin{pmatrix}
\left(4 \frac{d}{dr} g^{-1} \right)^2 \left(1 - \frac{1}{g^2} \right)^{-7/4} & 0 & 0 \\[3mm]
0 & \left(1 - \frac{1}{g^2} \right)^{-1/4} & 0 \\[3mm]
0 & 0 & \left(1 - \frac{1}{g^2} \right)^{-1/4} \sin^2 \theta
\end{pmatrix} =
$$

$$= -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\ 0 & \dfrac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')} & 0 \\ 0 & 0 & \dfrac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')}\sin^2\theta \end{pmatrix}.$$

Consistency of the conformal factor fixes the lapse function as

$$N^2 = \frac{1}{2}\,\sin^{-3}\theta\,. \tag{71}$$

The remaining consistency conditions are on g_{rr}, namely

$$4\frac{g'}{g^2}\left(1-\frac{1}{g^2}\right)^{-7/4} = \left(\frac{g'}{f'}\right)^2 \longrightarrow f' = \frac{1}{2}\,g^2\left(1-\frac{1}{g^2}\right)^{7/8}, \tag{72}$$

as well as on $g_{\theta\theta}$

$$\left(1-\frac{1}{g^2}\right)^{1/4} = \frac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'(f'/g')} \longrightarrow \left(\frac{f'}{g}\right)'f' = g'\left(1-\frac{1}{g^2}\right)^{3/4}. \tag{73}$$

Putting the result of (72) into (73) leads to the condition

$$g'\left(1-\frac{1}{g^2}\right)^{7/8}\left[1+\frac{7}{4g^2}\left(1-\frac{1}{g^2}\right)^{-1}-\frac{4}{g^2}\left(1-\frac{1}{g^2}\right)^{3/4}\right] = 0\,. \tag{74}$$

The solution to (74) is $g' = 0$, which means that g is a numerical constant given by the roots of the term in brackets. This is a seventh degree polynomial, which we will not attempt to solve in this paper. Note for g constant that $g_{rr} = 0$. If any of the roots of the polynomial are real, then they would yield the following metric

$$ds^2 = -\frac{1}{2}\,\sin^{-3}\theta\left[dt^2 + k_2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]. \tag{75}$$

The resulting metric is conformal to a 2-sphere radius $\sqrt{k_2}$, where $g = \left(1-k_2^4\right)^{1/2}$ is any one of the seven roots of (75). The metric resulting from $\vec{\lambda}_{(2)}$ is degenerate since $g_{rr} = 0$, and also not spherically symmetric on account of the θ-dependent conformal factor. The interpretation is that Birkhoff theorem still holds and does not apply to (75), which constitutes a new General Relativity solution.

§7. Third permutation of eigenvalues $\vec{\lambda}_{(3)}$. For the third permutation of eigenvalues $\vec{\lambda}_{(3)}$ we have $\vec{\varphi} = (\varphi, \varphi, \varphi_1)$, which leads to the Gauss' law constraint equations (31)

$$\mathbf{v}_3\{\ln\Phi\} = 3C_{[12]}, \quad \mathbf{v}_1\{\ln\psi\} = 3C_{23}, \quad \mathbf{v}_2\{\ln\psi\} = -3C_{13}. \tag{76}$$

The first equation from (76) is given by

$$\mathbf{v}_3\{\ln\Phi\} = 3C_{[12]} \;\longrightarrow\; \left(\frac{d}{dr}\,g^{-1}\right)\frac{\partial\ln\Phi}{\partial\phi} = 0\,, \qquad (77)$$

which implies that $\Phi = \Phi(r,\theta)$ is at this stage an arbitrary function of two variables. The second equation of (76) is given by

$$\mathbf{v}_1\{\ln\psi\} = 3C_{23} \;\longrightarrow$$
$$\longrightarrow\; -\sin\theta\left(1-\frac{1}{g^2}\right)\frac{\partial\ln\psi}{\partial r} = \frac{3}{2}\,\sin\theta\,\frac{\partial}{\partial r}\left(1-\frac{1}{g^2}\right), \quad (78)$$

which integrates to

$$\psi = k(\theta)\left(1-\frac{1}{g^2}\right)^{-3/2}. \qquad (79)$$

This is consistent with the results from (77), since there can be no ϕ dependence. The third equation of (76) is given by

$$\mathrm{v}_2\{\ln\psi\} = -3C_{13} \;\longrightarrow\; \frac{\partial}{\partial r}\left(\frac{\sin\theta}{g}\right)\frac{\partial\ln\psi}{\partial\theta} = -3\frac{\partial}{\partial r}\left(\frac{\cos\theta}{g}\right), \quad (80)$$

which integrates to

$$\psi = c(r)\sin^{-3}\theta\,, \qquad (81)$$

where c is at this stage an arbitrary function. From (77) $\Phi = \Phi(r,\theta)$ can be an arbitrary function of r and θ, and hence we are free to determine this dependence entirely from ψ. Consistency of (79) with (81) implies that

$$\psi = -\frac{\Lambda}{3}\,\Psi^{-1/3} + \Psi^{2/3} = \sin^{-3}\theta\left(1-\frac{1}{g^2}\right)^{-3/2}. \qquad (82)$$

Unlike the case for $\vec{\lambda}_{(2)}$ we are allowed to have a nonzero Λ in equation (82), since there is no longer a constraint on the functional dependence of Φ. Therefore we are free to solve equation (82) for Ψ, which enables us to fix $\Phi = \Phi(\psi)$. Equation (82) is a cubic polynomial equation for the quantity $\Psi^{1/3}$, which can be solved in closed form (see e.g. Appendix A for the derivation)

$$\left.\begin{aligned}
\Psi &= 2\sqrt{-\frac{\psi}{3}}\,\sin\left\{\frac{1}{3}\arcsin\left[\frac{\sqrt{3}\Lambda}{2}\,(-\psi)^{-3/2}\right]\right\}^3 \\
\psi &= \sin^{-3}\theta\left(1-\frac{1}{g^2}\right)^{-3/2}
\end{aligned}\right\}. \qquad (83)$$

For the purposes of constructing a 3-metric we will be content with the $\Lambda = 0$ case which follows from (82), yielding

$$\Psi = \Psi(r, \theta) = (\sin \theta)^{-9/2} \left(1 - \frac{1}{g^2} \right)^{-9/4}.$$
(84)

Using the previous configuration, equation (84) yields a 3-metric

$$(h_{ij})_{\vec{\lambda}_{(3)}} = \frac{1}{2} \left(1 - \frac{1}{g^2} \right)^{9/4} \sin^{9/2}\theta \begin{pmatrix} \dfrac{4\left(\frac{d}{dr} g^{-1} \right)^2}{1 - \frac{1}{g^2}} & 0 & 0 \\ 0 & 1 - \frac{1}{g^2} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2} \right) \sin^2\theta \end{pmatrix}.$$

§7.1. Hodge duality condition for $\lambda_{(3)}$ for $\Lambda = 0$. The initial value constraints imply the existence of a spatial 3-metric $(h_{ij})_{\vec{\lambda}_{(3)}}$. We must enforce the Hodge duality condition as a consistency condition, and examine the implications with respect to Birkhoff's theorem. From the Gauss' law constraint we can read off from (84) that

$$\Psi = \left(1 - \frac{1}{g^2} \right)^{3/4} \sin^3\theta.$$
(85)

So upon implementation of the Hodge duality condition, then the 3-metric must satisfy the condition $(h_{ij})_{\text{Constraints}} = (h_{ij})_{\text{Hodge}}$, given by

$$(h_{ij})_{\vec{\lambda}_{(3)}} = \frac{1}{2} \sin^{9/2}\theta \times$$

$$\times \begin{pmatrix} \left(4\frac{d}{dr} g^{-1} \right)^2 \left(1 - \frac{1}{g^2} \right)^{5/4} & 0 & 0 \\ 0 & \left(1 - \frac{1}{g^2} \right)^{13/4} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2} \right)^{13/4} \sin^2\theta \end{pmatrix} =$$

$$= -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\ 0 & \dfrac{\left(1 - \frac{1}{g^2} \right)}{(f'/g)' (f'/g')} & 0 \\ 0 & 0 & \dfrac{\left(1 - \frac{1}{g^2} \right)}{(f'/g)' (f'/g')} \sin^2\theta \end{pmatrix}.$$

Consistency of the conformal factor fixes the lapse function as

$$N^2 = \sin^{9/2}\theta.$$
(86)

The remaining consistency conditions are on g_{rr}, namely

$$4\frac{g'}{g^2}\left(1-\frac{1}{g^2}\right)^{5/4} = \left(\frac{g'}{f'}\right)^2 \;\longrightarrow\; f' = \frac{1}{2}\,g^2\left(1-\frac{1}{g^2}\right)^{-5/8}, \qquad (87)$$

as well as on $g_{\theta\theta}$

$$\left(1-\frac{1}{g^2}\right)^{13/4} = \frac{\left(1-\frac{1}{g^2}\right)}{(f'/g)'\,(f'/g')} \;\longrightarrow\; \left(\frac{f'}{g}\right)'f' = g'\left(1-\frac{1}{g^2}\right)^{-9/4}. \quad (88)$$

Putting the result of (87) into (88) leads to the condition

$$g'\left(1-\frac{1}{g^2}\right)^{-5/8}\left(1-\frac{37}{8g^2}+\frac{37}{8g^4}\right) = 0\,. \qquad (89)$$

The solution to (89) is $g'=0$, which means that g is a constant given by the roots of the quartic polynomial in brackets. The solution is

$$g = \pm\sqrt{\frac{37}{16}\pm\frac{1}{8}\sqrt{\frac{185}{2}}}\;. \qquad (90)$$

There are four roots, each of which corresponds to a 2-sphere

$$ds^2 = -\frac{1}{2}\sin^{9/2}\theta\left[dt^2 + k_3\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]. \qquad (91)$$

The resulting metric is conformal to a 2-sphere of radius $\sqrt{k_3}$, determined by any of the four roots (90). In direct analogy with the case from $\vec{\lambda}_{(2)}$, the solutions corresponding to $\vec{\lambda}_{(3)}$ are also degenerate and not spherically symmetric. Hence Birkhoff's theorem still holds and $\vec{\lambda}_{(1)}$ yields the unique static spherically symmetric vacuum solution.

§8. Conclusion. In this paper we have constructed some solutions to the Einstein equations using the instanton representation method. We have applied this method to spacetimes of Petrov Type D, producing some known solutions. We first constructed the Schwarzschild blackhole solution from a particular permutation $\vec{\lambda}_{(1)}$ of the eigenvalues of Ψ_{ae} solving the initial value constraints, by implementation of the Hodge duality condition. This was done to establish the validity of the method for a simple well-known case. Then using the remaining eigenvalue permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$, we constructed additional solutions which might be not as well-known, and perhaps even new. This would on the surface suggest that the instanton representation method be rendered inadmissible, since Birkhoff's theorem implies that any additional solutions besides the Schwarzschild solution must not exist. However,

upon further analysis we have shown that the Hodge duality condition applied to the $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ metrics imposed stringent restrictions on their form. These restrictions led to two new solutions for which Birkhoff's theorem does not apply. The metrics for $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ became conformally related to 2-spheres of fixed radius determined by the roots of certain polynomial equations. Since the conformal factor depends on θ, then these metrics are not spherically symmetric in the usual sense. This, combined with the observation that the metrics are degenerate, leads us to conclude that the instanton representation method as applied in this paper is fully consistent with Birkhoff's theorem, and also is indeed capable of producing General Relativity solutions. Our main results have been the validation of the instanton representation method for the Schwarzschild case, and as well the construction of two solutions (75) and (91) which to the present author's knowledge appear to be new. Having established the validity of the instanton representation method for a special situation governing the Petrov Type D case as a testing ground, we are now ready to apply the method to the construction of more general solutions.

Appendix A. Roots of the cubic polynomial in trigonometric form. We would like to solve the cubic equation

$$z^3 + pz = q, \tag{92}$$

Many techniques for solving the cubic involve complicated radicals, which introduce complex numbers which are not needed when the roots are real-valued. We prefer the trigonometric method, which avoids such complications. Define a transformation

$$z = u \sin\theta. \tag{93}$$

Substitution of (93) into (92) yields

$$\sin^3\theta + \frac{p}{u^2} \sin\theta = \frac{q}{u^3}. \tag{94}$$

Comparison of (94) with the trigonometric identity

$$\sin^3\theta - \frac{3}{4} \sin\theta = -\frac{1}{4} \sin(3\theta) \tag{95}$$

enables one to make the identifications

$$\frac{p}{u^2} = -\frac{3}{4}, \qquad \frac{q}{u^3} = -\frac{1}{4} \sin(3\theta). \tag{96}$$

This implies that

$$u = \frac{2}{\sqrt{3}} \, (-p)^{1/2}, \qquad \sin\bigl(3\theta\bigr) = -\frac{3\sqrt{3}}{2} \frac{q}{(-p)^{3/2}} \, . \tag{97}$$

We can now solve (97) for θ

$$\theta = \frac{1}{3} \, \arcsin\left[-\frac{3\sqrt{3}}{2} \frac{q}{(-p)^{3/2}} \right] + \frac{2\pi m}{3} \, , \qquad m = 0, 1, 2 \tag{98}$$

and in turn for z using (93). The solution is

$$z = \frac{1}{\sqrt{3}} \, (-p)^{1/2} \, T^m_{1/3} \left[-3\sqrt{3} \, q \, (-p)^{-3/2} \right] , \tag{99}$$

where we have defined

$$T^m_{1/3}(t) = 2 \sin\left[-\frac{1}{3} \, \arcsin\left(\frac{t}{2} \right) \right] . \tag{100}$$

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