On the Komar Energy and the Generalized Smarr Formula for a Charged Black Hole Inspired by Noncommutative Geometry

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Abstract: We calculate the Komar energy E for a charged black hole inspired by noncommutative geometry and identify the total mass (M_0) by considering the asymptotic limit. We also found the generalized Smarr formula, which shows a deformation from the well known relation $M_0 - Q_0^2/r_+ = 2ST$ depending on the noncommutative scale length ℓ.

Contents:

§1. Introduction. There is a deep connection between gravity and thermodynamics that has been known for a long time, from the works of Bekenstein and Hawking [1–3] to the recent research of Padmanabhan [4, 5]. In a thermodynamical system like Schwarzschild black hole, the entropy S, the Hawking temperature T and energy E are related by the first law of thermodynamics

$$dE = T\,dS\,, \tag{1}$$

where E is identified with the Komar energy [6, 7] and specifically for a Schwarzschild black hole it equals the total mass of the black hole, M. There is also an integral version of this equation

$$E = M = 2T\,S\,. \tag{2}$$

known as the Smarr formula [8] and it can be verified by using the expressions for temperature and entropy,

$$T = \frac{1}{8\pi M}\,, \tag{3}$$

$$S = \frac{A}{4} = 4\pi M^2. \tag{4}$$

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Eq. (2) has been obtained in different ways [5, 9] and the Komar energy is identified with the conserved charge associated with the Killing vector defined at the event horizon (see for example [10]). Recently, some generalized expressions for Smarr formula in different spacetimes have been studied [9–11] and in particular, the Kerr-Newman black hole with electric charge Q and angular momentum J satisfies the Smarr relation [12]

$$M = 2TS + \Phi_H Q + 2\Omega_H J\,, \tag{5}$$

where Φ_H and Ω_H are the electric potential and angular velocity at the horizon, respectively.

As a continuation of the research in black holes inspired by noncommutative geometry started in [13], in this paper we investigate the specific case of a 4-dimensional spherically symmetric charged black hole studied in [14–21]. This solution is obtained by introducing the noncommutativity effect through a coherent state formalism [22–24], which implies the replacement of the point distributions by smeared structures throughout a region of linear size ℓ. We perform the analysis by obtaining the Komar energy by direct integration and found the generalized Smarr formula, which shows a deformation from the usual relation depending on the noncommutative parameter ℓ.

§2. Komar energy of the charged noncommutative black hole. Many formulations of noncommutative field theory are based on the Weyl-Wigner-Moyal $*$-product [25–27] that lead to some important problems such as Lorentz invariance breaking, loss of unitarity or UV divergences of the quantum field theory. However, Smailagic and Spallucci [14–18, 20] explained recently a model of noncommutativity that can be free from the problems mentioned above. They assume that a point-like mass M and charge Q, instead of being quite localized at a point, must be described by a smeared structure throughout a region of linear size ℓ. The metric for this distribution is given by [21]

$$ds^2 = -f\left(r\right)dt^2 + \frac{dr^2}{f\left(r\right)} + r^2 d\Omega^2, \tag{6}$$

where

$$f(r) = 1 - \frac{2M\left(r\right)}{r} + \frac{Q^2\left(r\right)}{r^2}\,, \tag{7}$$

$$Q\left(r\right) = \frac{Q_0}{\sqrt{\pi}}\sqrt{\gamma^2\left(\frac{1}{2},\frac{r^2}{4\ell^2}\right) - \frac{r}{\sqrt{2}\ell}\gamma\left(\frac{1}{2},\frac{r^2}{2\ell^2}\right) + \frac{\sqrt{2}r}{\ell}\gamma\left(\frac{3}{2},\frac{r^2}{4\ell^2}\right)}\,, \tag{8}$$

$$M(r) = \frac{2M_0}{\sqrt{\pi}}\,\gamma\left(\frac{3}{2}, \frac{r^2}{4\ell^2}\right), \tag{9}$$

and

$$\gamma\left(\frac{a}{b}, x\right) = \int_0^x du\, u^{\frac{a}{b}-1} e^{-u} \tag{10}$$

is the lower incomplete gamma function. Considering a spatial 2-sphere V with boundary ∂V, the Komar integral for the energy is

$$E(V) = \frac{a}{16\pi} \oint_{\partial V} \nabla^{\mu} \xi^{\nu} d\Sigma_{\mu\nu}\,, \tag{11}$$

where the killing vector is $\xi = \frac{\partial}{\partial t}$, $d\Sigma_{\mu\nu}$ is the surface element at the boundary and the value of constant a will be found by comparison with the noncommutative Schwarzschild case. This is

$$E = \frac{2a}{16\pi} \oint_{\partial V} \nabla^{\mu} \xi^{t} d\Sigma_{\mu t}\,, \tag{12}$$

where the factor 2 appears because of the symmetry of the integrand. The covariant derivative involved is

$$\nabla_{\mu}\, \xi^t = \partial_{\mu} \xi^t + \Gamma^t_{\mu\sigma} \xi^{\sigma} = \Gamma^t_{\mu t}\,, \tag{13}$$

and for the noncommutative charged solution the nonvanishing connections are

$$\Gamma^t_{rt} = \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2}\frac{dQ^2}{dr} - Q^2}{r\left(r^2 - 2Mr + Q^2\right)}\,, \tag{14}$$

$$\Gamma^t_{tt} = \Gamma^t_{\theta t} = \Gamma^t_{\varphi t} = 0\,, \tag{15}$$

giving

$$E = \frac{a}{8\pi} \oint_{\partial V} \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2}\frac{dQ^2}{dr} - Q^2}{r^3}\, d\Sigma_{rt}\,. \tag{16}$$

The surface element corresponds to

$$d\Sigma_{rt} = -d\Sigma_{tr} = -r^2 \sin^2\theta\, d\theta d\varphi \tag{17}$$

and therefore

$$E = -\frac{a}{8\pi} \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2}\frac{dQ^2}{dr} - Q^2}{r} \oint_{\partial V} \sin^2\theta\, d\theta d\varphi\,, \tag{18}$$

$$E = \frac{a}{2}\left[\frac{dM}{dr} r - M - \frac{1}{2}\frac{dQ^2}{dr} + \frac{Q^2}{r}\right]. \tag{19}$$

By comparison with the Komar energy of the Schwarzschild black hole, we shall identify $a = -2$. Hence, the energy of the noncommutative charged black hole is finally given by

$$E = M - \frac{dM}{dr}\, r - \frac{Q^2}{r} + Q\, \frac{dQ}{dr}\, . \tag{20}$$

The horizons of the metric (6) can be found by setting $f(r_\pm) = 0$, i.e.

$$r_\pm^2 - 2r_\pm M\,(r_\pm) + Q^2\,(r_\pm) = 0\,, \tag{21}$$

which can be solved as

$$r_\pm = M\,(r_\pm) \pm \sqrt{M^2\,(r_\pm) - Q^2\,(r_\pm)}\ . \tag{22}$$

The Hawking temperature is defined in terms of the surface gravity at the event horizon by

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\, \partial_r f\,(r)\big|_{r=r_+}\,, \tag{23}$$

which gives in this case

$$T = \frac{1}{2\pi r_+^2}\left[M\,(r_+) - \frac{Q^2\,(r_+)}{r_+} - r_+ \frac{dM}{dr}\bigg|_{r=r_+} + Q\,(r_+)\, \frac{dQ}{dr}\bigg|_{r=r_+} \right]. \tag{24}$$

The entropy in terms of the area of the horizon is given by the well known relation

$$S = \frac{A}{4} = \pi r_+^2 \tag{25}$$

and therefore, the Komar energy (20) at the event horizon becomes

$$E = 2\pi r_+^2 T = 2ST. \tag{26}$$

Using the Reissner-Nordström values $r_\pm = M_0 \pm \sqrt{M_0^2 - Q_0^2}$ as a first approximation of the horizons (22) and putting them into the incomplete gamma functions of Eqs. (8) and (9) one obtains

$$r_\pm = M_\pm \pm \sqrt{M_\pm^2 - Q_\pm^2} \tag{27}$$

where we have defined $M_\pm$ and $Q_\pm$ in Page 112.

For a large value of its argument (i.e. large masses), function ε tends to unity while the exponential term goes to zero, giving the classical Reissner-Nordström horizons $r_\pm \to r_{RN\pm} = M_0 \pm \sqrt{M_0^2 - Q_0^2}$.

$$M_{\pm} = M_0 \left[\varepsilon \left(\frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{2\ell} \right) - \frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{\sqrt{\pi}\,\ell} \exp\left(- \frac{\left(M_0 \pm \sqrt{M_0^2 - Q_0^2}\right)^2}{4\ell^2} \right) \right],$$

$$Q_{\pm} = Q_0 \sqrt{\varepsilon^2 \left(\frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{2\ell} \right) - \frac{\left(M_0 \pm \sqrt{M_0^2 - Q_0^2}\right)^2}{\sqrt{2\pi}\,\ell^2} \exp\left(- \frac{\left(M_0 \pm \sqrt{M_0^2 - Q_0^2}\right)^2}{4\ell^2} \right)}\,,$$

and $\varepsilon\,(x)$ is the Gauss error function,

$$\varepsilon\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du\,.$$

Using the same first approximation for the event horizon r_+ in the Hawking temperature (23) one obtains [29]

$$T \approx \frac{1}{4\pi}\frac{r_+ - r_-}{r_+^2}\,. \tag{28}$$

This approximation permits us to write the Komar energy at the horizon, using Eqs. (26), (28) and (27), as

$$E = 2\pi r_+^2 T = \frac{r_+ - r_-}{2}\,, \tag{29}$$

$$E = \frac{1}{2}\left[M_+ + M_- + \sqrt{M_+^2 - Q_+^2} - \sqrt{M_-^2 - Q_-^2}\,\right]. \tag{30}$$

By considering the behavior of the functions $M_\pm$ and $Q_\pm$, it is easy to see that the limit of large masses of (30), as well as taking the limit $\ell \to 0$, recover the Reissner-Nordström energy while for $Q_0 = 0$, it gives the result of Banerjee and Gangopadhyay [28] for the noncommutative Schwarzschild black hole with the usual $E = M_0$. These results let us identify the quantity M_0 as the total mass of the black hole and Q_0 as its total electric charge.

With a similar procedure, the entropy can be approximated by

$$S = \pi r_+^2 \approx \pi \left(M_+ + \sqrt{M_+^2 - Q_+^2}\right)^2, \tag{31}$$

which give in the limit of large masses, or in the limit $\ell \to 0$, the usual result for the Reissner-Nordström black hole,

$$S \to S_{RN} = \pi \left(M_0 + \sqrt{M_0^2 - Q_0^2}\right)^2. \tag{32}$$

Using Eqs. (8) and (9) and the property of the gamma function

$$\frac{\partial}{\partial u}\,\gamma\left(\frac{a}{b}, u\right) = e^{-u} u^{-1+\frac{a}{b}} \tag{33}$$

to perform the derivatives, the Komar energy (20) for this spacetime yields

$$\begin{aligned} E = M(r) - \frac{Q^2(r)}{r} - \frac{M_0}{2\sqrt{\pi}}\frac{r^3}{\ell^3}\,e^{-\frac{r^2}{4\ell^2}} &+ \\ + \frac{Q_0^2}{2\pi}\Bigg[\frac{2}{\ell}\,e^{-\frac{r^2}{4\ell^2}}\,\gamma\left(\frac{1}{2}, \frac{r^2}{4\ell^2}\right) - \frac{1}{\sqrt{2}\,\ell}\,\gamma\left(\frac{1}{2}, \frac{r^2}{2\ell^2}\right) &+ \\ + \frac{\sqrt{2}}{\ell}\,\gamma\left(\frac{3}{2}, \frac{r^2}{4\ell^2}\right) - \frac{r}{\ell^2}\,e^{-\frac{r^2}{2\ell^2}} + \frac{\sqrt{2}}{4}\frac{r^3}{\ell^4}\,e^{-\frac{r^2}{4\ell^2}}\Bigg]&. \end{aligned} \tag{34}$$

Using the long distance approximations for the gamma functions

$$\gamma\left(\frac{3}{2},\frac{r^2}{4\ell^2}\right)\simeq\frac{\sqrt{\pi}}{2}-\frac{r}{2\ell}\,e^{-r^2/4\ell^2}\,, \tag{35}$$

$$\gamma\left(\frac{1}{2},\frac{r^2}{2\ell^2}\right)\simeq\sqrt{\pi}-\sqrt{2}\,\ell\,\frac{e^{-r^2/2\ell^2}}{r}\,, \tag{36}$$

$$\gamma\left(\frac{1}{2},\frac{r^2}{4\ell^2}\right)\simeq\sqrt{\pi}-2\,\ell\,\frac{e^{-r^2/4\ell^2}}{r}\,, \tag{37}$$

we obtain finally

$$\begin{aligned} M_0-\frac{Q_0^2}{r_+} &= 2TS+\frac{M_0}{\sqrt{\pi}}\frac{r_+}{\ell}\,e^{-\frac{r_+^2}{4\ell^2}}\left(1+\frac{r_+^2}{2\ell^2}\right)+\\ &\quad+\frac{Q_0^2}{\pi r_+}\left[e^{-\frac{r_+^2}{2\ell^2}}\left(\frac{5}{2}+\frac{r_+^2}{2\ell^2}+\frac{4\ell^2}{r_+^2}\right)-\right.\\ &\quad\left.-\,e^{-\frac{r_+^2}{4\ell^2}}\left(4\sqrt{\pi}\,\frac{\ell}{r_+}+\sqrt{\pi}\,\frac{r_+}{\ell}+\frac{\sqrt{2}}{4}\frac{r_+^2}{\ell^2}+\frac{\sqrt{2}}{8}\frac{r_+^4}{\ell^4}\right)\right]. \end{aligned} \tag{38}$$

Since M_0 and Q_0 have been identified as the mass and charge of the black hole, Eq. (38) corresponds to the generalization of the *Smarr formula for the noncommutative charged black hole*. Note that this relation deviates from the usual one (5) by the two last terms in the right hand side, but it is clear that in the limit $\ell\to 0$ these terms disappear. In the case $Q_0=0$ we recover the relation for the noncommutative Schwarzschild black hole presented in [28, 30, 31].

§3. Conclusion. We have computed the Komar energy for a charged black hole inspired in noncommutative geometry and its asymptotic limit that let us identify the constant M_0 as its total mass and Q_0 as its electric charge. With these results, we obtained the noncommutative version of the Smarr formula (38) which show a deformation from the usual relation and the new terms depend on the noncommutative parameter ℓ.

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