Earth Flyby Anomalies Explained by a Time-Retarded Causal Version of Newtonian Gravitational Theory

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Abstract: Classical Newtonian gravitational theory does not satisfy the causality principle because it is based on instantaneous action-at-a-distance. A causal version of Newtonian theory for a large rotating sphere is derived herein by time-retarding the distance between interior circulating point-mass sources and an exterior field-point. The resulting causal theory explains exactly the Earth flyby anomalies reported by NASA in 2008. It also explains exactly an anomalous decrease in the Moon's orbital speed. No other known theory has been shown to explain both the flyby anomalies and the lunar orbit anomaly.

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Introduction. It has long been known that electromagnetic fields propagate at or near the vacuum speed of light. The actual speed of light depends on whether the field is propagating in a vacuum or in a material medium. In either case, to calculate the electromagnetic fields of a moving point-charge, the concept of "time retardation" must be used [1]. The causality principle indicates that the "effect" of a "causal" physical field requires a certain amount of time to propagate from a point-source to a distant field-point. Classical Newtonian theory is "acausal" because the Newtonian gravitational field is based on instantaneous action-at-a-distance [2].

Gravitational fields are believed to propagate in empty space with exactly the same speed as the vacuum speed of light [3]. In 1898 the speed of the Sun's gravitational field was found by a high school math teacher, P. Gerber, by calculating what it would need to be to cause the (in 1898) "anomalous" advance of the perihelion of Mercury [4]. Gerber's value, 3.05500×10^8 m/s, is about 2% greater than the vacuum speed of light. In 2002 a group of radio astronomers measured the speed of Jupiter's gravitational field by detecting the rate of change in the gravitational bending of radio waves from a distant quasar as the giant planet crossed the line-of-sight [5]. They concluded that the speed of Jupiter's gravitational field is 1.06 ± 0.21 times the vacuum speed of light. These results suggest that the speed of propagation of the gravitational field near a massive central object may not be exactly the same as the vacuum speed of light.

The first terrestrial measurement that proved a connection between gravity and light, the gravitational red-shift, was carried out by R. V. Pound and G. A. Rebka in 1959 [6]. In 1972 J. C. Hafele and R. E. Keating reported the results of their experiments which detected the relativistic time dilation and the gravitational red-shift for precision clocks flown around the world using commercial jet flights [7]. This experiment showed conclusively that clocks at a deeper gravitational potential run slower and that moving clocks run slower. It also showed that the Sagnac effect [8], which originally was for electromagnetic fields, also applies to gravitational fields. To correct for these relativistic effects, the precision clocks used in the GPS system are adjusted before they are launched into space [9].

In 2008 Anderson *et al.* [10] reported that anomalous orbital-energy changes have been observed during six spacecraft flybys of the Earth. The reported speed-changes range from $+13.28$ mm/s for the NEAR flyby to -4.6 mm/s for the Galileo-II flyby. Anderson *et al.* state in their abstract:

"These anomalous energy changes are consistent with an empirical prediction formula which is proportional to the total orbital energy per unit mass and which involves the incoming and outgoing geocentric latitudes of the asymptotic spacecraft velocity vectors."

Let the calculated speed-change be designated by δv_{emp}. The empirical prediction formula found by Anderson *et al.* can be expressed as follows

$$\delta v_{\text{emp}} = \frac{2\,v_{\text{eq}}}{c}\,v_{\text{in}}\,(\cos\lambda_{\text{in}} - \cos\lambda_{\text{out}}) =$$
$$= -\frac{2\,v_{\text{eq}}}{c}\,v_{\text{in}}\int_{t_{\text{in}}}^{t_{\text{out}}}\sin\bigl(\lambda(t)\bigr)\frac{d\lambda}{dt}\,dt\,, \qquad (1.1)$$

where v_{eq} is the Earth's equatorial rotational surface speed, c is the vacuum speed of light, v_{in} is the initial asymptotic inbound speed, λ_{in} is the asymptotic inbound geocentric latitude, and λ_{out} is the asymptotic outbound geocentric latitude. If t is the observed coordinate time for the spacecraft in its trajectory, then $\lambda_{\text{in}} = \lambda(t_{\text{in}})$ and $\lambda_{\text{out}} = \lambda(t_{\text{out}})$. If $d\lambda/dt = 0$, then $\delta v_{\text{emp}} = 0$. An order of magnitude estimate for the maximum possible value for δv_{emp} is $2\,(5\times10^2/3\times10^8)\,v_{\text{in}} \sim 30\,\text{mm/s}$.

The following is a direct quote from the conclusions of Anderson *et al.* (the ODP means the Orbit Determination Program):

"Lämmerzahl *et al.* [11] studied and dismissed a number of possible explanations for the Earth flyby anomalies, including Earth atmosphere, ocean tides, solid Earth tides, spacecraft charging, magnetic moments, Earth albedo, solar wind, coupling of Earth's spin with rotation of the radio wave, Earth gravity, and relativistic effects predicted by Einstein's theory. All of these potential sources of systematic error, and more, are modeled in the ODP. None can account for the observed anomalies."

The article by Lämmerzahl *et al.* [11], which is entitled "Is the physics within the Solar system really understood?", was published in 2006.

A direct quote from the abstract for a more recent article, one published in 2009 by M. M. Nieto and J. D. Anderson, follows [12]:

"In a reference frame fixed to the solar system's center of mass, a satellite's energy will change as it is deflected by a planet. But a number of satellites flying by Earth have also experienced energy changes in the Earth-centered frame — and that's a mystery."

Nieto and Anderson then conclude their article with the following comments:

"Several physicists have proposed explanations of the Earth flyby anomalies. The least revolutionary invokes dark matter bound to Earth. Others include modifications of special relativity, of general relativity, or of the notion of inertia; a light speed anomaly; or anisotropy in the gravitational field — all of those, of course, deny concepts that have been well tested. And none of them have made comprehensive, precise predictions of Earth flyby effects. For now the anomalous energy changes observed in Earth flybys remain a puzzle. Are they the result of imperfect understandings of conventional physics and experimental systems, or are they the harbingers of exciting new physics?"

When the article by Nieto and Anderson was published, "conventional" or "mainstream" physics had not resolved the mystery of the Earth flyby anomalies. It appears that a new and possibly unconventional theory is needed.

The empirical prediction formula (1.1) found by Anderson *et al.* is not based on any mainstream theory of physics (it was simply "picked out of the air"). However, the empirical prediction formula is remarkably simple and gives calculated speed-change values that are surprisingly close to the observed speed-change values. The empirical prediction formula gives three clues for that which must be satisfied by any theory that is developed to explain the flyby anomaly:

1) the theory must produce a formula for the speed-change that is proportional to the ratio v_{eq}/c,

2) the anomalous force acting on the spacecraft must change the λ-component of the spacecraft's velocity, and

3) it must be proportional to v_{in}.

The objective of *this article* is threefold:

1) derive a new causal version of classical acausal Newtonian theory,

2) show that this new version is able to produce exact agreement with all six of the anomalous speed-changes reported by Anderson *et al.*, and

3) show that it is also able to explain exactly the "lunar orbit anomaly", an anomalous change in the Moon's orbital speed which will be described below.

This new version for Newtonian theory uses only mainstream physics:

1) classical Newtonian theory, and

2) the causality principle which requires time-retardation of the gravitational force.

It also satisfies the three requirements of the empirical prediction formula.

This article proposes a simple correction that converts Newton's acausal theory into a causal theory. The resulting causal theory has a new, previously overlooked, time-retarded transverse component, designated $\boldsymbol{g}_{\text{trt}}$, which depends on $1/c_{\text{g}}$, where c_{g} is the speed of gravity, which approximately equals the speed of light. The new total gravitational field for a large spinning sphere, $\boldsymbol{g}$, has two components, the standard well-known classical acausal radial component, $\boldsymbol{g}_{\text{r}}$, and a new relatively small previously undetected time-retarded transverse vortex component, $\boldsymbol{g}_{\text{trt}}$. The total vector field $\boldsymbol{g} = \boldsymbol{g}_{\text{r}} + \boldsymbol{g}_{\text{trt}}$. The zero-divergence vortex transverse vector field $\boldsymbol{g}_{\text{trt}}$ is orthogonal to the irrotational radial vector field $\boldsymbol{g}_{\text{r}}$.

This new vector field is consistent with Helmholtz's theorem, which states that any physical vector field can be expressed as the sum of the gradient of a zero-rotational scalar potential and the curl of a zero-divergence vector potential [13, p. 52]. This means that $\boldsymbol{g}_{\text{r}}$ can be derived in the standard way from the gradient of a scalar potential, and $\boldsymbol{g}_{\text{trt}}$ can be derived from the curl of a vector potential, but $\boldsymbol{g}_{\text{trt}}$ cannot be derived from the gradient of a scalar potential.

The time retarded gravitational fields for a moving point-mass can be derived by using the slow-speed weak-field approximation for Einstein's general relativity theory. Let $\boldsymbol{\varphi}$ be the time-retarded scalar potential, let $\boldsymbol{e}$ be the time-retarded "gravitoelectric" acceleration field, let $\boldsymbol{a}$ be the time-retarded vector potential, and let $\boldsymbol{h}$ be the time-retarded "gravitomagnetic" induction field. It is shown in §2 that the formulas for φ, $\boldsymbol{e}$, $\boldsymbol{a}$, and $\boldsymbol{h}$, have been derived by W. Rindler in his popular textbook, *Essential Relativity* [14]. They are as follows

$$\left. \begin{array}{ll} \varphi = G \iiint \left[\dfrac{\rho}{r''}\right] dV, & \boldsymbol{a} = \dfrac{G}{c} \iiint \left[\dfrac{\rho\,\boldsymbol{u}}{r''}\right] dV \\[2ex] \boldsymbol{e} = -\boldsymbol{\nabla}\varphi\,, & \boldsymbol{h} = \boldsymbol{\nabla} \times 4\boldsymbol{a} \end{array} \right\}, \qquad (1.2)$$

where ρ is the mass-density of the central object, $\boldsymbol{u}$ is the inertial velocity of a source-point-mass which is held solidly in the central rotating object by nongravitational forces (inertial velocity means the velocity in an inertial frame), $\boldsymbol{r}''$ is the vector distance from a source-point-mass to the field-point, the square brackets denote that the enclosed value is to be retarded by the light travel time from the source-point to the field-point, and dV is an element of volume of the central body.

Let the origin for an inertial (nonrotating and nonaccelerating)

frame-of-reference coincide with the center-of-mass of a contiguous central object. Let $\boldsymbol{r}'$ be the radial vector from the origin to a source-point-mass in the central object, and let $\boldsymbol{r}$ be the radial vector from the origin to an external field-point, so that the vector distance from the source-point to the field-point $\boldsymbol{r}'' = \boldsymbol{r} - \boldsymbol{r}'$. The triple integrals in (1.2) indicate that the retarded integrands $[\rho/r'']$ and $[\rho\,\boldsymbol{u}/r'']$ are to be integrated over the volume of the central object at the retarded time.

Let m be the mass of a test-mass that occupies the field-point at $\boldsymbol{r}$, and let $\boldsymbol{v}$ be the inertial velocity of the test mass. The analogous Lorentz force law, i.e., the formula for the time-retarded gravitational force $\boldsymbol{F}$ acting on m at $\boldsymbol{r}$, is [14]

$$\boldsymbol{F} = -\,m\left(\boldsymbol{e} + \frac{1}{c}\left(\boldsymbol{v} \times \boldsymbol{h}\right)\right) = -\,m\,\boldsymbol{\nabla}\left(G \iiint \left[\frac{\rho}{r''}\right] dV\right) - $$
$$ -\,m\left(\boldsymbol{v} \times \left(\boldsymbol{\nabla} \times \left(\frac{4G}{c^2} \iiint \left[\frac{\rho\,\boldsymbol{u}}{r''}\right] dV\right)\right)\right). \quad (1.3)$$

This shows that Rindler's time-retarded version for the slow-speed weak-field approximation gives a complete stand-alone time-retarded solution. The time-retarded fields were derived from general relativity theory, but there is at this point no further need for reference to the concepts and techniques of general relativity theory. Needed concepts and techniques are those of classical Newtonian theory.

Furthermore, Rindler's version satisfies the causality principle because the fields are time-retarded. It is valid as a first order approximation only if

$$v^2 \ll c^2, \qquad u^2 \ll c^2, \qquad \frac{GM}{r} = |\varphi| \ll c^2, \qquad (1.4)$$

where M is the total mass of the central object.

Notice in (1.3) that the acceleration caused by the gravitoelectric field $\boldsymbol{e}$ is independent of c, but the acceleration caused by the gravitomagnetic induction field $\boldsymbol{h}$ is reduced by the factor $1/c^2$. The numerical value for c is on the order of 3×10^8 m/s. If the magnitude for $\boldsymbol{e}$ is on the order of $10\,\mathrm{m/s}^2$ (the Earth's field at the surface), and the magnitudes for $\boldsymbol{u}$ and $\boldsymbol{v}$ are on the order of 10^4 m/s, the relative magnitude for the acceleration caused by $\boldsymbol{h}$ would be on the order of $10 \times 4\,(10^4/3 \times 10^8)^2$ m/s^2 $\sim 10^{-8}\,\mathrm{m/s}^2$. This estimate shows that, for most slow-speed weak-field practical applications in the real world, the acceleration caused by $\boldsymbol{h}$ is totally negligible compared to the acceleration caused by $\boldsymbol{e}$.

The empirical prediction formula (1.1) indicates that the flyby speed-

change is reduced by $1/c$, not by $1/c^2$, which rules out the gravitomagnetic induction field $\boldsymbol{h}$ as a possible cause for the flyby anomalies. The acceleration of the gravitomagnetic field is simply too small to explain the flyby anomalies.

Consequently, if the gravitomagnetic component is ignored for being negligible, the practicable version for Rindler's Lorentz force law (1.3) becomes the same as a time-retarded version for Newton's well-known inverse-square law

$$\boldsymbol{F} = -\, Gm \, \boldsymbol{\nabla} \iiint \left[\frac{\rho}{r''} \right] dV, \qquad (1.5)$$

where $\boldsymbol{F}$ is the time-retarded gravitational force acting on m.

Let $d^3\boldsymbol{F}$ be the time-retarded elemental force of an elemental point-mass source dm' at r'. With this notation, the time-retarded version for Newton's inverse-square law becomes

$$d^3\boldsymbol{F} = -\, Gm \, \frac{dm'}{r''^2} \, \frac{\boldsymbol{r}''}{r''}, \qquad (1.6)$$

where $\boldsymbol{r}''/r''$ is a unit vector directed towards increasing $\boldsymbol{r}''$. The notation $d^3\boldsymbol{F}$ indicates that the differential element of force must be integrated over 3-dimensional space to get the total force.

By definition, the gravitational field of a point-source dm' at $\boldsymbol{r}'$ is the gravitational force of the source that acts on a test-mass of mass m at $\boldsymbol{r}$ per unit mass of the test-mass.

The traditional symbol for the Newtonian gravitational vector field is $\boldsymbol{g}$. Therefore, the formula for the time-retarded elemental gravitational field $d^3\boldsymbol{g}$ of an elemental point-mass-source dm' at $\boldsymbol{r}'$ for a field-point at $\boldsymbol{r}$ occupied by a test-mass of mass m becomes

$$d^3\boldsymbol{g} = \frac{d^3\boldsymbol{F}}{m} = -\, G \, \frac{dm'}{r''^2} \, \frac{\boldsymbol{r}''}{r''}. \qquad (1.7)$$

The negative sign indicates that the gravitational force is attractive.

Let $\rho(\boldsymbol{r}')$ be the mass-density of the central object at $\boldsymbol{r}'$. Then

$$dm' = \rho(\boldsymbol{r}') \, dV. \qquad (1.8)$$

The resulting formula for the elemental gravitational field $d^3\boldsymbol{g}$, which consists of the radial component $d^3\boldsymbol{g}_{\mathrm{r}}$ and the transverse component $d^3\boldsymbol{g}_{\mathrm{trt}}$, becomes $d^3\boldsymbol{g} = d^3\boldsymbol{g}_{\mathrm{r}} + d^3\boldsymbol{g}_{\mathrm{trt}}$. The formula for each component becomes

$$d^3\boldsymbol{g}_{\mathrm{r}} = -\, G \, \frac{dm'}{r''^2} \left(\frac{\boldsymbol{r}''}{r''} \right)_{\mathrm{r}}, \qquad d^3\boldsymbol{g}_{\mathrm{trt}} = -\, G \, \frac{dm'}{r''^2} \left(\frac{\boldsymbol{r}''}{r''} \right)_{\mathrm{trt}}, \qquad (1.9)$$

where $(\boldsymbol{r}''/r'')_{\mathrm{r}}$ is the radial component of the unit vector and $(\boldsymbol{r}''/r'')_{\mathrm{trt}}$ is the transverse component of the unit vector. The total field is obtained by a triple integration over the volume of the central object at the retarded time.

The triple integral is rather easy to solve by numerical integration (such as by using the integration algorithm provided in Mathcad15) if the central object can be approximated by a large spinning isotropic sphere. To get a good first approximation for this article, the Earth is simulated by a large spinning isotropic sphere. It is shown in §3 that the triple integration for g_{trt} leads to the necessary factor $1/c_{\mathrm{g}}$, where c_{g} is the speed of propagation of the Earth's gravitational field.

It is also shown in the forthcoming §3 that the formula for the magnitude of $\boldsymbol{g}_{\mathrm{trt}}$ is

$$g_{\mathrm{trt}}(\theta) = - \, G \, \frac{I_{\mathrm{E}}}{r_{\mathrm{E}}^4} \, \frac{v_{\mathrm{eq}}}{c_{\mathrm{g}}} \, \frac{\Omega_\phi(\theta) - \Omega_{\mathrm{E}}}{\Omega_{\mathrm{E}}} \, \cos^2\!\big(\lambda(\theta)\big) \, PS\big(r(\theta)\big), \qquad (1.10)$$

where G is the gravity constant, I_{E} is the Earth's spherical moment of inertia, r_{E} is the Earth's spherical radius, Ω_{E} is the Earth's spin angular speed, v_{eq} is the Earth's equatorial surface speed, c_{g} is the speed of propagation of the Earth's gravitational field, θ is the spacecraft's parametric polar coordinate angle in the plane of the orbit or trajectory ($\Omega_\theta = d\theta/dt$ is the spacecraft's orbital angular speed), Ω_ϕ is the azimuthal ϕ-component of Ω_θ, λ is the spacecraft's geocentric latitude, r is the spacecraft's geocentric radial distance, and $PS(r)$ is an inverse-cube power series representation for the triple integral over the Earth's volume. If the magnitude is negative, i.e., if $\Omega_\phi > \Omega_{\mathrm{E}}$ (prograde), the vector field component $\boldsymbol{g}_{\mathrm{trt}}$ is directed towards the east. If $\Omega_\phi < 0$ (retrograde), it is directed towards the west.

The magnitude for $\boldsymbol{g}_{\mathrm{trt}}$ satisfies the first requirement of the empirical prediction formula. It is proportional to $v_{\mathrm{eq}}/c_{\mathrm{g}} \cong v_{\mathrm{eq}}/c$. But the empirical prediction formula also requires that the speed-change must be in the λ-component of the spacecraft's velocity, $\boldsymbol{v}_\lambda$. The magnitude for the λ-component is defined by

$$v_\lambda = r_\lambda \, \frac{d\lambda}{dt} = r_\lambda \, \frac{d\lambda}{d\theta} \, \frac{d\theta}{dt} = r_\lambda \Omega_\theta \, \frac{d\lambda}{d\theta}\,, \qquad (1.11)$$

where r_λ is the λ-component of r. The velocity component, $\boldsymbol{v}_\lambda$, is orthogonal to $\boldsymbol{g}_{\mathrm{trt}}$. Consequently, $\boldsymbol{g}_{\mathrm{trt}}$ cannot directly change the magnitude of $\boldsymbol{v}_\lambda$ (it only changes the direction).

However, a hypothesized "induction-like" field, designated $\boldsymbol{F}_\lambda$, can be directed perpendicularly to $\boldsymbol{g}_{\mathrm{trt}}$ in the $\boldsymbol{v}_\lambda$-direction. Assume that the

ϕ-component of the curl of $\boldsymbol{F}_\lambda$ equals $-k\, d\boldsymbol{g}_{\mathrm{trt}}/dt$, where k is a constant[*]. This induction-like field can cause a small change in the spacecraft's speed. The reciprocal of the constant k, $v_{\mathrm{k}} = 1/k$, called herein the "induction speed", becomes an adjustable parameter for each case. The average for all cases gives an overall constant for the causal version of Newton's theory.

The formula for the magnitude of $\boldsymbol{F}_\lambda$ is shown in §3 to be

$$F_\lambda(\theta) = \frac{v_{\mathrm{eq}}}{v_{\mathrm{k}}} \frac{r_{\mathrm{E}}}{r(\theta)} \int_0^\theta \frac{r(\theta)}{r_{\mathrm{E}}} \frac{\Omega_\theta(\theta)}{\Omega_{\mathrm{E}}} \frac{1}{r_{\mathrm{E}}} \frac{dr}{d\theta} \frac{dg_{\mathrm{trt}}}{d\theta}\, d\theta\,. \qquad (1.12)$$

The acceleration caused by $\boldsymbol{F}_\lambda$ satisfies the second requirement of the empirical prediction formula, the one that requires that the anomalous force must change the λ-component of the spacecraft's velocity. It needs to be emphasized at this point that the acceleration field $\boldsymbol{F}_\lambda$ is a hypothesis, and is subject to proof or disproof by the facts-of-observation. This hypothesis is needed to satisfy the requirement that the speed-change must be in the λ-component of the spacecraft's velocity.

The anomalous time rate of change in the spacecraft's kinetic energy is given by the dot product, $\boldsymbol{v} \cdot \boldsymbol{F}_\lambda$. It is shown in §3 that the calculated asymptotic speed-change, δv_{trt}, is given by

$$\delta v_{\mathrm{trt}} = \delta v_{\mathrm{in}} + \delta v_{\mathrm{out}}\,, \qquad (1.13)$$

where

$$\delta v_{\mathrm{in}} = \delta v\,(\theta_{\min})\,, \qquad \delta v_{\mathrm{out}} = \delta v\,(\theta_{\max})\,, \qquad (1.14)$$

and

$$\delta v\,(\theta) = \frac{v_{\mathrm{in}}}{2} \int_0^\theta \frac{r_\lambda(\theta)\, F_\lambda(\theta)}{v_{\mathrm{in}}^2} \frac{d\lambda}{d\theta}\, d\theta\,. \qquad (1.15)$$

The angles $\theta_{\min}$ and $\theta_{\max}$ are the minimum and maximum values for θ. The initial speed $v_{\mathrm{in}} = v\,(\theta_{\min})$. The speed-change $\delta v(\theta)$ is proportional to v_{in}, which satisfies the third requirement of the empirical prediction formula.

It is shown in §4 that this "neoclassical" causal version for acausal Newtonian theory explains exactly the flyby anomalies. Table 1 lists, for each of the six Earth flybys reported by Anderson *et al.*, the observed speed change from Appendix A, δv_{obs}, the calculated speed change from (1.13), δv_{trt}, the ratio that was used for the speed of gravity, c_{g}/c, the value for the induction speed ratio that gives exact agreement with the observed speed-change, $v_{\mathrm{k}}/v_{\mathrm{eq}}$, and the value for the eccentricity of the

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[*]For Maxwell's theory, the numerical value for the constant $k = 1/c$ [1].

trajectory, ε.

Notice in Table 1 that the required values for v_k cluster between $6\,v_{eq}$ and $17\,v_{eq}$. Also notice that the two high-precision flybys, NEAR and Rosetta, put very stringent limits on the speed of gravity, c_g. If the "true" value for v_k had been known with high precision, the two high-precision flybys would have provided first-ever measured values for the speed of propagation of the Earth's gravitational field.

In 1995, F. R. Stephenson and L. V. Morrison published a study of records of eclipses from 700 BC to 1990 AD [15]. They conclude (LOD means length-of-solar-day, ms cy^{-1} means milliseconds per century): 1) the LOD has been increasing on average during the past 2700 years at the rate of $+1.70 \pm 0.05$ ms cy^{-1}, i.e. $(+17.0 \pm 0.5) \times 10^{-6}$ s per year, 2) tidal braking causes an increase in the LOD of $+2.3 \pm 0.1$ ms cy^{-1}, i.e. $(+23 \pm 1) \times 10^{-6}$ s per year, and 3) there is a non-tidal decrease in the LOD, numerically -0.6 ± 0.1 ms cy^{-1}, i.e. $(-6 \pm 1) \times 10^{-6}$ s per year.

Stephenson and Morrison state that the non-tidal decrease in the LOD probably is caused by "post-glacial rebound". Post-glacial rebound decreases the Earth's moment of inertia, which increases the Earth's spin angular speed, and thereby decreases the LOD. But post-glacial rebound cannot change the Moon's orbital angular momentum.

According to Stephenson and Morrison, tidal braking causes an increase in the LOD of $(23 \pm 1) \times 10^{-6}$ seconds per year, which causes a decrease in the Earth's spin angular momentum, and by conservation of angular momentum causes an increase in the Moon's orbital angular momentum. It is shown in §5 that tidal braking alone would cause an increase in the Moon's orbital speed of $(19 \pm 1) \times 10^{-9}$ m/s per year, which corresponds to an increase in the radius of the Moon's orbit of (14 ± 1) mm per year.

But lunar-laser-ranging experiments have shown that the radius of the Moon's orbit is actually increasing at the rate of (38 ± 1) mm per year [16]. This rate for increase in the radius corresponds to an increase in the orbital speed of $(52 \pm 2) \times 10^{-9}$ m/s per year. Clearly there is an unexplained or anomalous difference in the change in the radius of the orbit of (-24 ± 2) mm per year $(38 - 14 = 24)$, and a corresponding anomalous difference in the change in the orbital speed of $(-33 \pm 3) \times 10^{-9}$ m/s per year $(52 - 19 = 33)$. This "lunar orbit anomaly" cannot be caused by post-glacial rebound, but it can be caused by the proposed neoclassical causal version of Newton's theory.

It is shown in §5 that the proposed neoclassical causal theory produces a change in the Moon's orbital speed of $(-33 \pm 3) \times 10^{-9}$ m/s per year if the value for the induction speed $v_k = (8 \pm 1)\,v_{eq}$. The eccen-
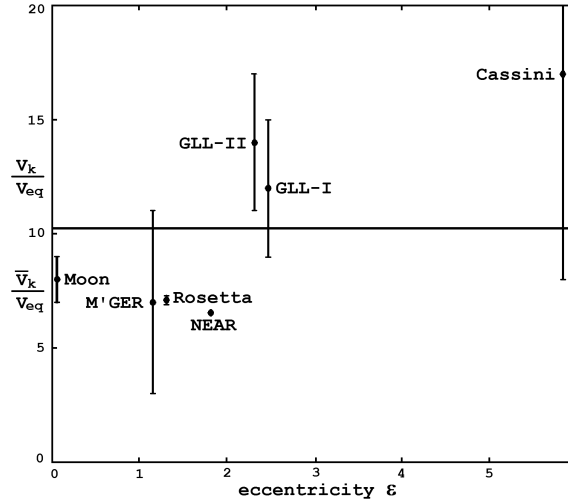
Fig. 1: Required induction speed ratio (designated by ●), $v_{\mathrm{k}}/v_{\mathrm{eq}} \pm a$ rough estimate for the uncertainty, versus eccentricity ε. The average value for all seven ratios, $\bar{v}_{\mathrm{k}} = 10.2\,v_{\mathrm{eq}}$, is shown by the horizontal line.

tricity for the Moon's orbit, $\varepsilon = 0.0554$, indicates that it revolves in a nearly circular closed orbit. Consequently, a new closed orbit case can be added to the open orbit flybys listed in Table 1. A graph of the required induction speed ratio, $v_{\mathrm{k}}/v_{\mathrm{eq}}$, versus eccentricity ε, Fig. 1, shows that the required value for the induction speed for the Moon is consistent with the required values for the induction speed for the six flyby anomalies.

The average $\pm$ standard deviation for the seven induction speed ratios in Fig. 1 is

$$\bar{v}_{\mathrm{k}} = (10.2 \pm 3.8)\,v_{\mathrm{eq}} = 4.8 \pm 1.8 \text{ km/s}. \qquad (1.16)$$

It will be interesting to compare this average value with parameter values for other theories which explain the flyby anomalies.

The neoclassical causal theory can be further tested by doing high precision Doppler-shift research studies of the orbital motions of spacecrafts in highly eccentric and inclined near-Earth orbits. The predicted annual speed-change δv_{yr} (prograde) and $\delta v r_{\mathrm{yr}}$ (retrograde) for orbits with eccentricity $\varepsilon = 0.5$, inclination $\alpha_{\mathrm{eq}} = 45°$, and geocentric latitude at perigee $\lambda_{\mathrm{p}} = 45°$, with the induction speed set equal to its maximum probable value $v_{\mathrm{k}} = 14\,v_{\mathrm{eq}}$, and with the radial distance at perigee r_{p} ranging from $2r_{\mathrm{E}}$ to $8r_{\mathrm{E}}$, are calculated in §6 and listed in Table 2.

Flyby	NEAR*	GLL-I	Rosetta*	M'GER	Cassini	GLL-II
δv_{obs} (mm/s)	+13.46 ±0.01	+3.92 ±0.30	+1.80 ±0.03	+0.02 ±0.01	−2 ±1	−4.6 ±1
δv_{trt} (mm/s)	+13.46 ±0.01	+3.92 ±0.30	+1.80 ±0.03	+0.02 ±0.01	−2 ±1	−4.6 ±1
c_{g}/c	1.000 ±0.001	1	1.00 ±0.02	1	1	1
$v_{\text{k}}/v_{\text{eq}}$	6.530 ±0.005	12 ±3	7.1 ±0.2	7 ±4	17 ±9	14 ±3
ε	1.8142	2.4731	1.3122	1.3596	5.8456	2.3186

Table 1: Listing of the observed speed-change, δv_{obs}, the calculated speed-change from (1.13), δv_{trt}, the ratio used for the speed of gravity, c_{g}/c, the required value for the induction speed ratio, $v_{\text{k}}/v_{\text{eq}}$, and the eccentricity ε, for each of the six Earth flybys reported by Anderson *et al.* [10]. Listed uncertainties are rough estimates based on the uncertainty estimates of Anderson *et al.* The induction speed, v_{k}, was adjusted to make the calculated speed change, δv_{trt}, be identically equal to the observed speed change, δv_{obs}. The two high-precision flybys are marked by an asterisk.

$r_{\text{p}}/r_{\text{E}}$	2	3	4	5	6	7	8
P	11.2	20.7	31.8	44.4	58.4	73.6	89.9
δv_{yr}	+315	+29.5	+3.93	+0.173	−0.422	−0.422	−0.362
δvr_{yr}	−517	−76.8	−21.0	−7.97	−3.69	−1.95	−1.14

Table 2: Predicted period, P in hours, the speed-change for prograde orbits, δv_{yr} in mm/s per year, and the speed-change for retrograde orbits, δvr_{yr} in mm/s per year, for a near-Earth orbiting spacecraft with orbital parameters $\varepsilon = 0.5$, $\alpha_{\text{eq}} = 45°$, and $\lambda_{\text{p}} = 45°$, with $v_{\text{k}} = 14\,v_{\text{eq}}$, and for r_{p} ranging from $2r_{\text{E}}$ to $8r_{\text{E}}$.

If the proposed neoclassical causal theory is to be viable, it cannot conflict with Einstein's general relativity theory. The only possible conflict is with the excess advance in the perihelion of the planet Mercury, $+43$ arc seconds per century, which is explained exactly by general relativity theory. The predicted rate for change in the angle for perihelion for the neoclassical causal theory is shown in §7 to be less than 0.04 arc seconds per century, which is very much less than the relativistic advance and is undetectable. Therefore, there is no conflict with general relativity theory. Furthermore, the neoclassical causal theory does not require any change of any kind for general relativity theory. In fact, it is derived from general relativity theory.

There are at least two other published theories that try to provide an explanation for the Earth flyby anomalies. These are: 1) the 3-space flow theory of R. T. Cahill [17] and 2) the exponential radial field theory authored by H. J. Busack [18].

In [17] Cahill reviews numerous Michelson interferometer and one-way light-speed experiments which clearly show an anisotropy in the velocity of light. His calculated flyby speed-changes depend on the direction and magnitude for 3-space inflow at the spacecraft on the date and time of the flyby. Cahill found that the average speed for 3-space inflow is $12 \pm 5\,\mathrm{km/s}$. Cahill's average, $12 - 5 = 7\,\mathrm{km/s}$, essentially equals the average value for v_k, see (1.16), $4.8 + 1.8 = 6.6\,\mathrm{km/s}$.

In [18] Busack applies a small exponential correction for the Earth's radial gravitational field. If $f(\boldsymbol{r}, \boldsymbol{v})$ is Busack's correction, the inverse-square law becomes

$$\boldsymbol{g}_r(\boldsymbol{r}, \boldsymbol{v}) = -\frac{GM_\mathrm{E}}{r^2}\,\frac{\boldsymbol{r}}{r}\,\bigl(1 + f(\boldsymbol{r}, \boldsymbol{v})\bigr),$$

where $f(\boldsymbol{r}, \boldsymbol{v})$ is expressed as

$$f(\boldsymbol{r}, \boldsymbol{v}) = A \exp\left(-\frac{r - r_\mathrm{E}}{B - C(\boldsymbol{r} \cdot \boldsymbol{v})/(\boldsymbol{r} \cdot \boldsymbol{v}_\mathrm{Sun})}\right).$$

The velocity $\boldsymbol{v}$ is the velocity of the field-point in the "gravitational rest frame in the cosmic microwave background", and $\boldsymbol{v}_\mathrm{Sun}$ is the Sun's velocity in the gravitational rest frame. Numerical values for the adjustable constants are approximately $A = 2.2 \times 10^{-4}$, $B = 2.9 \times 10^5\,\mathrm{m}$, and $C = 2.3 \times 10^5\,\mathrm{m}$. Busack found that these values produce rather good agreement with the observed values for the flyby speed-changes.

The maximum possible value, $f(\boldsymbol{r}, \boldsymbol{v}) = A$, occurs where $r = r_\mathrm{E}$. At this point, $g_\mathrm{r} = (GM_\mathrm{E}/r_\mathrm{E}^2)(1 + A) \sim 10(1 + 2 \times 10^{-4})\,\mathrm{m/s}^2$. Compare this estimate with an estimate for the peak value for the neoclassical causal

transverse field for the NEAR flyby, which is $g = g_\mathrm{r}\sqrt{1 + g_\mathrm{trt}^2/g_\mathrm{r}^2} \sim$
$\sim 10(1 + 8 \times 10^{-12})$ m/s^2.

Both of these alternative theories require a preferred frame-of-reference. Neither has been tested for the lunar orbit anomaly, and neither satisfies the causality principle because neither depends on the speed of gravity.

In conclusion, the proposed neoclassical causal version for acausal Newtonian theory has passed seven tests: 1) explanation of the six flyby anomalies, and 2) explanation of the lunar orbit anomaly. It will be very difficult if not impossible for any other rational theory to be causal and pass all seven of these tests.

§2. Slow-speed weak-field approximation for general relativity theory.
The following comment from F. Rohrlich's article tells us about one problem that Sir Isaac Newton could not solve [2]:

> "Historians tell us that Newton was quite unhappy over the fact that his law of gravitation implies an action-at-a-distance interaction over very large distances such as that between the sun and the earth. But he was unable to resolve this problem."

The great author of Newtonian theory stood on the shoulders of giants, but he was not able to see Maxwell's theory or the slow-speed weak-field approximation for Einstein's theory.

The time-retarded version for the slow-speed weak-field approximation for general relativity theory provides a valid first-order approximation for the gravitational field of a moving point mass and a moving field-point. This approximation applies for "slowly" moving particles in "weak" gravitational fields. The word "slowly" means that $|\boldsymbol{u}| \ll c$, where $|\boldsymbol{u}|$ is the maximum magnitude for the source-particle velocity, that $|\boldsymbol{v}| \ll c$, where $|\boldsymbol{v}|$ is the maximum magnitude for the field-point test-particle velocity, and the word "weak" means $|\varphi| \ll c^2$, where $|\varphi| = GM/r$ is the maximum absolute magnitude for the scalar gravitational potential.

The chapter entitled *The Linear Approximation to GR* in W. Rindler's popular textbook starts on page 188 [14]. The following is a direct quote from pages 190 and 191:

> "In the general case, Equations (8.180) can be integrated by standard methods. For example, the first yields as the physically relevant solution,

$$\gamma_{\mu\nu} = -\frac{4G}{c^4}\iiint \frac{[T_{\mu\nu}]\,dV}{r}\,, \qquad (8.184)$$

where [ ] denotes the value "retarded" by the light travel time to the origin of r.

As an example, consider a system of sources in stationary motion (e.g., a rotating mass shell). All γ's will then be time-independent. If we neglect stresses and products of source velocities (which is not really quite legitimate[14]), the energy tensor (8.128) becomes

$$T_{\mu\nu} = \begin{pmatrix} \mathbf{0}_3 & -c^2\boldsymbol{v} \\ -c^2\boldsymbol{v} & c^4\rho \end{pmatrix} \tag{8.185}$$

where $\mathbf{0}_3$ stands for the 3×3 zero matrix, and so, from (8.184),

$$\gamma_{ij} = 0\,, \qquad i,j = 1,2,3. \tag{8.186}$$

For slowly moving test particles, $ds = c\,dt$. If we denote differentiation with respect to t by dots, the first three geodesic equations of motion become [cf. (8.15)]

$$\ddot{x}^i = -\Gamma^i_{\;\mu\nu}\,\dot{x}^\mu\dot{x}^\nu \tag{8.187}$$

$$= -\left(\gamma^i_{\;\mu,\nu} - \frac{1}{2}\,\gamma_{\mu\nu,}{}^i - \frac{1}{4}\,\eta^i_{\;\mu}\gamma_{,\nu} - \frac{1}{4}\,\eta^i_{\;\nu}\gamma_{,\mu} + \frac{1}{4}\,\eta_{\mu\nu}\gamma_{,}{}^i\right)\dot{x}^\mu\dot{x}^\nu, \tag{8.188}$$

where we have substituted into (8.187) from (8.168) and (8.172) and used $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu} = -h$. Moreover, $\gamma = c^2\gamma_{44}$. Now if we let $\dot{x}^\mu = (u^i, 1)$ and neglect products of the u's, Equation (8.188) reduces to

$$\ddot{x}^i = -\gamma^i_{\;4,j}\,u^j + \gamma_{j4,}{}^i\,u^j + \frac{1}{4}\,\gamma_{44,}{}^i\,.$$

This can be written vectorially in the form

$$\ddot{\boldsymbol{r}} = \mathbf{grad}\,\varphi - \frac{1}{c}\,(\boldsymbol{u} \times \mathbf{curl}\,\boldsymbol{a}) = -\left[\boldsymbol{e} + \frac{1}{c}\,(\boldsymbol{u} \times \boldsymbol{h})\right], \quad (8.189)$$

where [cf. (8.184), (8.185)]

$$\varphi = -\frac{1}{4}\,\gamma_{44} = G\iiint \frac{[\rho]\,dV}{r}\,,$$

$$\boldsymbol{a} = -\frac{c}{4}\,\gamma_4^i = \frac{1}{c}\,G\iiint \frac{[\rho\,\boldsymbol{u}]\,dV}{r}\,, \tag{8.190}$$

and

$$\boldsymbol{e} = -\mathbf{grad}\,\varphi\,, \qquad \boldsymbol{h} = \mathbf{curl}\,4\boldsymbol{a}\,. \tag{8.191}$$

The formal similarity with Maxwell's theory is striking. The only differences are: the minus sign in (8.189) (because the force is attractive); the factor G in (8.190) (due to the choice of units); and the novel factor 4 in (8.191) (ii)."

We need to change from Rindler's symbols to the symbols being used in this article. For the distance from the source-point to the field-point: $r \to r''$. For the integrands: $[\rho] \to [\rho/r'']$, and $[\rho \boldsymbol{u}] \to [\rho \boldsymbol{u}/r'']$. For the gradient and the curl: $\mathbf{grad} \to \boldsymbol{\nabla}$, $\mathbf{curl} \to \boldsymbol{\nabla} \times$.

The converted formulas for φ and $\boldsymbol{e}$ give the time-retarded scalar gravitational potential and the time-retarded gravitoelectric field

$$\varphi = G \iiint \left[\frac{\rho}{r''}\right] dV, \qquad \boldsymbol{e} = -\boldsymbol{\nabla}\,\varphi\,. \tag{2.1}$$

The converted formulas for $\boldsymbol{a}$ and $\boldsymbol{h}$ give the time-retarded vector gravitational potential and the time-retarded gravitomagnetic field

$$\boldsymbol{a} = \frac{G}{c} \iiint \left[\frac{\rho \boldsymbol{u}}{r''}\right] dV, \qquad \boldsymbol{h} = -\boldsymbol{\nabla} \times 4\boldsymbol{a}\,. \tag{2.2}$$

Let m be the mass of a test-mass that occupies the field point. Then the time-retarded gravitational force $\boldsymbol{F}$ that acts on the test-mass m becomes

$$\boldsymbol{F} = -m \left(\boldsymbol{e} + \frac{1}{c}\,(\boldsymbol{u} \times \boldsymbol{h})\right). \tag{2.3}$$

§3. Derivation of the formulas for the speed-change caused by the neoclassical causal version of Newton's theory.

Let the Earth be simulated by a large rotating isotropic sphere of radius r_{E}, mass M_{E}, angular speed Ω_{E}, moment of inertia I_{E}, and radial mass-density distribution $\rho(r')$. The radial mass-density distribution and values for the Earth's parameters are shown in Appendix B.

Consider a spacecraft in an open or closed orbit around this sphere. Let (X, Y, Z) be the rectangular coordinate axes for an inertial frame-of-reference, let the sphere's center coincide with the origin, and let the (X, Y) plane coincide with the equatorial plane, so that the Z-axis coincides with the sphere's rotational axis. Let $\boldsymbol{r}''$ be the vector distance from $\boldsymbol{r}'$ to $\boldsymbol{r}$, $\boldsymbol{r}'' = \boldsymbol{r} - \boldsymbol{r}'$.

Let the elemental gravitational field of an interior circulating point-mass dm' at r' be designated by $d^3\boldsymbol{g}$. As depicted in Fig. 2, $d^3\boldsymbol{g}$ has two components, a radial component designated by $d^3\boldsymbol{g}_{\mathrm{r}}$, and a transverse component designated by $d^3\boldsymbol{g}_{\mathrm{trt}}$. Therefore, $d^3\boldsymbol{g} = d^3\boldsymbol{g}_{\mathrm{r}} + d^3\boldsymbol{g}_{\mathrm{trt}}$.
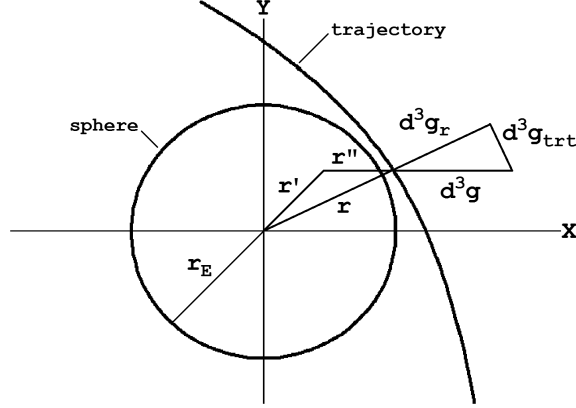
Fig. 2: Depiction of the vector distances $\boldsymbol{r}$, $\boldsymbol{r}'$, and $\boldsymbol{r}''$, and the components of the gravitational field at $\boldsymbol{r}$ of an elemental mass dm' at $\boldsymbol{r}'$ for a spacecraft flyby of a central spherical object of radius r_{E}. The vector distance from dm' at $\boldsymbol{r}'$ to the field point at $\boldsymbol{r}$ is $\boldsymbol{r}'' = \boldsymbol{r} - \boldsymbol{r}'$. The curved line labeled "trajectory" is the projection of the spacecraft's trajectory onto the (X, Y) equatorial plane. The elemental gravitational field $d^3\boldsymbol{g}$ has two components, a radial component $d^3\boldsymbol{g}_{\mathrm{r}}$ and a transverse component $d^3\boldsymbol{g}_{\mathrm{trt}}$, so that $d^3\boldsymbol{g} = d^3\boldsymbol{g}_{\mathrm{r}} + d^3\boldsymbol{g}_{\mathrm{trt}}$.

There are also two similar components of $\boldsymbol{r}''$, a relative radial component designated by RC, and a relative transverse Z-axis component designated by TC_Z. These components can be found by using the vector dot and cross products, as follows

$$\left.\begin{aligned}
RC &= \frac{\boldsymbol{r} \cdot \boldsymbol{r}''}{r''r} = \frac{\boldsymbol{r} \cdot (\boldsymbol{r} - \boldsymbol{r}')}{r''r} = \frac{r}{r''} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{r''r} \\
TC_Z &= \frac{(\boldsymbol{r} \times \boldsymbol{r}'')_Z}{r''r} = \frac{(\boldsymbol{r} \times (\boldsymbol{r} - \boldsymbol{r}'))_Z}{r''r} = \frac{(\boldsymbol{r}' \times \boldsymbol{r})_Z}{r''r}
\end{aligned}\right\}. \qquad (3.1)$$

Let t be the observed coordinate time for the spacecraft at $\boldsymbol{r}$ and let t' be the retarded-time at the interior circulating point-mass dm' at $\boldsymbol{r}'$. If the interior point-mass source dm' emits a gravitational signal at the retarded time t', the signal will arrive at the field-point at a slightly later time t. Let the speed at which the gravitational signal propagates be designated by c_{g}. Then the formulas that connect t to t' are

$$\left.\begin{aligned}
t &= t' + \frac{r''}{c_{\mathrm{g}}}, && t' = t - \frac{r''}{c_{\mathrm{g}}} \\
\frac{dt}{dt'} &= 1 + \frac{1}{c_{\mathrm{g}}}\frac{dr''}{dt'}, && \frac{dt'}{dt} = 1 - \frac{1}{c_{\mathrm{g}}}\frac{dr''}{dt}
\end{aligned}\right\}. \qquad (3.2)$$

Obviously, $t(t')$ is here a function of t', and *vice versa*, $t'(t)$ is a function of t.

By definition, the derivative dt/dt' is the Jacobian for the transformation from t to t'. Let $f(t, t')$ be an implicit function of t and t', and let $F(t)$ be the function that results from integration of $f(t, t')$ over t

$$F(t) = \int_0^t f(t, t')\, dt = \int_{t'(0)}^{t'(t)} f(t, t') \frac{dt}{dt'}\, dt' =$$

$$= \int_{t'(0)}^{t'(t)} f(t, t')\, (\text{Jacobian})\, dt'. \tag{3.3}$$

The formulas for the components $d^3 g_{\text{r}}$ and $d^3 g_{\text{trt}}$ can be found by substituting into the formulas of (1.9)

$$\left.\begin{array}{l} d^3 g_{\text{r}} = \left(-G \dfrac{dm'}{r''^2}\right)(RC)(\text{Jacobian}) \\[3mm] d^3 g_{\text{trt}} = \left(-G \dfrac{dm'}{r''^2}\right)(TC_Z)(\text{Jacobian}) \end{array}\right\}. \tag{3.4}$$

Because the speed of gravity $c_{\text{g}} \cong c$, it has a very large numerical value, and the derivative of r'' with respect to t' approximately equals the derivative of r'' with respect to t

$$\frac{dt}{dt'} = 1 + \frac{1}{c_{\text{g}}} \frac{dr''}{dt'} = 1 + \frac{1}{c_{\text{g}}} \frac{dr''}{dt} \frac{dt}{dt'} =$$

$$= 1 + \frac{1}{c_{\text{g}}} \frac{dr''}{dt} \left(1 + \frac{1}{c_{\text{g}}} \frac{dr''}{dt'}\right) \cong 1 + \frac{1}{c_{\text{g}}} \frac{dr''}{dt}. \tag{3.5}$$

The formulas for the geocentric radial distance to the field-point and its derivative are

$$\left.\begin{array}{l} r(\theta) = \dfrac{r_{\text{p}}(1 + \varepsilon)}{1 + \varepsilon \cos\theta} \\[3mm] \dfrac{dr}{d\theta} = \dfrac{r(\theta)^2}{r_{\text{p}}} \dfrac{\varepsilon}{1 + \varepsilon} \sin\theta \end{array}\right\}, \tag{3.6}$$

where θ is the parametric polar coordinate angle for the spacecraft, r_{p} is the geocentric radial distance at perigee, and ε is the eccentricity of the orbit or trajectory.

Let the rectangular coordinates for $\boldsymbol{r}$ be r_X, r_Y, r_Z and those for $\boldsymbol{r'}$ be r'_X, r'_Y, r'_Z. Let the spherical coordinates for $\boldsymbol{r}$ be r, λ, ϕ and those

for $\boldsymbol{r}'$ be r', λ', ϕ'. Then

$$
\left.\begin{aligned}
r_X &= r \cos \lambda \, \cos \phi \,, & r'_X &= r' \cos \lambda' \, \cos \phi' \\
r_Y &= r \cos \lambda \, \sin \phi \,, & r'_Y &= r' \cos \lambda' \, \sin \phi' \\
r_Z &= r \sin \lambda \,, & r'_Z &= r' \sin \lambda'
\end{aligned}\right\} . \qquad (3.7)
$$

The formula for dm' is

$$
dm' = \rho\,(r')\, r'^2 \cos \lambda' \, dr' \, d\lambda' \, d\phi' . \qquad (3.8)
$$

The square of the magnitude for $\boldsymbol{r}''$ is

$$
\begin{aligned}
r''^2 &= \left(r_X - r'_X\right)^2 + \left(r_Y - r'_Y\right)^2 + \left(r_Z - r'_Z\right)^2 = \\
&= \left(r \cos \lambda \, \cos \phi - r' \cos \lambda' \, \cos \phi'\right)^2 + \\
&\quad + \left(r \cos \lambda \, \sin \phi - r' \cos \lambda' \, \sin \phi'\right)^2 + \\
&\quad + \left(r \sin \lambda - r' \sin \lambda'\right)^2 .
\end{aligned}
$$

Expanding the square and using the trig identity for $\cos\left(\phi - \phi'\right)$ gives

$$
r''^2 = r^2 \left(1 + x\right), \qquad (3.9)
$$

where x is defined by

$$
x \equiv \frac{r'^2}{r^2} - 2\,\frac{r'}{r} \left(\cos \lambda \, \cos \lambda' \, \cos\left(\phi - \phi'\right) + \sin \lambda \, \sin \lambda'\right). \qquad (3.10)
$$

The derivative dr''/dt', which depends on the derivatives $(dr/dt',\ d\lambda/dt',\ d\phi/dt')$ and $(dr'/dt',\ d\lambda'/dt',\ d\phi'/dt')$, will be needed to find the Jacobian

$$
\left.\begin{aligned}
\frac{dr}{dt'} &\cong \frac{dr}{dt} = v_{\mathrm{r}} = \Omega_\theta \, \frac{dr}{d\theta} \,, & \frac{dr'}{dt'} &= 0 \\
\frac{d\lambda}{dt'} &\cong \frac{d\lambda}{dt} = \frac{v_\lambda}{r_\lambda} = \Omega_\lambda \,, & \frac{d\lambda'}{dt'} &= 0 \\
\frac{d\phi}{dt'} &\cong \frac{d\phi}{dt} = \frac{v_\phi}{r_\phi} = \Omega_\phi \,, & \frac{d\phi'}{dt'} &\cong \frac{d\phi'}{dt} = \Omega_{\mathrm{E}}
\end{aligned}\right\} . \qquad (3.11)
$$

We actually need the derivatives that contribute to the transverse motion of the projection of the orbit onto the equatorial plane. All but Ω_ϕ and Ω_{E} can be for now disregarded, because the other derivatives produce terms that will not survive the triple integration.

If the integrand is an odd function, e.g., $f(\phi') \sin \phi'$, where $f(\phi')$ is any even function, $f(-\phi') = f(\phi')$, the integral $\int f(\phi') \sin \phi' \, d\phi'$ over ϕ' from $-\pi$ to $+\pi$ vanishes, i.e., reduces to zero.

Consequently the derivative for the square of r'' reduces to

$$\frac{dr''^2}{dt'} = 2r''\frac{dr''}{dt'} =$$

$$= 2r'r\left(\frac{d\phi}{dt'} - \frac{d\phi'}{dt'}\right)\cos\lambda\,\cos\lambda'\,\sin\left(\phi - \phi'\right). \qquad (3.12)$$

The derivatives $d\phi/dt' \cong d\phi/dt = \Omega_\phi$ and $d\phi'/dt' \cong d\phi'/dt = \Omega_E$. The formula for the Jacobian reduces to

$$\text{Jacobian} = 1 + \frac{1}{c_g}\frac{dr''}{dt'} =$$

$$= 1 + \frac{r}{c_g}\frac{r'}{r''}\left(\Omega_\phi - \Omega_E\right)\cos\lambda\,\cos\lambda'\,\sin\left(\phi - \phi'\right). \qquad (3.13)$$

The formulas for RC and TC_Z, (2.1), reduce to

$$RC = \frac{r}{r''} - \frac{1}{r''r}\left(r_X r'_X + r_Y r'_Y + r_Z r'_Z\right) =$$

$$= \frac{r}{r''} - \frac{r'}{r''}\left(\cos\lambda\,\cos\lambda'\,\cos\left(\phi - \phi'\right) + \sin\lambda\,\sin\lambda'\right), \qquad (3.14)$$

and also

$$TC_Z = \frac{1}{r''r}\left(r'_X r_Y - r_X r'_Y\right) = \frac{r'}{r''}\cos\lambda\,\cos\lambda'\,\sin\left(\phi - \phi'\right). \qquad (3.15)$$

Firstly consider the radial component g_r. Substituting (3.8), (3.13), and (3.14) into (3.4) gives

$$d^3 g_r = \left(-G\frac{dm'}{r''^2}\right)(RC)\,(\text{Jacobian}) =$$

$$= \left(-G\frac{\rho\left(r'\right)}{r''^2}\cos\lambda'\,dr'\,d\lambda'\,d\phi'\right)\times$$

$$\times\frac{r}{r''}\left(1 - \frac{r'}{r}\sin\lambda\,\sin\lambda' - \frac{r'}{r}\cos\lambda\cos\lambda'\,\cos\left(\phi - \phi'\right)\right)\times$$

$$\times\left(1 + \frac{r}{c_g}\frac{r'}{r''}\left(\Omega_\phi - \Omega_E\right)\cos\lambda\,\cos\lambda'\,\sin\left(\phi - \phi'\right)\right). \qquad (3.16)$$

This formula contains both time-retarded or "causal" terms and non-time-retarded or "acausal" terms. The causal terms, those which contain the factor $1/c_g$, contain either $\sin\left(\phi - \phi'\right)$ or $\sin\left(\phi - \phi'\right)\cos\left(\phi - \phi'\right)$. These terms will vanish upon integration over ϕ' from $-\pi$ to $+\pi$, which means that all the effects of time-retardation cancel out for the radial

component. The only terms that survive the integration are the acausal terms. In other words, the radial component can be regarded as being acausal. It can be found by using the standard methods. Gauss' theorem gives the standard well-known inverse square law [13, p. 37]

$$\boldsymbol{g}_{\mathrm{r}} = -\, G\, \frac{M_{\mathrm{E}}}{r^2} \frac{\boldsymbol{r}}{r}\,. \tag{3.17}$$

The radial component, until recently the only known component, has been studied by many researchers for more than 300 years. It is well-known that $\boldsymbol{g}_{\mathrm{r}}$ obeys the conservation laws for orbital energy and orbital angular momentum. The orbital energy is conserved for an isotropic solid central sphere (with no tidal bulges) because the central force is conservative, i.e., there is no mechanism (e.g., friction) by which orbital kinetic and potential energy can be dissipated into another form of energy. The orbital angular momentum is conserved because the central radial force cannot exert a torque on the orbiting body.

Now consider the transverse component. Substituting (3.8), (3.13), and (3.15) into (3.4) gives

$$
\begin{aligned}
d^3 g_{\mathrm{trt}} \left(- G\, \frac{dm'}{r''^2}\right) &(TC_Z)\, (\text{Jacobian}) = \\
&= \left(- G\, \frac{\rho\,(r')}{r''^2}\, \cos\lambda'\, dr'\, d\lambda'\, d\phi'\right) \times \\
&\times \left(\frac{r'}{r''}\, \cos\lambda\, \cos\lambda'\, \sin\,(\phi - \phi')\right) \times \\
&\times \left(1 + \frac{r}{c_{\mathrm{g}}} \frac{r'}{r''}\, (\Omega_\phi - \Omega_{\mathrm{E}})\, \cos\lambda\, \cos\lambda'\, \sin\,(\phi - \phi')\right). \tag{3.18}
\end{aligned}
$$

This formula contains one causal term and one acausal term. The acausal term contains $\sin(\phi - \phi')$, which vanishes upon integration over ϕ' from $-\pi$ to $+\pi$. The surviving term, which contains $\sin^2(\phi - \phi')$, does not vanish upon the integration. Consequently, the time-retarded transverse field is causal, and it can be found by using (Jacobian-1).

Substituting (Jacobian-1) for (Jacobian) in (3.18) gives

$$
\begin{aligned}
d^3 g_{\mathrm{trt}} = &\left(- G\, \frac{\rho\,(r')}{r''^2}\, \cos\lambda'\, dr'\, d\lambda'\, d\phi'\right) \times \\
&\times \left(\frac{r'}{r''}\, \cos\lambda\, \cos\lambda'\, \sin\,(\phi - \phi')\right) \times \\
&\times \left(\frac{r}{c_{\mathrm{g}}} \frac{r'}{r''}\, (\Omega_\phi - \Omega_{\mathrm{E}})\, \cos\lambda\, \cos\lambda'\, \sin\,(\phi - \phi')\right). \tag{3.19}
\end{aligned}
$$

Rearranging and combining factors gives

$$d^3 g_{\text{trt}} = -A \left(\frac{\Omega_\phi - \Omega_{\text{E}}}{\Omega_{\text{E}}} \right) \cos^2(\lambda) \, IG \, \frac{dr'}{r_{\text{E}}} \, d\lambda' \, d\phi', \qquad (3.20)$$

where the definitions for the equatorial surface speed v_{eq}, and for the coefficient A, are

$$v_{\text{eq}} \equiv r_{\text{E}} \Omega_{\text{E}}, \qquad A \equiv G \bar{\rho} \, r_{\text{E}} \, \frac{v_{\text{eq}}}{c_{\text{g}}}, \qquad (3.21)$$

and the integrand for the triple integration is

$$IG \equiv \left(\frac{r_{\text{E}}}{r} \right)^3 \left(\frac{\rho(r')}{\bar{\rho}} \frac{r'^4}{r_{\text{E}}^4} \right) (\cos^3 \lambda') \left(\frac{\sin^2(\phi - \phi')}{(1 + x)^2} \right), \qquad (3.22)$$

where $\bar{\rho}$ is the mean value for $\rho(r')$. The formula for $\rho(r')$ and value for $\bar{\rho}$ can be found in Appendix B.

The solution for g_{trt} becomes

$$g_{\text{trt}} = -A \left(\frac{\Omega_\phi - \Omega_{\text{E}}}{\Omega_{\text{E}}} \right) \cos^2(\lambda) \, TI, \qquad (3.23)$$

where the triple integral TI is defined by

$$TI \equiv \int_0^{r_{\text{E}}} \frac{dr'}{r_{\text{E}}} \int_{-\pi/2}^{\pi/2} d\lambda' \int_{-\pi}^{\pi} IG \, d\phi'. \qquad (3.24)$$

Most of the integrals in this article are solved by using the numerical integration algorithm in Mathcad15. It can be shown that the solution for the triple integral TI is independent of λ and ϕ, which means that it can be solved with $\lambda = 0$ and $\phi = 0$. But solving a triple integral by numerical integration takes a lot of computer time, particularly if r is near the singularity at $r = r_{\text{E}}$, which must be avoided.

A suitable power series approximation for the triple integral is needed. Let $PS'(r)$ be a four-term power series, defined as follows

$$PS'(r) \equiv \frac{I_{\text{E}}}{\bar{\rho} r_{\text{E}}^5} \left(\frac{r_{\text{E}}}{r} \right)^3 \left(C_0 + C_2 \left(\frac{r_{\text{E}}}{r} \right)^2 + C_4 \left(\frac{r_{\text{E}}}{r} \right)^4 + C_6 \left(\frac{r_{\text{E}}}{r} \right)^6 \right). \quad (3.25)$$

Let $PS(r)$ be the same power series without the unitless coefficient, which for the Earth has the value 1.3856 (see Appendix B)

$$PS(r) \equiv \left(\frac{r_{\text{E}}}{r} \right)^3 \left(C_0 + C_2 \left(\frac{r_{\text{E}}}{r} \right)^2 + C_4 \left(\frac{r_{\text{E}}}{r} \right)^4 + C_6 \left(\frac{r_{\text{E}}}{r} \right)^6 \right). \quad (3.26)$$
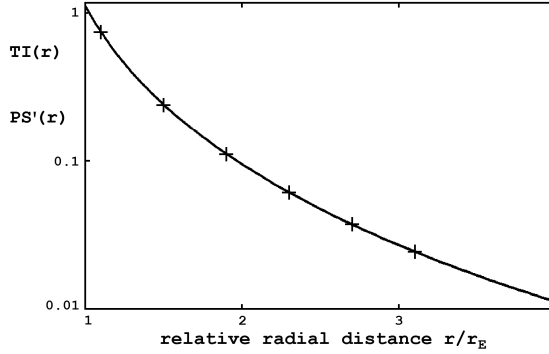
Fig. 3: Semilog graph of the triple integral $TI(r)$ of (3.24) designated by $+$, and the power series $PS'(r)$ of (3.25) designated by the solid curve, versus $r/r_{\rm E}$, using the coefficients of (3.27). The maximum relative difference $(TI - PS')/TI$ is less than 2×10^{-4}. Notice that there is no singularity in the power series and it can be extrapolated all the way down to the surface where $r = r_{\rm E}$.

By using the least-squares fitting routine in Mathcad15, the following values for the coefficients were found to give an excellent fit of $PS'(r)$ to the volume integral $TI(r)$

$$\left.\begin{array}{ll} C_0 = 0.50889, & C_2 = 0.13931 \\ C_4 = 0.01013, & C_6 = 0.14671 \end{array}\right\} . \qquad (3.27)$$

The quality of the fit using these coefficients is shown in Fig. 3. The maximum relative difference at the values for r shown by $+$ in Fig. 3 is less than 2×10^{-4}.

The solution for $g_{\rm trt}(\theta)$ can now be rewritten with $PS(r)$ as follows

$$g_{\rm trt}(\theta) = - G \, \frac{I_{\rm E}}{r_{\rm E}^4} \, \frac{v_{\rm eq}}{c_{\rm g}} \left(\frac{\Omega_\phi(\theta) - \Omega_{\rm E}}{\Omega_{\rm E}} \right) \cos^2\!\big(\lambda(\theta)\big) \, PS\big(r\,(\theta)\big) . \qquad (3.28)$$

The radial component $g_{\rm r}$ satisfies the conservation laws, but the relatively small transverse component $g_{\rm trt}$ does not satisfy the conservation laws. Because the strength of $g_{\rm trt} \sim 10^{-6} \, g_{\rm r}$, a good first approximation is obtained by applying the conservation laws.

Let L be the magnitude for the spacecraft's orbital angular momentum. Then

$$\text{constant} = L = m v_{\rm p} r_{\rm p} = m \, (r_{\rm p} \Omega_{\rm p}) \, r_{\rm p} =$$
$$= m r_{\rm p}^2 \Omega_{\rm p} = m r(\theta)^2 \Omega_\theta(\theta) .$$

Here r_p is the value for r at perigee, v_p is the orbital speed at perigee, and Ω_p is the orbital angular speed at perigee. Therefore, by conservation of orbital angular momentum, the formula for the spacecraft's orbital angular speed becomes

$$\Omega_\theta(\theta) \equiv \frac{d\theta}{dt} = \frac{r_\mathrm{p} v_\mathrm{p}}{r(\theta)^2} = \frac{r_\mathrm{p}^2}{r(\theta)^2}\, \Omega_\mathrm{p}\,. \tag{3.29}$$

Let E be the spacecraft's orbital kinetic energy plus the orbital potential energy. Then

$$\mathrm{constant} = E = \frac{1}{2}\, m\, v(\theta)^2 - \frac{GM_\mathrm{E}\, m}{r(\theta)} =$$
$$= \frac{1}{2}\, m v_\infty^2 + \frac{1}{2}\, m v_\mathrm{p}^2 - \frac{GM_\mathrm{E}\, m}{r_\mathrm{p}}\,.$$

Here v_∞ is the spacecraft's speed as $r \to \infty$. Therefore, by conservation of energy, the formula for the orbital speed becomes

$$v(\theta) = \sqrt{v_\infty^2 + v_\mathrm{p}^2 + 2\frac{GM_\mathrm{E}}{r(\theta)} - 2\frac{GM_\mathrm{E}}{r_\mathrm{p}}}\,. \tag{3.30}$$

Let (x, y, z) be the rectangular coordinates for an inertial frame with the origin at the center of the sphere and with the (x, y) plane coinciding with the plane of the orbit. Let θ_p be the the angle which rotates the (x, y) plane so that perigee occurs at $\theta = \theta_\mathrm{p}$. The formulas for r_x and r_y are

$$\left.\begin{aligned} r_x(\theta) &= r(\theta) \cos(\theta - \theta_\mathrm{p}) \\ r_y(\theta) &= r(\theta) \sin(\theta - \theta_\mathrm{p}) \end{aligned}\right\}. \tag{3.31}$$

The formulas for v_x and v_y are

$$\left.\begin{aligned} v_x &= \frac{dr_x}{dt} = v_\mathrm{r} \cos(\theta - \theta_\mathrm{p}) - r\, \Omega_\theta \sin(\theta - \theta_\mathrm{p}) \\ v_y &= \frac{dr_y}{dt} = v_\mathrm{r} \sin(\theta - \theta_\mathrm{p}) + r\, \Omega_\theta \cos(\theta - \theta_\mathrm{p}) \end{aligned}\right\}. \tag{3.32}$$

The radial component v_r is given by (3.6) and (3.11)

$$v_\mathrm{r} = \Omega_\theta\, \frac{dr}{d\theta} = \Omega_\theta\, \frac{r(\theta)^2}{r_\mathrm{p}}\, \frac{\varepsilon}{1 + \varepsilon} \sin\theta\,,$$
$$r(\theta) = \frac{r_\mathrm{p}(1 + \varepsilon)}{1 + \varepsilon \cos\theta}\,.$$

Let α_{eq} be the inclination of the orbital plane to the equatorial plane, and let λ_{p} be the geocentric latitude for perigee. If $\theta_{\mathrm{p}} = 0$, let the x-axis of the orbital frame coincide with the X-axis of the equatorial plane. Then the formulas for the transformation from the (x, y, z) orbital frame to the (X, Y, Z) equatorial frame are

$$\left. \begin{aligned} r_X &= r\,(\theta)\,\cos\,(\theta - \theta_{\mathrm{p}}) \\ r_Y &= r\,(\theta)\cos\alpha_{\mathrm{eq}}\,\sin\,(\theta - \theta_{\mathrm{p}}) \\ r_Z &= -r\,(\theta)\sin\alpha_{\mathrm{eq}}\,\sin\,(\theta - \theta_{\mathrm{p}}) \end{aligned} \right\} . \qquad (3.33)$$

The formulas for the velocity components in the (X, Y, Z) frame are

$$\left. \begin{aligned} v_X &= \frac{dr_X}{dt} = v_{\mathrm{r}}\,\cos\,(\theta - \theta_{\mathrm{p}}) - r\,\Omega_\theta\,\sin\,(\theta - \theta_{\mathrm{p}}) \\ v_Y &= \frac{dr_Y}{dt} = v_{\mathrm{r}}\cos\alpha_{\mathrm{eq}}\,\sin\,(\theta - \theta_{\mathrm{p}}) + r\,\Omega_\theta\cos\alpha_{\mathrm{eq}}\,\cos\,(\theta - \theta_{\mathrm{p}}) \\ v_Z &= \frac{dr_Z}{dt} = -v_{\mathrm{r}}\sin\alpha_{\mathrm{eq}}\,\sin\,(\theta - \theta_{\mathrm{p}}) - r\,\Omega_\theta\sin\alpha_{\mathrm{eq}}\,\cos\,(\theta - \theta_{\mathrm{p}}) \end{aligned} \right\} . \quad (3.34)$$

Let $r_\phi(\theta)$ be the geocentric radial distance to the projection of the field-point onto the (X, Y) equatorial plane, and let $r_\lambda(\theta)$ be the geocentric radial distance to the projection of the field point onto a vertical (X, Z) plane. Then

$$r_\phi(\theta) = \sqrt{r_X(\theta)^2 + r_Y(\theta)^2} \;, \qquad (3.35)$$

and

$$r_\lambda(\theta) = \sqrt{r_X(\theta)^2 + r_Z(\theta)^2} \;. \qquad (3.36)$$

Let v_ϕ be the speed of the projection of the field-point onto the (X, Y) equatorial plane, and let v_λ be the speed of the projection of the field point onto a vertical (X, Z) plane. Then

$$v_\phi(\theta) = \sqrt{v_X(\theta)^2 + v_Y(\theta)^2} \;, \qquad (3.37)$$

and

$$v_\lambda(\theta) = \sqrt{v_X(\theta)^2 + v_Z(\theta)^2} \;. \qquad (3.38)$$

The formula for the tangent of the azimuthal angle ϕ is

$$\tan\phi(\theta) = \frac{r_Y(\theta)}{\sqrt{r_X(\theta)^2 + r_Z(\theta)^2}} \;.$$

Solving for ϕ gives

$$\phi(\theta) = \tan^{-1}\left(\frac{r_Y(\theta)}{\sqrt{r_X(\theta)^2 + r_Z(\theta)^2}}\right). \qquad (3.39)$$

The formula for the tangent of the geocentric latitude is

$$\tan\lambda(\theta) = \frac{r_Z(\theta)}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}}.$$

Solving for λ gives

$$\lambda(\theta) = \tan^{-1}\left(\frac{r_Z(\theta)}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}}\right). \qquad (3.40)$$

The formula for Ω_ϕ becomes

$$\Omega_\phi(\theta) = \Omega_\theta(\theta)\frac{d\phi}{d\theta} = \pm\frac{v_\phi(\theta)}{r_\phi(\theta)}. \qquad (3.41)$$

Use the $+$ sign if α_{eq} takes numerical values in the range $0° < \alpha_{\mathrm{eq}} < 90°$, and the $-$ sign if $90° < \alpha_{\mathrm{eq}} < 180°$.

The value for the angle θ_{p} depends on the latitude for perigee λ_{p}, which ranges from $-90°$ to $+90°$, and α_{eq}, which ranges from $0°$ to $+180°$. If $\alpha_{\mathrm{eq}} = 0°$ or $180°$, then $\theta_{\mathrm{p}} = 0°$. If $\alpha_{\mathrm{eq}} = 90°$, then $\theta_{\mathrm{p}} = \lambda_{\mathrm{p}}$. If $0 < \alpha_{\mathrm{eq}} < \pi$ radians and $\alpha_{\mathrm{eq}} \neq \frac{\pi}{2}$ and $\sin\lambda_{\mathrm{p}} \leqslant \sin\alpha_{\mathrm{eq}}$, the formula for θ_{p} (the angle is taken here in radians) is

$$\theta_{\mathrm{p}} = \sin^{-1}\left(\frac{\sin\lambda_{\mathrm{p}}}{\sin\alpha_{\mathrm{eq}}}\right). \qquad (3.42)$$

If $\sin\lambda_{\mathrm{p}} > \sin\alpha_{\mathrm{eq}}$, the inverse sine function is shifted from the primary branch and the value for θ_{p} is greater than $90°$. Of the six flybys reported by Anderson *et al.* [10], only the MESSENGER flyby has a value for θ_{p} that is greater than $90°$; (from Appendix A) $\alpha_{\mathrm{eq}} = 133.1°$, $\lambda_{\mathrm{p}} = 46.95°$, which gives $\theta_{\mathrm{p}} = 90.0467°$.

We need a method to determine the numerical values for the minimum and maximum permissible values for θ, designated $\theta_{\min}$ and $\theta_{\max}$. One method is to solve $r(\theta)$ (3.6) for the value for θ which causes the denominator to be zero. Let θ_∞ be that value

$$\theta_\infty = \cos^{-1}\left(\frac{-1}{\varepsilon}\right). \qquad (3.43)$$

Consequently, to avoid integration over the singularity in $r(\theta)$, the value for $\theta_{\min}$ must be greater than $-\theta_\infty$ and the value for $\theta_{\max}$ must be less than $+\theta_\infty$.

Let $t_{\rm in}(\theta_{\rm in})$ be the (negative) time for the start of inbound data accumulation before $t = 0$ at $\theta = 0$ at perigee, and let $t_{\rm out}(\theta_{\rm out})$ be the (positive) time for the end of outbound data accumulation after $t = 0$ at $\theta = 0$ at perigee.

The formulas for calculating $t_{\rm in}$ and $t_{\rm out}$ are

$$t_{\rm in} = \int_{t(0)}^{t(\theta_{\rm in})} dt = \int_0^{\theta_{\rm in}} \frac{dt}{d\theta}\, d\theta = \int_0^{\theta_{\rm in}} \frac{1}{\Omega_\theta(\theta)}\, d\theta =$$

$$= \int_0^{\theta_{\rm in}} \frac{r(\theta)^2}{v_{\rm p} r_{\rm p}}\, d\theta \longrightarrow -\infty \;\; \text{if} \;\; \theta_{\rm in} = -\theta_\infty\,,$$

and

$$t_{\rm out} = \int_{t(0)}^{t(\theta_{\rm out})} dt = \int_0^{\theta_{\rm out}} \frac{dt}{d\theta}\, d\theta = \int_0^{\theta_{\rm out}} \frac{1}{\Omega_\theta(\theta)}\, d\theta =$$

$$= \int_0^{\theta_{\rm out}} \frac{r(\theta)^2}{v_{\rm p} r_{\rm p}}\, d\theta \longrightarrow +\infty \;\; \text{if} \;\; \theta_{\rm out} = +\theta_\infty\,. \tag{3.44}$$

Numerical values for $t_{\rm in}$ and $t_{\rm out}$ were included in the report of Anderson *et al.* [10] only for the NEAR flyby (see Appendix A).

Let a and b be the semimajor and semiminor axes for an elliptical (closed) orbit ($0 \leqslant \varepsilon < 1$). Kepler's laws give the orbital angular speed in terms of a, b, and the period P [19]

$$\left.\begin{aligned} a &= \frac{1}{2}(r_{\rm a} + r_{\rm p}) && \text{(semimajor axis)} \\ b &= a\sqrt{1 - \varepsilon^2} && \text{(semiminor axis)} \\ P &= \frac{a^{3/2}}{\sqrt{GM_{\rm E}}} && \text{(Kepler's 3rd law)} \\ \Omega_\theta(\theta) &= \frac{2\pi}{P}\frac{a\,b}{r(\theta)^2} && \text{(Kepler's 2nd law)} \end{aligned}\right\}, \tag{3.45}$$

where $r_{\rm a}$ is the geocentric radial distance at apogee.

The equivalent circular orbit for an elliptical orbit will be needed. Let $r_{\rm co}$, $v_{\rm co}$ and $\Omega_{\rm co}$ be the radius, orbital speed, and orbital angular speed for an equivalent circular orbit which has the period P, respectively. The formulas for $r_{\rm co}$, $v_{\rm co}$, and $\Omega_{\rm co}$ can be found by rearranging

Kepler's 3rd law

$$P = \frac{a}{\sqrt{GM_{\mathrm{E}}}} \sqrt{a} = \frac{a}{\sqrt{GM_{\mathrm{E}}/a}} = \frac{r_{\mathrm{co}}}{v_{\mathrm{co}}} = \frac{1}{\Omega_{\mathrm{co}}}$$
$$v_{\mathrm{co}} = \sqrt{\frac{GM_{\mathrm{E}}}{a}} = r_{\mathrm{co}}\,\Omega_{\mathrm{co}}$$ $$\left.\right\}.$$ $$\quad (3.46)$$
$$r_{\mathrm{co}} = a\,, \qquad \Omega_{\mathrm{co}} = \frac{v_{\mathrm{co}}}{a}$$

The formulas for $\lambda(\theta)$ and $\Omega_\phi(\theta)$ are greatly simplified for elliptical orbits if $\alpha_{\mathrm{eq}} = \lambda_{\mathrm{p}}$. In this case, continue to use (3.6) for $r(\theta)$, and use (3.45) for $\Omega_\theta(\theta)$. Then use the following formulas for $\lambda(\theta)$, $\Omega_\phi(\theta)$, and $r_\lambda(\theta)$

$$\lambda(\theta) = \tan^{-1}\big(\tan\alpha_{\mathrm{eq}}\cos\theta\big)$$
$$\Omega_\phi(\theta) = \Omega_\theta(\theta)\cos\alpha_{\mathrm{eq}} \qquad \left.\right\}.$$ $$\quad (3.47)$$
$$r_\lambda(\theta) = r(\theta)\cos\theta$$

For closed orbits, the value for $\theta_{\min} = -\pi$ radians and the value for $\theta_{\max} = +\pi$ radians.

Let $\boldsymbol{F}_\lambda$ be an induction-like field, and let the ϕ-component of the curl of $\boldsymbol{F}_\lambda$ equal $-k\,d\boldsymbol{g}_{\mathrm{trt}}/dt$, where k is a constant. The formula for the curl operator in spherical coordinates can be found in J. D. Jackson's textbook [1]

$$\boldsymbol{\nabla} \times \boldsymbol{F}_\lambda = \boldsymbol{e}_\phi \frac{1}{r} \frac{\partial}{\partial r}\left(r\,F_\lambda\right) = -\,k\,\frac{d\boldsymbol{g}_{\mathrm{trt}}}{dt} = \boldsymbol{e}_\phi k\,\frac{dg_{\mathrm{trt}}}{dt}\,, \qquad (3.48)$$

where $\boldsymbol{e}_\phi$ is a unit vector directed towards the east. Solving for $\partial(r\,F_\lambda)/\partial r$ and integrating both sides from $t(0)$ to $t(\theta)$ gives

$$\int_{t(0)}^{t(\theta)} \frac{\partial}{\partial r}\left(r\,F_\lambda\right) dt = \int_0^\theta \frac{\partial}{\partial r}\left(r\,F_\lambda\right) \frac{dt}{d\theta}\,d\theta = \int_0^\theta \frac{d}{d\theta}\left(r\,F_\lambda\right) \frac{d\theta}{dr}\frac{dt}{d\theta}\,d\theta =$$
$$= k\int_{t(0)}^{t(\theta)} r\,\frac{dg_{\mathrm{trt}}}{dt}\,dt = k\int_0^\theta r\,\frac{dg_{trt}}{d\theta}\,d\theta\,. \qquad (3.49)$$

Therefore,

$$\int_0^\theta \left(\frac{d}{d\theta}\left(r\,F_\lambda\right)\frac{d\theta}{dr}\frac{dt}{d\theta} - k\,r\,\frac{dg_{\mathrm{trt}}}{d\theta}\right) d\theta = 0\,. \qquad (3.50)$$

This equation is satisfied for all values of θ, if and only if,

$$\frac{d}{d\theta}\left(r\,F_\lambda\right) = k\,r\,\frac{dr}{d\theta}\frac{d\theta}{dt}\frac{dg_{\mathrm{trt}}}{d\theta}\,. \qquad (3.51)$$

Integrating both sides from 0 to θ gives

$$F_\lambda(\theta) = \frac{k}{r(\theta)} \int_0^\theta r(\theta)\, \Omega_\theta(\theta)\, \frac{dr}{d\theta}\, \frac{dg_{\mathrm{trt}}}{d\theta}\, d\theta\,. \tag{3.52}$$

Units for F_λ are the units for acceleration, m/s². The constant k has units of $(\mathrm{m/s})^{-1}$. Let v_{k} be the reciprocal of k, $v_{\mathrm{k}} \equiv 1/k$, which will be called the "induction speed". Regard v_{k} as an adjustable parameter for each case, and regard the average for all cases as a fixed parameter for the neoclassical causal theory. The formula for F_λ can be rewritten in terms of v_{k} and v_{eq}

$$F_\lambda(\theta) = \frac{v_{\mathrm{eq}}}{v_{\mathrm{k}}} \frac{r_{\mathrm{E}}}{r(\theta)} \int_0^\theta \frac{r(\theta)}{r_{\mathrm{E}}} \frac{\Omega_\theta(\theta)}{\Omega_{\mathrm{E}}} \frac{1}{r_{\mathrm{E}}} \frac{dr}{d\theta} \frac{dg_{\mathrm{trt}}}{d\theta}\, d\theta\,. \tag{3.53}$$

Let $\delta v / v_{\mathrm{in}} \ll 1$ be the relative change in the magnitude of $\boldsymbol{v}$ due to $\boldsymbol{F}_\lambda$, where v_{in} is the initial speed. The dot product $\boldsymbol{v} \cdot \boldsymbol{F}_\lambda$ gives the time rate at which the orbital energy is changed. Therefore,

$$\left(1 + \frac{\delta v}{v_{\mathrm{in}}}\right)^2 \cong 1 + 2\frac{\delta v}{v_{\mathrm{in}}} =$$

$$= 1 + \frac{1}{v_{\mathrm{in}}^2} \int_{t(0)}^{t(\theta)} F_\lambda\, v_\lambda\, dt =$$

$$= 1 + \frac{1}{v_{\mathrm{in}}^2} \int_0^\theta r_\lambda F_\lambda\, \frac{d\lambda}{d\theta}\, d\theta\,. \tag{3.54}$$

Therefore,

$$\delta v(\theta) = \frac{v_{\mathrm{in}}}{2} \int_0^\theta \frac{r_\lambda(\theta)\, F_\lambda(\theta)}{v_{\mathrm{in}}^2}\, \frac{d\lambda}{d\theta}\, d\theta\,. \tag{3.55}$$

As previously defined, $\theta_{\min}$ and $\theta_{\max}$ are the minimum and maximum values for θ. Let δv_{trt} be the speed-change for a flyby or for one revolution. Then

$$\delta v_{\mathrm{in}} = \delta v(\theta_{\min})\,, \quad \delta v_{\mathrm{out}} = \delta v(\theta_{\max})\,, \quad \delta v_{\mathrm{trt}} = \delta v_{\mathrm{in}} + \delta v_{\mathrm{out}}\,. \tag{3.56}$$

These formulas will be used in §4 to calculate the speed-changes listed in Table 1.

§4. Calculated speed-changes for six Earth flybys caused by the neoclassical causal version of Newton's theory. The trajectory parameters given by Anderson *et al.* [10] are listed in Appendix A.

The parameters for the NEAR spacecraft flyby will be used for the following example calculation. The same method will be applied to derive the time-retarded speed-change for each of the remaining five flybys.

Numerical values for the Earth's parameters and the Earth's radial mass-density distribution are given in Appendix B.

As previously defined, $r(\theta)$ is the geocentric radial distance to the spacecraft in the plane of the trajectory. The formulas for $r(\theta)$ and its derivative (3.6) are

$$\left.\begin{aligned} r(\theta) &= \frac{r_\mathrm{p}(1+\varepsilon)}{1+\varepsilon\cos\theta} \\ \frac{dr}{d\theta} &= \frac{r(\theta)^2}{r_\mathrm{p}}\frac{\varepsilon}{1+\varepsilon}\sin\theta \end{aligned}\right\}, \tag{4.1}$$

where θ is the parametric polar coordinate angle, ε is the eccentricity for the hyperbolic trajectory, and r_p is the geocentric radial distance to the spacecraft at perigee (at $\theta = 0$).

The following method will be used to find ε. The asymptotic angle α_asm, Fig. 4, depends on the deflection angle,

$$\alpha_\mathrm{asm} = \frac{1}{2}\left(180° - DA\right) = \frac{1}{2}\left(180° - 66.9°\right) = 56.55°. \tag{4.2}$$

The radial distance at perigee, r_p, depends on the altitude at perigee h_p, through r_E, as follows

$$r_\mathrm{p} = r_\mathrm{E} + h_\mathrm{p} = r_\mathrm{E} + 539 \text{ km} = 1.0846\, r_\mathrm{E}. \tag{4.3}$$

The impact parameter FP is given by conservation of angular momentum,

$$FP\, v_\infty = r_\mathrm{p} v_\mathrm{p}, \tag{4.4}$$

where v_∞ and v_p are values listed in Appendix A and r_p is given by (4.3). Given numerical values for the NEAR flyby are

$$v_\infty = 6.851 \text{ km/s}, \qquad v_\mathrm{p} = 12.739 \text{ km/s}.$$

The numerical value for FP becomes

$$FP = r_\mathrm{p}\frac{v_\mathrm{p}}{v_\infty} = 2.0167\, r_\mathrm{E}. \tag{4.5}$$

The ratio $FP/OF = \sin\alpha_\mathrm{asm}$. Therefore,

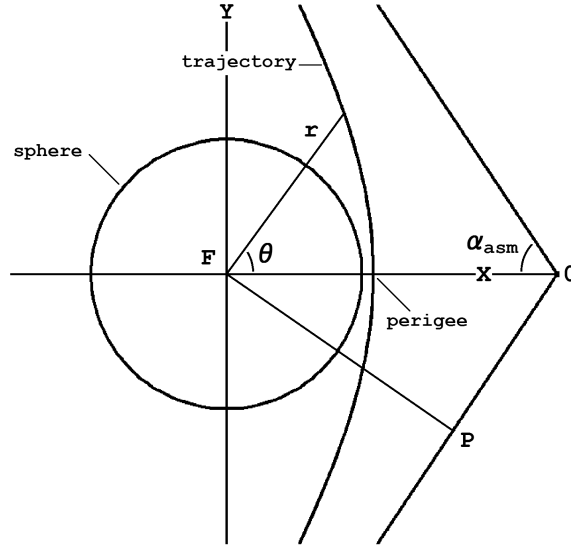$$OF = \frac{FP}{\sin\alpha_\mathrm{asm}} = 2.4171\, r_\mathrm{E}. \tag{4.6}$$

Fig. 4: Hyperbolic trajectory for the NEAR spacecraft flyby in the (x, y) trajectory plane for a central sphere of radius r_{E} (using (3.31) with $\theta_{\mathrm{p}} = 0$). The geocentric radial distance to the spacecraft is $r(\theta)$ at the parametric angle θ. The least geocentric distance r_{p} is at perigee. The asymptote angle α_{asm} is defined by the deflection angle. The center of the sphere is at the focus F. The impact parameter is the distance FP. Another trajectory parameter is the distance OF.

The parameter a is the distance $OF - r_{\mathrm{p}}$,

$$a = OF - r_{\mathrm{p}} = 1.3325\, r_{\mathrm{E}}. \tag{4.7}$$

The parameter b depends on the asymptotic angle α_{asm},

$$b = a \tan \alpha_{\mathrm{asm}} = 2.0170\, r_{\mathrm{E}}. \tag{4.8}$$

The eccentricity ε depends on a and b,

$$\varepsilon = \frac{\sqrt{a^2 + b^2}}{a} = 1.8142. \tag{4.9}$$

This gives the numerical value for ε to be used in $r(\theta)$ (4.1), which is the geocentric radial distance to the spacecraft in the plane of the trajectory.

The value for θ_∞, (3.43), for the NEAR flyby is

$$\theta_\infty = \cos^{-1}\left(\frac{-1}{\varepsilon}\right) = 123.45°. \tag{4.10}$$

Notice that $180° - \alpha_{\mathrm{asm}}$ also equals θ_∞.

For the NEAR spacecraft flyby, $\alpha_{\mathrm{eq}} = 108.0°$ and $\lambda_{\mathrm{p}} = 33.0°$, which from (3.42) gives

$$\theta_{\mathrm{p}} = \sin^{-1}\left(\frac{\sin \lambda_{\mathrm{p}}}{\sin \alpha_{\mathrm{eq}}}\right) = 34.9364°. \qquad (4.11)$$

Numerical values for r_{p}, α_{asm}, ε, θ_∞, and θ_{p} for each of the six flybys reported by Anderson *et al.* are listed in Table 3.

The formula for $\lambda(\theta)$ is given by (3.40),

$$\lambda(\theta) = \tan^{-1}\left(\frac{r_Z(\theta)}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}}\right), \qquad (4.12)$$

where r_X, r_Y, and r_Z are given by (3.33),

$$\left.\begin{array}{l} r_X(\theta) = r(\theta)\cos(\theta - \theta_{\mathrm{p}}) \\[4pt] r_Y(\theta) = r(\theta)\cos\alpha_{\mathrm{eq}}\sin(\theta - \theta_{\mathrm{p}}) \\[4pt] r_Z(\theta) = -r(\theta)\sin\alpha_{\mathrm{eq}}\sin(\theta - \theta_{\mathrm{p}}) \end{array}\right\}. \qquad (4.13)$$

Table 4 compares the listed inbound and outbound asymptotic latitudes from Appendix A, λ_{in} and λ_{out}, with the calculated asymptotic latitudes $\lambda(-0.9999\,\theta_\infty)$ and $\lambda(+0.9999\,\theta_\infty)$ using (4.12). This table shows that some of the listed latitudes are inconsistent with the maximum and minimum permissible calculated values.

The starting and ending times for the NEAR flyby are known from Appendix A to be -88.4 hours and $+95.6$ hours. The calculated values for t_{in} and t_{out} are given by (3.44)

$$\left.\begin{array}{l} \text{if } \theta_{\mathrm{in}} = -0.9973\,\theta_\infty = -123.1192° \\[10pt] t_{\mathrm{in}} = \displaystyle\int_0^{\theta_{\mathrm{in}}} \frac{r(\theta)^2}{v_{\mathrm{p}} r_{\mathrm{p}}}\, d\theta = -87.75 \text{ hours} \\[14pt] \text{if } \theta_{\mathrm{out}} = +0.9975\,\theta_\infty = +123.1437° \\[10pt] t_{\mathrm{out}} = \displaystyle\int_0^{\theta_{\mathrm{out}}} \frac{r(\theta)^2}{v_{\mathrm{p}} r_{\mathrm{p}}}\, d\theta = +94.88 \text{ hours} \end{array}\right\}. \qquad (4.14)$$

This shows that known values for t_{in} and t_{out} can be used to calculate precise values for θ_{in} and θ_{out}. Values for t_{in} and t_{out} for the other flybys were not listed, so another method is used herein to get estimated values

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
$r_\mathrm{p}/r_\mathrm{E}$	1.0846	1.1507	1.3070	1.3684	1.1844	1.0476
α_asm	56.55°	66.15°	40.35°	42.65°	80.15°	64.45°
ε	1.8142	2.4731	1.3122	1.3596	5.8456	2.3186
θ_∞	123.45°	113.85°	139.65°	137.35°	99.85°	115.55°
θ_p	+34.9364°	+44.8989°	+36.9067°	+90.0468°	−68.3765°	−57.4444°

Table 3: Trajectory parameter values for each of the six Earth flybys reported by Anderson *et al.* [10]. The ratio $r_\mathrm{p}/r_\mathrm{E}$ is the geocentric radial distance at perigee relative to the Earth's radius, α_asm is the angle for the asymptotes (see Fig. 4), ε is the eccentricity for the trajectory, θ_∞ is the value for θ which makes $r(\theta_\infty)$ go to infinity, and θ_p is the value for θ which rotates the (x, y) orbital plane so that the latitude for perigee equals the value for λ_p listed in Appendix A.

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
λ_in	+20.76°?	+12.52°	+2.81°?	−31.44°	+12.92°	+34.26°?
$\lambda\left(-0.9999\,\theta_\infty\right)$	+20.52°	+12.63°	+1.99°	−32.50°	+12.94°	+34.08°
λ_out	−71.96°?	−34.15°	−34.29°?	−31.92°	−4.99°	−4.87°?
$\lambda\left(+0.9999\,\theta_\infty\right)$	−71.94°	−34.26°	−34.12°	−32.45°	−5.02°	−4.62°

Table 4: Comparison of the listed asymptotic inbound and outbound geocentric latitudes, λ_in and λ_out (from Appendix A) with the calculated latitudes, $\lambda\left(-0.9999\,\theta_\infty\right)$ and $\lambda\left(+0.9999\,\theta_\infty\right)$ by using (4.12). Cases where the listed latitude is incompatible with the calculated latitude are marked with "?".

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
$\theta_{\text{in}}/\theta_\infty$	-0.9973	-0.998	-0.983	-0.993	-0.999	-0.998
$\theta_{\text{out}}/\theta_\infty$	$+0.9975$	$+0.998$	$+0.993$	$+0.994$	$+0.999$	$+0.998$
t_{in} (hours)	-88	-87	-88	-85	-89	-81
t_{out} (hours)	$+95$	$+87$	$+88$	$+100$	$+89$	$+81$
$r_{\text{in}}/r_{\text{E}}$	346	444	206	207	808	412
$r_{\text{out}}/r_{\text{E}}$	374	444	206	241	808	412
v_{in}/v_∞	$+1.416$	$+1.415$	-1.426	$+1.427$	$+1.414$	-1.415

Table 5: The above values for θ_{in} and θ_{out} for the NEAR flyby are based on listed values for $t_{\text{in}} = 88.4$ hours and $t_{\text{out}} = 95.6$ hours. The values for θ_{in} and θ_{out} for the other flybys are rough estimates by using plausible values for t_{in} and t_{out}. The last three rows list the corresponding ratios for $r_{\text{in}}/r_{\text{E}}$, $r_{\text{out}}/r_{\text{E}}$, and v_{in}/v_∞. The curious required sign reversal for v_{in} for the Rosetta and GLL-II flybys may be a manifestation of the covariance and contravariance of vectors [20].

for θ_{in} and θ_{out}. Table 5 lists the NEAR values and the estimated values for θ_{in} and θ_{out}, corresponding values for t_{in} and t_{out}, and corresponding values for $r_{\text{in}}/r_{\text{E}}$, $r_{\text{out}}/r_{\text{E}}$, and v_{in}/v_∞, where $r_{\text{in}} = r(\theta_{\text{min}})$, $r_{\text{out}} = r(\theta_{\text{out}})$, and $v_{\text{in}} = v(\theta_{\text{min}})$ by using (3.30). The required sign change for v_{in} for the Rosetta and GLL-II flybys may be a manifestation of the covariance and contravariance of vectors [20].

The formula for Ω_θ is given by (3.29)

$$\Omega_\theta(\theta) = \frac{r_{\text{p}} v_{\text{p}}}{r(\theta)^2}. \qquad (4.15)$$

The formula for $\Omega_\phi(\theta)$ is given by (3.41)

$$\Omega_\phi(\theta) = \pm \frac{v_\phi(\theta)}{r_\phi(\theta)} = \pm \frac{\sqrt{v_X(\theta)^2 + v_Y(\theta)^2}}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}}, \qquad (4.16)$$

where the X and Y components of r are given by (4.13). The X, Y, and Z components of v, given by (3.34), are

$$\left. \begin{aligned} v_X(\theta) &= v_{\text{r}} \cos(\theta - \theta_{\text{p}}) - r(\theta)\Omega_\theta(\theta)\sin(\theta - \theta_{\text{p}}) \\ v_Y(\theta) &= v_{\text{r}} \cos\alpha_{\text{eq}} \sin(\theta - \theta_{\text{p}}) + r(\theta)\Omega_\theta(\theta)\cos\alpha_{\text{eq}}\cos(\theta - \theta_{\text{p}}) \\ v_Z(\theta) &= -v_{\text{r}}\sin\alpha_{\text{eq}}\sin(\theta - \theta_{\text{p}}) - r(\theta)\Omega_\theta(\theta)\sin\alpha_{\text{eq}}\cos(\theta - \theta_{\text{p}}) \end{aligned} \right\} . \quad (4.17)$$

The formula for $v_{\rm r}$, given by (3.6) and (3.11), is

$$v_{\rm r}(\theta) = \Omega_\theta(\theta)\,\frac{dr}{d\theta} = \Omega_\theta(\theta)\,\frac{r(\theta)^2}{r_{\rm p}}\,\frac{\varepsilon}{1+\varepsilon}\,\sin\theta\,. \qquad (4.18)$$

A graph of Ω_ϕ relative to $\Omega_{\rm E}$, Fig. 5, shows that this component of the angular speed is negative (retrograde) with a minimum value of about 50 times the Earth's angular speed.

The formula for the time-retarded transverse gravitational field is given by (3.28)

$$g_{\rm trt}(\theta) = -\,G\,\frac{I_{\rm E}}{r_{\rm E}^4}\,\frac{v_{\rm eq}}{c_{\rm g}}\left(\frac{\Omega_\phi(\theta) - \Omega_{\rm E}}{\Omega_{\rm E}}\right)\cos^2\!\big(\lambda(\theta)\big)\,PS\big(r(\theta)\big)\,. \qquad (4.19)$$

Numerical values for G, $I_{\rm E}$, $r_{\rm E}$, $\Omega_{\rm E}$, and $v_{\rm eq}$ are listed in Appendix B. To start, let's assume that $c_{\rm g} = 1.000\,c$ for the NEAR flyby. The formula for Ω_ϕ is given by (4.16). The formula for λ is given by (4.12). The formula for $PS(r)$ is given by (3.26)

$$PS(r) \equiv \left(\frac{r_{\rm E}}{r}\right)^3\left(C_0 + C_2\left(\frac{r_{\rm E}}{r}\right)^2 + C_4\left(\frac{r_{\rm E}}{r}\right)^4 + C_6\left(\frac{r_{\rm E}}{r}\right)^6\right). \qquad (4.20)$$

Numerical values for the coefficients are given by (3.27)

$$\left.\begin{array}{ll} C_0 = 0.50889, & C_2 = 0.13931 \\ C_4 = 0.01013, & C_6 = 0.14671 \end{array}\right\}. \qquad (4.21)$$

The formula for $r(\theta)$ is given by (4.1).

A graph of $g_{\rm trt}(\theta)$ versus θ with $c_{\rm g} = 1.000\,c$, Fig. 6, shows that the transverse field rises from zero to a sharp peak near $\theta = 0°$, then decreases to zero. An expanded view near the peak, Fig. 7, shows a significant difference in the peak values for $c_{\rm g} = 1.000\,c$ and $c_{\rm g} = 1.060\,c$.

The formula for $F_\lambda(\theta)$ given by (3.53) is

$$F_\lambda(\theta) = \frac{v_{\rm eq}}{v_{\rm k}}\,\frac{r_{\rm E}}{r(\theta)}\int_0^\theta \frac{r(\theta)}{r_{\rm E}}\,\frac{\Omega_\theta(\theta)}{\Omega_{\rm E}}\,\frac{1}{r_{\rm E}}\,\frac{dr}{d\theta}\,\frac{dg_{\rm trt}}{d\theta}\,d\theta\,, \qquad (4.22)$$

where $v_{\rm k}$ is the induction speed (an adjustable parameter), $r(\theta)$ and $dr/d\theta$ are given by (4.1), $\Omega_\theta(\theta)$ is given by (4.15), and $dg_{\rm trt}/d\theta$ is found by using the numerical differentiation algorithm in Mathcad15 for the derivative of the time-retarded transverse field $g_{\rm trt}$ given by (4.19).

A couple of trial calculations indicated that for the NEAR flyby $v_{\rm k} = 6.530\,v_{\rm eq}$ gives a speed-change that agrees exactly with the observed
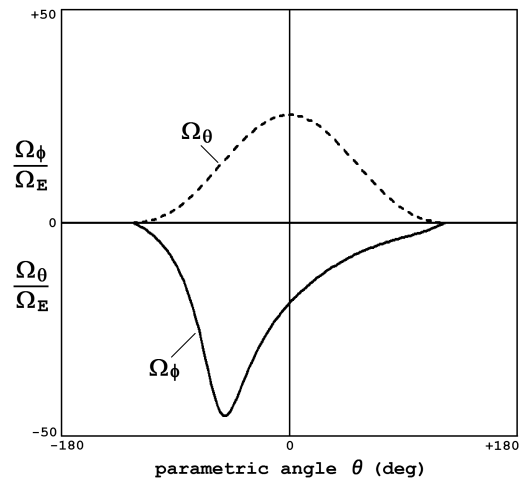
Fig. 5: Graph of the ratio $\Omega_\phi/\Omega_{\mathrm{E}}$ versus θ for the NEAR flyby. The relative angular speed is negative (retrograde) because the inclination $\alpha_{\mathrm{eq}} = 108.0° > 90°$. The ratio $\Omega_\theta/\Omega_{\mathrm{E}}$ is shown for reference.
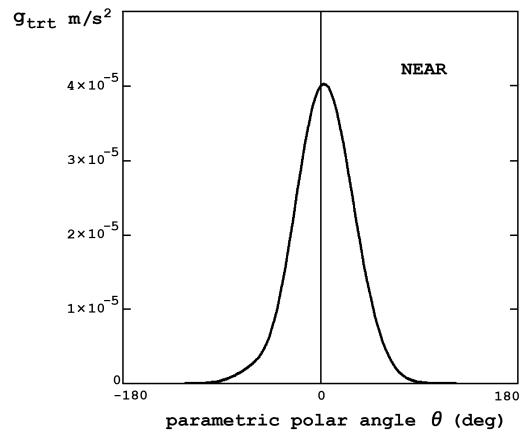
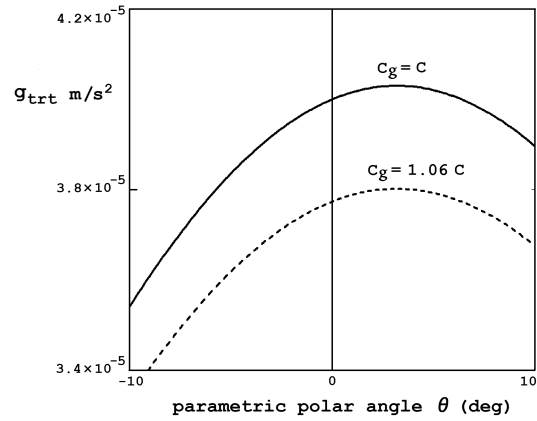Fig. 6: Time-retarded transverse gravitational field $g_{\mathrm{trt}}(\theta)$ versus θ for the NEAR flyby using $c_{\mathrm{g}} = 1.000\,c$.

Fig. 7: Expanded view near the peak for $g_{\mathrm{trt}}(\theta)$ versus θ for the NEAR flyby using $c_{\mathrm{g}} = 1.000\,c$ and $c_{\mathrm{g}} = 1.060\,c$. There is a 6% difference in the peak values.
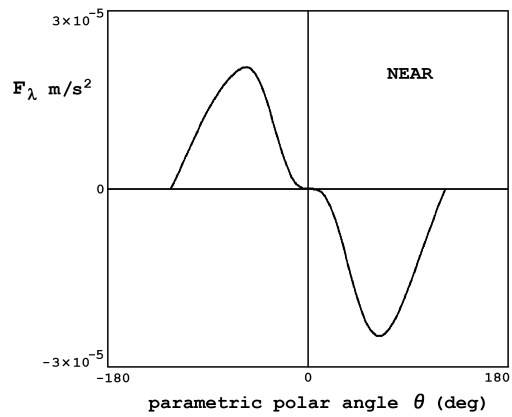
Fig. 8: Graph of the induction field F_λ versus θ for the NEAR flyby with $v_{\mathrm{k}}/v_{\mathrm{eq}} = 6.530$. During the inbound there is a positive peak and during the outbound there is a slightly stronger negative peak.

speed-change. A graph of F_λ versus θ, Fig. 8, shows a positive peak during the inbound and a slightly stronger negative peak during the outbound.

The calculated speed-change is given by (3.55)

$$\delta v\,(\theta) = \frac{v_{\text{in}}}{2} \int_0^\theta \frac{r_\lambda(\theta)\,F_\lambda(\theta)}{v_{\text{in}}^2} \frac{d\lambda}{d\theta}\, d\theta \,, \tag{4.23}$$

where $v_{\text{in}} = v\,(\theta_{\text{min}}) = 1.416\,v_\infty$ (from Table 5), $r_\lambda(\theta)$ is given by (3.36), $F_\lambda(\theta)$ is given by (4.22), and $d\lambda/d\theta$ is found by using the numerical differentiation algorithm in Mathcad15 for the derivative of the latitude $\lambda(\theta)$ given by (4.12).

For the NEAR flyby, $\theta_{\text{min}} = -123.1192°$ and $\theta_{\text{max}} = +123.1437°$, given by (4.14). Then by (3.56),

$$\left.\begin{aligned}
\delta v_{\text{in}} &= \delta v\,(\theta_{\text{min}}) = -15.9577 \text{ mm/s} \\
\delta v_{\text{out}} &= \delta v\,(\theta_{\text{max}}) = +29.4184 \text{ mm/s} \\
\delta v_{\text{trt}} &= \delta v_{\text{in}} + \delta v_{\text{out}} = +13.4607 \text{ mm/sec}
\end{aligned}\right\}. \tag{4.24}$$

The observed speed-change for the NEAR flyby (Appendix A) is

$$\delta v_{\text{obs}} = (+13.46 \pm 0.01) \text{ mm/s}. \tag{4.25}$$

The calculated value δv_{trt} equals exactly the observed value δv_{obs} if

$$\delta v_{\text{k}} = (6.530 \pm 0.005)\,v_{\text{eq}}\,, \qquad \text{with } c_{\text{g}} = 1.000\,c. \tag{4.26}$$

Repeating the calculation with $c_{\text{g}} = 1.060\,c$ requires a slightly smaller value for v_{k} to make $\delta v_{\text{trt}} = \delta v_{\text{obs}}$,

$$v_{\text{k}} = (6.160 \pm 0.005)\,v_{\text{eq}}\,, \qquad \text{if } c_{\text{g}} = (1.060 \pm 0.001)\,c. \tag{4.27}$$

If the "true" value for v_{k} were known with a precision of 1 part in a thousand, 0.1%, this calculation for the NEAR flyby speed-change would provide a first-ever measured value for the Earth's speed of gravity!

Results for all six flybys using the parameter values of Table 3 and Table 5 are listed in Table 1. Table 1 lists the observed anomalous speed change, δv_{obs}, with the reported uncertainty, the calculated time-retarded speed change, δv_{trt}, with the corresponding uncertainty, the ratio gravity-speed/light-speed that was used in the calculation, c_{g}/c, the required relative induction speed, $v_{\text{k}}/v_{\text{eq}}$, with the corresponding uncertainty, and the calculated value for the eccentricity for the trajectory, ε.

§5. Anomalous decrease in the Moon's orbital speed caused by the neoclassical causal version of Newton's theory. Numerical values for the Earth's parameters, M_E, r_E, Ω_E, I_E, and v_{eq}, are listed in the Appendix B. Needed numerical values for the Moon, M_M, r_p, r_a, ε, and α_{eq}, are also listed in the Appendix B.

Let $r(\theta)$ be the radial distance from the center of the Earth to the center of the Moon

$$\left.\begin{array}{l} r(\theta) = \dfrac{r_p\,(1+\varepsilon)}{1+\varepsilon\cos\theta} \\[3mm] \dfrac{dr}{d\theta} = \dfrac{r(\theta)^2}{r_p}\,\dfrac{\varepsilon}{1+\varepsilon}\,\sin\theta \end{array}\right\}. \tag{5.1}$$

Let $rM(\theta)$ be the radial distance from the origin of a barycentric frame to the Moon, and let rM_a and rM_p be the value for rM at apogee and at perigee. Let a_M and b_M be the semimajor and semiminor axes for the Moon's elliptical orbit

$$\left.\begin{array}{l} rM(\theta) = \dfrac{M_E}{M_E + M_M}\,r(\theta) = 0.9879\,r(\theta) \\[4mm] \dfrac{drM}{d\theta} = \dfrac{M_E}{M_E + M_M}\,\dfrac{dr}{d\theta} \\[4mm] rM_a = \dfrac{M_E}{M_E + M_M}\,r_a = 62.905\,r_E \\[4mm] rM_p = \dfrac{M_E}{M_E + M_M}\,r_p = 56.301\,r_E \\[4mm] a_M = \dfrac{1}{2}\,(rM_a + rM_p) = 59.603\,r_E \\[4mm] b_M = a_M\sqrt{1-\varepsilon^2} = 59.511\,r_E \end{array}\right\}. \tag{5.2}$$

By Kepler's 3rd law, (3.45), the calculated lunar period PM is

$$PM = \dfrac{2\pi\,a_M^{3/2}}{\sqrt{G\,(M_E + M_M)}} = 26.78 \text{ days.} \tag{5.3}$$

By Kepler's 2nd law, (3.45), the orbital angular speed is

$$\Omega M(\theta) = \dfrac{2\pi}{PM}\,\dfrac{a_M b_M}{rM(\theta)^2}\,. \tag{5.4}$$

Let the Moon's orbital speed at perigee be vM_p and at apogee be

vM_a. Numerical values are

$$\left.\begin{array}{l} vM_p = rM_p\Omega M(0) = 10.90\times10^2 \text{ m/s} \\ vM_a = rM_a\Omega M(-\pi) = 9.755\times10^2 \text{ m/s} \end{array}\right\}. \tag{5.5}$$

Let v_{co} and Ω_{co} be the orbital speed and orbital angular speed for an equivalent circular orbit which has the radius a_M and the period PM (3.46). We then have, respectively,

$$\left.\begin{array}{l} v_{co} = \sqrt{\dfrac{G\left(M_E + M_M\right)}{a_M}} = 1.031\times10^3 \text{ m/s} \\[2mm] \Omega_{co} = \dfrac{v_{co}}{a_M} = 2.715\times10^{-6} \text{ rad/s} \end{array}\right\}. \tag{5.6}$$

Let $\delta v_{co} \ll v_{co}$ be a small change in the orbital speed, let $\delta a_M \ll a_M$ be the corresponding change in the radius of the orbit, and let $\delta\Omega_{co} \ll \Omega_{co}$ be the corresponding change in the angular speed. Then

$$\left.\begin{array}{l} v_{co}^2 = \dfrac{\text{constant}}{a_M} \\[3mm] \left(1 + \dfrac{\delta v_{co}}{v_{co}}\right)^2 \cong 1 + 2\dfrac{\delta v_{co}}{v_{co}} = \dfrac{1}{1 + \delta a_M/a_M} \cong 1 - \dfrac{\delta a_M}{a_M} \\[3mm] 2\dfrac{\delta v_{co}}{v_{co}} \cong \dfrac{\delta a_M}{a_M} \\[3mm] \dfrac{\delta\Omega_{co}}{\Omega_{co}} \cong \dfrac{\delta v_{co}}{v_{co}} - \dfrac{\delta a_M}{a_M} = -\dfrac{\delta v_{co}}{v_{co}} \end{array}\right\}. \tag{5.7}$$

According to Stephenson and Morrison, tidal braking increases the LOD by 23×10^{-6} seconds per year [15]. Let $\delta LOD \ll LOD$ be this change in the LOD

$$\left.\begin{array}{l} LOD = 60 \times 60 \times 24 = 86400 \text{ s} \\ \delta LOD = 23\times10^{-6} \text{ s per year} \end{array}\right\}. \tag{5.8}$$

The Earth's sidereal rotational period in seconds is

$$\frac{2\pi}{\Omega_E} = 86164.1 \text{ s.} \tag{5.9}$$

Therefore,

$$LOD = \frac{2\pi}{\Omega_E} 1.002738 = 86400 \text{ s.} \tag{5.10}$$

Let $\delta\Omega_E$ be a small change in Ω_E. Then

$$\left.\begin{aligned}
1 + \frac{\delta LOD}{LOD} &= \frac{1}{1 + \delta\Omega_E/\Omega_E} \cong 1 - \frac{\delta\Omega_E}{\Omega_E} \\
\delta\Omega_E &= -\Omega_E \frac{\delta LOD}{LOD} = -1.941 \times 10^{-14} \text{ rad/s per year}
\end{aligned}\right\}. \quad (5.11)$$

Let S_E be the magnitude for the Earth's spin angular momentum, and let δS_E be a small change in S_E. Assume there is no change in I_E. Then we have

$$\left.\begin{aligned}
S_E &= I_E\Omega_E = 5.851 \times 10^{33} \text{ kg} \times \text{m}^2/\text{s} \\
\delta S_E &= S_E \frac{\delta\Omega_E}{\Omega_E} = -1.558 \times 10^{24} \text{ kg} \times \text{m}^2/\text{s per year}
\end{aligned}\right\}. \quad (5.12)$$

Let L_M be the magnitude for the Moon's orbital angular momentum, and let δL_M be a small change in L_M. By conservation of the Earth's spin angular momentum and the Moon's orbital angular momentum,

$$\left.\begin{aligned}
L_M &= M_M v_{co} a_M = 2.877 \times 10^{34} \text{ kg} \times \text{m}^2/\text{s} \\
\frac{\delta L_M}{L_M} &= \frac{\delta v_{co}}{v_{co}} + \frac{\delta a_M}{a_M} = 3\frac{\delta v_{co}}{v_{co}} \\
L_M + S_M &= \text{constant} \\
\delta L_M &= -\delta S_E = +1.558 \times 10^{24} \text{ kg} \times \text{m}^2/\text{s per year}
\end{aligned}\right\}. \quad (5.13)$$

The resulting change in the orbital speed is

$$\delta v_{co} = \frac{v_{co}}{3}\frac{\delta L_M}{L_M} = +18.6 \times 10^{-9} \text{ m/s per year.} \quad (5.14)$$

Equation (5.5) gives

$$\delta a_M = 2a_M \frac{\delta v_{co}}{v_{co}} = +13.7 \times 10^{-3} \text{ m per year.} \quad (5.15)$$

This shows that tidal braking alone causes an increase in the radius of 14 mm per year, and a corresponding increase in the orbital speed of 19×10^{-9} m/s per year.

But lunar-laser-ranging experiments have shown that the radius is actually increasing by [16]

$$\delta a_M = +38 \text{ mm per year.} \quad (5.16)$$

The corresponding increase in the orbital speed (5.7) takes the following numerical value

$$\delta v_{\mathrm{co}} = \frac{v_{\mathrm{co}}}{2} \frac{\delta a_{\mathrm{M}}}{a_{\mathrm{M}}} = +51.6 \times 10^{-9} \text{ m/s per year.} \qquad (5.17)$$

There is an obvious difference! An unexplained hidden action is causing the rate for change in the radius to decrease from 38 to 14 mm per year (-24 mm per year). The corresponding rate for change in the orbital speed is decreased from 52×10^{-9} to 19×10^{-9} m/s per year (-33×10^{-9} m/s per year). This unexplained difference is the "lunar orbit anomaly".

The lunar orbit anomaly can be explained exactly by the neoclassical causal theory. Let α_{eq} be the average inclination of the Moon's orbital plane (Appendix B)

$$\alpha_{\mathrm{eq}} = 23° \pm 5°. \qquad (5.18)$$

Let the latitude for perigee, λ_{p}, equal α_{eq}. Then by (3.47),

$$\lambda M(\theta) = \tan^{-1}\left(\tan \alpha_{\mathrm{eq}} \cos \theta\right), \qquad (5.19)$$

and the ϕ-component of ΩM becomes

$$\Omega_\phi(\theta) = \Omega M(\theta) \cos \alpha_{\mathrm{eq}}. \qquad (5.20)$$

By (3.47), the formula for the λ-component of $r M$ becomes

$$r_\lambda(\theta) = r M(\theta) \cos \theta. \qquad (5.21)$$

The formula for the Earth's transverse field at the Moon with $c_{\mathrm{g}} = c$ (3.28) is

$$g_{\mathrm{trt}}(\theta) = -G \frac{I_{\mathrm{E}}}{r_{\mathrm{E}}^4} \frac{v_{\mathrm{eq}}}{c} \left(\frac{\Omega_\phi(\theta) - \Omega_{\mathrm{E}}}{\Omega_{\mathrm{E}}} \right) \cos^2\left(\lambda M(\theta)\right) PS\left(r(\theta)\right). \qquad (5.22)$$

The formula for $PS(r)$ is given by (3.26).

A trial run gave the following value for the induction speed which gives the observed speed-change

$$v_{\mathrm{k}} = 7.94 \, v_{\mathrm{eq}}. \qquad (5.23)$$

The formula for the transverse induction-like field (3.53) becomes

$$F_\lambda(\theta) = \frac{v_{\mathrm{eq}}}{v_{\mathrm{k}}} \frac{r_{\mathrm{E}}}{r(\theta)} \int_0^\theta \frac{r(\theta)}{r_{\mathrm{E}}} \frac{\Omega M(\theta)}{\Omega_{\mathrm{E}}} \frac{1}{r_{\mathrm{E}}} \frac{dr}{d\theta} \frac{dg_{\mathrm{trt}}}{d\theta} \, d\theta. \qquad (5.24)$$
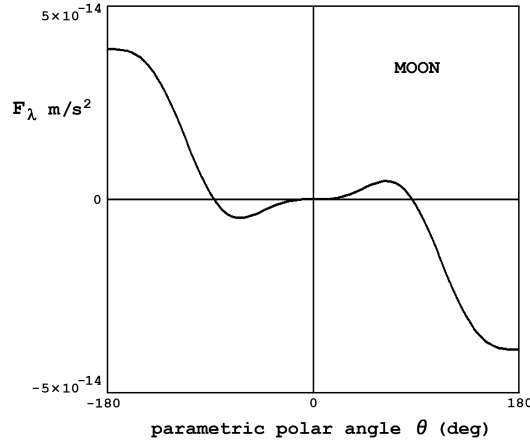
Fig. 9: Induction-like field F_λ versus the parametric angle θ for the Moon with $v_{\rm k} = 7.94\, v_{\rm eq}$.

The derivative $dg_{\rm trt}/d\theta$ is found by using the differentiation algorithm in Mathcad15. A graph of F_λ versus θ for the Moon, Fig. 9, shows that there is asymmetry about $\theta = 0$.

The formula for the speed change, (3.55), becomes

$$\delta v(\theta) = \frac{v M_{\rm a}}{2} \int_0^\theta \frac{r_\lambda(\theta)\, F_\lambda(\theta)}{v M_{\rm a}^2} \frac{d\lambda}{d\theta}\, d\theta\,. \tag{5.25}$$

By numerical differentiation and integration,

$$\left.\begin{aligned}
\delta v_{\rm in} &= -\delta v(-\pi) = -1.21 \times 10^{-9}\ \text{m/s} \\
\delta v_{\rm out} &= +\delta v(+\pi) = -1.21 \times 10^{-9}\ \text{m/s} \\
\delta v_{\rm trt} &= \delta v_{\rm in} + \delta v_{\rm out} = -2.42 \times 10^{-9}\ \text{m/s per revolution}
\end{aligned}\right\}. \tag{5.26}$$

Let $N_{\rm rev}$ be the number of lunar revolutions per year, let yr be the number of seconds in a year, and let $\delta v M$ be the accumulated orbital speed-change per year. Then

$$\left.\begin{aligned}
N_{\rm rev} &= yr/PM = 13.64 \\
\delta v M &= N_{\rm rev}\delta v_{\rm trt} = -33.0 \times 10^{-9}\ \text{m/s per year}
\end{aligned}\right\}. \tag{5.27}$$

This shows that, with $v_{\rm k} = 7.94\, v_{\rm eq}$, the calculated value for the Moon's orbital speed-change is -33×10^{-9} m/s per year, which explains exactly

the lunar orbit anomaly. The final value for v_k with uncertainty reduces to

$$\frac{v_k}{v_{eq}} = 8 \pm 1. \tag{5.28}$$

§6. Predicted annual speed-change for spacecrafts in highly eccentric and inclined near-Earth orbits.

The speed-change caused by the causal version of Newton's theory depends on the speed of propagation of the gravitational field, c_g, the properties of the central sphere; M_E, r_E, Ω_E, I_E, and v_{eq}, the orbital properties of the spacecraft; r_p, ε, α_{eq}, and λ_p, and the induction speed, v_k. If $\varepsilon = 0$, the speed-change $\delta v_{trt} = 0$, regardless of the value for α_{eq}. If $\alpha_{eq} = 0$, the speed-change $\delta v_{trt} = 0$, regardless of the value for ε. Even if both ε and α_{eq} are not zero, the speed-change is still zero if perigee is over the equator or one of the poles. The maximum speed-change occurs for spacecrafts with highly eccentric and inclined near-Earth orbits, such as with the inclination $\alpha_{eq} = 45°$ and the latitude at perigee $\lambda_p = 45°$.

Suppose the orbital properties for a spacecraft are $\varepsilon = 0.5$, $\alpha_{eq} = 45°$, and $\lambda_p = 45°$. Let r_p range from $2r_E$ to $8r_E$. Shown below are the numerical values for $r_p = 2r_E$. The period is given by Kepler's 3rd law (3.45)

$$P = \frac{2\pi a^{3/2}}{\sqrt{GM_E}} = 11.2 \text{ hours}, \tag{6.1}$$

where

$$\left.\begin{array}{l} r_a = r_p \dfrac{1+\varepsilon}{1-\varepsilon} = 6r_E \\[2mm] a = \dfrac{1}{2}\left(r_a + r_p\right) = 4r_E \\[2mm] b = a\sqrt{1-\varepsilon^2} = 3.464\,r_E \end{array}\right\}. \tag{6.2}$$

The formula for Ω_θ is given by (3.45)

$$\Omega_\theta(\theta) = \frac{2\pi}{P}\,\frac{ab}{r(\theta)^2}\,, \tag{6.3}$$

where

$$\left.\begin{array}{l} r(\theta) = \dfrac{r_p(1+\varepsilon)}{1+\varepsilon\cos\theta} \\[3mm] \dfrac{dr}{d\theta} = \dfrac{r(\theta)^2}{r_p}\,\dfrac{\varepsilon}{1+\varepsilon}\,\sin\theta \end{array}\right\}. \tag{6.4}$$

Let Ω_a be the spacecraft's orbital angular speed at apogee and let

v_{a} be the orbital speed at apogee. Then

$$\left.\begin{aligned}\Omega_{\mathrm{a}} &= \Omega_\theta(-\pi) = 0.819\,\Omega_{\mathrm{E}} \\ v_{\mathrm{a}} &= \Omega_{\mathrm{a}} r_{\mathrm{a}} = 4.92\,v_{\mathrm{eq}}\end{aligned}\right\}. \qquad (6.5)$$

If the latitude for perigee $\lambda_{\mathrm{p}} = \alpha_{\mathrm{eq}}$, and if the value for θ at perigee is zero, then by (3.47)

$$\lambda(\theta) = \tan^{-1}\left(\tan\alpha_{\mathrm{eq}} \cos\theta\right), \qquad (6.6)$$

and the ϕ-component of Ω_θ becomes

$$\left.\begin{aligned}\Omega_\phi(\theta) &= \Omega_\theta(\theta)\cos\alpha_{\mathrm{eq}} \quad \text{for prograde orbits} \\ \Omega r\phi(\theta) &= -\Omega_\phi(\theta) \qquad\quad \text{for retrograde orbits}\end{aligned}\right\}. \qquad (6.7)$$

The projection of "prograde" orbits onto the equatorial plane revolves in the same direction as the Earth's spin, and the projection of "retrograde" orbits onto the equatorial plane revolves opposite to the direction of the Earth's spin.

The formula for the λ-component of r becomes

$$r_\lambda(\theta) = r(\theta)\cos\theta. \qquad (6.8)$$

The formula for the Earth's time-retarded transverse field with $c_{\mathrm{g}} = c$, (3.28), is

$$g_{\mathrm{trt}}(\theta) = -G\,\frac{I_{\mathrm{E}}}{r_{\mathrm{E}}^4}\,\frac{v_{\mathrm{eq}}}{c}\left(\frac{\Omega_\phi(\theta) - \Omega_{\mathrm{E}}}{\Omega_{\mathrm{E}}}\right)\cos^2\!\big(\lambda(\theta)\big)\,PS\big(r(\theta)\big), \qquad (6.9)$$

where $PS(r)$ is given by (3.26).

The "true" value for v_{k} probably lies between $10\,v_{\mathrm{eq}}$ and $14\,v_{\mathrm{eq}}$ (1.16). To minimize the predicted speed-change, choose its maximum probable value, $v_{\mathrm{k}} = 14\,v_{\mathrm{eq}}$. The formula for the induction-like field is given by (3.53)

$$F_\lambda(\theta) = \frac{v_{\mathrm{eq}}}{v_{\mathrm{k}}}\,\frac{r_{\mathrm{E}}}{r(\theta)}\int_0^\theta \frac{r(\theta)}{r_{\mathrm{E}}}\,\frac{\Omega_\theta(\theta)}{\Omega_{\mathrm{E}}}\,\frac{1}{r_{\mathrm{E}}}\,\frac{dr}{d\theta}\,\frac{dg_{\mathrm{trt}}}{d\theta}\,d\theta. \qquad (6.10)$$

A graph of F_λ for $r_{\mathrm{p}} = 2r_{\mathrm{E}}$ with $v_{\mathrm{k}} = 14\,v_{\mathrm{eq}}$ is shown in Fig. 10.

The speed-change for one period is given by (3.56)

$$\left.\begin{aligned}\delta v(\theta) &= \frac{v_{\mathrm{a}}}{2}\int_0^\theta \frac{r_\lambda(\theta)\,F_\lambda(\theta)}{v_{\mathrm{a}}^2} \\ \delta v_{\mathrm{trt}} &= \delta v_{\mathrm{in}} + \delta v_{\mathrm{out}} = -\delta v(-\pi) + \delta v(+\pi) = \\ &\qquad\qquad = 28.3 \text{ mm/s per revolution}\end{aligned}\right\}. \qquad (6.11)$$
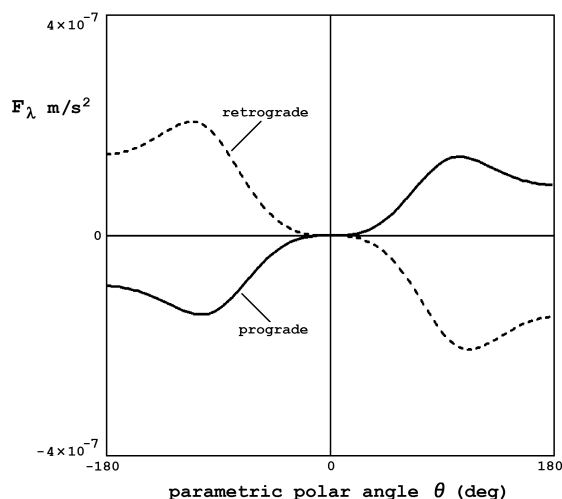
Fig. 10: Induction-like field F_λ versus the parametric angle θ for a spacecraft in a near-Earth orbit with $\varepsilon = 0.5$, $\alpha_{eq} = 45°$, $\lambda_p = 45°$, and $r_p = 2r_E$ with $v_k = 14v_{eq}$. The solid curve is for a prograde orbit and the dashed curve is for a retrograde orbit.

Let N_{rev} be the number of revolutions in one year, let δv_{yr} be the total speed-change accumulated during one year, and let yr be the number of seconds in a year (P is the period in seconds)

$$\left. \begin{array}{l} N_{rev} = yr/P = 780 \text{ revolutions per year} \\ \delta v_{yr} = N_{rev}\,\delta v_{trt} = 315 \text{ mm/s per year} \end{array} \right\}. \tag{6.12}$$

The resulting calculated periods and speed-changes for r_p ranging from $2r_E$ to $8r_E$ are listed in Table 2.

§7. Is there a conflict between the neoclassical causal theory and general relativity theory?   The only possible case where there could be a conflict is the excess for the advance in the perihelion of Mercury [21]. This section shows that the Sun's time-retarded transverse gravitational field causes a change in the angle for Mercury's perihelion of less than 0.04 arc seconds per century, which is negligibly less than the relativistic advance of 43 arc seconds per century and therefore is undetectable.

Let M_S, r_S, Ω_S, I_S, and v_{eq} be the Sun's mass, radius, spin angular speed, moment of inertia, and equatorial surface speed. Numerical

values for the Sun are listed in Appendix B. A 4-term power series approximation for the triple integral over the Sun's volume also can be found in Appendix B.

Let r_a and r_p be Mercury's heliocentric radial distance at aphelion and at perihelion, let ε be the eccentricity, let P_M be the observed sidereal orbital period, let α_{eq} be the inclination to the Sun's equatorial plane, and let λ_p be the heliocentric latitude at perihelion. Numerical values from Appendix B are

$$\left.\begin{aligned}
r_a &= 69816900 \times 10^3 \text{ m} \\
r_p &= 46001200 \times 10^3 \text{ m} \\
\varepsilon &= 0.205630 \\
P_M &= 87.969 \text{ days} = 7.6005 \times 10^6 \text{ s} \\
\alpha_{eq} &= 3.38° \\
\lambda_p &= 3.38°
\end{aligned}\right\}. \tag{7.1}$$

The semimajor and semiminor axes are

$$\left.\begin{aligned}
a_M &= \frac{1}{2}\left(r_a + r_p\right) = 5.791 \times 10^{10} \text{ m} \\
b_M &= a_M \sqrt{1 - \varepsilon^2} = 5.667 \times 10^{10} \text{ m}
\end{aligned}\right\}. \tag{7.2}$$

The calculated period given by Kepler's 3rd law is

$$P = \frac{2\pi a_M^{3/2}}{\sqrt{GM_S}} = 7.5998 \times 10^6 \text{ s} = 1.01\, P_M. \tag{7.3}$$

The formula for Ω_θ, given by (2.28), is

$$\Omega_\theta(\theta) = \frac{2\pi}{P}\frac{a_M\, b_M}{r(\theta)^2}, \tag{7.4}$$

where

$$\left.\begin{aligned}
r(\theta) &= \frac{r_p\,(1+\varepsilon)}{1+\varepsilon\cos\theta} \\
\frac{dr}{d\theta} &= \frac{r(\theta)^2}{r_p}\frac{\varepsilon}{1+\varepsilon}\,\sin\theta
\end{aligned}\right\}. \tag{7.5}$$

Let Ω_a be Mercury's orbital angular speed at aphelion and let v_a be the orbital speed at aphelion. Then

$$\left.\begin{aligned}
\Omega_a &= \Omega_\theta(-\pi) = 5.559 \times 10^{-7} \text{ rad/s} \\
v_a &= \Omega_a r_a = 3.881 \times 10^4 \text{ m/s}
\end{aligned}\right\}. \tag{7.6}$$

Mercury's heliocentric latitude is

$$\lambda(\theta) = \tan^{-1}\left(\tan\alpha_{\text{eq}}\cos\theta\right), \tag{7.7}$$

and the ϕ-component of Ω_θ is

$$\Omega_\phi(\theta) = \Omega_\theta(\theta)\cos\alpha_{\text{eq}}. \tag{7.8}$$

The λ-component of r is

$$r_\lambda(\theta) = r(\theta)\cos\theta. \tag{7.9}$$

The formula for the Sun's time-retarded transverse field with $c_{\text{g}} = c$ substituted is

$$g_{\text{trt}}(\theta) = -G\frac{I_{\text{S}}}{r_{\text{S}}^4}\frac{v_{\text{eq}}}{c}\left(\frac{\Omega_\phi(\theta)-\Omega_{\text{S}}}{\Omega_{\text{S}}}\right)\cos^2\left(\lambda(\theta)\right)PS\left(r(\theta)\right). \tag{7.10}$$

The average value for v_{k} is $10\,v_{\text{eq}}$. The formula for the induction-like field becomes

$$F_\lambda(\theta) = \frac{v_{\text{eq}}}{v_{\text{k}}}\frac{r_{\text{S}}}{r(\theta)}\int_0^\theta \frac{r(\theta)}{r_{\text{S}}}\frac{\Omega_\theta(\theta)}{\Omega_{\text{S}}}\frac{1}{r_{\text{S}}}\frac{dr}{d\theta}\frac{dg_{\text{trt}}}{d\theta}\,d\theta. \tag{7.11}$$

By numerical differentiation and integration, the speed-change becomes

$$\delta v_{\text{trt}} = -4.71\times10^{-7}\text{ m/s per revolution.} \tag{7.12}$$

Numerical values for the orbital speed v_{co} and the angular speed Ω_{co} for an equivalent circular orbit for Mercury, by (3.46), are

$$\left.\begin{array}{c} v_{\text{co}} = \sqrt{\dfrac{GM_{\text{S}}}{a_{\text{M}}}} = 4.788\times10^4\text{ m/s} \\[4mm] \Omega_{\text{co}} = \dfrac{v_{\text{co}}}{a_{\text{M}}} = 8.268\times10^{-7}\text{ rad/s} \end{array}\right\}. \tag{7.13}$$

Let θ_{p} be the value for θ at perihelion, let $\delta\theta_{\text{p}}$ be the change in θ_{p} per revolution, and set $\delta v_{\text{co}} = \delta v_{\text{trt}}$. Then

$$\left.\begin{array}{c} \dfrac{\delta\theta_{\text{p}}}{2\pi} = \dfrac{\delta\Omega_{\text{co}}}{\Omega_{\text{co}}} = 3\dfrac{\delta v_{\text{co}}}{v_{\text{co}}} = 3\dfrac{\delta v_{\text{trt}}}{v_{\text{co}}} \\[4mm] \delta\theta_{\text{p}} = 6\pi\dfrac{\delta v_{\text{trt}}}{v_{\text{co}}} = -1.86\times10^{-10}\text{ rad per revolution} \end{array}\right\}. \tag{7.14}$$

Let N_{rev} be the number of Mercury's revolutions in one year, let $\Delta\theta_{\text{p}}$ be the accumulated angular change of Mercury during one year, and let

yr be the number of seconds in a year

$$\left.\begin{aligned}
N_{\mathrm{rev}} &= \frac{yr}{P_{\mathrm{M}}} = 4.125 \\[2mm]
\Delta\theta_{\mathrm{p}} &= N_{\mathrm{rev}}\delta\theta_{\mathrm{p}} = -7.71\times10^{-9} \text{ rad per year} \\[2mm]
\Delta\theta_{\mathrm{p}} &\times \frac{180}{\pi} \times 60 \times 60 \times 100 = -0.016 \text{ arc sec per century}
\end{aligned}\right\} . \quad (7.15)$$

Thus we find that the absolute magnitude for the change in the angle for perihelion is less than 0.04 arc seconds per century, which is totally negligible compared with the relativistic change of 43 arc seconds per century.

§8. Conclusions and recommendations. There is here within conclusive evidence that the proposed neoclassical causal version of Newton's theory agrees with the facts-of-observation to the extent that such facts are currently available. The proposed causal version is a natural rational extension of Newton's acausal theory. It applies only for slow-speeds and weak-fields, i.e., for $v^2 \ll c^2$ and $GM/r \ll c^2$. Effects of time retardation appear at the relatively large first-order v/c_{g} level, but they are normally very small and are previously undetected because they decrease inversely with the cube of the bypass distance. If the bypass is very close, however, time retardation effects can be relatively large. It is recommended that future research projects utilize various available methods to detect new first-order effects of the causality principle.

Appendix A. Parameter values for six Earth flybys
Table A1 lists needed parameter values which can be found in the report by Anderson *et al.* [10]. The symbols are changed to be those used for this article.

   The start time for the incoming and the end time for the outgoing data intervals for the NEAR flyby are stated in the caption for Fig. 3 of the report

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
h_p (km)	539	960	1956	2347	1175	303
λ_p (deg)	+33.0	+25.2	+20.20	+46.95	−23.5	−33.8
v_p (km/s)	12.739	13.740	10.517	10.389	19.026	14.080
v_∞ (km/s)	6.851	8.949	3.863	4.056	16.010	8.877
DA (deg)	66.9	47.7	99.3	94.7	19.7	51.1
α_eq (deg)	+108.0	+142.9	+144.9	+133.1	+25.4	+138.7
λ_in (deg)	+20.76	+12.52	+2.81	−31.44	+12.92	+34.26
λ_out (deg)	−71.96	−34.15	−34.29	−31.92	−4.99	−4.87
δv_obs (mm/s)	+13.46 ±0.01	+3.92 ±0.3	+1.80 ±0.03	+0.02 ±0.01	−2 ±1	−4.6 ±1
δv_emp (mm/s)	+13.28	+4.12	+2.07	+0.06	−1.07	−4.67

Table A1: Earth flyby parameter values for the NEAR, Galileo-I, Rosetta, MESSENGER (M'GER), Cassini, and Galileo-II spacecraft flybys. The altitude at perigee h_p is referenced to the Earth geoid, λ_p is the geocentric latitude at perigee, v_p is the magnitude of the spacecraft's inertial velocity at perigee, v_∞ is the magnitude for the osculating hyperbolic excess velocity, DA is the deflection angle between the incoming and outgoing asymptotic velocity vectors, α_eq is the inclination of the orbital plane to the Earth's equatorial plane, λ_in and λ_out are the geocentric latitudes for the incoming and outgoing osculating asymptotic velocity vectors, and δv_obs is the measured change in the spacecraft's orbital speed with an estimated realistic uncertainty for the measured value. The last row gives the calculated speed-change values from the empirical prediction formula δv_emp.

by Anderson *et al.*

$$t_\text{in} = -88.4 \text{ hours}, \qquad t_\text{out} = +95.6 \text{ hours}. \tag{A.1}$$

The data time intervals for the other flybys were not given.

Anderson *et al.* report the asymptotic flyby "declinations" instead of the asymptotic geocentric latitudes. From Fig. 1 of their report, there is no doubt that the inbound asymptotic latitude λ_in is positive for the NEAR flyby (+ for northern latitudes) and the outbound asymptotic latitude λ_out is negative (− for southern latitudes). This recognition for the correct signs is applied in Table A1.

Notice in Table A1 that both of the flybys which have negative speed-changes (the flybys in the case of Cassini and GLL-II) have negative values for the latitude at perigee λ_p.

Appendix B. Various numerical values and radial mass-density distributions.

Various numerical values are needed to evaluate the formulas for the transverse gravitational field. The following values were found in [22, 23]:

$G = 6.6732 \times 10^{-11} \text{ m}^3/\text{kg} \times \text{s}^2$	Gravity constant,
$c = 2.997925 \times 10^8 \text{ m/s}$	Vacuum speed of light,
$\Omega_{\text{E}} = 7.292115 \times 10^{-5} \text{ rad/s}$	Earth's sidereal angular speed,
$M_{\text{E}} = 5.9761 \times 10^{24} \text{ kg}$	Earth's total mass,
$r_{\text{E}} = 6,371,034 \text{ m}$	Earth's equivalent spherical radius,
$v_{\text{E}} = r_{\text{E}}\Omega_{\text{E}} = 464.58 \text{ m/s}$	Earth's equatorial surface speed,
$V_{\text{E}} = 1.08322 \times 10^{21} \text{ m}^3$	Earth's volume,
$\bar{\rho}_{\text{E}} = 5.517 \times 10^3 \text{ kg/m}^3$	Earth's mean mass-density,
$I_{\text{E}} = 8.0238 \times 10^{37} \text{ kg/m}^2$	Earth's spherical moment of inertia,
$I_{\text{E}}/\bar{\rho}\, r_{\text{E}}^5 = 1.3856$	Unitless ratio for the moment of inertia.

The Earth's interior consists of four major regions: inner core, outer core, mantle, and crust [22]. The formula for the radial mass-density distribution, derived from seismic data, is

$$\rho\left(r'\right) = \text{if}\left(r' < r_{\text{ic}}, \rho_{\text{ic}}, \text{if}\left(r' < r_{\text{oc}}, \rho_{\text{oc}}(r'), \text{if}\left(r' < r_{\text{man}}, \rho_{\text{man}}(r'), \rho_{\text{cst}}(r')\right)\right)\right),$$

where (radii are in meters and densities are in kg/m^3)

$$r_{\text{ic}} = 1230 \times 10^3,$$

$$\rho_{\text{ic}} = 13 \times 10^3,$$

$$r_{\text{oc}} = 3486 \times 10^3,$$

$$\rho_{\text{oc}}(r') = 12 \times 10^3 + 2.0 \times 10^3 \left(\frac{r_{\text{ic}} - r'}{r_{\text{oc}} - r_{\text{ic}}}\right) - 0.6 \times 10^3 \left(\frac{r_{\text{ic}} - r'}{r_{\text{oc}} - r_{\text{ic}}}\right)^2,$$

$$r_{\text{man}} = 6321 \times 10^3,$$

$$\rho_{\text{man}}(r') = 5.75 \times 10^3 + 0.4 \times 10^3 \left(\frac{r_{\text{oc}} - r'}{r_{\text{man}} - r_{\text{oc}}}\right) - 2.05 \times 10^3 \left(\frac{r_{\text{oc}} - r'}{r_{\text{man}} - r_{\text{oc}}}\right)^2,$$

$$r_{\text{cst}} = r_{\text{E}} = 6378 \times 10^3,$$

$$\rho_{\text{cst}}(r') = 3.3 \times 10^3 + 0.6 \times 10^3 \left(\frac{r_{\text{man}} - r'}{r_{\text{cst}} - r_{\text{man}}}\right) - 0.5 \times 10^3 \left(\frac{r_{\text{man}} - r'}{r_{\text{cst}} - r_{\text{man}}}\right)^2.$$

The following numerical values for the Moon are taken from [24]:

$M_{\text{M}} = 7.3477 \times 10^{22} \text{ kg}$	Moon's mass,
$r_{\text{p}} = 363,104 \times 10^3 \text{ m}$	Moon's radial distance at perigee,
$r_{\text{a}} = 405,696 \times 10^3 \text{ m}$	Moon's radial distance at apogee,

$$\varepsilon = 0.0554 \qquad \text{Eccentricity,}$$

$$P_{\text{sid}} = 27.321582 \text{ days} \qquad \text{Moon's sidereal orbital period,}$$

$$P_{\text{sol}} = 29.530589 \text{ days} \qquad \text{Moon's solar orbital period,}$$

$$\alpha_{\text{eq}} = 23.4° \pm 5.2° \qquad \text{Moon's inclination (average} \pm \text{variation).}$$

The following numerical values for the Sun are taken from [25]:

$$M_{\text{S}} = 1.9891 \times 10^{30} \text{ kg} \qquad \text{Sun's mass,}$$

$$r_{\text{S}} = 6.955 \times 10^{8} \text{ m} \qquad \text{Sun's equatorial radius,}$$

$$\bar{\rho} = 1.408 \times 10^{3} \text{ kg/m}^3 \qquad \text{Sun's mean mass-density,}$$

$$\rho_{\text{ctr}} = 1.622 \times 10^{5} \text{ kg/m}^3 \qquad \text{Mass-density at the center,}$$

$$\rho_{\text{photo}} = 2 \times 10^{-4} \text{ kg/m}^3 \qquad \text{Mass-density at the photosphere,}$$

$$P_{\text{S}} = 25.05 \text{ days} = 2.158 \times 10^{4} \text{ s} \qquad \text{Equatorial rotational period,}$$

$$\Omega_{\text{S}} = 2\pi/P_{\text{S}} = 2.911 \times 10^{-4} \text{ rad/s} \qquad \text{Equatorial angular speed,}$$

$$v_{\text{eq}} = r_{\text{S}}\,\Omega_{\text{S}} = 2.025 \times 10^{3} \text{ m/s} \qquad \text{Equatorial rotational surface speed,}$$

$$I_{\text{S}} = 3.367 \times 10^{46} \text{ kg} \times \text{m}^2 \qquad \text{Sun's moment of inertia,}$$

$$I_{\text{S}}/\bar{\rho}\,r_{\text{S}}^5 = 0.147 \qquad \text{Unitless ratio for the moment of inertia,}$$

$$\alpha_{\text{S}} = 7.25° \qquad \text{Obliquity to the ecliptic.}$$

An exponential function provides a reasonably valid approximation for the Sun's radial mass-density distribution

$$\rho\left(r'\right) = \text{if}\left(r' \leqslant r_{\text{S}}, \ \rho_{\text{ctr}} \exp\left(-\left(\frac{r'}{r_{\text{core}}}\right)^2\right), \ 0\right),$$

where the numerical value for r_{core} is

$$r_{\text{core}} = 0.18707\, r_{\text{S}}\,.$$

The following four-term power series

$$PS(r) = \left(\frac{r_{\text{S}}}{r}\right)^3\left(C_0 + C_2\left(\frac{r_{\text{S}}}{r}\right)^2 + C_4\left(\frac{r_{\text{S}}}{r}\right)^4 + C_6\left(\frac{r_{\text{S}}}{r}\right)^6\right),$$

provides an excellent fit to the triple integral over the Sun's volume with the following values for the coefficients

$$C_0 = 0.500000, \qquad C_2 = 0.017498,$$

$$C_4 = 0.001376, \qquad C_6 = 0.000173.$$

The following numerical values for the planet Mercury are taken from [21]:

$$r_{\text{p}} = 46,001,200 \times 10^{3} \text{ m} \qquad \text{Radial distance at perihelion,}$$

$$r_{\text{a}} = 69,816,900 \times 10^{3} \text{ m} \qquad \text{Radial distance at aphelion,}$$

$$\varepsilon = 0.205630 \qquad \text{Eccentricity,}$$

$P_\mathrm{M} = 87.969$ days   Mercury's sidereal orbital period,

$\alpha_\mathrm{eq} = 3.38°$ Inclination to the Sun's equator,

$\lambda_\mathrm{p} = 3.38°$ Heliocentric latitude at perihelion.

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