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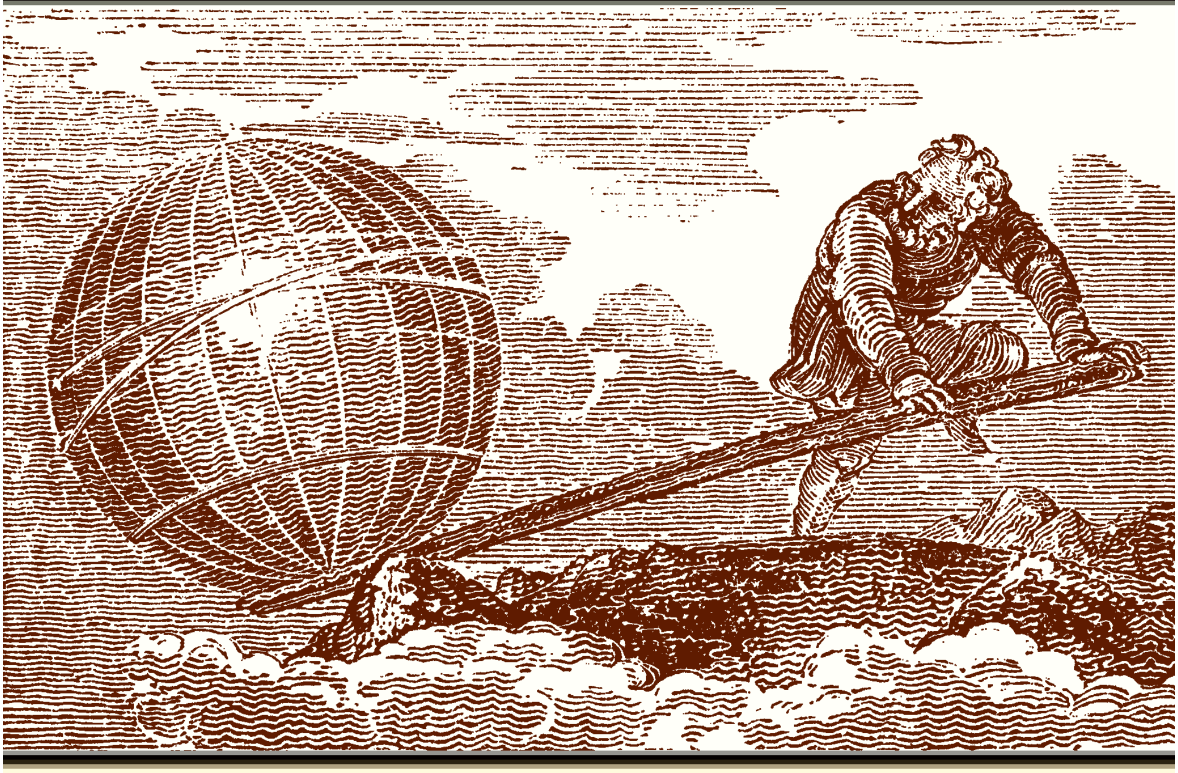

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A Presentation Concerning the Propagation of Light Determined by Monsieur Rømer of the Royal Academy of Sciences

Ole Rømer

December 7, 1676

Abstract: This is a translation of the paper, where the Danish astronomer Ole Rømer presents his original method for measuring the velocity of light. Rømer was the first person to determine a finite velocity for the propagation of light, and determined the velocity with an appropriate precision. This paper was originally published in French: Demonstration touchant le mouvement de la lumiere trouvé par monsieur Römer de l'Academie Royale des Sciences. *Journal des Sçavans*, du lundy, 7 Decembre 1676, pages 233–236. Herein the original Danish transcription of the name of the author is used instead the French, Römer, printed in *Journal des Sçavans*. In this paper an anonymous reporter of *Journal des Sçavans*, who actually wrote the text from Rømer's words, referred Rømer in the third person, according to the academic tone and traditions usual in the 17th century. Translated into English in 2008 by Dmitri Rabounski.

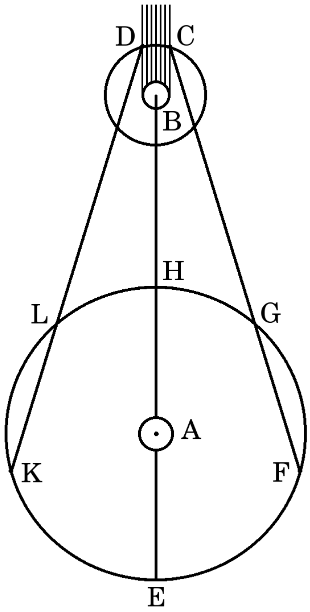
For a long time philosophers were troubled to find an experiment resolving the following problem: is the action of light transferred in an instant at any distance, or does it require some time? Monsieur Rømer of the Royal Academy of Science* found such a method, based on observations of the first satellite of Jupiter. Using this method, he showed that light travels a distance of about 3,000 leagues†, i.e. approximately the diameter of the Earth, in less than one second.

Let A be the Sun, B — Jupiter, C — the first satellite, which moves into the shadow of Jupiter, then appears again from the shadow at the point D, while E, F, G, H, L, K denote the locations of the Earth at different distances from Jupiter (see Figure).

Suppose someone of the Earth, which is located at the point L (near the second quadrature of Jupiter), observes the first satellite of Jupiter at the moment when it appears from the shadow at the point D. Then,

*This is *l'Academie Royale des Sciences*, established in 1666 in Paris by Jean-Baptiste Colbert. The Academy was closed due to the French Revolution of 1789, as well were all the Royal institutions in France. — Editor's comment. D.R.

†Rømer means *la lieue de Paris* (3,898 m) known also as *la nouvelle lieue*. It was introduced in 1674 instead of *l'ancienne lieue* (3,248 m), and remained a standard for distances until 1793. — Editor's comment. D.R.



approximately $42\frac{1}{2}$ hours later, after one full orbital revolution of the satellite, the Earth arrives at the point K, he observes the satellite coming back at the point D. Obviously, if light travels along the path LK during a finite time, the satellite should be observed coming back at the point D with a delay relative to the Earth at the point L. Thus the orbital revolutions of the satellite, recorded according to its appearance from the shadow, should be delayed by the time that light travels from L to K. In the reverse case, in the quadrature FG where the Earth moves towards the light, the orbital revolutions recorded according the penetration into shadow should seem shorter as the revolutions recorded according to the appearance from the shadow are longer. In a duration of $42\frac{1}{2}$ hours, in which this satellite undergoes approximately one full revolution, the distance from the Earth to Jupiter changes, in both quadratures, by at least 210 diameters of the Earth. Hence, if light would travel the diameter of the Earth in one second, it would travel each interval FG and KL in $3\frac{1}{2}$ minutes. This should lead to a deviation of about half an hour between two revolutions of this satellite observed in FG and KL respectively. On the other hand, nothing of such a substantial difference has been found.

This fact however does not mean that light requires no time for travel: in his precise study of this subject monsieur Rømer determined that such a deviation, inaccessible to recordings on two revolutions, becomes very substantial for many revolutions taken altogether. For instance, 40 revolutions observed from the side F should be substantially shorter that 40 other revolutions, observed from the opposite side (this effect is independent of any position in the Zodiac where Jupiter would be located), and this is in a ratio of 22 to the interval HE, which is twice the distance from us to the Sun.

The necessity of the new equation of delayed light was established by all the observations obtained at the Royal Academy and the Observatory during the past 8 years. This was verified later, by the appearance of the first satellite from the shadow of Jupiter, observed in the evening $5^{\rm h}35^{\rm m}45^{\rm s}$ on November 9 of this year, in Paris, that occurred 10 minutes later than it was expected on the basis of the observations produced in August, when the Earth was much closer to Jupiter; this was predicted by monsieur Rømer, at the Academy in the beginning of September*.

To remove all doubts that this inequality originates in the delay of light, he shows that this effect cannot appear due to any eccentricity or any other source which are usually employed for explanation of the irregularities in the motion of the Moon and the other planets: through all these he is sure that the first satellite of Jupiter is eccentric, and also that the satellite is orbiting faster or slower while Jupiter approaches the Sun or moves away from it, hence revolutions of this machine are unequal; so he is sure that the last three causes of inequality do not affect the first cause which is obvious.

*Rømer means the Observatory of Paris where he was employed, commencing in 1672, as Giovanni Cassini's assistant, observing eclipses of the satellites of Jupiter. Before that, in 1671, working at the Uraniborg Observatory in Denmark, Rømer, with another astronomer, Jean Picard, recorded the periods of about 140 eclipses of Io, the first satellite of Jupiter. Cassini also recorded the periods of eclipses of the satellites of Jupiter at the Observatory of Paris in the years 1666–1668. In his calculation of 1976 Rømer used his own observations of 1672–1676, the Danish observations of 1671, and also Cassini's data of 1666–1668. — Editor's comment. D.R.

On the Gravitation Produced by the Earth on Different Substances

Loránd Eötvös

The presentation held at the Hungarian Academy of Sciences
on January 20, 1889

Abstract: This is a translation of the celebrated presentation held by Loránd Eötvös at the Hungarian Academy of Sciences on January 20, 1889. Here Eötvös discusses the historical foundations of his claim about the universal equality of inertial and passive-gravitational mass, and gives a survey of his own many-year experimental geophysical studies verifying the equivalence principle with high precision. The famous Eötvös experiment verifying the equivalence principle, first given in this short presentation, was cited many times by Albert Einstein as one of the basics to his General Theory of Relativity. Later the Eötvös experiment became a key point for all following experimental research verifying the equivalence principle, which are still continuing till now, with much increased measuerement precision. This short presentation was originally published in 1890, in the *Mathematical and Natural Science Proceedings of Hungary*, which were issued in German (the official language of the Austro-Hungarian Empire): Roland von Eötvös. Über die Anziehung der Erde auf verschiedene Substanzen. *Mathematische und Naturwissenschaftliche Berichte aus Ungarn*, 1890, Bd. 8, S. 65–68. In this translation we use the original Hungarian transcription of the name of the author instead the German version, Roland von Eötvös, printed in that journal. Translated from the German in 2008 by Larissa Borissova and Dmitri Rabounski. The translators thank Péter Király and Istvánné (Kati) Szalay of the Research Institute for Particle and Nuclear Physics, the Hungarian Academy of Sciences, for assistance with the original Eötvös paper.

Of the suppositions used by Newton as the foundations of his theory of gravitation, the most important is the one which claims that the gravitation produced by the Earth on an Earth-bound body is proportional to the mass of the body, and is independent of the structure of the substance composing it.

Newton has already verified this supposition of him by experiment. He was unsatisfied with the scholarly experiments, well-known to him, which revealed the fact that a feather and a coin fell equally fast in emptiness. Targeting this purpose, he used motions of a pendulum which could be registered with much precision. Once he made a pendulum, where the same-weight-bodies consisting of different substances such as gold, silver, lead, glass, sand, table salt, water, corn, and wood, were moving along the arcs of circle, each of which possessing the same radius, and where he registered the duration of the oscillation, he was able to conclude that there was no difference between them.

No doubt, those experiments produced by Newton were much more precise than the aforementioned scholarly experiments; on the other hand, the measurement precision of those experiments was only 1/1,000, so they, strictly speaking, proved only the fact that the difference between the accelerations did not exceed 1/1,000 of their numerical value. This measurement precision which he used in such an important problem could not be deemed satisfactory. Bessel therefore concluded that repetitions of such a classical experiment on a pendulum were necessary.

Proceeding from his measurements produced from the oscillation losses in gold, silver, lead, iron, zinc, brass, marble, clay, quartz, and meteorite substance, he had unambiguously proved that the gravitational accelerations of these bodies did not possess deviations larger than 1/50,000 from each other. This however was insufficient as well. Bessel pointed out very well that it would always be very interesting to check the validity of this assumption with increasing precision provided by the permanently developing instruments of each of the future generations.

Such a research is desirable due to two reasons. First, this is due to the fact that Newton's supposition led to such a foundation, according to which we can find the mass of a body through its weight measured by a balance. It is required by the logic that the truth of this supposition should be proven upto at least such a precision, which can be reached in the weight, and this is much higher than 1/50,000 part, even more than than 1/1,000,000 part. Second, this is due to the fact that the research produced by Newton and Bessel covered only bodies whose material structure was similar to each other, and manifested a small difference, while this problem is still remaining open for many liquid and gaseous bodies. Proceeding from Bessel's experiments, we can conclude at most that the gravity of the air differs from that of a solid body no greater than 1/50 part.*

Since in the process of my research of the gravity of mass my attention was turned towards this problem, and since I resolved it in an absolutely different way than Newton and Bessel did, and since I reached

*In the original manuscript in German here is a typing mistake "Tein fünfzigtausendstelt", i.e. 1/50,000, while it should obvious be "Tein fünfzigstelt", i.e. 1/50. It is doubtful that, on the most lightweight body of those in this research, the clay ground, the mass of the ousted air is more than almost 1/2,000 part. Thus, we obtain the measurement precision for the air much lesser than the mentioned due to the typing mistake, namely: $2{,}000 \times 1/50{,}000 = 1/25$. — Comment due to Pál Selényi, the corresponding member of the Hungarian Academy of Sciences, Budapest, who studied the original Eötvös papers in 1953.

much higher measurement precision than they had, I found the way of my considerations and the results of my experiment to be worthy of presentation to the respected Academy.

The force due to which the bodies located in the empty space fall onto the Earth, and which is known as gravity, is a sum of two components, namely — the gravitation of the Earth and the centrifugal force, which is due to the rotation of the Earth. These two components, in general, are neither equal to each other nor oppositely directed at each other; they create an angle with respect to each other, which is approximately the same as the angle of the geographical latitude. The direction of the resulting sum depends on these components; it is also clear that, at the same point on the Earth, since the centrifugal force of the same-mass-bodies is the same, the gravity of these bodies should be different if the force of gravitation attracting each of these bodies is different.

At Budapest the centrifugal force results in a deviation towards the South for approximately $5'56''$, i.e. $356''$ from the direction of the attraction of the Earth. We obtain by calculation that, if the attraction from the side of the Earth on two bodies of the same mass, but consisting of different substances, would differ as $^1/_{1,000}$ part, these two gravities were directed at an angle of $0.365''$ (that is approximately $^1/_{3''}$) with respect to each other, while if the difference in the force of gravity would be $^1/_{20,000,000}$ part, the angle was $^{356''}/_{20,000,000}$ that results a little more than $^1/_{60,000''}$.

The lead lot and the libelle* of the torsion balance are not enough sensitive to the very small deviation in the direction of the force of gravity, which is expected in this observation. However this torsion balance as a whole is applicable to such an observation very well, because I already registered small deviations in the direction of the force of gravity in other observations with it.

I fixed a body, the weight of which was approximately 30 g, at the end of the shoulder of the balance. The shoulder, the length of which varied from 25 to 50 cm, was suspended through a platinum thread. Once the shoulder was directed orthogonally towards the meridian, I registered its position relative to the box of the whole instrument precisely by a system

*Consider a mirror fixed to the torsion thread. The light beam falling onto it, then reflected, may swivel around the zero point of the scale. Specialists call this *die Libelle*, in German, that means a *dragon-fly* in English, because such two light beams, being swivelling, seems like the large wings of a dragon fly in flight. Relative to technics in general, *die Libelle* is the decisive part of a water-level. It is a small glass container filled with liquid and a gas bubble. The gas bubble indicates whether the water-level is exactly horizontal or not. — Editor's comment. D.R. (The editor is thankful to Ulrich Neumann, Germany, for discussion.)

of two mirrors, one of which was moved in common with the shoulder, while another one was fixed on the box. Then I turned out the whole instrument, in common with the box, at 180° in such a way that the body, located initially at the Eastern end of the shoulder, arrived at the Western end of it. Then I registered this new position of the shoulder relative to the instrument. If the gravity of the body at both sides was differently directed, a twist of the suspending thread appeared. At the same time, such an effect was not registered in the case where a brass ball was fixed at one end of the shoulder, while the other end was equipped with a glass, corkwood, or antimonite crystal; meanwhile the deviation of $1/60{,}000''$ in the direction of the force of gravity should yield a twist of $1'$, which is surely accessed.

Later I also studied, especially, this situation in the case of the air. A body moving in the air was acted by the force, caused by the ousted air. The force was equal to the gravity of the ousted air, but directed oppositely towards it. If the gravity of the air was directed similarly to that produced on the other bodies, this circumstance manifested itself as a twist of the thread in the aforementioned experiments. Of course, this twist was proportional to the weight of the ousted air, not the weight of the body in the air. In order to increase the aforementioned twist as much as possible, I fixed, at one end of the shoulder, an empty glass ball, whose volume was $120\,\mathrm{cm}^3$ volume, while its weight was 30 g, so the drift of the air was approximately $1/200$ of the last one. All these had required much accuracy: the deviating effect of the air stream on the body of so large volume should be removed so that the shoulder was in the state of sure equilibrium. This task was realized only in the resting underground floor of the Institute of Physics of the Budapest University, at night and only due to the fact that I had registered the state of equilibrium by a photo camera.

I was unable to also consider the twisting in the fall. So my experiments, which are still 400 times more precise than those produced by Bessel, showed no difference from Newton's supposition.

I therefore have to claim by right that, in general, the difference between the gravity of the bodies, which have equal masses but consist of different substances, is lesser than $1/20{,}000{,}000$ in the case of brass, glass, antimonite, and corkwood, but it is undoubtedly less than $1/100{,}000$ in the case of air.

On the Gravitational Field of a Point-Mass, According to Einstein's Theory

Karl Schwarzschild

Submitted on January 13, 1916

Abstract: This is a translation of the paper *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie* by Karl Schwarzschild, where he obtained the metric of a space due to the gravitational field of a point-mass. The paper was originally published in 1916, in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, S. 189–196. Translated from the German in 2008 by Larissa Borissova and Dmitri Rabounski.

§1. In his study on the motion of the perihelion of Mercury (see his presentation given on November 18, 1915*) Einstein set up the following problem: a point moves according to the requirement

$$\left.\begin{aligned} &\delta\int ds = 0, \\ \text{where}\qquad &ds = \sqrt{\Sigma\, g_{\mu\nu}\, dx^\mu dx^\nu}\,, \quad \mu,\nu = 1,2,3,4 \end{aligned}\right\}, \tag{1}$$

where $g_{\mu\nu}$ are functions of the variables x, and, in the framework of this variation, these variables are fixed in the start and the end of the path of integration. Hence, in short, this point moves along a geodesic line, where the manifold is characterized by the line-element ds.

Taking this variation gives the equations of this point

$$\frac{d^2x^\alpha}{ds^2} = \sum_{\mu,\nu} \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\,, \quad \alpha,\beta = 1,2,3,4, \tag{2}$$

where

$$\Gamma^\alpha_{\mu\nu} = -\frac{1}{2}\sum_\beta g^{\alpha\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right), \tag{3}$$

while $g^{\alpha\beta}$, which are introduced and normed with respect to $g_{\alpha\beta}$, mean the reciprocal determinant† to the determinant $|g_{\mu\nu}|$.

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*Schwarzschild means the article: Einstein A. Erklärung der Perihelbewegung der Merkur aus der allgemeinen Relativitätstheorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1915, S. 831–839. — Editor's comment. D.R.

†This is the determinant of the reciprocal matrix, i.e. a matrix whose indices are raised to the given matrix. One referred to the reciprocal matrix as the *subdeterminant*, in those years. — Editor's comment. D.R.

Commencing now and so forth, according to Einstein's theory, a test-particle moves in the gravitational field of the mass located at the point $x^1 = x^2 = x^3 = 0$, if the "components of the gravitational field" Γ satisfy the "field equations"

$$\sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\nu\alpha} = 0 \qquad (4)$$

everywhere except the point $x^1 = x^2 = x^3 = 0$ itself, and also if the determinant equation

$$|\, g_{\mu\nu} | = -1 \qquad (5)$$

is satisfied.

These field equations in common with the determinant equation possess the fundamental property, according to which their form remains unchanged in the framework of substitution of any other variables instead of x^1, x^2, x^3, x^4, if the substitution of the determinant equals 1.

Assume the curvilinear coordinates x^1, x^2, x^3, while x^4 is time. We assume that the mass located at the origin of the coordinates remains unchanged with time, and also the motion is uniform and linear up to infinity. In such a case, according to the calculation by Einstein (see page 833*) the following requirements should be satisfied:

1. All the components should be independent of the time coordinate x^4;
2. The equalities $g_{\rho 4} = g_{4\rho} = 0$ are satisfied exactly for $\rho = 1, 2, 3$;
3. The solution is spatially symmetric at the origin of the coordinate frame in that sense that it comes to the same solution after the orthogonal transformation (rotation) of x^1, x^2, x^3;
4. These $g_{\mu\nu}$ vanish at infinity, except the next four boundary conditions, which are nonzero

$$g_{44} = 1\,, \qquad g_{11} = g_{22} = g_{33} = -1\,.$$

The task is to find such a line-element, possessing such coefficients, that the field equations, the determinant equation, and these four requirements would be satisfied.

§2. Einstein showed that this problem in the framework of the first order approximation leads to Newton's law, and also that the second order approximation covers the anomaly in the motion of the perihelion

*Schwarzschild means page 833 in the aforementioned Einstein paper of 1915 published in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften.* — Editor's comment. D.R.

of Mercury. The following calculation provides an exact solution of this problem. As supposed, in any case, an exact solution should have a simple form. It is important that the resulting calculation shows the uniqueness of this solution, while Einstein's approach gives ambiguity, and also that the method shown below gives (with some difficulty) the same good approximation. The following text leads to the representation of Einstein's result with increasing precision.

§3. We denote time t, while the rectangular coordinates* are denoted x, y, z. Thus the well-known line-element, satisfying the requirements 1–3, has the obvious form

$$ds^2 = F dt^2 - G\left(dx^2 + dy^2 + dz^2\right) - H\left(xdx + ydy + zdz\right)^2,$$

where F, G, H are functions of $r = \sqrt{x^2 + y^2 + x^2}$.

The condition (4) requires, at $r = \infty$: $F = G = 1$, $H = 0$.

Moving to the spherical coordinates† $x = r\sin\vartheta\cos\varphi$, $y = r\sin\vartheta\sin\varphi$, $z = r\cos\vartheta$, the same line-element is

$$ds^2 = F dt^2 - G\left(dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2\right) - Hr^2 dr^2 =$$
$$= F dt^2 - \left(G + Hr^2\right) dr^2 - Gr^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right). \qquad (6)$$

In the spherical coordinates the volume element is $r^2\sin\vartheta\, dr\, d\vartheta\, d\varphi$, the determinant of transformation from the old coordinates to the new ones $r^2\sin\vartheta$ differs from 1; the field equations are still to be unchanged and, with use the spherical coordinates, we need to process complicated transformations. However the following simple method allows us to avoid this difficulty. Assume

$$x^1 = \frac{r^3}{3}, \quad x^2 = -\cos\vartheta, \quad x^3 = \varphi, \qquad (7)$$

then the equality $r^2 dr \sin\vartheta\, d\vartheta\, d\varphi = dx^1 dx^2 dx^3$ is true in the whole volume element. These new variables also represent spherical coordinates in the framework of this unit determinant. They have obvious advantages to the old spherical coordinates in this problem, and, at the same time, they still remain valid in the framework of the considerations. In addition to these, assuming $t = x^4$, the field equations and the determinant equation remain unchanged in form.

*The Cartesian coordinates. — Editor's comment. D.R.

†In the original — "polar coordinates". However it is obvious that Schwarzschild means the three-dimensional spherical coordinates. — Editor's comment. D.R.

In these new spherical coordinates the line-element has the form

$$ds^2 = F(dx^4)^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right)(dx^1)^2 - \\ - G\,r^2\left[\frac{(dx^2)^2}{1-(x^2)^2} + (dx^3)^2\left[1-(x^2)^2\right]\right], \quad (8)$$

on the basis of which we write

$$ds^2 = f_4\,(dx^4)^2 - f_1\,(dx^1)^2 - f_2\,\frac{(dx^2)^2}{1-(x^2)^2} - f_3\,(dx^3)^2\left[1-(x^2)^2\right]. \quad (9)$$

In such a case $f_1, f_2 = f_3, f_4$ are three functions of x^1, which satisfy the following conditions

1. At $x^1 = \infty$: $f_1 = \frac{1}{r^4} = \left(3x^1\right)^{-\frac{4}{3}}$, $f_2 = f_3 = r^2 = \left(3x^1\right)^{\frac{2}{3}}$, $f_4 = 1$;
2. The determinant equation $f_1 \cdot f_2 \cdot f_3 \cdot f_4 = 1$;
3. The field equations;
4. The function f is continuous everywhere except $x^1 = 0$.

§4. To obtain the field equations we need first to construct the components of the gravitational field according to the line-element (9). The simplest way to do this is by directly taking the variation, which gives the differential equations of the geodesic line, then the components will be seen from the equations. The differential equations of the geodesic line along the line-element (9) are obtained by directly taking this variation in the form

$$f_1\,\frac{d^2x^1}{ds^2} + \frac{1}{2}\,\frac{\partial f_4}{\partial x^1}\left(\frac{dx^4}{ds}\right)^2 + \frac{1}{2}\,\frac{\partial f_1}{\partial x^1}\left(\frac{dx^1}{ds}\right)^2 - \\ - \frac{1}{2}\,\frac{\partial f_2}{\partial x^1}\left[\frac{1}{1-(x^2)^2}\left(\frac{dx^2}{ds}\right)^2 + \left[1-(x^2)^2\right]\left(\frac{dx^3}{ds}\right)^2\right] = 0\,,$$

$$\frac{f_2}{1-(x^2)^2}\,\frac{d^2x^2}{ds^2} + \frac{\partial f_2}{\partial x^1}\,\frac{1}{1-(x^2)^2}\,\frac{dx^1}{ds}\,\frac{dx^2}{ds} + \\ + \frac{f_2\,x^2}{\left[1-(x^2)^2\right]^2}\left(\frac{dx^2}{ds}\right)^2 + f_2\,x^2\left(\frac{dx^3}{ds}\right)^2 = 0\,,$$

$$f_2\left[1-(x^2)^2\right]\frac{d^2x^3}{ds^2} + \frac{\partial f_2}{\partial x^1}\left[1-(x^2)^2\right]\frac{dx^1}{ds}\,\frac{dx^3}{ds} - 2f_2\,x_2\,\frac{dx^2}{ds}\,\frac{dx^3}{ds} = 0\,,$$

$$f_4\,\frac{d^2x^4}{ds^2} + \frac{\partial f_4}{\partial x^1}\,\frac{dx^1}{ds}\,\frac{dx^4}{ds} = 0\,.$$

Comparing these equations to (2) gives the components of the gravitational field

$$\Gamma_{11}^{1} = -\frac{1}{2}\frac{1}{f_1}\frac{\partial f_1}{\partial x^1}\,, \quad \Gamma_{22}^{1} = +\frac{1}{2}\frac{1}{f_1}\frac{\partial f_2}{\partial x^1}\frac{1}{1-(x^2)^2}\,,$$

$$\Gamma_{33}^{1} = +\frac{1}{2}\frac{1}{f_1}\frac{\partial f_2}{\partial x^1}\left[1-(x^2)^2\right],$$

$$\Gamma_{44}^{1} = -\frac{1}{2}\frac{1}{f_1}\frac{\partial f_4}{\partial x^1}\,,$$

$$\Gamma_{21}^{2} = -\frac{1}{2}\frac{1}{f_2}\frac{\partial f_2}{\partial x^1}\,, \quad \Gamma_{22}^{2} = -\frac{x^2}{1-(x^2)^2}\,, \quad \Gamma_{33}^{2} = -x^2\left[1-(x^2)^2\right],$$

$$\Gamma_{31}^{3} = -\frac{1}{2}\frac{1}{f_2}\frac{\partial f_2}{\partial x^1}\,, \quad \Gamma_{32}^{3} = +\frac{x^2}{1-(x^2)^2}\,,$$

$$\Gamma_{41}^{4} = -\frac{1}{2}\frac{1}{f_4}\frac{\partial f_4}{\partial x^1}\,,$$

while the rest of the components of it are zero.

Due to the symmetry of rotation around the origin of the coordinates, it is sufficient to construct the field equations at only the equator ($x^2=0$): once they are differentiated, we can substitute 1 instead of $1-(x^2)^2$ everywhere into the above obtained formulae. Thus, after this algebra, we obtain the field equations

$$a)\quad \frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x^1}\right) = \frac{1}{2}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x^1}\right)^2 + \left(\frac{1}{f_2}\frac{\partial f_2}{\partial x^1}\right)^2 + \frac{1}{2}\left(\frac{1}{f_4}\frac{\partial f_4}{\partial x^1}\right)^2,$$

$$b)\quad \frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_2}{\partial x^1}\right) = 2 + \frac{1}{f_1 f_2}\left(\frac{\partial f_2}{\partial x^1}\right)^2,$$

$$c)\quad \frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_4}{\partial x^1}\right) = \frac{1}{f_1 f_4}\left(\frac{\partial f_4}{\partial x^1}\right)^2.$$

Besides these three equations, the functions f_1, f_2, f_3 should satisfy the determinant equation

$$d)\quad f_1(f_2)^2 f_4 = 1 \quad \text{or} \quad \frac{1}{f_1}\frac{\partial f_1}{\partial x^1} + \frac{2}{f_2}\frac{\partial f_2}{\partial x^1} + \frac{1}{f_4}\frac{\partial f_4}{\partial f_1} = 0\,.$$

First of all I remove *b*). So three functions f_1, f_2, f_4 of *a*), *c*), and *d*) still remain. The equation *c*) takes the form

$$c')\quad \frac{\partial}{\partial x^1}\left(\frac{1}{f_4}\frac{\partial f_4}{\partial x^1}\right) = \frac{1}{f_1 f_4}\frac{\partial f_1}{\partial x^1}\frac{\partial f_4}{\partial x^1}\,.$$

Integration of it gives

$$c'')\quad \frac{1}{f_4}\frac{\partial f_4}{\partial x^1} = \alpha f_1\,,$$

where α is the constant of integration. Summation of a) and c') gives

$$\frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x^1} + \frac{1}{f_4}\frac{\partial f_4}{\partial x^1}\right) = \left(\frac{1}{f_2}\frac{\partial f_2}{\partial x^1}\right)^2 + \frac{1}{2}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x^1} + \frac{1}{f_4}\frac{\partial f_4}{\partial x^1}\right)^2.$$

With taking d) into account, it follows that

$$-\,2\,\frac{\partial}{\partial x^1}\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x^1}\right) = 3\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x^1}\right)^2.$$

After integration, we obtain

$$\frac{1}{\frac{1}{f_2}\frac{\partial f_2}{\partial x^1}} = \frac{3}{2}\,x^1 + \frac{\rho}{2}\,,$$

where ρ is the constant of integration. Or

$$\frac{1}{f_2}\frac{\partial f_2}{\partial x^1} = \frac{2}{3x^1+\rho}\,.$$

We integrate it once again:

$$f_2 = \lambda\left(3x^1+\rho\right)^{\frac{2}{3}},$$

where λ is the constant of integration. The condition at infinity requires: $\lambda = 1$. Hence

$$f_2 = (3x_1+\rho)^{\frac{2}{3}}\,. \tag{10}$$

Next, it follows from c'') and d) that

$$\frac{\partial f_4}{\partial x^1} = \alpha f_1 f_4 = \frac{\alpha}{(f_2)^2} = \frac{\alpha}{(3x_1+\rho)^{\frac{4}{3}}}\,.$$

We integrate it, taking the condition at infinity into account:

$$f_4 = 1 - \alpha\left(3x^1+\rho\right)^{-\frac{1}{3}}. \tag{11}$$

Finally, it follows from d) that

$$f_1 = \frac{\left(3x^1+\rho\right)^{-\frac{4}{3}}}{1-\alpha\,(3x^1+\rho)^{-\frac{1}{3}}}\,. \tag{12}$$

As easy to check, the equation b) corresponds to the found formulae for f_1 and f_2.

This satisfies all the requirements up to the continuity condition. The function f_1 remains continuous, if

$$1 = \alpha\left(3x^1 + \rho\right)^{-\frac{1}{3}}, \qquad 3x^1 = \alpha^3 - \rho\,.$$

In order to break the continuity at the origin of the coordinates, there should be

$$\rho = \alpha^3. \tag{13}$$

The continuity condition connects, by the the same method, both constants of integration ρ and α.

Now, the complete solution of our problem has the form

$$f_1 = \frac{1}{R^4}\,\frac{1}{1-\frac{\alpha}{R}}\,, \qquad f_2 = f_3 = R^2, \qquad f_4 = 1 - \frac{\alpha}{R}\,,$$

where the auxiliary quantity R has been introduced

$$R = \left(3x^1 + \rho\right)^{\frac{1}{3}} = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}.$$

If substituting the formulae of these functions f into the formula of the line-element (9), and coming back to the regular spherical coordinates, we arrive at such a formula for the line-element

$$\left.\begin{aligned} ds^2 = \left(1-\frac{\alpha}{R}\right)dt^2 - \frac{dR^2}{1-\frac{\alpha}{R}} - R^2\left(d\vartheta^2 + \sin^2\!\vartheta\, d\varphi^2\right) \\ R = \left(r^3+\alpha^3\right)^{\frac{1}{3}} \end{aligned}\right\}, \tag{14}$$

which is the exact solution of the Einstein problem.

This formula contains the sole constant of integration α, which is dependent on the numerical value of the mass located at the origin of the coordinates.

§5. The uniqueness of this solution follows from the aforementioned calculations. For one who is troubled with the uniqueness of Einstein's method, followed from this, we consider the following example. There above, from the continuity condition, the formula

$$f_1 = \frac{\left(3x^1+\rho\right)^{-\frac{4}{3}}}{1-\alpha\left(3x^1+\rho\right)^{-\frac{1}{3}}} = \frac{\left(r^3+\rho\right)^{-\frac{4}{3}}}{1-\alpha\left(r^3+\rho\right)^{-\frac{1}{3}}}$$

was obtained. In the case where α and ρ are small values, the second order term and the higher order terms vanish from the series so that

$$f_1 = \frac{1}{r^4}\left[1 + \frac{\alpha}{r} - \frac{4}{3}\,\frac{\rho}{r^3}\right].$$

This exception, in common with the respective exceptions for f_1, f_2, f_4 taken to within the same precision, satisfies all the requirements of this problem. The continuity requirement added nothing in the framework of this approximation, but only a break at the point of the origin of the coordinates. Both constants α and ρ are arbitrarily determined, so the physical side of this problem in not determined. The exact solution of this problem manifests that in a real case, with the approximations, a break appears not at the point of the origin of the coordinates, but in the region $r = \left(\alpha^3 - \rho\right)^{\frac{1}{3}}$, and we should suppose $\rho = \alpha^3$ in order to move the break to the origin of the coordinates. In the framework of such an approximation through the exponents of α and ρ, we need to know very well the law which rules these coefficients, and also be masters in the whole situation, in order to understand the necessity of connexion between α and ρ.

§6. In the end, we are looking for the equation of a point moving along the geodesic line in the gravitational field related to the line-element (14). Proceeding from the three circumstances according to which the line-element is homogeneous, differentiable, and its coefficients are independent of t and ρ, we take the variation so we obtain three intermediate integrals. Because the motion is limited to the equatorial plane ($\vartheta = 90°$, $d\vartheta = 0$), these intermediate integrals have the form

$$\left(1 - \frac{\alpha}{R}\right)\left(\frac{dt}{ds}\right)^2 - \frac{1}{1 - \frac{\alpha}{R}}\left(\frac{dR}{ds}\right)^2 - R^2\left(\frac{d\varphi}{ds}\right)^2 = const = h\,, \quad (15)$$

$$R^2\,\frac{d\varphi}{ds} = const = c\,, \quad (16)$$

$$\left(1 - \frac{\alpha}{R}\right)\frac{dt}{ds} = const = 1\,, \quad (17)$$

where the third integral means definition of the unit of time.

From here it follows that

$$\left(\frac{dR}{d\varphi}\right)^2 + R^2\left(1 - \frac{\alpha}{R}\right) = \frac{R^4}{c^2}\left[1 - h\left(1 - \frac{\alpha}{R}\right)\right]$$

or, for $\frac{1}{R} = x$,

$$\left(\frac{dx}{d\varphi}\right)^2 = \frac{1-h}{c^2} + \frac{h\alpha}{c^2}\,x - x^2 + \alpha x^3. \tag{18}$$

We denote $\frac{c^2}{h} = B$, $\frac{1-h}{h} = 2A$ that is identical to Einstein's equations (11) in the cited presentation*, and gives the observed anomaly of the perihelion of Mercury.

In a general case Einstein's approximation for a curved trajectory meets the exact solution, only if we introduce

$$R = \left(r^3 + \alpha^3\right)^{\frac{1}{3}} = r\left(1 + \frac{\alpha^3}{r^3}\right)^{\frac{1}{3}} \tag{19}$$

instead of r. Because $\frac{\alpha}{r}$ is close to twice the square of the velocity of the planet (the velocity of light is 1), the expression within the brackets, in the case of Mercury, is different from 1 by a value of the order 10^{-12}. The quantities R and r are actually identical, so Einstein's approximation satisfies the practical requirements of even very distant future.

In the end it is required to obtain the exact form of Kepler's third law for circular trajectories. Given an angular velocity $n = \frac{d\varphi}{dt}$, according to (16) and (17), and introducing $x = \frac{1}{R}$, we have

$$n = cx^2\left(1 - \alpha x\right).$$

In a circle both $\frac{dx}{d\varphi}$ and $\frac{d^2x}{d\varphi^2}$ should be zero. This gives, according to (18), that

$$\frac{1-h}{c^2} + \frac{h\alpha}{c^2}\,x - x^2 + \alpha x^3 = 0\,, \qquad \frac{h\alpha}{c^2} - 2x + 3\alpha x^2 = 0\,.$$

Removing h from both circles gives

$$\alpha = 2c^2 x\left(1 - \alpha x\right)^2.$$

From here it follows that

$$n^2 = \frac{\alpha}{2}\,x^3 = \frac{\alpha}{2R^3} = \frac{\alpha}{2\left(r^3 + \alpha^3\right)}\,.$$

Deviation of this formula from Kepler's third law is absolutely invisible up to the surface of the Sun. However given an ideal point-mass, the

*Einstein A. Erklärung der Perihelbewegung der Merkur aus der allgemeinen Relativitätstheorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1915, S. 831–839. — Editor's comment. D.R.

angular velocity does not experience unbounded increase with lowering of the orbital radius (such an unbounded increase should be experienced according to Newton's law), but approximates to a finite limit

$$n_0 = \frac{1}{\alpha\sqrt{2}}\,.$$

(For a mass which is in the order of the mass of the Sun, this boundary frequency should be about 10^4 per second.) This circumstance should be interesting in the case where a similar law rules the molecular forces.

On the Gravitational Field of a Sphere of Incompressible Liquid, According to Einstein's Theory

Karl Schwarzschild

Submitted on February 24, 1916

Abstract: This is a translation of the paper *Über das Gravitationsfeld einer Kugel aus incompressiebler Flüssigkeit nach der Einsteinschen Theorie* published by Karl Schwarzschild, in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 424–435. Here Schwarzschild expounds his previously obtained metric for the spherically symmetric gravitational field produced by a point-mass, to the case where the source of the field is represented by a sphere of incompressible fluid. Schwarzschild formulates the physical condition of degeneration of such a field. Translated from the German in 2008 by Larissa Borissova and Dmitri Rabounski.

§1. As the next step of my study concerning Einstein's theory of gravitation, I calculated the gravitational field of a homogeneous sphere of a finite radius, consisting of incompressible fluid. This clarification, "consisting of incompressible fluid", is necessary to be added, due to the fact that gravitation, in the framework of the relativistic theory, depends on not only the quantity of the matter, but also on its energy. For instance, a solid body having a specific state of internal stress would produce a gravitation other than that of a liquid.

This calculation is a direct continuation of my presentation concerning the gravitational field of a point-mass (see *Sitzungsberichte*, 1916, S. 189*), to which I will refer here in short†.

§2. Einstein's equations of gravitation (see *Sitzungsberichte*, 1915, S. 845‡) in the general form manifest that

$$\sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\nu\alpha} = G_{\mu\nu} \,. \tag{1}$$

*Schwarzschild K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

†Schwarzschild means that, somewhere in this paper, he will refer to his formulae deduced in his first publication of 1916. — Editor's comment. D.R.

‡Einstein A. Die Feldgleichungen der Gravitation. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1915, S. 844–847. — Editor's comment. D.R.

The quantities $G_{\mu\nu}$ vanish where is no matter. Inside an incompressible liquid they are determined in the following way: the "mixed tensor of the energy" of an incompressible liquid, according to Einstein (see *Sitzungsberichte*, 1914, S. 1062*) is equal to

$$T_1^1 = T_2^2 = T_3^3 = -\,p\,, \qquad T_4^4 = \rho_0\,, \tag{2}$$

while the rest of the T_μ^ν are zero. Here p is the pressure, ρ is the constant density of the liquid.

The "covariant tensor of the energy" will be

$$T_{\mu\nu} = \sum_\tau T_\mu^\tau\, g_{\nu\tau}\,. \tag{3}$$

Besides

$$T = \sum_\tau T_\tau^\tau = \rho_0 - 3p \tag{4}$$

and also

$$\varkappa = 8\pi k^2,$$

where $\varkappa$ is Gauss' gravitational constant. Then, according to Einstein (see *Sitzungsberichte*, 1915, S. 845, Gleichung 2a†), the right sides of the equations have the form

$$G_{\mu\nu} = -\,\varkappa\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T\right). \tag{5}$$

To be in the state of equilibrium, such a liquid should satisfy the conditions (see equation 7a ibidem†)

$$\sum_\alpha \frac{\partial T_\tau^\alpha}{\partial x^\alpha} + \sum_{\mu\nu} \Gamma_{\tau\nu}^\mu\, T_\mu^\nu = 0\,. \tag{6}$$

§3. In the case of such a sphere, as well as in the case of a point-mass, these general equations should be normalized for the symmetrical rotation around the origin of the coordinates. As in the case of a point-mass, it is recommended to move to the spherical coordinates chosen

*Einstein A. Die formale Grundlage der allgemeinen Relativitätstheorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1914, S. 1030–1085. This is a bulky paper concerning the formal basics of the General Theory of Relativity, wherein Einstein considered his equations of gravitation. — Editor's comment. D.R.

†Einstein A. Die Feldgleichungen der Gravitation. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1915, S. 844–847. — Editor's comment. D.R.

so that the determinant equals 1*

$$x^1 = \frac{r^3}{2}\,, \quad x^2 = -\cos\vartheta\,, \quad x^3 = \varphi\,, \quad x^4 = t\,. \tag{7}$$

The line-element should have the same form

$$ds^2 = f_4\,(dx^4)^2 - f_1\,(dx^1)^2 - f_2\,\frac{(dx^2)^2}{1-(x^2)^2} - f_3\,(dx^3)^2\left[1-(x^2)^2\right], \tag{8}$$

so that we have

$$g_{11} = -\,f_1\,, \quad g_{22} = -\,\frac{f_2}{1-(x^2)^2}\,, \quad g_{33} = -\,f_2\left[1-(x^2)^2\right], \quad g_{44} = f_4\,,$$

while the other $g_{\mu\nu}$ are zero. These f are functions dependent only on x^1.

In the space outside this sphere, the solutions (10), (11), (12) were found†

$$f_4 = 1 - \alpha\left(3x^1+\rho\right)^{-\frac{1}{3}}, \quad f_2 = \left(3x^1+\rho\right)^{\frac{2}{3}}, \quad f_1(f_2)^2 f_4 = 1\,, \tag{9}$$

where α and ρ are two arbitrary constants, which should be determined on the basis of the mass and the radius of the sphere.

We are going to construct the field equations for the internal space of this sphere with use of the formula (8) for the line-element, then solve these equations. Concerning the right sides, we obtain

$$T_{11} = T_1^1\,g_{11} = -\,pf_1\,, \qquad T_{22} = T_2^2\,g_{22} = -\,\frac{pf_2}{1-(x^2)^2}\,,$$

$$T_{33} = T_3^3\,g_{33} = -\,pf_2\left[1-(x^2)^2\right], \qquad T_{44} = T_4^4\,g_{44} = \rho_0 f_4\,,$$

$$G_{11} = \frac{\varkappa f_1}{2}\,(p-\rho_0)\,, \qquad G_{22} = \frac{\varkappa f_2}{2}\,\frac{1}{1-(x^2)^2}\,(p-\rho_0)\,,$$

$$G_{33} = \frac{\varkappa f_2}{2}\left[1-(x^2)^2\right](p-\rho_0), \qquad G_{44} = -\,\frac{\varkappa f_4}{2}\,(\rho_0+3p)\,.$$

*In the original — "polar coordinates". The same formulation was used in his first paper of 1916. Obviously Schwarzschild means the three-dimensional spherical coordinates, whose origin meets the centre of the sphere of incompressible liquid. — Editor's comment. D.R.

†Here Schwarzschild refers to the formulae (10), (11), and (12) obtained in his first paper: Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

We can assume that the components $\Gamma^{\alpha}_{\mu\nu}$ of the gravitational field expressed through these functions f, and also the left sides of the field equations are independent of the point-mass (see §4). Limiting our task again by considering the equator ($x^2 = 0$), we obtain the following system of equations.

First, these are three field equations

$$a)\quad -\frac{1}{2}\frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x^1}\right)+\frac{1}{4}\frac{1}{(f_1)^2}\left(\frac{\partial f_1}{\partial x^1}\right)^2+\frac{1}{2}\frac{1}{(f_2)^2}\left(\frac{\partial f_2}{\partial x^1}\right)^2+ \\ +\frac{1}{4}\frac{1}{(f_4)^2}\left(\frac{\partial f_4}{\partial x^1}\right)^2=-\frac{\varkappa}{2}f_1(\rho_0-p)\,,$$

$$b)\quad +\frac{1}{2}\frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_2}{\partial x^1}\right)-\frac{1}{2}\frac{1}{f_1f_2}\left(\frac{\partial f_2}{\partial x^1}\right)^2=-\frac{\varkappa}{2}f_2(\rho_0-p)\,,$$

$$c)\quad -\frac{1}{2}\frac{\partial}{\partial x^1}\left(\frac{1}{f_1}\frac{\partial f_4}{\partial x^1}\right)+\frac{1}{2}\frac{1}{f_1f_4}\left(\frac{\partial f_4}{\partial x^1}\right)^2=-\frac{\varkappa}{2}f_4(\rho_0+3p)\,.$$

We should add to these the determinant equation

$$d)\quad f_1(f_2)^2f_4=1\,.$$

The equilibrium conditions provide just one equation

$$e)\quad -\frac{\partial p}{\partial x^1}=-\frac{p}{2}\left[\frac{1}{f_1}\frac{\partial f_1}{\partial x^1}+\frac{2}{f_2}\frac{\partial f_2}{\partial x^1}\right]+\frac{\rho_0}{2}\frac{1}{f_4}\frac{\partial f_4}{\partial x^1}\,.$$

Proceeding from the common consideration of Einstein's equations, it follows that the aforementioned 5 equations with respect to 4 variables f_1, f_2, f_4, p are consistent with each other.

We should find solutions of these 5 equations, which would be free of singularity inside the sphere. There on the surface of the sphere $p = 0$ should be true, the functions f in the neighbourhood of their derivatives should be continuous, and be transferred into the quantities (9) which are true outside the sphere.

We will omit the index 1 in x^1, for simplicity.

§4. The equation e), due to the determinant equation, transforms into

$$-\frac{\partial p}{\partial x}=\frac{\rho_0+p}{2}\frac{1}{f_4}\frac{\partial f_4}{\partial x}\,.$$

It can be easy integrated, and gives

$$(\rho_0+p)\sqrt{f_4}=const=\gamma\,. \tag{10}$$

The field equations a), b), c), after multiplication by the factors -2, $+2\,\frac{f_1}{f_2}$, $-2\,\frac{f_1}{f_4}$, transform into

$$a')\quad \frac{\partial}{\partial x}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x}\right)=\frac{1}{2\,(f_1)^2}\left(\frac{\partial f_1}{\partial x}\right)^2+\frac{1}{(f_2)^2}\left(\frac{\partial f_2}{\partial x}\right)^2+ + \frac{1}{2\,(f_4)^2}\left(\frac{\partial f_4}{\partial x}\right)^2+\varkappa f_1\,(\rho_0-p)\,,$$

$$b')\quad \frac{\partial}{\partial x}\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x}\right)=2\,\frac{f_1}{f_2}+\frac{1}{f_1f_2}\,\frac{\partial f_1}{\partial x}\,\frac{\partial f_2}{\partial x}-\varkappa f_1\,(\rho_0-p)\,,$$

$$c')\quad \frac{\partial}{\partial x}\left(\frac{1}{f_4}\,\frac{\partial f_4}{\partial x}\right)=\frac{1}{f_1f_4}\,\frac{\partial f_1}{\partial x}\,\frac{\partial f_4}{\partial x}+\varkappa f_1\,(\rho_0+3p)\,.$$

Forming the combinations $a'+2b'+c'$ and $a'+c'$, and using the determinant equation, we obtain, finally,

$$0=4\,\frac{f_1}{f_2}-\frac{1}{(f_2)^2}\left(\frac{\partial f_2}{\partial x}\right)^2-\frac{2}{f_2f_4}\,\frac{\partial f_2}{\partial x}\,\frac{\partial f_4}{\partial x}+4\varkappa f_1p\,, \tag{11}$$

$$0=2\,\frac{\partial}{\partial x}\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x}\right)+\frac{3}{(f_2)^2}\left(\frac{\partial f_2}{\partial x}\right)^2+2\varkappa f_1\,(\rho_0+p)\,. \tag{12}$$

Now we introduce new variables, which are desirable due to the fact that, according to the results obtained for the point-mass, such variables behave simply outside the sphere as they are independent of the terms of these equations which contain ρ and p. So the equations, being expressed with the new variables, should have a simple form as well.

The new variables are

$$f_2=\eta^{\frac{2}{3}},\qquad f_4=\zeta\,\eta^{-\frac{1}{3}},\qquad f_1=\frac{1}{\zeta\eta}\,. \tag{13}$$

Then, according to (9) outside the sphere,

$$\eta=3x+\rho\,,\qquad \zeta=\eta^{\frac{1}{3}}-\alpha\,, \tag{14}$$

$$\frac{\partial\eta}{\partial x}=3\,,\qquad \frac{\partial\zeta}{\partial x}=\eta^{-\frac{2}{3}}. \tag{15}$$

We introduce these new variables and, at the same time, remove ρ_0+p with $\gamma f_4^{-\frac{1}{2}}$ according to (10). As a result the equations (11) and (12) transform into

$$\frac{\partial\eta}{\partial x}\,\frac{\partial\zeta}{\partial x}=3\eta^{-\frac{2}{3}}+3\varkappa\gamma\zeta^{-\frac{1}{2}}\eta^{\frac{1}{6}}-3\varkappa\rho_0\,, \tag{16}$$

$$2\zeta\,\frac{\partial^2\eta}{\partial x^2} = -\,3\,\varkappa\gamma\zeta^{-\frac{1}{2}}\,\eta^{\frac{1}{6}}\,. \tag{17}$$

Summation of these two equations gives

$$2\zeta\,\frac{\partial^2\eta}{\partial x^2} + \frac{\partial\eta}{\partial x}\,\frac{\partial\zeta}{\partial x} = 3\eta^{-\frac{2}{3}} - 3\,\varkappa\rho_0\,.$$

The integrating multiplier of this equation is $\frac{\partial\eta}{\partial x}$. We obtain, after integration,

$$\zeta\left(\frac{\partial\eta}{\partial x}\right)^2 = 9\,\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\,\eta + 9\,\lambda\,, \tag{18}$$

where λ is the constant of integration.

Raising it to a power of $\frac{3}{2}$ gives

$$\zeta^{\frac{3}{2}}\left(\frac{\partial\eta}{\partial x}\right)^3 = \left(9\,\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\eta + 9\lambda\right)^{\frac{3}{2}}.$$

Dividing (17) by this equation, we obtain that ζ vanishes so that the following differential equations with respect to η is obtained

$$\frac{2\,\frac{\partial^2\eta}{\partial x^2}}{\left(\frac{\partial\eta}{\partial x}\right)^3} = -\frac{3\,\varkappa\gamma\,\eta^{\frac{1}{6}}}{\left(9\,\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\eta + \lambda\right)^{\frac{3}{2}}}\,.$$

Again, $\frac{\partial\eta}{\partial x}$ is the integrating multiplier here. We obtain, after integration,

$$\frac{2}{\left(\frac{\partial\eta}{\partial x}\right)} = 3\,\varkappa\gamma\int\frac{\eta^{\frac{1}{6}}\,d\eta}{\left(9\,\eta^{\frac{1}{3}} - 3\,\varkappa\rho_0\eta + \lambda\right)^{\frac{3}{2}}} \tag{19}$$

and, because

$$\frac{2}{\frac{\delta\eta}{\delta x}} = \frac{2\,\delta x}{\delta\eta}\,,$$

iterated integration gives

$$x = \frac{\varkappa\gamma}{18}\int d\eta\int\frac{\eta^{\frac{1}{6}}\,d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\,\eta + \lambda\right)^{\frac{3}{2}}}\,. \tag{20}$$

It follows from here that x is a function of η and, vice versa, that η is a function of x. Besides, ζ, due to (18), (19), and also (13), is a function of f. Thus our problem has came back to quadratures.

§5. Now we should find the constants of integration so that the internal region of the sphere would be free of singularity, and also the continuous transfer from the values of the functions f and their derivatives inside the sphere to the respective values outside it would be allowed in the surface.

There in the surface of the sphere $r = r_a$, $x = x_a$, $\eta = \eta_a$, etc. The continuity of η and ζ can be satisfied in any case through the respective choice of the constants α and ρ. If also, according to it, the derivatives remain continuous, and, due to (15), $\left(\frac{d\eta}{dx}\right)_a = 3$ and $\left(\frac{d\zeta}{dx}\right)_a = \eta_a^{-\frac{2}{3}}$, the equations (16) and (18) should be

$$\gamma = \rho_0 \zeta_a^{\frac{1}{2}} \eta_a^{-\frac{1}{6}}, \qquad \zeta_a = \eta_a^{\frac{1}{3}} - \frac{\varkappa \rho_0}{3} \eta_a + \lambda . \tag{21}$$

It follows from here that

$$\zeta_a \eta_a^{-\frac{1}{3}} = (f_4)_a = 1 - \frac{\varkappa \rho_0}{3} \eta_a^{\frac{2}{3}} + \lambda \eta_a^{-\frac{1}{3}} .$$

Thus we have

$$\gamma = \rho_0 \sqrt{(f_4)_a} . \tag{22}$$

Comparing it to (10), we see that it satisfies the condition $p = 0$ in the surface. The requirement $\left(\frac{d\eta}{dx}\right)_a = 3$ leads to the following determination of the limits of integration in (19)

$$\frac{3\,dx}{d\eta} = 1 - \frac{\varkappa \gamma}{6} \int_{\eta}^{\eta_a} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa \rho_0}{3} \eta + \lambda\right)^{\frac{3}{2}}} \tag{23}$$

so that, with taking (20) into account, we arrive at the determination of the limits of integration

$$3\,(x - x_a) = \eta - \eta_a + \frac{\varkappa \gamma}{6} \int_{\eta}^{\eta_a} d\eta \int_{\eta}^{\eta_a} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa \rho_0}{3} \eta + \lambda\right)^{\frac{3}{2}}} . \tag{24}$$

The surface conditions are satisfied completely. The constants η_a and λ are still undetermined; we will determine the constants through the continuity conditions at the origin of the coordinates.

First, we should require that $\eta = 0$ at $x = 0$. If this condition were wrong, f_2 would take a finite numerical value at the origin of the coordinates, so the change of the angle $d\varphi = dx^3$ at the origin of the coordinates (that does not mean a real motion) would give a meaning to the line-element. Thus, as follows from (24), the following condition connects x_a and η_a

$$3x_a = \eta_a - \frac{\varkappa\gamma}{6}\int\limits_0^{\eta_a} d\eta \int\limits_{\eta}^{\eta_a} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\,\eta + \lambda\right)^{\frac{3}{2}}}\,. \tag{25}$$

Finally, λ is determined by the condition, according to which the pressure inside the sphere should be finite and positive, as follows from (10), and also f_4 should be finite and nonzero. Proceeding from (13), (18) and (23), we have

$$f_4 = \zeta\,\eta^{-\frac{1}{3}} = \left(1 - \frac{\varkappa\rho_0}{3}\,\eta^{\frac{2}{3}} + \lambda\eta^{-\frac{1}{3}}\right) \times$$

$$\times \left[1 - \frac{\varkappa\gamma}{6}\int\limits_{\eta}^{\eta_0} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\,\eta + \lambda\right)^{\frac{3}{2}}}\right]^2. \tag{26}$$

First, it is supposed here that $\lambda \gtrless 0$. Then, for very small numerical values of η we obtain

$$f_4 = \frac{\lambda}{\eta^{\frac{1}{3}}}\left[K + \frac{\varkappa\gamma}{7}\,\frac{\eta^{\frac{7}{6}}}{\lambda^{\frac{3}{2}}}\right]^2,$$

where

$$K = 1 - \frac{\varkappa\gamma}{6}\int\limits_0^{\eta_0} \frac{\eta^{\frac{1}{6}}\, d\eta}{\left(\eta^{\frac{1}{3}} - \frac{\varkappa\rho_0}{3}\,\eta + \lambda\right)^{\frac{3}{2}}}\,. \tag{27}$$

At the middle point $(\eta = 0)$ f_4 is also infinite, with an exception under the condition $K = 0$ where f_4 vanishes at $\eta = 0$. There is no such case where there could be a finite and nonzero value of f_4 at $\eta = 0$. We see from here that the assumption $\lambda \gtrless 0$ does not lead to physically useful solutions. Hence, we should assume $\lambda = 0$.

§6. Now the condition $\lambda = 0$ constitutes all the constants of integration. If we introduce a new variable χ instead η as follows

$$\sin\chi = \sqrt{\frac{\varkappa\rho_0}{3}}\,\eta^{\frac{1}{3}}, \quad \text{where} \quad \sin\chi_a = \sqrt{\frac{\varkappa\rho_0}{3}}\,\eta_a^{\frac{1}{3}}\,, \tag{28}$$

the equations (13), (26), (10), (24), (25) after elementary algebra take the following form

$$f_2 = \frac{3}{\varkappa\rho_0}\,\sin^2\chi\,, \quad f_4 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2, \quad f_1(f_2)^2 f_4 = 1\,, \tag{29}$$

$$\rho_0 + p = \rho_0 \, \frac{2\cos\chi_a}{3\cos\chi_a - \cos\chi}\,, \tag{30}$$

$$3x = r^3 = \left(\frac{\varkappa\rho_0}{3}\right)^{-\frac{3}{2}} \left[\frac{9}{4}\cos\chi_a \left(\chi - \frac{1}{2}\sin 2\chi\right) - \frac{1}{2}\sin^3\chi\right]. \tag{31}$$

The constant χ_a is determined, through the density ρ_0 and the radius r_a of the sphere, by the ratio

$$\left(\frac{\varkappa\rho_0}{3}\right)^{\frac{3}{2}} r_a^3 = \frac{9}{4}\cos\chi_a \left(\chi_a - \frac{1}{2}\sin 2\chi_a\right) - \frac{1}{2}\sin^3\chi_a\,. \tag{32}$$

The constants α and ρ, in the case of the solution attributed to the external region, follow from (14) as

$$\rho = \eta_a - 3x_a\,, \qquad \alpha = \eta^{\frac{1}{3}} - \zeta_a$$

and take the form

$$\rho = \left(\frac{\varkappa\rho_0}{3}\right)^{-\frac{3}{2}} \left[\frac{3}{2}\sin^3\chi_a - \frac{9}{4}\cos\chi_a \left(\chi_a - \frac{1}{2}\sin 2\chi_a\right)\right], \tag{33}$$

$$\alpha = \left(\frac{\varkappa\rho_0}{3}\right)^{-\frac{1}{2}} \sin^3\chi_a\,. \tag{34}$$

If using the variables χ, ϑ, φ instead of x^1, x^2, x^3, the line-element in the region inside the sphere takes the simple form

$$ds^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 dt^2 - \\ - \frac{3}{\varkappa\rho_0}\left[d\chi^2 + \sin^2\chi\, d\vartheta^2 + \sin^2\chi\,\sin^2\vartheta\, d\varphi^2\right]. \tag{35}$$

Outside the sphere the line-element is still has the same form as that for a point-mass

$$\left.\begin{aligned} & ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) \\ & \text{where} \\ & R^3 = r^3 + \rho \end{aligned}\right\}. \tag{36}$$

Here ρ is determined according to (33), while it was $\rho = \alpha^3$ in the case of a point-mass.

§7. A few following notes should be given on the complete solution of our problem, presented in the previous Paragraph.

1. The spatial element ($dt = 0$) inside the sphere is

$$-\,ds^2 = \frac{3}{\varkappa\rho_0}\left[\,d\chi^2 + \sin^2\!\chi\, d\vartheta^2 + \sin^2\!\chi\,\sin^2\!\vartheta\, d\varphi^2\,\right].$$

This is the line-element of the so-called non-Euclidean geometry of a spherical space. The spherical space geometry holds also in the internal region of our sphere. The curvature radius of such a spherical space is $\sqrt{\frac{3}{\varkappa\rho_0}}$. Our sphere has formed not all of the spherical space, but only a region in it; this is because χ cannot grow up to $\frac{\pi}{2}$, but grows up only to the boundary limit χ_a. Concerning the Sun the curvature radius of the spherical space, which determine the geometry of the interior of the Sun, would be equal to about 500 radii of the Sun (see equations 39 and 42).

This is a very interesting sequel to Einstein's theory, which manifests the fact that this theory is demanded for the geometry of a spherical space as the reality inside a gravitating sphere (this geometry had the power of a purely theoretical consideration before that).

Inside the sphere the "naturally measurable" quantities of length are

$$\sqrt{\frac{3}{\varkappa\rho_0}}\, d\chi\,, \qquad \sqrt{\frac{3}{\varkappa\rho_0}}\,\sin\chi\, d\vartheta\,, \qquad \sqrt{\frac{3}{\varkappa\rho_0}}\,\sin\chi\,\sin\vartheta\, d\varphi\,. \tag{37}$$

The radius of the sphere, "measured from within" to the surface, is

$$P_i = \sqrt{\frac{3}{\varkappa\rho_0}}\,\chi_a\,. \tag{38}$$

The circumference of the sphere, measured along the meridian (or any other great circle) then divided by 2π, should be referred as the "measured-from-outside" radius P_a. It is*

$$P_a = \sqrt{\frac{3}{\varkappa\rho_0}}\,\sin\chi_a\,. \tag{39}$$

According to the formula (36) describing the line-element outside the sphere, this formula for P_a is obviously identical to $R_a = \left(r_a^3 + \rho\right)^{\frac{1}{3}}$ the variable R takes in the surface of the sphere.

*Schwarzschild denoted by i ("innen gemessene") the radius "measured from within", while a (that means "außen gemessene") was used for the radius "measured from outside" due to the original pronunciation of these terms in German. — Editor's comment. D.R.

The following simple relations were obtained for α from (34) through the radius P_a

$$\frac{\alpha}{P_\alpha} = \sin^2\chi_a\,, \qquad \alpha = \frac{\varkappa\rho_0}{3}\,P_a^3\,. \tag{40}$$

Then the volume of our sphere is

$$V = \left(\sqrt{\frac{3}{\varkappa\rho_0}}\,\right)^3 \int\limits_0^{\chi_a} d\chi\,\sin^2\chi \int\limits_0^{\pi} d\vartheta\,\sin\vartheta \int\limits_0^{2\pi} d\varphi = \\ = 2\pi\left(\sqrt{\frac{3}{\varkappa\rho_0}}\,\right)^3\left(\chi_a - \frac{1}{2}\sin 2\chi_a\right).$$

Proceeding from here, the mass M of our sphere is

$$M = \rho_0 V = \frac{3}{4k^2}\sqrt{\frac{3}{\varkappa\rho_0}}\left(\chi_a - \frac{1}{2}\sin 2\chi_a\right), \tag{41}$$

where $\varkappa = 8\pi k^2$.

2. The following notes are related to the equations of motion of a point of infinitely small mass, located outside our sphere. These equations have the same form as those for a point-mass (see equations 15–17 for that*).

At large distances the point moves according to Newton's law, where $\frac{\alpha}{2k^2}$ plays a rôle of the attracting mass. Therefore we can refer to $\frac{\alpha}{2k^2}$ as the "gravitational mass" of our sphere.

If such a point moves from the rest state at infinity up to the surface of the sphere, the "naturally measurable" velocity of fall of this point we obtain is

$$v_a = \frac{1}{\sqrt{1-\frac{\alpha}{R}}}\,\frac{dR}{ds} = \sqrt{\frac{\alpha}{R_a}}\,.$$

Then, according to (40),

$$v_a = \sin\chi_a\,. \tag{42}$$

Concerning the Sun, the velocity of the fall is about $\frac{1}{500}$ of the velocity of light. As easy to see in the case of the small numerical values of χ_a and χ (which is $\chi < \chi_a$) following from this velocity, all our equations

*Here Schwarzschild refers to the equations obtained by him in his first paper: Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

(up to the Einsteinian effects of the second order) transform into to the effects of Newton's theory.

3. For the ratio of the gravitational mass $\frac{\alpha}{2k^2}$ to the mass of matter M we obtain

$$\frac{\alpha}{2k^2 M} = \frac{2}{3}\,\frac{\sin^3\chi_a}{\chi_a - \frac{1}{2}\sin 2\chi_a}\,. \tag{43}$$

With the growing velocity of the fall $v_a = (\sin\chi_a)$ the growing concentration of the mass lowers the ratio of the gravitational mass to the mass of matter. This fact explains that, for instance, at a constant mass and growing density the body approaches the lesser radius than earlier due to the drainage of energy (the lowering of temperature due to radiation).

4. The velocity of light inside our sphere becomes

$$v = \frac{2}{3\cos\chi_a - \cos\chi}\,, \tag{44}$$

and it grows up from the value $\frac{1}{\cos\chi_a}$ in the surface to the value $\frac{2}{3\cos\chi_a - 1}$ at the central point. The value of the density $\rho_0 + p$ grows, according to (10) and (30), proportional to the velocity of light.

At the centre of the sphere ($\chi = 0$) the velocity of light and the density become infinity. Once $\cos\chi_a = \frac{1}{3}$ the velocity of fall reaches $\sqrt{\frac{8}{9}}$ of the (naturally measurable) velocity of light. This value sets the upper limit of the concentration; a sphere of incompressible liquid cannot be denser than this. If we like to apply our equations to the values $\cos\chi < \frac{1}{3}$, we obtain the break just out of the centre of the sphere. At the same time it is possible to find solutions of this problem on the greater values of χ_a continuous at least out of the centre of the sphere, if we move to the case where $\lambda \gtrless 0$ and the condition $K = 0$ (see equation 27) is true. On the path to these solutions, which are however nonsense in physics due to that fact that they give infinite density at the centre of the sphere, we can move to the boundary case where a mass is concentrated in a point, then find, again, the relation $\rho = \alpha^3$ which, according to the earlier study*, is true for a point-mass. We also note that it is possible to talk about only one point-mass in so far as we use the variable r, which in the opposite case (amazingly) does not play a rôle for the geometry and motion in the gravitational field. For

*Schwarzschild K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1916, S. 189–196. — Editor's comment. D.R.

an external observer, as follows from (40), a sphere of the gravitational mass $\frac{\alpha}{2k^2}$ cannot have a radius measured from outside whose numerical value is less than

$$P_a = \alpha\,.$$

Concerning a sphere of incompressible liquid such a border should be $\frac{9}{8}\alpha$. (In the case of the Sun it should be 3 km, while for a mass of 1 gramme it should be 1.5×10^{-28} cm.)

On the Relativistic Theory of an Anisotropic Inhomogeneous Universe

Abraham Zelmanov

Abstract: Here the General Theory of Relativity is expounded from the point of view of space-time as a continuous medium, and the mathematical apparatus for calculation of physically observable quantities (the theory of chronometric invariants) is constructed. Then this mathematical apparatus is applied to set up the basics of the theory of an inhomogeneous anisotropic universe, which profitably contrasts the self-limited theories of a homogeneous universe (most commonly used in modern relativistic cosmology). Owing to such an extension of the relativistic cosmology, we determine the whole range of cosmological models (scenarios of evolution) which could be theoretically conceivable in the space-time of the General Theory of Relativity. Translated from the original Russian manuscript of 1957, in 2008 by Dmitri Rabounski.

§1. The question "is the Universe homogeneous and isotropic, or not" is connected with the question about the scale of the Universe. Let l be a length which is in the order of the upper limit of the space regions meant, by us, to be infinite small. Then $L \gg l$ is a length, which is in the order of the size of the whole region of space we observe. As obvious, in connexion to the question about the scale, two different understandings about homogeneity and isotropy are possible. In other words, two questions can be asked: 1) are the conditions of homogeneity and isotropy satisfied at the numerical values of l and L, assumed by us; 2) is there a large enough l that, under any $L \gg l$, the conditions of homogeneity and isotropy are satisfied.

In comparing the theory to observations, the first of the above understandings of homogeneity and isotropy plays a rôle. In such a case the numerical values of l and L should be determined at least in the order of these values. In consideration of questions such as those related to the infinity of space, the second understanding of homogeneity and isotropy is important. Observational data give no direct answer to the question about homogeneity and isotropy of the Universe with respect to the second meaning. I don't provide the references to the observational data here. On the other hand, much information about the distribution of masses, provided for instance by Ambarzumian in his presentation [1], allows us to be sure in the fact that, at any $l \ll L$, the Universe is inhomogeneous, in the first meaning of this term. Ambarzumian was absolutely right in his note that the Metagalactic redshift should be interpret as the Doppler effect, and, when considering the scale of the Metagalaxy, we should take into account the effects of the General Theory of Relativity. Thus a relativistic theory of an inhomogeneous anisotropic universe — a theory, the results of which would be able to be compared to the observational data, and which, generally speaking, gives a model of the whole Universe — should be our task.

As will be shown in §10, inhomogeneity leads to anisotropy.* On the other hand, at least some factors of anisotropy bear a tendency to decrease with the expansion of the Metagalaxy (see §13). So the anisotropy, being weak in the current epoch, was probable a valuable factor which played an important rôle billions of years ago.

So a relativistic theory of an inhomogeneous anisotropic universe is our actual task. The necessity of such a theory was pointed out, aside for the special studies on this theme, in [6, 7] and also in [8]. Below, only a few problems related to the formal mathematical basics of this theory will be considered. It should be noted that the basic equations and the deduced equations of our theory do not depend on homogeneity and isotropy of the Universe in the second meaning: the equations are independent of the numerical values of l and L.

Assume that matter on the scale we are considering is a continuous medium, which moves laminarely in common with a continuous field of sub-luminal velocities. So there are coordinate frames which everywhere accompany the medium. On the other hand, all that will be said in §3–§8 does not depend on these assumptions. Of course, in the modern epoch of observation, the most interesting case is such a scale of consideration where the "molecules" of this medium are galactic clusters. Suppose also that, in the scale we are considering, the thermodynamical terms are meaningful, and the laws of relativistic thermodynamics hold. We also assume that Einstein's equations

$$G_{\mu\nu} = -\varkappa \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T \right) + \Lambda\, g_{\mu\nu} \tag{1}$$

are true everywhere in the four-dimensional region we are considering. Here in the equations $G_{\mu\nu}$ is the contracted world-tensor of the curvature, $g_{\mu\nu}$ is the metric world-tensor, $T_{\mu\nu}$ is the energy-momentum tensor, $T = T^{\alpha}_{\alpha}$, $\varkappa$ is Einstein's constant of gravitation ($\varkappa = 8\pi\gamma/c^2$, where γ is Newton's constant of gravitation and c is the fundamental velocity), while Λ is the cosmological constant. We keep the cosmological constant

*In addition to it, there are observational data about anisotropy of the redshift in a small region of the Metagalaxy near us [2–4]. Of course, this information should be checked on the basis of the newest observational data [5].

in the equations, because we like to have a possibility to compare our results to those known in the literature.

§2. We always mean spatial (three-dimensional) homogeneity and isotropy, not four-dimensional world-quantities. The presence or the absence of spatial homogeneity and spatial isotropy depends on the frame of reference. For instance, as obvious, the isotropy can be attributed to only those reference frames, which accompany continuous matter and masses, because both the flow of matter and moving masses break the isotropy. On the one hand, the question about the presence of homogeneity and isotropy can be set up as the question about the possibility of such reference frames, where the said homogeneity and isotropy take a place. We all know the homogeneous isotropic relativistic models. In such a model, a frame consisting of the four coordinates can be chosen, wherein

$$\left.\begin{aligned} ds^2 = c^2dt^2 - R^2\,\frac{d\xi^2 + d\eta^2 + d\zeta^2}{\left[1 + \frac{k}{4}\left(\xi^2 + \eta^2 + \zeta^2\right)\right]^2} \\ R = R(t), \qquad k = 0, \pm 1 \end{aligned}\right\}. \tag{2}$$

Such a reference frame can be the necessary and sufficient indication of homogeneity and isotropy in cosmology. On the other hand, the question about the presence of homogeneity and isotropy can be set up in a frame of the accompanying coordinates. Such a statement of this problem will be realized in the next Paragraphs.

The theory of an inhomogeneous anisotropic universe has two main directions, which are characterized as follows: a) the search for exact particular solutions of the equations of gravitation, and the consideration of such models which bear the properties of symmetry; b) as common as possible, the qualitative study of the behaviour (evolution) of matter and the metric under different physical assumptions.

The models, which are spherically symmetric under the vanishing of the pressure, viscosity, and the flow of energy, the models with a spherically symmetric distribution of matter concentrated in a centre (core), and the models filled with a limited spherical distribution of matter were studied by McVittie [9], Tolman [10, 12], Datt [11], Oppenheimer and Volkoff [13], Oppenheimer and Snyder [14], Järnefelt [15, 16], Einstein and Strauss [17], Bondi [18], Omer [19], Just [20]. The models, which are axially symmetric and rotating, were considered in the studies of Kobushkin [21] and Gödel [22]. There are main studies produced in the research direction a), or connected to it.

Among the studies produced in the research direction b), McCrea's study [23] remains aloof, where the problem of the observable properties of an inhomogeneous anisotropic universe was considered. The behaviour (evolution) of matter and the metric in such a universe was qualitatively considered in studies of mine [24, 26], Raychaudhury [25] and Komar [27]. In the study [24] I introduced chronometrically invariant quantities (using another terminology), and considered applications of them in the General Theory of Relativity to the problem we are now interested in, in the framework of the particular conditions, where the flow of energy, viscosity, pressure, and, hence, the power field were neglected. A few years later, Raychaudhury [25] considered particular aspects of the same problem in the case where $\Lambda = 0$, with the neglection of the same factors. The quantities and equations derived by him, and also his conclusions [25] are the same as that which was found by me earlier [24]. Raychaudhury however did not introduce chronometrically invariant quantities, and used the incorrect definition (12) of the observable spatial metric instead of the correct formula (7) given below. As a result, his equations, generally speaking, don't possess a direct physical interpretation in the framework of the considered problem. Meanwhile, using [24, 26] one can show that his results concerning the effects produced by, in our terminology, the absolute rotation and the anisotropy of the deformations in a) the behaviour of the changes of a space volume and b) the scale of time are correct in the considered case. His results in the research direction a) repeated some results obtained earlier by me in [24]. The research direction b) was not considered in my study [24]. My newest paper [26] constituted supplement and generalization of the results, which were obtained earlier in [24] under lower assumptions. Komar [27] showed that special states are inevitable in the case of $\Lambda = 0$ under the absence of, in our terminology, the power field, absolute rotation, pressure, viscosity and the flow of energy. This conclusion repeats one of the results obtained earlier in [24] and [25].

In the next Paragraphs I give the further generalization and development of some results initially obtained by me in [24, 26].

This is Gödel's solution [22], which will be required in our research:

$$\left.\begin{array}{c} ds^2 = a^2\left[(dx^0)^2 + 2\,e^{x^1}dx^0dx^2 - (dx^1)^2 + \dfrac{e^{2x^1}}{2}(dx^2)^2 - (dx^3)^2\right] \\ 0 < a = const \end{array}\right\}. \tag{3}$$

In cosmology, accompanying coordinates are commonly used. The necessary and sufficient condition for such coordinates requires that the numerical value of the three-dimensional velocity should be lower than the velocity of light, while the components of the velocity should be finite, simple and continuous functions of the four coordinates.

§3. We denote space-time indices 0, 1, 2, 3 in Greek (where 0 corresponds to the time dimension), while spatial indices 1, 2, 3 are denoted in Roman. We assume that summation takes a place on two same indices met in the same term. We assume that

$$x^0 = ct, \qquad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu,$$

and, in a locally Galilean reference frame, we have

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

We assume also that the components of the metric world-tensor $g_{\mu\nu}$ are continuous along the coordinates x^α in common with their first derivatives and second derivatives. In common, we assume that all quantities used here satisfy, in this part and in the rest parts, the requirements of the General Theory of Relativity.* Besides, while talking about three-dimensional (spatial) tensors and the other three-dimensional geometrical quantities (e.g. Christoffel's symbols), we will omit the notion about the number of the dimensions.

Four-dimensional coordinate systems resting with respect to the same reference body (which is deforming, in a general case) are connected to each other by the transformations

$$\tilde{x}^0 = \tilde{x}^0(x^0, x^1, x^2, x^3)\,, \tag{4a}$$

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3)\,, \qquad \frac{\partial \tilde{x}^i}{\partial x^0} = 0\,. \tag{4b}$$

The choice of a body of reference is equivalent to the choice of the congruence of the time lines $x^i = const$. Suppose that a reference body has been chosen. Then, of all the quantities non-covariant to the general transformations

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^0, x^1, x^2, x^3)\,, \tag{5}$$

those quantities are physically preferred which are covariant with respect to the transformations (4a) and (4b). Hence, such physically preferred quantities are invariant with respect to the transformations (4a), and are covariant to the transformations (4b). We therefore call such physically preferred quantities *chronometric invariants*. Such chrono-

*The formulae (1), (2) and (3) satisfy all these requirements as well.

metrically invariant quantities can be considered as three-dimensional tensors in any of the given spatial sections $x^0 = const$. They can be also considered as tensors in a space, all elements of which (i.e. three-dimensional local spaces) are definitely orthogonal to the time lines under any given coordinate of time. We mean, by a three-dimensional space of a given body of reference (a reference space), a space determined in this way. Such a space is, generally speaking, non-holonomic. This means that, with a body of reference given in a general case, it is impossible to find such a spatial section which could be everywhere orthogonal to the time lines: in such a general case it is impossible to find, by the transformation (4a), such a coordinate of time x^0 that $g_{0i} = 0$ would be everywhere true in the spatial section.

With chronometrically invariant quantities and chronometrically invariant operators, we remove a difficulty proceeding from the fact that many non-chronometrically invariant quantities and relations (the conditions of homogeneity and isotropy, for instance) depend on the arbitrarity of our choice of the time coordinate. In a general case (in Gödel model, for instance), this difficulty can neither be avoided by the choice of a preferred coordinate of time satisfying the conditions $g_{00} = 1$ and $g_{0i} = 0$ (as for the homogeneous isotropic models) nor the substantial easing of this situation due to the choice of a preferred coordinate of time such that the weak condition $g_{0i} = 0$ satisfies everywhere.

Let $Q^{ik\dots p}_{00\dots 0}$ be the components of a world-tensor of the rank n, all upper indices of which are nonzero, while all m lower indices are zero. For such a tensor, the quantities

$$T^{ik\dots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\dots p}_{00\dots 0}$$

are the components of a chronometrically invariant contravariant (three-dimensional) tensor of the rank $n - m$. Using this rule, we can easily find the chronometrically invariant form for quantities and operators, if we know the formulae of them under a specially chosen coordinate of time according to the transformations (4a), for instance, if $g_{00} = 1$ and $g_{0i} = 0$ at the given world-point.

§4. Targeting the chronometrically invariant formulae for the elementary length $d\sigma$, the metric tensors h_{ik} and h^{ik}, and the fundamental determinant $h = |h_{ik}|$, we obtain

$$d\sigma^2 = h_{ik}\, dx^i dx^k, \tag{6}$$

$$h_{ik} = -\,g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}}\,, \quad h^{ik} = -\,g^{ik}, \quad h = -\frac{g}{g_{00}}\,, \tag{7}$$

where $g = |g_{\mu\nu}|$. The spatial metric determined in such a way coincides with that assumed by Landau and Lifshitz, see (82.5) and (82.6) in [28], and that assumed by Fock, see (55.20) in [29]. For the elementary chronometrically invariant interval of time $d\tau$ and the elementary world-interval ds, we obtain

$$c\,d\tau = \frac{g_{0\alpha}dx^{\alpha}}{\sqrt{g_{00}}}\,, \qquad ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{8}$$

For the chronometrically invariant velocity v^i of the motion of a test-particle, we have

$$\mathrm{v}^i = \frac{dx^i}{d\tau}\,, \qquad h_{ik}\,\mathrm{v}^i\mathrm{v}^k = \left(\frac{d\sigma}{d\tau}\right)^2.$$

If $ds = 0$, $h_{ik}\,\mathrm{v}^i\mathrm{v}^k = c^2$: the chronometrically invariant velocity of light in vacuum is always equal the fundamental velocity.

We mark the chronometrically invariant differential operators by the asterisk. For such operators (they coincide with d/dt, $\partial/\partial t$ and $\partial/\partial x^i$ under the conditions $g_{00} = 1$ and $g_{0i} = 0$) we obtain

$$\frac{{}^*d}{dt} = \frac{d}{d\tau}\,, \qquad \frac{{}^*\partial}{\partial t} = \frac{c}{\sqrt{g_{00}}}\frac{\partial}{\partial x^0}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0}\,. \tag{9}$$

For the chronometrically invariant generalizations of Christoffel's symbols and the operator of general covariant differentiation, we have*

$$\Delta_{ij,k} = \frac{1}{2}\left(\frac{{}^*\partial h_{jk}}{\partial x^i} + \frac{{}^*\partial h_{ik}}{\partial x^j} - \frac{{}^*\partial h_{ij}}{\partial x^k}\right), \quad \Delta^k_{ij} = h^{kl}\Delta_{ij,l}\,, \tag{10}$$

$${}^*\nabla_i\, Q^{\cdot\cdot k}_{j\cdots} = \frac{{}^*\partial Q^{\cdot\cdot k}_{j\cdots}}{\partial x^i} - \Delta^l_{ij}Q^{\cdot\cdot k}_{l\cdots} - \dots + \Delta^k_{il}Q^{\cdot\cdot l}_{j\cdots}\,. \tag{11}$$

As can be easily seen,

$${}^*\nabla_i\, h_{jk} = 0\,, \qquad {}^*\nabla_i\, h^k_j = 0\,, \qquad {}^*\nabla_i\, h^{jk} = 0\,.$$

The metric of a spatial section $x^0 = const$ is determined by the tensor

$$y_{ik} = -\,g_{ik}\,, \quad y^{ik} = -\,g^{ik} + \frac{g^{0i}g^{0k}}{g^{00}}\,, \quad y = -\,g\,g^{00}, \tag{12}$$

where $y = |y^{ik}|$ is the determinant of the tensor.

The metric (7) is chronometrically invariant, space-like everywhere, and the length of an "unchangeable" elementary rest-scale in this metric equals the "proper" length. On the other hand, the metric (12) does not

*See also formula (17).

bear these properties: see, for instance, Gödel's model (3). In particular, in contrast to h, y may become negative (in general, $y \leqslant h$) that leads to the negative numerical value of the volume of the region of the spatial section where this happens. Therefore, even with a given coordinate of time, and fixation of the numerical value of the coordinate, a true physical meaning is attributed to not the metric (12) of a spatial section $x^0 = const$, but to the metric (7) of the space (space-time) where the value $x^0 = const$ is fixed.

§5. We assume that all differentiable quantities bear the properties which allow them to change the order in the usual (not chronometrically invariant or generally covariant) differentiation. In such a case,

$$\frac{{}^*\partial^2}{\partial x^i\,\partial t} - \frac{{}^*\partial^2}{\partial t\,\partial x^i} = \frac{F_i}{c^2}\frac{{}^*\partial}{\partial t}\,, \qquad \frac{{}^*\partial^2}{\partial x^i\,\partial x^k} - \frac{{}^*\partial^2}{\partial x^k\,\partial x^i} = \frac{2\,A_{ik}}{c^2}\frac{{}^*\partial}{\partial t}\,. \tag{13}$$

These chronometrically invariant vector F_i and chronometrically invariant antisymmetric tensor A_{ik}, determined by the equalities (9) and (13), satisfy the identities

$$\frac{{}^*\partial A_{jk}}{\partial x^i} + \frac{{}^*\partial A_{ki}}{\partial x^j} + \frac{{}^*\partial A_{ij}}{\partial x^k} + \frac{1}{c^2}\left(F_i A_{jk} + F_j A_{ki} + F_k A_{ij}\right) = 0\,, \tag{14}$$

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{15}$$

and also to the identities (17).

The identity satisfying the three equalities $A_{ik} = 0$ in a given four-dimensional region is the necessary and sufficient condition for the reducing of all g_{0i} to zero everywhere in this region by the transformation (4a): in such a case $d\tau$ has an integration multiplier, i.e. time is allowed to be integrated along a path in this region (time is integrable). In other words, the identity satisfying the equalities $A_{ik} = 0$ is the necessary and sufficient condition of holonomity of the given space of reference. Thus A_{ik} is the chronometrically invariant tensor of the space non-holonomity. The identity satisfying all six equalities $F_i = 0$ and $A_{ik} = 0$ in a given four-dimensional region is the necessary and sufficient condition for the reducing of all g_{00} to 1 and of all g_{0i} to zero by the transformation (4a). In other words, this is necessary and sufficient for $d\tau$ to be a total differential.

At any world-point O, one can set up a four-dimensional locally geodesic frame of reference $\tilde{\Sigma}_0$, which satisfies the following condition: at this point, the chronometrically invariant velocity of a given reference frame Σ with respect to the locally geodesic reference frame $\tilde{\Sigma}_0$

is zero $(\tilde{\rm v}^j)_0 = 0$. Considering the reference frame $\tilde{\Sigma}_0$, we introduce in it the chronometrically invariant quantities which characterize the motion of our reference frame Σ with respect to $\tilde{\Sigma}_0$ in a four-dimensional neighbourhood of the point O: we take the generally covariant characteristics of the motion such as the acceleration vector $(\tilde{w}_j)_0$, the tensor of angular velocity of the rotation $(\tilde{a}_{jl})_0$ and the tensor of the rate of the deformation $(\tilde{d}_{jl})_0$, then express them through the chronometrically invariant velocity by the removing of regular derivatives with chronometrically invariant derivatives. Using the general transformations (5), we obtain that the equalities

$$F_i = -\frac{\partial \tilde{x}^j}{\partial x^i}\,(\tilde{w}_j)_0\,, \qquad A_{ik} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^k}\,(\tilde{a}_{jl})_0$$

are true at any world-point O.

We introduce also a chronometrically invariant tensor D_{ik}, which satisfies the equality

$$D_{ik} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^k}\,(\tilde{d}_{jl})_0$$

at any world-point O.

In this context, F_i is the vector of acceleration of our reference space Σ with respect to the locally geodesic reference space $\tilde{\Sigma}_0$, taken with the opposite sign, A_{ik} is the tensor of angular velocity of the rotation of our reference space Σ with respect to $\tilde{\Sigma}_0$, while D_{ik} is the tensor of the rate of deformation of our reference space Σ with respect to $\tilde{\Sigma}_0$. It is possible to prove that

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \quad D = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \tag{16}$$

where $D = D_j^j$ has the meaning of the speed of the relative expansion of the volume element of the space.*

Denote by $\Gamma^\sigma_{\mu\nu}$ the four-dimensional Christoffel symbols of the 2nd kind. Then we have the identities

$$\left.\begin{aligned} &\frac{\Gamma^i_{00}}{g_{00}} = -\frac{F^i}{c^2}\,, \qquad \frac{g^{i\alpha}\Gamma^k_{\alpha 0}}{\sqrt{g_{00}}} = -\frac{1}{c}\left(A^{ik} + D^{ik}\right) \\ &g^{i\alpha}g^{j\beta}\Gamma^k_{\alpha\beta} = h^{il}h^{jm}\Delta^k_{lm} \end{aligned}\right\}, \tag{17}$$

*The volume of an element of the space we are considering can be represented as an integral from $\sqrt{h}\,dx^1dx^2dx^3$, where dx^i and also the region of the change of x^1 along which the integration is processed are independent of x^0.

which allow us to find F_i, A_{ik}, D_{ik} and Δ^k_{lm} through $\Gamma^\sigma_{\mu\nu}$.

The study of the equations of motion of a particle, presented in [26], manifested that F^k can be interpreted as the sum of the force of gravity and the force of inertia (the latter is derived from the carrying acceleration), both calculated for the unit of mass, while A_{ik} is the angular velocity of the absolute rotation of our reference frame derived from Coriolis' effect.

§6. For a covariant vector Q_l, with the note on the properties of the differentiable quantities we made in the beginning of §5, we obtain

$$\left({}^*\nabla_{ik} - {}^*\nabla_{ki}\right)Q_l = \frac{2\,A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdots j}\,Q_j\;, \tag{18}$$

$$H_{lki\cdot}^{\cdots j} = \frac{{}^*\partial\Delta^j_{il}}{\partial x^k} - \frac{{}^*\partial\Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}\;. \tag{19}$$

The chronometrically invariant tensor (19), which is analogous to Schouten's tensor, is different in its properties in a general case from the Riemann-Christoffel tensor. We introduce the chronometrically invariant tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \tag{20}$$

which possesses all the algebraical properties of the Riemann-Christoffel tensor. There are identity correlations between the quantities H_{lkij}, from one side, and also the quantities C_{lkij}, D_{mn} and A_{mn} from the other side. As easy to see,

$$H_{lkij} = C_{lkij} + \frac{1}{c^2}\left(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + A_{kl}D_{ij} + A_{li}D_{jk}\right). \tag{21}$$

As obvious, if $A_{mn}=0$ or $D_{mn}=0$, we have $H_{lkij}=C_{lkij}$. We introduce also $H_{lk}=H_{lki\cdot}^{\cdots i}$, $H=H^k_k$ and $C_{lk}=C_{lki\cdot}^{\cdots i}$, $C=C^k_k$. Then

$$H_{lk} = C_{lk} + \frac{1}{c^2}\left(A_{kj}D^j_l + A_{lj}D^j_k + A_{kl}D\right), \quad H = C\,. \tag{22}$$

The metric of any spatial section is determined by (12). The curvature of a spatial section is characterized by the regular Riemann-Christoffel tensor K_{lkij} corresponding to the metric (12). Pave such spatial sections $x^0 = const$ through a world-point O, but at different coordinates of time, that they satisfy (at this point O) the conditions

$$g_{0i} = 0\,, \qquad \frac{\partial g_{0i}}{\partial x^k} + \frac{\partial g_{0k}}{\partial x^i} = 0\,. \tag{23}$$

We call such spatial sections maximally orthogonal to the time line in a neighbourhood of the given world-point. Each of the spatial sections possesses its own Riemann-Christoffel tensor K_{lkij}. These tensors coincide with each other at this point O, and satisfy the equality

$$C_{lkij} = K_{lkij} + \frac{2}{c^2}\left(A_{ij}A_{kl} + A_{jk}A_{il} + 2A_{ik}A_{jl}\right). \tag{24}$$

In each of these maximally orthogonal spatial sections, which cross the space at the point O, we introduce the regular Riemann-Christoffel tensor corresponding to the metric (7), not to (12).* This tensor can be considered as the Riemann-Christoffel tensor of a space, wherein the coordinate of time is fixed at a numerical value $x^0 = const$, satisfying the conditions (23). At the world-point O, the tensors coincide with each other in all the spatial sections, and are equal to C_{lkij}. Let x^{mn} be a chronometrically invariant unit bivector, which fixes a two-dimensional direction in a given spatial section. In such a case, for the Riemannian curvature in this two-dimensional direction, we have $K_{lkij}\,x^{ik}x^{lj}$ in the metric (12) and $C_{lkij}\,x^{ik}x^{lj}$ in the metric (7). Due to (21) and (24),

$$H_{lkij}\,x^{ik}x^{lj} = C_{lkij}\,x^{ik}x^{lj} = K_{lkij}\,x^{ik}x^{lj} - \frac{12}{c^2}\left(A_{ij}x^{ij}\right)^2. \tag{25}$$

We introduce also $K_{lk} = K_{lki\cdot}^{\cdots i}$ and $K = K_k^k$. In such a case,

$$C_{lk} = K_{lk} + \frac{6}{c^2}A_{li}A_{k\cdot}^{\cdot i}\,, \qquad C = K + \frac{6}{c^2}A_{ki}A^{ki}. \tag{26}$$

For the Gaussian curvatures, we have, respectively: $-\frac{1}{6}C$ and $-\frac{1}{6}K$. As obvious,

$$C_{lkij}\,x^{ik}x^{lj} \leqslant K_{lkij}\,x^{ik}x^{lj}, \qquad -\frac{1}{6}\,C \leqslant -\frac{1}{6}\,K\,.$$

Thus, with a fixed A_{mn}, the space curvature is characterized by the quantities C_{lkij}, C_{lk} and C, which are connected to the metric (7), and also by the quantities K_{lkij}, K_{lk} and K, connected to the metric (12). Because the metric (7) is physically preferred, we will use those quantities, which are connected to it.

§7. Here we introduce auxiliary quantities and relations.

Let ε_{ijk} and ε^{ijk} be such antisymmetric unit chronometrically invariant tensors that $\varepsilon_{123} = \sqrt{h}$ and $\varepsilon^{123} = 1/\sqrt{h}$. As easy to see,

$$^*\nabla_l\,\varepsilon_{ijk} = 0\,, \quad ^*\nabla_l\,\varepsilon^{ijk} = 0\,, \quad \frac{^*\partial\varepsilon_{ijk}}{\partial t} = \varepsilon_{ijk}\,D\,, \quad \frac{^*\partial\varepsilon^{ijk}}{\partial t} = -\,\varepsilon^{ijk}D\,.$$

*Generally speaking, the metrics (7) and (12) coincide only at the point O.

We introduce the chronometrically invariant vector of angular velocity of the space rotation

$$\Omega^i = \frac{1}{2}\,\varepsilon^{ijk} A_{jk}\,, \qquad \Omega_i = \frac{1}{2}\,\varepsilon_{ijk}\, A^{jk}. \tag{27}$$

The identities (14) and (15) can be represented, respectively, as

$$^*\nabla_j\,\Omega^j + \frac{1}{c^2}\,F_j\,\Omega^j = 0\,, \tag{28}$$

$$\frac{2}{\sqrt{h}}\,\frac{^*\partial}{\partial t}\left(\sqrt{h}\,\Omega^i\right) + \varepsilon^{ijk}\,^*\nabla_j\,F_k = 0\,. \tag{29}$$

As obvious, any axial vector field is anisotropic. The field of a tensor Z_{ik} is isotropic, if $Z_{ik} = \frac{1}{3}\,Zh_{ik}$, where $Z = Z_j^j$. We characterize the anisotropy of the space deformation and the anisotropy of the space curvature by the quantities, respectively,

$$\Pi_{ik} = D_{ik} - \frac{1}{3}\,Dh_{ik}\,, \quad \Pi_i^k\Pi_k^i = D_i^k D_k^i - \frac{1}{3}\,D^2 \geqslant 0\,, \tag{30}$$

$$\Sigma_{ik} = C_{ik} - \frac{1}{3}\,C\,h_{ik}\,. \tag{31}$$

The condition of homogeneity of the field of any tensor $Z_{i\dots}^{\cdot\cdot k}$ can be written in the form: $^*\nabla_j\,Z_{i\dots}^{\cdot\cdot k} = 0$.

We assume the notations

$$\dot{Z} = \frac{\partial Z}{\partial t}\,, \qquad ^*\dot{Z} = \frac{^*\partial Z}{\partial t}\,. \tag{32}$$

As obvious, having any chosen coordinate of time, the conditions $^*\dot{Z} = 0$ and $^*\ddot{Z} > 0$ are equivalent to the conditions $\dot{Z} = 0$ and $\ddot{Z} > 0$, the conditions $^*\dot{Z} = 0$ and $^*\ddot{Z} = 0$ are equivalent to the conditions $\dot{Z} = 0$ and $\ddot{Z} = 0$, while the conditions $^*\dot{Z} = 0$ and $^*\ddot{Z} < 0$ are equivalent to the conditions $\dot{Z} = 0$ and $\ddot{Z} < 0$. Thus, marking the time derivatives by the asterisk, we can write the conditions of the extrema in the chronometrically invariant form.

§8. We denote by ρ the density of mass, by J^i the vector of the density of the flow of mass (this quantity is the same that the vector of the density of momentum), by U^{ik} the tensor of the density of the flow of momentum, and $U = U_j^j$. As obvious, ρc^2 is the density of energy, while $J^i c^2$ is the vector of the density of the flow of energy. Let these notations be attributed to chronometrically invariant quantities.

In such a case,

$$\frac{T_{00}}{g_{00}} = \rho\,, \qquad \frac{c\,T_0^i}{\sqrt{g_{00}}} = J^i, \quad c^2 T^{ik} = U^{ik}, \quad T = \rho - \frac{U}{c^2}\,.$$

The equations of the conservation of energy and momentum can be written as follows

$$\frac{^*\partial\rho}{\partial t} + D\rho + \frac{1}{c^2}\,D_{ij}\,U^{ij} + \left[\left(^*\nabla_j - \frac{1}{c^2}\,F_j\right) J^j\right] - \frac{1}{c^2}\,F_j\,J^j = 0\,, \qquad (33)$$

$$\frac{^*\partial J^k}{\partial t} + DJ^k + 2\big(D_i^k + A_{i\cdot}^{\cdot k}\big)J^i + \left[\left(^*\nabla_i - \frac{1}{c^2}\,F_i\right) U^{ik}\right] - \rho F^k = 0\,. \quad (34)$$

All the terms contained on the left side of the equations (33) and (34) have obvious physical meanings. The third term and the fifth (last) term in (33) are the relativistic terms, proceeding from the connexion between mass and energy. These terms take into account the change of the density of mass, which is due to the surface forces working while the volume element of space deforms (the third term on the left side), and the change of the flowing energy due to the acting gravitational and inertial forces (the fifth term). The fourth terms of (33) and (34) (they are taken into square brackets) constitute the "physical divergence" of J^i and U^{ik} respectively. The fact that physical divergence differs from mathematical divergence originates in the circumstance that, with the same dt, the intervals $d\tau$ are different at different coordinate points on the boundary of the elementary volume. As is obvious, (33) and (34) are the actual equations for mass and momentum. Multiplying (33) and (34) term-by-term by c^2, we are able to obtain the actual equations for energy and the flow of energy.

With all the above, Einstein's equations (1) take the form

$$\frac{^*\partial D}{\partial t} + D_{jl}D^{lj} + A_{jl}A^{lj} + {}^*\nabla_j\,F^j - \frac{1}{c^2}\,F_jF^j = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \Lambda c^2, \qquad (35)$$

$$^*\nabla_j\big(h^{ij}D - D^{ij} - A^{ij}\big) + \frac{2}{c^2}\,F_jA^{ij} = \varkappa J^i, \qquad (36)$$

$$\frac{^*\partial D_{ik}}{\partial t} - \big(D_{ij} + A_{ij}\big)\big(D_k^j + A_{k\cdot}^{\cdot j}\big) + DD_{ik} - D_{ij}D_k^j + 3A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left(^*\nabla_i\,F_k + {}^*\nabla_k\,F_i\right) - \frac{1}{c^2}\,F_iF_k - c^2C_{ik} = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right) + \Lambda c^2 h_{ik}\,. \qquad (37)$$

As obvious, all these ten equations (35), (36) and (37) are connected by four relations (33) and (34).

In a coordinate frame accompanying a medium, this medium plays a rôle of the body of reference, while the world-lines of the elements of this medium are the lines of time. In such a reference frame, the tensors D_{ik} and A_{ik} characterize the rate of deformation and the velocity of rotation of the medium. The equations (35), (36) and (37) in such accompanying coordinates can be considered as the equations of motion of a continuous medium. These equations, in common with the equations (33) and (34) and also the identities (14) and (15), allow a far-reaching analogy with the non-relativistic equations of motion of a continuous medium which satisfy the unitary interpretation [30]. This analogy permits the non-relativistic equations to be used for the quasi-Newtonian approximation to the relativistic theory (see §18).

§9. So forth, we use a coordinate frame, which accompanies a medium. We assume also that there are no other forces produced by masses, aside for the gravitational and inertial forces. We characterize the following chronometrically invariant quantities: the equilibrium pressure p_0 (it is determined by the equation of state), the true pressure p, the tensor of the viscous tension α_{ik}, the anisotropic part $\beta_{ik} = \alpha_{ik} - \frac{1}{3}\alpha h_{ik}$ of the viscosity tensor α_{ik}, and also $\alpha = \alpha_j^j$. With these notations,

$$p = p_0 - \frac{1}{3}\,\alpha\,, \quad U_{ik} = p_0 h_{ik} - \alpha_{ik} = p h_{ik} - \beta_{ik}\,.$$

Using (33) and thermodynamical considerations, we obtain

$$\beta_{jl} D^{jl} = \beta_{jl}\,\Pi^{jl} \geqslant 0\,, \qquad \alpha\, D \geqslant 0\,. \tag{38}$$

In the coordinates, accompanying the medium, $c^2 J^i = q^i$, where q^i is the vector of density of the flow of any form of energy (radiation or heat) with respect to this medium. The viscosity, characterized by the tensor β_{ik}, and the viscosity, characterized by the scalar α, can be considered as the viscosity of the 1st kind and that of the 2nd kind, respectively. Thus the conservation equations (33) and (34) take the form

$$\frac{{}^*\partial\rho}{\partial t} + D\left(\rho + \frac{p}{c^2}\right) = \frac{1}{c^2}\left[\beta_{jl}\,\Pi^{jl} - \left({}^*\nabla_j\, q^j - \frac{2}{c^2}\,F_j\, q^j\right)\right], \tag{39}$$

$$\frac{1}{c^2}\left(\frac{{}^*\partial q_i}{\partial t} + D q_i - 2A_{i\cdot}^{\cdot j} q_j\right) - \left({}^*\nabla_j - \frac{1}{c^2}\,F_j\right)\beta_i^j + \\ + \left(\frac{{}^*\partial p}{\partial x^i} - \frac{1}{c^2}\,F_i\, p\right) - \rho F_i = 0\,, \tag{40}$$

while the system of Einstein's equations (35), (36) and (37) is equivalent to the system

$$\frac{{}^*\partial D}{\partial t} + \frac{1}{3}D^2 + \Pi_{jl}\Pi^{jl} - A_{jl}A^{jl} + \\ + {}^*\nabla_j F^j - \frac{1}{c^2}F_jF^j = -\frac{\varkappa}{2}\left(\rho c^2 + 3p\right) + \Lambda c^2, \quad (41)$$

$$\frac{4}{3}\frac{{}^*\partial D}{\partial x^i} - {}^*\nabla_j\left(\Pi_i^j + A_{i\cdot}^{\cdot j}\right) + \frac{2}{c^2}F_jA_{i\cdot}^{\cdot j} = \frac{\varkappa}{c^2}q_i\,, \quad (42)$$

$$\frac{1}{3}D^3 - \frac{1}{2}\Pi_{jl}\Pi^{jl} + \frac{3}{2}A_{jl}A^{jl} - \frac{1}{2}c^2C = \left(\varkappa\rho + \Lambda\right)c^2, \quad (43)$$

$$\frac{{}^*\partial\Pi_i^k}{\partial t} + D\,\Pi_i^k + \Pi_{ij}A^{kj} + \Pi^{kj}A_{ij} + 2\left(A_{ij}A^{kj} - \frac{1}{3}A_{jl}A^{jl}h_i^k\right) + \\ + \left[\frac{1}{2}\left({}^*\nabla_i F^k + {}^*\nabla^k F_i\right) - \frac{1}{3}\left({}^*\nabla_j F^j\right)h_i^k\right] - \\ - \frac{1}{c^2}\left(F_iF^k - \frac{1}{3}F_jF^jh_i^k\right) - c^2\Sigma_i^k + \varkappa\beta_i^k = 0\,. \quad (44)$$

The equations (41) and (42) are the actual equations (35) and (36) transformed with (30). The equations (43) and (44) were obtained from (35) and (37) with the use of (30) and (31). The left side of (44) is a tensor, whose trace is identically equal to zero. Thus all six equations, which constitute (44), are connected by the same algebraic relation.

§10. The equations (41) and (44) set up a connexion between the fields F_i, A_{ik}, D_{ik} and C_{ik}, from the one side, and the fields ρ, p, β_{ik} and q_i, from the other side. It is natural to determine the homogeneity of the Universe, in a given local region of it, by the conditions

$$\left.\begin{array}{llll} {}^*\nabla_j F_i = 0\,, & {}^*\nabla_j A_{ik} = 0\,, & {}^*\nabla_j D_{ik} = 0\,, & {}^*\nabla_j C_{ik} = 0 \\ \dfrac{{}^*\partial\rho}{\partial x^i} = 0\,, & \dfrac{{}^*\partial p}{\partial x^i} = 0\,, & {}^*\nabla_j\beta_{ik} = 0\,, & {}^*\nabla_j q_i = 0 \end{array}\right\}, \quad (45)$$

while the isotropy of the Universe, in a given local region, can be determined by the conditions

$$F_i = 0\,, \quad A_{ik} = 0\,, \quad \Pi_{ik} = 0\,, \quad \Sigma_{ik} = 0\,, \quad \beta_{ik} = 0\,, \quad q_i = 0\,. \quad (46)$$

As obvious, if we remove C_{ik} with K_{ik} in (45), the new conditions of the homogeneity will be equivalent to the initial conditions. Thus, if we remove the fourth condition of (46) with the requirement

$$K_{ik} - \frac{1}{3} K h_{ik} = 0\,,$$

the new conditions of the isotropy will be equivalent to the initial conditions. As easy to see, there are six factors of the anisotropy: the power field, the absolute rotation, the anisotropy of the deformation, the anisotropy of the curvature, the viscosity and the 1st kind, and the flow of the energy. The first five factors are connected among themselves by the relations (44).

Let the conditions (46) be true everywhere in a finite or infinite four-dimensional region. In such a case, in the same region, the equations (44) are satisfied identically, while the equations (39–43) take the form

$$\frac{{}^*\partial \rho}{\partial t} + D\left(\rho + \frac{p}{c^2}\right), \tag{47}$$

$$\frac{{}^*\partial p}{\partial x^i} = 0\,, \tag{48}$$

$$\frac{{}^*\partial D}{\partial t} + \frac{1}{3} D^2 = -\frac{\varkappa}{2}\left(\rho c^2 + 3p\right) + \Lambda c^2, \tag{49}$$

$$\frac{{}^*\partial D}{\partial x^i} = 0\,, \tag{50}$$

$$\frac{1}{3} D^2 - \frac{1}{2} c^2 C = \left(\varkappa \rho + \Lambda\right) c^2. \tag{51}$$

It follows, from (48), (49) and (50), that

$$\frac{{}^*\partial \rho}{\partial x^i} = 0\,, \qquad \frac{{}^*\partial C}{\partial x^i} = 0\,, \tag{52}$$

where the last equality can be obtained also in a direct way, on the basis of Schur's theorem, due to the holonomity of this space, and the isotropy of its curvature.

The equations (48), (50), (52) and (46) lead immediately to the conditions of the homogeneity (45). On the basis of (47) and also (49), (50) and (51), while taking the third equality of (16) into account, we obtain

$$\frac{{}^*\partial \left(C \sqrt[3]{h}\right)}{\partial t} = 0\,. \tag{53}$$

If the condition (53), the first four conditions of the isotropy (46), and the second condition of (52) satisfy, there among the accompanying coordinate frames is such a frame, wherein the homogeneous isotropic metric (2) is true and also

$$D = 3\,\frac{\dot{R}}{R}\,, \qquad C = -\frac{6\,k}{R^2}\,. \tag{54}$$

For Gödel's model (3), we obtain

$$\left.\begin{array}{lllll} h_{11} = a^2, & h_{22} = \dfrac{a^2}{2}\,e^{2x^1}, & h_{33} = a^2, & h_{ik} = 0 & (i \neq k) \\ F_i = 0\,, & A_{12} = -\dfrac{ac}{2}\,e^{x^1}, & A_{23} = 0\,, & A_{31} = 0\,, & D_{ik} = 0 \\ C_{11} = 1\,, & C_{22} = \dfrac{1}{2}\,e^{2x^1}, & C_{33} = 0\,, & C_{ik} = 0 & (i \neq k) \\ \varkappa\rho = \dfrac{1}{a^2} = -2\Lambda\,, & & p = 0\,, & \beta_{ik} = 0\,, & q_i = 0 \end{array}\right\}. \tag{55}$$

As can be seen, here the second and fourth conditions of the conditions of the anisotropy (46) do not satisfy, while all the conditions of the homogeneity (45) are satisfied.

So, in a general case, we formulate the following conclusions about a four-dimension region of space: 1) the isotropy leads to the homogeneity, hence 2) the inhomogeneity leads to the anisotropy; 3) the anisotropy does not require inhomogeneity; 4) as aforementioned in this Paragraph, in the understanding of the homogeneity and isotropy, only the models (2) are homogeneous and isotropic, while the model (3) is homogeneous, but anisotropic.

§11. The vectorial equation of conservation (40) expresses the law of the change of the flow of energy with time. In the absence of such a flow, this equation expresses the equilibrium condition between the surface forces and the gravitational inertial force (it is originated in masses). The chronometrically invariant rotor of the vector of the gravitational inertial force, i.e. the tensor

$${}^*\nabla_i\, F_k - {}^*\nabla_k\, F_i = \frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}$$

or the vector $\varepsilon^{ijk}\,{}^*\nabla_j\, F_k$ is nonzero in a general case.

A local centre of gravitational attraction can be determined by the conditions ${}^*\nabla_j\, F^j < 0$ and $F_i = 0$. As obvious, at such a point and also in a neighbourhood surrounding it, the following condition is true

$${}^*\nabla_j\, F^j - \frac{1}{c^2}\, F_j F^j < 0\,. \tag{56}$$

A local centre of radiation can be determined by the conditions ${}^*\nabla_j\, q^j < 0$ and $q_i = 0$. Hence, at such a point and also in a neighbourhood surrounding it, the following condition is true

$$ {}^*\nabla_j\, q^j - \frac{2}{c^2}\, F_j\, q^j > 0\,. \tag{57}$$

The scalar equation of conservation (39) expresses the law of the change of the mass or, equivalently, the energy of the volume element of the medium with time. We introduce the volume V and the energy $E = V\rho c^2$ of such an element. Taking into account that $D = {}^*\partial \ln V/\partial t$, we reduce (39) to the form

$$ dE + p\,dV = \left[\beta_{jl}\, \Pi^{jl} - \left({}^*\nabla_j\, q^j - \frac{2}{c^2}\, F_j\, q^j\right)\right] V d\tau\,, \tag{58}$$

where $p\,dV = p_0\, dV - \alpha D V d\tau$. As the inequalities (38) and (57) satisfy, and the sign of $d\tau$ is definitely given, the right side of (58) may reach, generally speaking, any sign. At the moment of an extremum of the volume of the element, obviously $D = 0$. At the moment of an extremum of the density of the volume, ${}^*\partial\rho/\partial t = 0$. As easy to see, from the scalar equation of conservation (39), these moments of time do not coincide in a general case.

In the absence of the viscosity of the 1st kind and also the flow of the energy, the equations (40) and (58) take the form, respectively,

$$ \frac{{}^*\partial p}{\partial x^i} = \left(\rho + \frac{p}{c^2}\right) F_i\,, \qquad dE + p\,dV = 0\,. \tag{59}$$

As well-known, the second of these equations was obtained earlier in the case of the metric (2), i.e. in the framework of the theory of a homogeneous isotropic universe.

§12. Consider the identities (14) and (15), and also the identities (28) and (29) which are equivalent to the previous. We see in (28) that, in a general case, neither the mathematical chronometrically invariant divergence nor the physical chronometrically invariant divergence of the vector of angular velocity of the absolute rotation of the space are non-equal to zero. In a particular case, in the absence of the power field, both divergences coincide, and are equal to zero. The identities (15) and (29), in the framework of the accompanying coordinates, represent the equations of the change of a vortex. In the case of a non-viscous barotropic medium free of the flow of energy, these identities give

$$ \frac{{}^*\partial}{\partial t}\left[A_{jk}\,(E + pV)\right] = 0\,, \qquad \frac{{}^*\partial}{\partial t}\left[\Omega^i \sqrt{h}\,(E + pV)\right] = 0\,. \tag{60}$$

These equalities are equivalent to each other. The second of them manifests the conservation of the vortical lines. The stress of the vortical tube is expressed by a surface integral from the quantity

$$\varepsilon_{ijk}\,\Omega^i dx^j \delta x^k = \frac{1}{2}\,A_{jk}\left(dx^j \delta x^k - dx^k \delta x^j\right),$$

where the components of the vectors dx^k and δx^j are independent of time. Having the vortical lines conserved, the region of the change of the space coordinates, with respect to which we perform integration, does not depend on time. Therefore each of two tensor equalities (60) is the necessary and sufficient condition for the synchronous conservation of a) the vortical lines and b) the product of the multiplication of the stress of the vortical tubes by the relativistic heat function $E+pV$. In the absence of the power field, the identities (15) and (29) give

$$\frac{{}^*\partial A_{ik}}{\partial t} = 0\,, \qquad \frac{{}^*\partial}{\partial t}\left(\Omega^i\sqrt{h}\right) = 0\,. \tag{61}$$

In both cases (60) and (61), the conditions of the holonomity or the non-holonomity of the accompanying space remain unchanged: these conditions are free to be realized in both cases. If we suppose that the space is holonomic and the holonomity remains unchanged, this supposition leads to the other limitations, most artificial of which are the requirements for the non-viscous and barotropic properties of the medium in the absence of the flow of energy. These requirements satisfy, with high precision, the observed values of the density, the pressure and the parameters of expansion of that part of the Metagalaxy, which is accessible to our observation in the present epoch. On the other hand, these requirements satisfy the worse; the more earlier stage of the expansion is under our consideration. This is because, with the expansion of the Metagalaxy, the pressure decreases faster than the density. Hence, considering the ancient age of the Metagalaxy, we should mean the accompanying space of the Metagalaxy to be non-holonomic, that is equivalent to the supposition that the Metagalaxy rotates.

§13. Instead of the change of the volume V of an element of the medium, we will consider the change of the quantity $R = f\sqrt[3]{V}$, where $f>0$ and $\partial f/\partial t = 0$. In such a case, $D = 3\,{}^*\dot{R}/R$. It is obvious that this quantity R, in contrast to the same named quantity of the formulae (2) and (54), considered under the condition $k \neq 0$, is determined at every point of the space to within a constant positive multiplier.

Our task is a general bound of the evolution of some characteristics of the space in the process of the expansion of the medium. We therefore consider this problem under some simplifications. We consider evolution of the factors of the anisotropy in the case where the rest factors of the anisotropy are neglected. Preliminary, we consider the change of the curvature scalar and the density under simplest assumptions.

In the case where the space is completely isotropic, as can be seen from (53) and (54), we have

$$C \sim R^{-2}. \tag{62}$$

If $p=0$, $\beta_{ik}=0$ and $q_i=0$, as seen from (39), we have

$$\rho \sim R^{-3}. \tag{63}$$

If $F_i=0$ and $\Pi_{ik}=0$, with use of (61) we obtain

$$\Omega_j\,\Omega^j \sim R^{-4}. \tag{64}$$

If $F_i=0$, $A_{ik}=0$, $\Sigma_{ik}=0$ and $\beta_{ik}=0$, it follows from (44) that

$$\Pi_i^k\,\Pi_k^i \sim R^{-6}. \tag{65}$$

As easy to see from (44), in the case where $F_i=0$, $A_{ik}=0$, $\Sigma_{ik}=0$ and $\beta_{ik}\neq 0$, the quantity $\Pi_i^k\Pi_k^i$ changes faster with the increasing R and slower with the decreasing R than according to the law (65). At $\beta_{ik}=2\eta\Pi_{ik}$, where η is the viscosity coefficient of the 1st kind, the quantity $\beta_i^k\beta_k^i$ changes faster than $\Pi_i^k\Pi_k^i$. If $F_i=0$, $p=0$ and $\beta_{ik}=0$, we obtain from (40) that

$$q_j\,q^j \sim R^{-8}. \tag{66}$$

Thus, according to our bound, the expansion of the Metagalaxy should be accompanied by a so fast decrease of the factors of the anisotropy such that the fact of the invisibility of the factors in the modern epoch does not allow us to ignore the presence of the factors in the past.

§14. We introduce the quantities

$$Q = \frac{2}{3}\,R\left(\Pi_i^k\,\Pi_k^i - 2\,\Omega_j\,\Omega^j + {}^*\nabla_j\,F^j - \frac{1}{c^2}\,F_j F^j\right), \tag{67}$$

$$S = \frac{1}{3}\,R^2\left(3\,\Omega_j\,\Omega^j - \frac{1}{2}\,\Pi_i^k\,\Pi_k^i - \frac{c^2}{2}\,C\right). \tag{68}$$

With these, the equations (39), (41) and (43) can be represented in the form, respectively,

$${}^*\dot{\rho} + 3\,\frac{{}^*\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = \frac{1}{c^2}\left[\beta_{jl}\,\Pi^{jl} - \left({}^*\nabla_j\,q^j - \frac{2}{c^2}\,F_j\,q^j\right)\right], \tag{69}$$

$$3\,\frac{{}^*\ddot{R}}{c^2 R} + \frac{3}{2}\,\frac{Q}{c^2 R} = -\frac{\varkappa}{2}\left(\rho + 3\,\frac{p}{c^2}\right) + \Lambda\,, \tag{70}$$

$$3\,\frac{{}^*\dot{R}^2}{c^2 R^2} + 3\,\frac{S}{c^2 R^2} = \varkappa\rho + \Lambda\,. \tag{71}$$

The equation (69), which connects the equations (70) and (71), can be transformed into the form

$${}^*\dot{S} = {}^*\dot{R}\,Q + \frac{\varkappa}{3}\,R^2\left[\beta_{jl}\,\Pi^{jl} - \left({}^*\nabla_j\, q^j - \frac{2}{c^2}\,F_j\, q^j\right)\right]. \tag{72}$$

As obvious, having any initially chosen moment of time, we can set up such an initial value of R such that the initial value of S is equal to $kc^2 = 0, \pm c^2$. In the case where the space is completely isotropic, the right side of (69) and the second term on the left side of (70) are zero, while S still retains its initial numerical value. In such a case, we, omitting the asterisk, obtain the well-known equations for the homogeneous isotropic models (2). With the metric (2) the equations (40), (42) and (44) become identities. In such a case, two equations (70) and (71) of the whole scope of ten equations are sufficient, under some additional physical assumptions, for the investigation about the possible evolution of R with time. In a general case, we can also find the kinds of evolution of R permitted by the equations (70) and (71). We however should take into account the fact that, in the consideration of the whole system of ten equations of gravitation, we can find some of the kinds of the evolution to be impossible. Following in this way, we, obviously, narrow the circle of the conceivable possibilities step-by-step.

From cosmological and cosmogonical points of view, most interesting are the principal possibility and the physical conditions in a) the models of the kind O_2 that points to an oscillation between two regular extrema of R at finite numerical values of the density (the so-called "oscillation of the 2nd kind"), or at least the principal possibility and the physical conditions of b) a regular minimum of R at a finite numerical value of the density. There is also an important question about the principal possibility and the physical conditions of c) the accelerating increase of R, because such a growth at the current velocity of the expansion of the Metagalaxy leads to the prolongation of the past part of the epoch of the expansion, i.e. to the prolongation of the whole scale of time. As known, for the homogeneous isotropic models (2) considered in the framework of the suppositions

$$0 < \rho c^2 \geqslant 3p \geqslant 0\,, \qquad \frac{\partial p}{\partial R} \leqslant 0\,, \tag{73}$$

the case a) is impossible, while the cases b) and c) are permitted with only a positive numerical value of the cosmological constant.

Let the cosmological constant be zero. In such a case, on the basis of (67–71), we obtain that in the absence of the absolute rotation and the inequality (56)* the cases b) and c) are impossible and, hence, the case a) is impossible as well. The numerical value of R increases either monotonically and, at $\tau \to \infty$, becomes unbounded, or it increases till a regular maximum, and then decreases. If in addition to it, the medium is free of viscosity and the flow of energy, the beginning of the increase and the end of the decrease of R is so-called a "special state", where the density and the speed of the change of R are infinite. In such a case, we obtain the same two kinds of evolution as those known in the models (2): M_1 that means the "monotonic change of the 1st kind", and also O_1, i.e. the "oscillation of the 1st kind". In this process, the anisotropy of the deformation leads to more braking of the expansion and, hence, to the shortening the whole scale of time. Thus, in concern of the accelerating expansion, the regular minima and the oscillation of the 2nd kind, the most important is the taking of the power field and the absolute rotation into account. Concerning the irregular minima free of the special states, most important is the taking of the viscosity and the flow of energy into account.

§15. In this Paragraph we consider the kinds of evolution of R in complete as permitted by the equations (70) and (71) in the framework of the suppositions (73). We consider the case of a barotropic non-viscous medium, which is free of the flows of energy. In such a case, the density and the pressure at each point can be considered as functions of R. From (69), we obtain

$$\frac{\partial\rho}{\partial R} + \frac{3}{R}\left(\rho + \frac{p}{c^2}\right) = 0\,, \qquad \frac{\partial}{\partial R}\left(\rho R^3\right) = -\,3R^2\,\frac{p}{c^2}\,. \tag{74}$$

In the absence of the pressure, the density changes according to (69), i.e. this process goes faster under the positive pressure. It is obvious that, if $R \to \infty$, $\rho R^n \to 0$ and $pR^n \to 0$ (here $0 \leqslant n \leqslant 3$). We define R_∞ as $R \to R_\infty$ under $\rho \to \infty$. With this definition we see that $R_\infty \geqslant 0$. As obvious, at the value $R \to R_\infty$ we have $\rho R^n \to \infty$ $(0 \leqslant n \leqslant 3)$.

Given the plane RS, we consider the area of the real changes of the volume of the space element, i.e. such an area wherein $R \geqslant R_\infty$ and ${}^*\dot{R}^2 \geqslant 0$. This area is bounded by the ultimate lines: the straight line

*This can be if, for instance, in addition to the absence of the absolute rotation, there is no the power field.

$R = R_\infty$ and the ultimate curve ${}^*\dot{R}^2 = 0$. Denoting by S_0 the ordinate of a point on this ultimate curve ${}^*\dot{R}^2 = 0$, we express the equation of this curve through (71) as follows

$$S_0 = \frac{c^2}{3}\left(\varkappa\rho + \Lambda\right)R^2, \qquad R \geqslant 0\,. \tag{75}$$

While taking (74) into account, we obtain

$$\frac{\partial S_0}{\partial R} = \frac{c^2}{3}\left[-\varkappa\left(\rho + 3\,\frac{p}{c^2}\right) + 2\Lambda\right]R\,, \tag{76}$$

$$\frac{\partial^2 S_0}{\partial R^2} = \frac{c^2}{3}\left[\varkappa\left(2\rho - \frac{3}{c^2}\frac{\partial p}{\partial R}R\right) + 2\Lambda\right] = 2\,\frac{S_0}{R^2} - \varkappa R\,\frac{\partial p}{\partial R}\,, \tag{77}$$

where, due to the second of the suppositions (73),

$$2\rho - \frac{3}{c^2}\frac{\partial p}{\partial R}R > 0\,. \tag{78}$$

At any value of Λ we obtain $S_0 \to +\infty$, while $\partial S_0/\partial R \to -\infty$ at $R \to R_\infty$: the ultimate straight line is the asymptote of the ultimate curve. If $\Lambda > 0$, with the increasing of R to the value R_∞ the value of S_0 decreases up to its minimal value $\frac{\varkappa}{2}\left(\rho c^2 + p\right)R^2$ then monotonically increases: $S_0 \to +\infty$ and $\partial S_0/\partial R \to +\infty$ at the value $R \to \infty$. In such a case the ultimate curve lies above the axis of abscisses, and is convex everywhere to the axis. If $\Lambda = 0$, with the increasing of R to the value R_∞ the value of S_0 monotonically decreases: $S_0 \to 0$ and $\partial S_0/\partial R \to 0$ at the value $R \to \infty$. In such a case, the axis of abscisses is the asymptote: the ultimate curve lies above this axis, and is convex everywhere to it. If $\Lambda < 0$, with the increasing of R to the value R_∞ the value of S_0 monotonically decreases: $S_0 \to -\infty$ and $\partial S_0/\partial R \to -\infty$ at the value $R \to \infty$. In such a case, in the area higher than the axis of abscisses the ultimate curve is everywhere convex to it, while in the area lower than the axis of abscisses the ultimate curve has a point of inflection (in a general case, there is an odd number of such points).

For the curves, which sketch the permitted changes of R in the plane RS, we write down, according to (72),

$$\frac{\partial S}{\partial R} = Q\,. \tag{79}$$

For those points of these curves, which coincide with the points of the ultimate curve, we obtain, from (71) and (75), (70), (76) and (79),

$${}^*\dot{R}^2 = S_0 - S\,, \qquad {}^*\ddot{R} = \frac{1}{2}\left(\frac{\partial S_0}{\partial R} - \frac{\partial S}{\partial R}\right). \tag{80}$$

§16. Split the considered interval of time into the minimal number of the intervals of monotonic change of R. There on the opposite boundaries of each interval (such an interval can be finite or infinite) the quantity R has the minimal and the maximal numerical values along all the values attributed to R in it. We recognize four kinds of states conceivable for such a volume element at the minimal value of R: the kind m means the states of finite density at a regular minimum of R; the kind a means the states of finite density at an asymptotic value of R; the kind c means the states of infinitely high density at zero or finite speed of the change of R (in particular, this happens at the minimal or asymptotic value of R coinciding with R_∞); the kind s means the states of infinitely high density at the infinitely high speed of the change of R (these are so-called "special states"). We recognize also three kinds of states conceivable for such a volume element at the maximal value of R: the kind M means the states of finite density at a regular maximum of R; the kind A means the states of finite density at an asymptotic value of R; the kind D means the asymptotic states of zero density at $R \to \infty$.

The states m, a, M, A are attributed to all the points of the ultimate curve. The states D are attributed to all the points of an infinitely distant straight line $R = +\infty$, which lie not higher than the ultimate curve. As obvious, this is the whole straight line $R = +\infty$ in the case where $\Lambda > 0$, this is the half-line $R = +\infty$, $S \leqslant 0$ in the case where $\Lambda = 0$, and this is just a single point $R = +\infty$, $S = -\infty$ in the case where $\Lambda < 0$. The states c constitute just a point $R = R_\infty$, $S = +\infty$. The states c are attributed to all the points of the ultimate curve.

We denote each kind of evolution of R by a row of characters, which mean the states transited by a volume element with time along the time interval of the monotonic change of R. According to the notions, the kinds of evolution of a volume element, which are met in the theory of a homogeneous isotropic universe, should be recognized as follows: the kind A_1 as sA (expansion) or As (contraction); the kind A_2 as aD (expansion) or Da (contraction); the kind M_1 as sD (expansion) or Ds (contraction); the kind M_2 as DmD; the kind O_1 as sMs.

The conceivable kinds of evolution of a volume element in the intervals of the monotonic increase of R, permitted under different conditions, are given in the Table below.

Aside for the trivial case of the homogeneous isotropic models (2), the condition $Q = 0$ satisfies, for instance, at the centre of spherical symmetry in the absence of the power field. The condition $Q > 0$ satisfies, for instance, outside this centre and, in a general case, at all points where there is no power field, as well as no the absolute rotation, while

	$Q=0$	$Q \geqslant 0$	$Q \gtreqless 0$
$\Lambda > 0$	sD, aD, mD sA sM	sD, aD, mD sA, aA, mA sM, aM, mM	sD, cD, aD, mD sA, cA, aA, mA sM, cM, aM, mM
$\Lambda = 0$	sD sM	sD sM	sD, cD, aD, mD sA, cA, aA, mA sM, cM, aM, mM
$\Lambda < 0$	sM	sM	sD, cD, aD, mD sA, cA, aA, mA sM, cM, aM, mM

the space deformation is anisotropic. The condition $Q<0$ satisfies, for instance, at the centre of spherical symmetry, which is the local centre of gravitational attraction in the sense of §11; and $Q<0$ satisfies also in the neighbourhood of such a centre of attraction.

§17. Here we provide some additional notes and comments to the previous results.

The solutions can have a physical meaning only outside the states of infinitely high density. It is meaningful to continue the solutions up to the states of infinitely high density. The formal conclusion about such states, obtained through the known equations of gravitation, should be considered, following Einstein, as a note on the inapplicability of these equations to the states of extremely high density such as the density inside atomic nuclei.

In the case of the homogeneous isotropic models, all the kinds shown in the Table are permitted. In the other cases, because we took into account not all of the equations of gravitation, we conclude that those kinds which are absent in this Table are impossible.

In the case of the homogeneous isotropic models, all that has been concluded about the evolution of any single volume element is also true for all elements of the considered three-dimensional region (which can be both finite and infinite). In a general case, these conclusions give a possibility to judge about the evolution of the rest volume elements, because of the continuity of space.

Along each interval of the monotonic change of R, all the rest quantities can be considered as functions of R in any case, not only in the case considered in §15. All the equations of §15 are true in the case of a barocline medium, which bear the viscosity of the 2nd kind, but is free of the 1st kind viscosity and the flows of energy. This allows us to distribute all the results of our Table onto this case, which is the most common for which this Table works.* In this Table we give only the permitted kinds of expansion. There are also the respective kinds of contraction corresponding to each of the kinds of expansion provided by this Table: the kind Ds corresponds to the kind sD, the kind Da corresponds to the kind aD, the kind Dm corresponds to the kind mD, and so forth. We consider the kinds of evolution of R in two adjacent intervals of the monotonic change, which are connected through a regular extremum of finite density. As is obvious, both kinds (expansion and contraction) should be in the row of the permitted kinds in all cases. However, in the case of a barotropic non-viscous medium, which is free of the flow of energy, and only in this case, we can assert that these two kinds are inverse to each other.

The behaviour of a homogeneous, isotropic model with time is valuable dependent on the Gaussian curvature of the space. In such a space, the numerical value of the Gaussian curvature is the same numerical value at all points, while the sign of the curvature remains unchanged with time, and is directly connected with the conditions of infiniteness of the space. In a general case, the correlation between the behaviour of a volume element and the Gaussian curvature is set up by the relations (68), (71) and (72). However there in the case of a homogeneous isotropic universe: 1) the Gaussian curvature changes from point to point, 2) it is impossible to assert that the sign of the Gaussian curvature remains unchanged at any point, 3) even if the space is holonomic, there is no direct connexion between the sign of the Gaussian curvature and the infiniteness of the space. In a general case, we should take into account the totality of the Riemannian curvatures at all points of the space, and along all two-dimensional directions in it (there in the homogeneous isotropic models they are everywhere equal to the Gaussian curvature). Using these curvatures, we are able to obtain the sufficient conditions of the infiniteness of a space, for instance

$$\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} \leqslant 0\,, \qquad \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} \geqslant 0\,, \quad B_{11} \leqslant 0\,, \tag{81}$$

where

$$B_{ik} = C_{ik} - \frac{1}{2}\,C h_{ik}\,. \tag{82}$$

*This is because that fact that the viscosity of the 1st kind and the flow of energy multiply the number of the allowed kinds of evolution.

It is possible to say that the simple statement of the problem about the infiniteness or the finiteness of space, which is specific to the theory of a homogeneous isotropic universe, is insufficient in the theory of such an inhomogeneous anisotropic universe whose space is holonomic, and is impossible in that case where the space is non-holonomic.

§18. In this Paragraph we consider a quasi-Newtonian approximation in cosmology, in the accompanying coordinates. In the framework of such an approximation, we use the equations of Newtonian mechanics (in Euclidean space) and Poisson's equation (or the generalization $\nabla_j^j \Phi = -4\pi\gamma\rho + \Lambda c^2$ of it, where Φ is the gravitational potential), without any universal ultimate conditions for the infiniteness of space. The reference frame accompanying the medium refines this approximation, because the velocity of macroscopic motions and some relativistic effects connected to it are zero in such coordinates. Therefore, the non-relativistic equations, constructed in the framework of the unitary interpretation of motion of a continuous medium [30], together with Poisson's equation (or its generalization given above) in the accompanying coordinates are both reasonable to be used as the quasi-Newtonian approximation to the chronometrically invariant relativistic equations.

Use the accompanying coordinates x^i and Newtonian mechanics in the pseudo-Euclidean space. Let t be Newtonian time. Let h_{ik}, h, D_{ik}, D, A_{ik} be the chronometrically invariant quantities which characterize the space of the accompanying frame of reference: the metric tensor, the fundamental determinant, the tensor of the rate of the space deformation, the speed of the relative volume expansion of the space, the tensor of angular velocity of the absolute rotation of the space. In such a quasi-Newtonian case,

$$D_{ik} = \frac{1}{2}\frac{\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{\partial h^{ik}}{\partial t}\,, \quad D = \frac{\partial \ln\sqrt{h}}{\partial t}\,, \tag{83}$$

$$\nabla_i \left(D_{jk} + A_{jk}\right) - \nabla_j \left(D_{ik} + A_{ik}\right) = 0\,, \tag{84}$$

$$\frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} + \frac{\partial A_{ij}}{\partial x^k} = 0\,. \tag{85}$$

Let F^k be the gravitational inertial force, acting per unit mass, which puts the surface forces into equilibrium. Let U^{ik} be the tensor of the density of the flow of momentum, while ρ is the density of mass. In such a case,

$$\nabla_i U^{ik} - \rho F^k = 0\,, \tag{86}$$

$$\frac{\partial\rho}{\partial t} + D\rho = 0\,, \qquad \frac{\partial}{\partial t}\left(\rho\sqrt{h}\right) = 0\,, \tag{87}$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{\partial F_k}{\partial x^i} - \frac{\partial F_i}{\partial x^k}\right) = 0\,, \tag{88}$$

$$\frac{\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D^j_k + A^{\cdot j}_{k\cdot}\right) + \frac{1}{2}\left(\nabla_i F_k + \nabla_k F_i\right) = \nabla_{ik}\Phi\,. \tag{89}$$

Contracting (84) term-by-term, we obtain

$$\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) = 0\,. \tag{90}$$

Contracting (89) term-by-term, while taking the equation of the potential into account, we obtain

$$\frac{\partial D}{\partial t} + D_{jl}D^{lj} + A_{jl}A^{lj} + \nabla_j F^j = -\,4\pi\gamma\rho + \Lambda c^2. \tag{91}$$

It is obvious that the relativistic relations (14), (15), (16), the relativistic law of energy and momentum (33), (34), and the relativistic equations of gravitation (35), (36), (37) have the non-relativistic analogy in, respectively, the relations (85), (88), the equations (83) and (87), and the equations (86), (91), (90), (89). According to their physical meanings, (85) and (90) are identities like (84), (86) constitutes the equations of equilibrium, (87) is the continuity equation, (88) and (89) are the equations of motion of the medium, while (91), while taking (89) into account, substitutes instead the equation of the potential. These equations allow us to find the desired quasi-Newtonian approximation for the curvature. Comparing (37) and (89), we obtain

$$c^2 C_{ik} = DD_{ik} - D_{ij}D^j_k + 3A_{ij}A^{\cdot j}_{k\cdot} + \nabla_{ik}\Phi - \left(4\pi\gamma\rho + \Lambda c^2\right)h_{ik} \tag{92}$$

that leads to

$$c^2 C = D^2 - D_{jl}D^{jl} + 3A_{jl}A^{jl} - 16\pi\gamma\rho - 2\Lambda c^2. \tag{93}$$

As seen, in the framework of the quasi-Newtonian (non-relativistic) approximation, the equality (92) should be considered as the definition of the curvature tensor C_{ik}. At the same time, emphasizing the expansion of this formula by which comes the relativistic theory, we can calculate, through the equality (92) and its sequel (93), the Riemannian curvature and the Gaussian curvature of the accompanying space.

§19. Numerous researchers considered (and used) the similarity and analogy between the relativistic equations, obtained in the framework of different cosmological models, and the non-relativistic equations, obtained for the respective distribution and motion of masses. The first persons who did it were Milne and McCrea [31–33], who used this analogy for the homogeneous isotropic models, Bondi [18], who applied this analogy for the spherically symmetric models, and also Heckmann and Schücking [37], in the case of the axially symmetric homogeneous models (see also Heckmann [7], for this case). They all considered the cases with no pressure, viscosity and, flow of energy.* They removed Newtonian law of gravitation with a generalization of it, where the cosmological constant has been included. Such an application of Newtonian mechanics and Newtonian law of gravitation, based on the aforementioned analogy, is known as *Newtonian cosmology.* In such a cosmology, the uncertainty of the field of gravitation (the non-relativistic gravitational paradox) was either ignored or removed, in a hidden form, by some additional requirements, which are not usual in Newtonian theory. Neither the chronometric invariants in the relativistic equations nor the accompanying coordinates in the non-relativistic equations were applied by the aforementioned researchers. Almost all of them (see [34, 36, 37]) and also Layzer [35] discussed the question about the legitimacy of such a Newtonian cosmology. For instance, Heckmann and Schücking [38] supposed some changes on the ultimate conditions on the Newtonian potential at spatial infinity.

In contrast to the aforementioned authors, our method, which shows how to use this analogy (we proposed this method in §18), works in the framework of the following requirements:

1) Cancel any universal ultimate conditions on the potential at spatial infinity. (Considering every particular problem, such ultimate conditions or limitations used in the non-relativistic theory should meet analogous conditions or limitations assumed in the same problem considered in the relativistic theory);
2) Interpret the non-relativistic solutions as an approximation to the relativistic solutions. The use of the non-relativistic equations as the quasi-Newtonian approximation to the relativistic equations, includes the calculation for the space curvature;
3) Use of the chronometrically invariant quantities and operators in the relativistic equations. Such a use makes the relativistic equations look very similar to the non-relativistic equations;
4) Apply the accompanying coordinates in the non-relativistic equations. This makes the equations not only look similar to the rel-

*The presence of the factors leads to the lowering of the aforementioned analogy.

ativistic equations, but is also profitable to the quasi-Newtonian approximation itself;

5) Consideration of not only particular models, but also, and mainly, the general cases of relativistic cosmology.

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1. Ambarzumian V. A. Presentation given at *The 6th Soviet Meeting on Cosmogony*, Moscow, June 5–7, 1957.
2. Shapley H. *Proc. Nat. Acad. Sci. USA*, 1938, vol. 24, no. 7, 282, and vol. 24, no. 12, 527.
3. Rubin V. C. *Astron. J.*, 1951, vol. 56, no. 2, 47.
4. Ogorodnikov K. F. In: *Problems of Cosmogony*, Acad. Sci. USSR Publ., 1952, vol. 1, 150.
5. Humason M. L., Mayall N. U. and Sandage A. R. *Astron. J.*, 1956, vol. 61, no. 3, 97.
6. Tolman R. C. Relativity, Thermodynamics and Cosmology. Oxford Univ. Press, Oxford, 1934.
7. Heckmann O. Theorien der Kosmologie. Springer, Berlin, 1942.
8. Zelmanov A. L. Cosmology. In: *Astronomy in the USSR over the Past 30 Years*, Technical Literature Publ. (Gostechizdat), Moscow, 1948.
9. McVittie G. C. *Monthly Notices Roy. Astron. Soc.*, 1933, vol. 93, no. 5, 325.
10. Tolman R. C. *Proc. Nat. Acad. Sci. USA*, 1934, vol. 20, no. 3, 169.
11. Datt B. *Z. Phys.*, 1938, vol. 108, no. 5–6, 314.
12. Tolman R. C. *Phys. Rev.*, 1938, vol. 55, no. 4, 364.
13. Oppenheimer J. and Volkoff G. *Phys. Rev.*, 1938, vol. 55, no. 4, 374.
14. Oppenheimer J. and Snyder H. S. *Phys. Rev.*, 1939, vol. 56, no. 5, 455
15. Järnefelt G. *Ann. Acad. Sci. Fennicae*, Ser. A, 1940, vol. 45, no. 3.
16. Järnefelt G. *Ann. Acad. Sci. Fennicae*, Ser. A, 1942, vol. 45, no. 12.
17. Einstein A. and Strauss E. G. *Rev. Mod. Phys.*, 1945, vol. 17, 120.
18. Bondi H. *Monthly Notices Roy. Astron. Soc.*, 1947, vol. 107, no. 5–6, 410.
19. Omer G. C. *Astrophys. J.*, 1949, vol. 109, no. 1, 164.
20. Just K. *Z. Phys.*, 1956, vol. 145, no. 2, 235.
21. Kobushkin P. K. Exact Solutions of Einstein's Gravitational Equations in the Case of Axial Symmetry. PhD thesis, Moscow, 1941.
22. Gödel K. *Rev. Mod. Phys.*, 1949, vol. 21, no. 3, 447.
23. McCrea W. H. *Z. Astrophys.*, 1939, vol. 18, 98.
24. Zelmanov A. L. On Deformations and the Curvature of Accompanying Space. PhD thesis, preprint of the Sternberg Astron. Institute, Moscow, 1944.
25. Raychaudhury A. *Phys. Rev.*, 1955, vol. 98, no. 4, 1123.
26. Zelmanov A. L. *Soviet Physics Doklady*, 1956, vol. 1, 227 (translated from *Doklady Academii Nauk USSR*, 1956, vol. 107, no. 6, 815).
27. Komar A. *Phys. Rev.*, 1956, vol. 104, no. 2, 544.

28. Landay L. D. and Lifshitz E. M. The Classical Theory of Fields. Technical Literature Publ. (Gostechizdat), Moscow, 1948.

29. Fock V. A. The Theory of Space, Time and Gravitation. Technical Literature Publ. (Gostechizdat), Moscow, 1955.

30. Zelmanov A. L. *Doklady Academii Nauk USSR*, 1948, vol. 61, no. 6, 993.

31. Milne E. A. *Quarterly Journal of Mathematics*, Oxford Series, 1934, vol. 5, 64.

32. McCrea W. H. and Milne E. A. *Quarterly Journal of Mathematics*, Oxford Series, 1934, vol. 5, 73.

33. Milne E. A. Relativity, Gravitation and World-Structure. Oxford Univ. Press, Oxford, 1935.

34. Bondi H. Cosmology. Cambridge Univ. Press, Cambridge, 1952.

35. Layzer D. *Astron. J.*, 1954, vol. 59, no. 7, 268.

36. McCrea W. H. *Nature*, 1955, vol. 175, no. 4454, 466.

37. Heckmann O. und Schücking E. *Z. Astrophys.*, 1955, vol. 38, no. 2, 95.

38. Heckmann O. und Schücking E. *Z. Astrophys.*, 1956, vol. 40, no. 2, 81.

On the Permissible Numerical Value of the Curvature of Space

Karl Schwarzschild

The presentation held at the German Astronomical Society
Annual Meeting in Heidelberg, July 8–11, 1900

Abstract: This is a translation of Schwarzschild's pioneering presentation where he pondered upon the possible non-Euclidean structure of space and gave a lower limit for the measurable radius of curvature of space as 4,000,000 astronomical units (supposing the space to be hyperbolic) or 100,000,000 astronomical units (elliptic space). The paper was originally published as: Schwarzschild K. Über das zulässige Krümmungsmaaß des Raumes. *Vierteljahrsschrift der Astronomische Gesellschaft*, 1900, Bd. 35, S. 337–347. Translated into English in 2008 by Dmitri Rabounski. The translator thanks Ulrich Neumann, Germany, for a copy of the Schwarzschild manuscript in German, and also Stephen J. Crothers, Australia, for assistance.

Permitting myself to call your attention for this presentation, which has neither practical purpose nor mathematical meaning, I should be excused due to the theme of the presentation itself. This theme is obviously very attractive to most of you due to the fact that it is related to the expansion of our views to boundaries far away from our everyday experience, and opens beautiful horizons for possible experiments in the future. The fact that all these lead us to the failure of numerous traditional views which are most hard rooted in the heads of astronomers, is just an advantage of this new theme from the view of everyone who believes in the relativity of our knowledge.

This talk is on the permissibility of curved space. You all know that in the 19th century along with the Euclidean geometry numerous other non-Euclidean geometrical systems were developed, which were headed by the geometrical systems of so-called spherical space and of so-called pseudo-spherical space (we will deal mainly with these two systems here). It is possible to develop in detail a picture of what would be observed in a spherically curved space or a pseudo-spherically curved space. I however limit myself by only a reference to Helmholtz' paper *The Origin and Meaning of the Geometrical Axioms**. Here we

*Hermann von Helmholtz. Über den Ursprung und die Bedeutung der geometrischen Axiome. *Vortrag gehalten im Docentenverein zu Heidelberg*, 1870, Universitätsbibliothek Heidelberg. Published in English as: Hermann Helmholtz. The origin and meaning of geometrical axioms. *Mind*, July 1876, vol. 1, no. 3, pages 301–321. — Editor's comment. D.R.

are come into a fairyland of geometry, which is especially beautiful due to the fact that it may relate to our real world, and finally we are unsure in the impossibility of it. Here we consider how wide the boundaries of this fairyland can be expanded, *what is the largest numerical value of the permissible curvature of space, what is the smallest radius of the space curvature.*

One usually answers this question unsatisfactorily, at least unsatisfactorily from the viewpoint of an astronomer. In Euclidean geometry the sum of the angles in a triangle is $2d$; while in the case of the non-Euclidean geometry the larger the triangle we are considering, the more this sum differs from $2d$. One may point out that even in the case of the largest of the measured triangles (the apex of such a triangle is a star, while the base is drawn by the diameter of the Earth's orbit) the sum of the angles in each of them wasn't found to be different from $2d$. Hence the curvature of space should be negligible. In such an answer people overlook just one circumstance. They don't take into account the circumstance that the angle at the star isn't a subject of measurement, but is obtained as a calculation resulting from the theorems of Euclidean geometry, the correctness of which will be the subject of our consideration here. Besides, an astronomer shouldn't be satisfied by the note, according to which he should neglect the curvature of space in the scale of the nearest stars, whose parallax is accessible to measurement; to obtain a picture of the interior of the world of stars, he should take into account the distances to even the weakest stars, which are far relative to us.

I begin consideration of this problem from the point of view which gives the possibility of talking about the theoretical meaning of non-Euclidean geometry. In order to measure the positions of three vertices of a triangle, we will employ the light beams coming from one of the vertices. The lengths of the sides a, b, c of this triangle will be measured according to the duration of time required for the light beam to travel along the lengths, while the angles α, β, γ will be measured by a regular astronomical instrument. Our everyday experience manifests in plane trigonometry, true on all triangles within the precision of measurement. Suppose that the regular trigonometry is not absolutely precise and that in reality the sides and the angles of triangles are connected by the following relations

$$\sin\alpha : \sin\beta : \sin\gamma = \sin\frac{a}{R} : \sin\frac{b}{R} : \sin\frac{c}{R}\,, \tag{a}$$

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cdot\cos\frac{b}{R} + \sin\frac{a}{R}\cdot\sin\frac{b}{R}\cdot\cos\gamma\,. \tag{b}$$

Here R is a very large interval we will refer to as the *curvature radius of space*, by which we mean no close analogy to the curvature radius known in the geometry of two dimensions. The aforementioned formulae coincide with the main formulae of spherical trigonometry which, as well known, transform into the regular trigonometric formulae in the case where the sides of the triangle are small relative to the radius R of the sphere. However taking R sufficiently large, the sides of any triangle we are measuring become small relative to R. Therefore, by increasing R we can always arrive at a case where the formulae (a) and (b) meet the regular trigonometric formulae within the measurement precision. In order words, it is enough to increase R to reduce the formulae (a) and (b) into coincidence with our everyday experience.

Here we don't consider a purely mathematical problem on the grounds of acceptance of formulae (a) and (b) for any triangle, without internal contradiction. As we know this question has been answered positively. Besides, as shown by research, the requirement that spherical trigonometry be applicable to all triangles in a space provides no exact information about coherence of the space. Among the possible forms of space which permit spherical trigonometry, the simplest and most well-known are the so-called *spherical space* and the *elliptic space.* The following common properties are attributed to a spherical space and an elliptic space: such a space is finite, the volume of it is finite as well and is dependent on the curvature radius. By following a path in such a space, we arrive at the initial point. Relations given in a plane of such a space are absolutely the same as those on the surface of a sphere according to usual views. Besides that, a plane located in a curved space is determined, as usual and everywhere, by all straight lines — all beams of light which pass through two crossed light beams. Any straight line in a plane of a such curved space is similar to a great circle on the surface of a sphere. For two parallel straight lines, i.e. two straight lines crossing a third straight line at equal angles (two right angles, for instance), these straight lines are similar to two meridians crossing the equator at right angles. Similarly for the crossing meridians at the point of the pole, the straight lines cross each other in a curved space at the distance $\frac{\pi}{2}R$ in a curved space. One may say that, concerning a plane of a curved space, two parallel straight lines should cross each other twice like two great circles on a sphere. This hypothesis lies at the foundation of spherical space. However it is possible that two parallel straight lines cross each other only once; this assumption leads us to "elliptic" space. It is possible to map a plane in a curved space onto a usual spherical surface in such a way that each point of the plane conjugated, not only with the radius, but also the diameter and, hence, two diametrically opposite points of the spherical surface. Therefore, if taking great circles passing through a point of a spherical surface, and crossing each other at the diametrically opposite point, the incoming point and the point diametrically opposite to it are similar to a single point of a plane in a curved space, where the respective straight lines cross each other. From this we conclude that we, travelling by way of the length πR (not $2\pi R$), arrive at the initial point and, at the same time, the maximum distance between two points in such a space is $\frac{\pi}{2}R$. Similarly we study *elliptic space*, which is the simplest of spherical trigonometry spaces. (In talking above about spherical space instead elliptic space, we merely used the more common and usual term.)

But first we should mention another very simple generalization of non-Euclidean geometry. If in (a) and (b) we replace R with an imaginary quantity iR, we obtain

$$\sin\alpha : \sin\beta : \sin\gamma = \mathrm{Sin}\,\frac{a}{R} : \mathrm{Sin}\,\frac{b}{R} : \mathrm{Sin}\,\frac{c}{R}\,, \qquad (a')$$

$$\mathrm{Cos}\,\frac{c}{R} = \mathrm{Cos}\,\frac{a}{R}\cdot\mathrm{Cos}\,\frac{b}{R} + \mathrm{Sin}\,\frac{a}{R}\cdot\mathrm{Sin}\,\frac{b}{R}\cdot\cos\gamma\,, \qquad (b')$$

where capital letters denote hyperbolic functions. These equalities transform into the equalities of plane trigonometry with the increase of R. There are various spatial forms wherein the special trigonometry based on formulae (a') and (b') are true. The simplest of these spatial forms is the so-called "*pseudo-Riemannian*" or "*hyperbolic*" *space*. Such a space is infinite: therein each point is crossed by a couple of straight lines without intersecting another given straight line. The geometry on any of the planes of such a space is analogous to the geometry on a so-called pseudo-sphere, which is constant negative curvature surface.

Now we turn our attention to the problem of how to determine the *parallax* in the cases of both elliptic and hyperbolic spaces. Any of the definitions of parallax can be reduced to the following: given two times of observation separated by a half year duration, we measure the angle created at the Earth by two straight lines which connect it with two stars we observe. Assume, for simplicity, that one of these two stars, S_1, is positioned exactly in the continuation of the diameter of the Earth's orbit, while the other star, S_2, is positioned in the line which is approximately orthogonal to this direction. Denoting E_1 and E_2 as the positions of the Earth at the times of observation ($E_1E_2 = r$ is the diameter of the Earth's orbit), the observations give both angles $S_1E_1S_2 = \alpha$ and $S_1E_2S_2 = \beta$. The quantity $p = \frac{\alpha-\beta}{2}$ is known as the parallax of the star S_2. The problem is as follows: having the elements α, β, $2r$, how to calculate the distances $E_1S_2 = a$ and $E_2S_2 = b$ from the star S_2 to both locations of the Earth in the cases of spherical trigonometry and pseudo-spherical trigonometry. Because the straight line directed at S_2 should be approximately orthogonal to $E_2E_1S_1$, we can assume $a = b = d$ where d is the distance from the star. We take into account that fact that the parallax p is a very small angle, and the curvature radius of the space should be undoubtedly much larger than the diameter of the Earth's orbit. With these we easily obtain the following formulae for the distance in the case of an elliptic space

$$\operatorname{cotg}\frac{d}{R} = \frac{R}{r}\,p \quad \text{or} \quad \sin\frac{d}{R} = \frac{1}{\sqrt{p^2R^2 + r^2}}\,, \tag{c}$$

and that in a hyperbolic space

$$\operatorname{Cotg}\frac{d}{R} = \frac{R}{r}\,p \quad \text{or} \quad \operatorname{Sin}\frac{d}{R} = \frac{1}{\sqrt{p^2R^2 - r^2}}\,. \tag{c'}$$

The last of these formulae leads to a conclusion concerning *hyperbolic space*. Naturally, given each real distance d, the inequality $pR > r$ should hold. Therefore there is a minimum parallax, which is $p = \frac{r}{R}$, that should be observed for even very distant stars. On the other hand we know of many stars which don't have a parallax of even $0.05''$. Hence the numerical value of the minimal parallax should be lesser than $0.05''$. We obtain the lower limit of the curvature radius of the hyperbolic space

$$R > \frac{r}{\operatorname{arc} .050''} \quad \text{that is} \quad R > 4,000,000 \text{ radii of the Earth's orbit.}$$

According to this the curvature of the hyperbolic space is so small that it doesn't manifest in any measurements on the scale of the planetary system. Besides, because any hyperbolic space is infinite, as is any Euclidean space, it is impossible to find unusual phenomena by observation of stars in the sky.

Before consideration of *elliptic space*, I remark that it was recently shown by Prof. Seeliger that the most accurate representation of our stellar system, on the basis of the observational data, concludes that all stars we observe (the number of the stars is no greater than 40 million) are located inside the space, the diameter of which is a few hundred million times larger than the radius of the Earth's orbit, beyond which a large and approximately empty space begins. This concept bears somewhat comfortably upon our minds, because according to it the complete study of the limited stellar system is an special stage in the evolution of our knowledge about the world. But this comfort and satisfaction would be much more effective if we imagined the space enclosed in itself, in a finite and complete manner, or approximately filled with this stellar system. Naturally, if so, we could reach a stage when the space has been studied completely, like the surface of the Earth has been studied, so that any macroscopic studies of the space have ceased being subordinate to microscopic studies. These very advanced studies may explain, in my view, that strong interest that attracts us to the hypothesis of elliptic space.

Now we look at the results of *calculation of the parallax in the elliptic space*. Employing the aforementioned formula

$$\operatorname{cotg}\frac{d}{R} = \frac{R}{r}\,p\,,$$

we can obtain, concerning any measured parallax p of a star, a specific real numerical value of the distance d to the star at *any numerical value of the curvature radius* R. Thus we see that it would be erroneous to think that the limit of R was found proceeding from only our measurements of the stellar parallaxes. According to these measurements, it would be possible that the space was so strongly curved that, travelling along a path equal to approximately 1,000 distances from the Earth to the Sun (i.e. the distance travelled by light during a few days), we would arrived at the initial point of our journey. Therefore, not purely metric reasons but physical reasons lead us to a conclude that the curvature radius is much larger than that suggested.

A very small curvature radius would lead to the metric inconsistencies in the planetary system. Because we further find a greater upper limit of it, it is enough to say that, in the case of the curvature radius equal to 30,000 radii of the Earth's orbit, it produces an imperceptible effect even in triangles which are as large as the distance to the orbit of Neptune. This radius of the space curvature corresponds to the length which is no larger than 1/10 part of the distance to the nearest stars.

So, assume $R = 30,000$ radii of the Earth's orbit. According to formula (c), we calculate the distance to the stars at different numerical values of the parallax. We obtain

$$\text{for } p = 1.0'' \qquad 0.908\,\frac{R\cdot\pi}{2} = 42,800 \text{ radii of the Earth's orbit,}$$

$$\text{for } p = 0.1'' \qquad 0.991\,\frac{R\cdot\pi}{2} = 46,700 \text{ radii of the Earth's orbit,}$$

$$\text{for } p = 0.0'' \qquad 1.000\,\frac{R\cdot\pi}{2} = 47,100 \text{ radii of the Earth's orbit.}$$

It is easy to see that we have arrived at quite ridiculous results. There are maybe a hundred stars whose parallax is $p > 0.1''$. Thus these hundred stars should be scattered at distances between one another no larger than 46,700 radii of the Earth's orbit, while the rest of the space at only 400 radii of the Earth's orbit is reserved for the remaining millions of stars. In such a case the Sun would be located in a space of exceptionally small stellar density, while everywhere beyond a certain distance from it there is an exceptionally large density of stars. To show this density of stars, I calculated the volume of the space limited by 46,700 radii of the Earth's orbit, and also the volume of the remaining part of the space, then I calculated the average distance between two stars assuming that there is exactly 100 million stars in total. I found that in the approximately empty space near the Sun the average distance between two stars is about 15,000 radii of the Earth's orbit, while in the high density inhabited rest of the space the average distance is only 40 radii of the Earth's orbit. Of course, it is impossible to accept such a calculation result that stars are so close to each other; otherwise it would be found in the physical interactions among the stars. It follows that the supposed curvature radius of 30,000 radii of the Earth's orbit is too small.

It is clear that by increasing R we may overcome all these difficulties, because they all vanish at $R = \infty$ (this is an obvious assumption). It is enough to take R so large that those 100 million stars with parallaxes less than $0.1''$ we assumed inhabit the space, which is a million times larger than the space inhabited by 100 million stars with parallaxes bigger than $0.1''$. Simple algebra shows that this takes place for

$$R = 160,000,000 \text{ radii of the Earth's orbit.}$$

In the case of a similar radius of the space curvature, light would travel around the whole space, along the path πR, in 8,000 years. However the size of the respective elliptic space is approximate the same as that suggested by Seeliger for the finite system of resting stars, not yet so large a size as that of the stellar system known according to the usual bounds. One could suggest R to be two or three times less than the above, but even such a reduction of R doesn't lead to the suggested abnormal emptiness of stars in the neighbourhood of the Sun and their high density at large distances from it.

Thus we arrive at the conclusion that the assumption, according to which R is equal to approximately 100,000,000 radii of the Earth's orbit, doesn't contradict the observational data. In the case of such a numerical value of R the whole finite space is homogeneous, filled with the observable stars.

One more fact should also be noted here. Given an elliptic space, any light beam arrives back at its initial point after travelling across the whole space. So light beams emitted into such a space from the opposite (invisible to us) side of the Sun should travel across the space then also meet the Earth, then create an anti-image of the Sun from the opposite side of the real image of it. This anti-image shouldn't fade with respect to the real image of the Sun, because light beams compress upon returning to the initial point of travel, becoming such ones as they travel in the least direct way from the source of light. But due to that fact that such an anti-image of the Sun was never observed we are enforced to suppose that light, travelling across the whole space, experiences absorption which is so strong that the anti-image is invisible. This supposition is true if supposing the absorption to be approximately 40 stellar magnitudes. There is no facts against the supposition of such a numerical value of the absorption, which seems small compared to the scale of the Earth.

In conclusion: *it is possible to imagine, with no contradiction of the experimental data, that the world is closed within a hyperbolic (pseudo-spherical) space, the curvature radius of which is larger than 4,000,000 radii of the Earth's orbit, or, alternatively — within an elliptic space, the curvature radius of which is larger than 100,000,000 radii of the Earth's orbit. In addition, in the second case, it should be supposed an absorption of light equal to 40 stellar magnitudes per around space travel.*

Now we should limit ourselves by these. At least, I see no other way to make a principal step in this direction with use of the contemporary methods of research, i.e. how to prove that the volume of the space is so large with respect to the volume of the stellar system we observe, or that the space has a really positive or negative curvature. On the other hand, I can provide some considerations which, despite providing no definite solution, may bring us to a specific preferential numerical value of R within the aforementioned scale of the values.

It is well known that astronomers, in their study of the distribution of stars in space, proceed from the simplest possible rational hypotheses about the average luminosities of stars, then they distribute the stars at different distances from the Sun by such methods that arrive at the numbers of stars of each stellar magnitude obtained in astronomical observations. Such research — the main result of which was mentioned above — was already produced by Prof. Seeliger. It could be produced in the same way in the cases of a pseudo-spherical space or of an elliptic space. I have calculated, in the cases of both spatial forms, the dependence of the number of stars from their stellar magnitude in the approximation that the luminosities of all stars are the same, and also that the density of the stellar population in all regions of the space is uniform. I have found, using the same physical assumptions, that the number of stars increases with the increase of their luminosity more slowly in the pseudo-spherical space in contrast to that in the Euclidean space, while in an elliptic space it increases faster than in the Euclidean space. In the real situation, as is well known, the number of stars increases with their luminosity slower than expected on the basis of the simple hypotheses about the Euclidean space. Proceeding from this fact, it could be concluded that the pseudo-spherical space is real. But, of course, no serious meaning can be attributed to these speculations, because the hypotheses of the equal luminosities and the equal density of stars take, as probable, no place in the real situation. However, as I have already said, this theory could be developed in the case of a curved space on the same bases used by Prof. Seeliger, who developed the theory in Euclidean space. Comparing the conclusion with the observational data, one could say then that the simplest picture of the distribution of stars is obtained on the assumption that space has a non-zero curvature. Of course, it is impossible to expect that a definite and final answer will be obtained here. We therefore have to accept that sad fact that there is little hope for a solid proof to the finitude of space.

Appendix. In the above, of all the spatial forms where "free motion of solid bodies" is possible, only the main types were considered (as has been noted by F. Klein). In order to finalize this theme, the other spaces which have this property should be compared to the astronomical data. I would exclude from consideration "spherical space" and other so-called "double-spaces", where all light emitted from a point travels to another point, collecting all the light anew. This is because we have no reason for introduction of such a complicated hypothesis. Therefore we have to settle for the so-called "*simple Clifford-Klein spatial forms*".

Of all these spatial forms, special is the one which amplifies the fact that the acceptance of Euclidean geometry is not equivalent, as one usually thinks, to the indefiniteness of the space. Imagine that we, after greatly enhanced astronomical data, found that our universe consists of countless copies of our Milky Way, that the infinite space can be split into many cubes, each of which contains a stellar system that is absolutely equivalent to the system of our Milky Way. Do we really stop at the assumption of an infinite number of identical copies of the same world-entity? To understand the absurdity of this, think about just one sequel: in such a case we ourselves, the observing objects, should exist in an infinite number of manifestations. We would better go to the assumption that these copies are only imaginary images, while the real space permits such coherences due to which we, being left the cube from one side and travelling always along a straight path, arrive at the cube from its opposite side. Such a space as we have supposed is nothing but the simplest of the Clifford-Klein spatial forms: a finite space of Euclidean geometry. It is easy to see the sole condition which should be attributed to such a Clifford-Klein space: because as yet nothing has been found concerning the (imaginary) copies of the system of the Milky Way, the volume of the space should be bigger than the volume we attribute to the Milky Way on the basis of the theorems of Euclidean geometry.

About the other simple Clifford-Klein spatial forms, we limit ourselves by only a few words, due to that fact that these spaces aren't sufficiently studied as yet. All these forms are obtained in analogous way by the identical imaginary copying of the same world-entity in a Euclidean space, in an elliptic space, or in a hyperbolic space. Experimental data lead us, again, to the condition according to which the volume of any such spaces should be bigger than the volume of the stellar system we observe.

The Luminiferous Ether is Detected as a Wind Effect Relative to the Ether Using a Uniformly Rotating Interferometer

Georges Sagnac

Abstract: This is English translation of Georges Sagnac's paper, where he gives a presentation for his "rotating interferometer experiment" which manifested the phenomenon called later the *Sagnac effect*. This paper was originally published, in French, as: L'éther lumineux démontré par l'effet du vent relatif d'éther dans un interféromètre en rotation uniforme. Note de G. Sagnac, présentée par E. Bouty. *Comptes rendus*, tome 157, 1913, pages 708–710. Translated from the French in 2008 by William Lonc, Canada. The Editor of *The Abraham Zelmanov Journal* thanks William Lonc for this effort, and also Ioannis Haranas, Canada, for assistance. Special thank go to the *National Library of France* and Nadège Danet in person for the permission to reproduce the originally Sagnac paper in English.

§1. The Method. — I uniformly *rotated*, at a speed of one or two turns per second around a vertical axis, a horizontal plate (50 cm in diameter) on which the various components of an interferometer were firmly anchored, analogous to the one I used in previous research and described in 1910 (*Comptes rendus*, tome 150, page 1676). The two interfering beams, after reflection from 4 mirrors placed at the edges of the rotating platform, were superimposed and travelled in opposite directions around exactly the same horizontal *circuit circumscribing the area* S. The rotating system also contained the light source: a small electric lamp, and the detector L, a fine-grained photographic plate that registered the interference fringes at the focal point of a lens.

In the images d and s, obtained successively during a *right-hand* rotation of the platform and then a *left-hand* rotation, both at the same rotation frequency, the central fringe was observed to occur at two different positions. I measure the difference between the centres of the fringes.

First method. — I mark on image d, and then on image s, the position of the central fringe relative to the image of a micrometer's vertical graduations placed in the focal plane of a collimator.

Second method. — I measure directly the distance from the vertical central fringe of the image d to the central fringe in image s precisely contiguous to the first but below a thin horizontal line separating the two. I obtain these two contiguous images without touching the photographic plate-holder, by opening-prior to obtaining the images d and s — the two contiguous positions corresponding to the illuminated slit on the edges of the horizontal edges (razor blades) in the collimator's focal plane.

§2. Optical rotation effect. — Measured from the fring-spacing, the displacement z from the interference centre that I observed with the preceding method is a particular case of the optical rotation effect that I have defined earlier (*Congrès de Bruxelles de septembre*, 1910, tome 1, page 217; *Comptes rendus*, tome 152, 1911, page 310; *Le Radium*, tome VIII, 1911, page 1), and which, in the context of current ideas, should be construed as a direct observation of the luminiferous ether.

In a system moving as a whole relative to the ether, the propagation time between any two points of the system should change in a way similar to a stationary system subjected to an ether wind, the relative speed of which at each point of the system will be the same and directly opposite to the speed of any point, and would contain light waves in a manner similar to atmospheric wind carrying sound waves. The observation of the optical effect of such an *ether wind relative to the* [stationary] *ether* will constitute a proof of the ether's existence, just as the observation of a wind relative to the atmosphere on the speed of sound in a moving system would constitute — everything else being equal — a proof of the existence of a stationary atmosphere enveloping the moving system.

The need to bring to one *common luminous point* oscillations that are combined at *another point* and to thereby produce interference, reduces to zero the first-order interference effect of the linear translation of the entire optical system, if the matter constituting the ether does not produce a *circular motion* C of the ether within the optical circuit of area S; that is to say, a *rotation* or *circulation* bS in the ether (*Comptes rendus*, tome 141, 1905, page 1220; 1910 and 1911, *loc. cit.*). I have shown interferometrically (1910 and 1911, *loc. cit.*) with an optical path enclosing $20\,\mathrm{m}^2$ in *vertical* projection, that ether drag in the Sun's neighbourhood does not produce a *rotational density* b of more than $1/1000$ rad. per second in the ether.

In a *horizontally mounted* optical circuit, at Latitude a, the diurnal rotation of Earth should, if the ether is stationary, produce a rotation relative to the ether with a density of $\frac{4\pi \sin\alpha}{T}$ or $\frac{4\pi \sin\alpha}{86164}$ rad. per sec, where T is the duration of the sidereal day; a very small quantity compared with $1/1000$, the upper limit that I established for a vertically mounted optical circuit. I hope to be able to determine whether a corresponding small optical rotation exists or not.

It was easier for me to first find a proof for the ether's existence by rotating a small optical circuit. A rotational frequency N of two turns per second gave me a rotational density of $4\pi N$ relative to the ether for a rotation of 25 rad. per second. A uniform *left-hand rotation* of the interferometer produces a *left-handed ether wind*; and delays by x the phase of the beam (T) whose motion around the area S is *right-handed*, and advances by the other beam R by the same amount, thus displacing the fringes by $2x$ units. The displacement z that I observe between images s and d should be twice that of the former*. On the basis of the value of x observed earlier (*loc. cit.*, 1910 and 1911), we have

$$z = 4x = 4\,\frac{bS}{\lambda V_0} = \frac{16\pi NS}{\lambda V_0}\,;$$

where V_0 is the speed of light in vacuum, and λ is the operating wavelength.

For a rotational frequency of $N = 2$ per sec., and the path area S being $860\,\text{cm}^2$, the observed value of z is 0.07 when using indigo light, and is easily visible in the photographs I attach to this Note and where the fringe-spacing is between 0.5 and 1.0 mm.

The interference displacement z, a constant fringe-spacing for the same value of rotation frequency N, disappears on the photographs when the fringes were made sufficiently narrow; this shows that the observed effect is very much due to a *phase difference* related to the rotational motion of the system and that (thanks to counter-screws that prevent movement of the mounting screws of the optical components) the displacement of the interferogram, observed in the comparison of image s with image d, does not arise from accidental relative displacements or elastic effects in the optical components during rotation.

Turbulent air produced above the interferometer by a fan rotating about a vertical axis and blowing downwards does not produce any displacement of the interferogram's centre, given a careful superposition of the two opposite beams. Any turbulent air, analogous and less intense, produced during rotation of the system does not affect the experiment.

The observed interference effect is very much the effect of optical rotation due to the motion of the system relative to the ether, and directly shows the existence of the ether, a necessary condition for the luminiferous waves proposed by Huygens and Fresnel.

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*That is, twice that of $2x$. — Translator's comment. W. L.

Regarding the Proof for the Existence of a Luminiferous Ether Using a Rotating Inteferometer Experiment

Georges Sagnac

Abstract: This is English translation of Georges Sagnac's second paper, which presents his "rotating interferometer experiment" where the phenomenon known as the *Sagnac effect* manifests itself. This paper was originally published, in French, as: Sur la preuve de la réalité de l'éther lumineux par l'expérience de l'interférographe tournant. Note de G. Sagnac, présentée par E. Bouty. *Comptes rendus*, 1913, tome 157, pages 1410–1413. Translated from the French in 2008 by William Lonc, Canada. The Editor of *The Abraham Zelmanov Journal* thanks William Lonc for this effort, and also Ioannis Haranas, Canada, for assistance. Special thank go to the *National Library of France* and Nadège Danet in person for the permission to reproduce the originally Sagnac paper in English.

In *Comptes rendus* of October 27 last (page 708 of this Volume 157), I showed that an interferometer using a closed optical path enclosing a given *area* and rotating in the plane of the path, detects the movement of the system relative to the ether in space.

§1. The interferometer, described elsewhere in detail, is sketched in the diagram below: a plate revolving horizontally (50 cm diameter) carries with it, solidly attached (mounting screws fitted with counter-screws) all the optical components and the luminous source O: a small electric lamp with a horizontal metallic filament. The microscope objective C_o projects the image of the filament, through the Nicol prism N, onto the horizontal slit F in the focal plane of the collimator objective C; m is a mirror. The parallel polarized beam, with vertical Fresnel vibrations, is split at the thin layer b of air $\mathfrak{J}$, as in most of the interferometers in my research (*Comptes rendus*, tome 150, 1910, page 1676) that I used for an optical study of the Earth's motion (*Congrès de Bruxelles*, Sept., 1910, tome 1, page 207; *Comptes rendus*, tome 152, 1911, page 310; *Le Radium*, tome VIII, 1911, page 1). Beam T, propagated through the air layer $\mathfrak{J}$, reflects successively from 4 mirrors and travels around the path $\mathfrak{J}-a_1-a_2-a_3-a_4-\mathfrak{J}$ with an area S. Beam R, reflected at the same air-layer J, goes around the same path but in the opposite sense. When the two beams return to $\mathfrak{J}$, T is propagated again, and R is reflected again. They now travel in the same direction as T^2 and

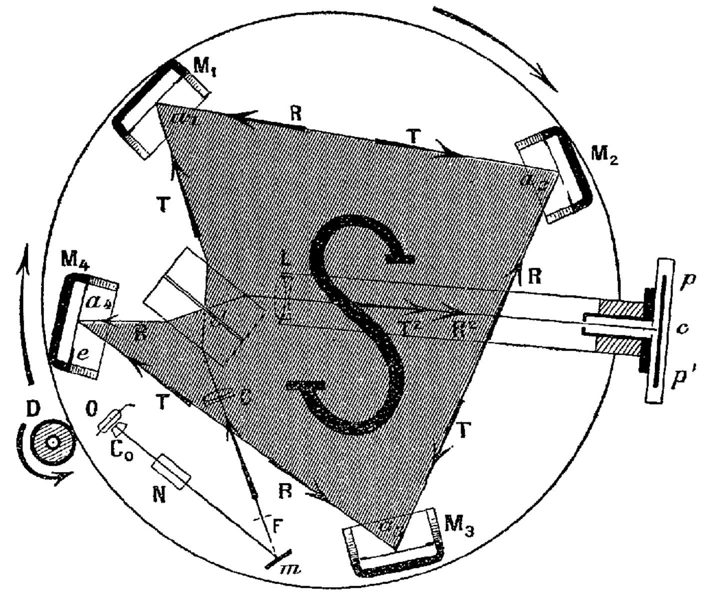
R^2, and interfere at the principal focus of lens L on the fine-grained photographic plate pp'.

§2. Procedure. — I remind the reader that the perfect superposition of the two opposing beams T and R results in an extinction in the lens's field of view for the lamp's indigo radiation, close in wavelength to the radiation from a mercury-arc lamp. In addition, a small rotation ε of the beam-splitter $\mathfrak{J}$ about a vertical axis in a right-handed sense (D) or left-handed (S) changes the dark field into a vertical central fringe accompanied by parallel fringes on both sides.

Once the fringes are suitably adjusted, and the photographic plate pp' installed in the holder in red light, I slowly activate an electric motor, the vertical axis of which has a horizontal disk D attached to it. The disk has a leather rim that is in contact with the rim of the circular plate. Once the desired rotational frequency N is reached, I take a photograph by sending a current to the small lamp O via slip rings on the axle of the circular plate.

§3. Direction and magnitude of the optical rotation effect. — In Fresnel's hypothesis of the ether, the luminous waves T and R propagate in the ether with a speed V_0 independent of the motion of the interferometer. The phase of the waves T in the right-handed sense (*see* the diagram) is changed along the closed path, as if the luminiferous ether had a left-handed rotation when the system rotates in the sense d [right-handed] and magnitude $4\pi NS$ of *this rotation*, or *relative circular motion* C of the ether within the closed optical path gives, according to the expression $\frac{C}{\lambda V_0}$, a lag x in the phase of the waves in beam T, and advances by the same amount the phase of the waves in beam R propagating in the reverse direction. The fringes should then move by $2x$ divisions. The absolute direction of this displacement y of the fringes should be pp', that is, d, like the rotation of the interferometer (the effect is in the *positive* direction) if the drive wheel rotates in the D direction [right-handed]. The displacement z equal to $2y$ or $4x$, measured by comparing image s with image d, should therefore be in the sense d. If the drive-wheel is rotating in the S direction [left-handed], then displacements y and z should change sense.

After many runs, I have always observed the sense to change as expected. The fact that the effect z reverses when I rotate the beam-splitter $\mathfrak{J}$ by even a fraction of a degree when reversing the rotation direction of D, identifies the effect as a phase-difference associated with the circular motion of the interferometer, and allows for isolation from



the effect of deformation in the optical component s.

I now offer examples of the measurement of z compared with values calculated from the expression $\frac{16\pi NS}{\lambda V_0}$; I determined the wavelength λ corresponding to the fringe-spacing obtained with the small lamp O and compared it with the fringe-spacing for the 436 nμ radiation from a mercury arc-lamp; there was little difference. The measurements were made by one of the two methods described in my Note of October 27th last. The central fringe c, well-defined in the negative image that I studied, and the weak lateral fringes f, are outlined only by a narrow half-light, conducive to a precise measurement of the points obtained by a slight enlargement while positioning the sharp fringe between the two parallel wires of an ocular micrometer.

	Sense	N	z from c	z from f	z calc.
Method 1	S [left]	0.86	−0.026	$\gg$	−0.029
($S = 863$ cm^2)	D [right]	1.88	+0.070	$\gg$	+0.065
Method 2	S [left]	2.21	−0.072	−0.078	−0.075
($S = 866$ cm^2)	S [left]	2.35	−0.077	−0.080	−0.079

The interferometer produces and records, from the expression $\frac{1}{2}z$, the rotation effect in first order, of the assembly's movement as a whole without importing any external reference marks.

The outcome of these measurements shows that in ambient space, light propagates with speed V_0 independent of the motion of the apparatus, the light source O and the optical system. This property of space describes the luminiferous ether experimentally. The interferometer measures, according to the expression $\frac{1}{4}z\lambda V_0$, the relative circular motion of the luminiferous ether within the closed optical path $\mathfrak{J}-a_1-a_2-a_3-a_4-\mathfrak{J}$.

The Classification of Spaces Defining Gravitational Fields

Alexei Petrov

Abstract: In this paper written in 1954 Alexei Petrov describes his famous classification of spaces according to the algebraical structure of the curvature tensor, that determines the classes of the gravitational fields permitted therein. Now this classification of spaces (and, respectively, of the gravitational fields) is known as *Petrov's classification*. This paper was originally published, in Russian, in Scientific Transactions of Kazan State University: Petrov A. Z. Klassifikazija prostranstv, opredelajuschikh polja tjagotenia. *Uchenye Zapiski Kazanskogo Gosudarstvennogo Universiteta*, 1954, vol. 114, book 8, pages 55–69. Translated from Russian in 2008 by Vladimir Yershov, England–Pulkovo.

In this paper, the detailed proof of results obtained and published by the author earlier in 1951 [1]. Namely, it is shown that by examining the algebraic structure of the curvature tensor V_4 one can establish a classification of the gravitational fields defined by this tensor and given in the form

$$ds^2 = g_{ij}\,dx^i dx^j, \tag{1}$$

with the fundamental tensor satisfying the field equations

$$R_{ij} = \varkappa\, g_{ij} \tag{2}$$

(we shall refer to the corresponding manifolds as T_4).

§1. Bivector space. Let us consider a point P of the manifold T_4, and associate it with a local center-affine geometry E_4. In this E_4 let us select those tensors that satisfy the following conditions: 1) the number of both covariant and contravariant indices must be even; and 2) the covariant and contravariant indices can be grouped in separate antisymmetric pairs. We shall regard each of these pairs as a single collective index, denoting it with a Greek letter in order to distinguish it from the indices corresponding to T_4 and E_4, for which we shall continue using Latin letters. Thus, according to the number of possible values for these collective indices, we shall get an $N = \frac{n(n-1)}{2}$ - dimensional manifold (6 dimensions for $n = 4$), the tensors E_4 with these properties defining on this manifold tensors with one-half rank.

One can say that *each point of* T_4 *is assigned to a local 6-dimensional centre-affine geometry with the group*

$$\left.\begin{aligned} \eta^{\alpha'} = A^{\alpha'}_{\alpha}\,\eta^{\alpha}, \quad \eta^{\alpha} = A^{\alpha}_{\alpha'}\,\eta^{\alpha'} \\ |A^{\alpha'}_{\alpha}| \neq 0\,, \qquad A^{\alpha}_{\beta}A^{\beta}_{\gamma} = \delta^{\alpha}_{\gamma} \end{aligned}\right\}. \tag{3}$$

Indeed, by ordering the collective indices (while selecting a single pair from the two possible, ij and ji), we shall get six possible collective indices. Let us take, for example, the following indexing:

$$1-14, \quad 2-24, \quad 3-34, \quad 4-23, \quad 5-31, \quad 6-12.$$

Let us now consider the transformation of the components T^{ij} of, generally speaking, a nonsimple bivector

$$T^{i'j'} = A^{i'j'}_{ij}\,T^{ij},$$

assuming

$$A^{\alpha'}_{\alpha} = 2A^{[i'j']}_{ij}, \quad \text{where } A^{i'}_{i} = \left(\frac{\partial x^{i'}}{\partial x^{i}}\right)_{P}.$$

In terms of collective indices, this gives

$$T^{\alpha'} = A^{\alpha'}_{\alpha}\,T^{\alpha};$$

i.e., the set of bivectors T_n determines a set of contravariant vectors in E_N (in this case the dimensionality does not matter), assuming that the relations (3) are satisfied. The validity of these relations can be checked directly by passing to the Latin indices.

Let us call the manifold obtained a *bivector space*. Of a special interest for our further consideration will be the curvature tensor T_4. In the bivector space this tensor corresponds to a symmetric tensor of the second rank

$$R_{ijkl} \longrightarrow R_{\alpha\beta} = R_{\beta\alpha}\,.$$

In any local E_6 one can define a metric by using for this purpose any tensor in T_4 with the properties

$$M_{klij} = M_{jikl} = -M_{ijkl} = -M_{ijlk}\,,$$

given that the corresponding second-rank tensor in E_6 is nonsingular. Let the tensor

$$g_{ikjl} = g_{ij}g_{kl} - g_{il}g_{kj} \longrightarrow g_{\alpha\beta} = g_{\beta\alpha} \tag{4}$$

be such a fundamental tensor in E_6. It is plain to see that $g_{\alpha\beta}$ gives a nondegenerate metrization because $|g_{ij}| \neq 0$, and

$$|g_{\alpha\beta}| = p\,|g_{ij}|^{2n}, \qquad p \neq 0\,.$$

For a definite g_{ij} the tensor $g_{\alpha\beta}$ will be definite; and for an indefinite g_{ij} the tensor $g_{\alpha\beta}$ will also, in general, be indefinite. Let us note, that here we shall consider only those fields of gravity that correspond to a real distribution of matter in space, which would require [2] the fundamental tensor g_{ij} be reducible to the form

$$(g_{ij}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \tag{5}$$

in the real coordinate system in any given point of T_4, that is, we have arrived at the so-called Minkowski space. Then it follows from (4) that for the frame corresponding to the matrix (5) the fundamental tensor R_6 will be of the following form:

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \qquad |g_{\alpha\beta}| = -1, \tag{6}$$

i.e., the tensor $g_{\alpha\beta}$ is, in fact, indefinite.

§2. Classification of $\boldsymbol{T_4}$. A series of the most interesting problems arising in the study of the Riemannian manifolds is related to the curvature tensor V_n. As is known, this tensor is used for introducing the notion of curvature of V_n at a given point along a given two-dimensional direction or, which is the same, of the Gaussian curvature of a two-dimensional geodesic surface at a given point:

$$K = \frac{R_{ijkl}V^{ij}V^{kl}}{g_{pqrs}V^{pq}V^{rs}}\,; \tag{7}$$

where g_{pqrs} has the form (4), and the two-dimensional direction, which is defined by the vectors $\underset{1}{V}{}^i$ and $\underset{2}{V}{}^i$, is characterized by the simple bivector $V^{ij} = \underset{[1}{V}{}^i \underset{2]}{V}{}^j$. Let us introduce the notion of *generalized curvature* of V_n, which could be obtained from (7) by dropping the requirement of simplicity of the bivector V^{ij}. At some point of V_n this generalized invariant K will be a homogeneous zero-degree function of the components of the (generally, not simple) bivector V^{ij}. And, of course, this invariant will be meaningful in the bivector space, where it can be written as

$$K = \frac{R_{\alpha\beta}\, V^{\alpha}V^{\beta}}{g_{\alpha\beta}\, V^{\alpha}V^{\beta}}\,. \tag{8}$$

Let us find the critical values of K that will be equivalent to finding those vectors V^{α} in R_{N}, for which K takes critical values. Let us call these critical values of K *stationary curvatures* of V_n, and the corresponding bivectors V^{α} — *the stationary directions* in V_n. Thus, our task consists in finding *the unconditionally stationary vectors* V^{α} in the bivector space using the necessary and sufficient conditions for stationarity:

$$\frac{\partial K}{\partial V^{\alpha}} = 0\,. \tag{9}$$

We have to take into account that for an indefinite g_{ij} the tensor $g_{\alpha\beta}$ is also indefinite and, hence, it is possible to have isotropic stationary directions

$$g_{\alpha\beta}\, V^{\alpha}V^{\beta} = 0\,. \tag{10}$$

Let us first exclude this case, returning to it below.

If (10) does not hold then the conditions (9) result in

$$(R_{\alpha\beta} - K g_{\alpha\beta})\, V^{\beta} = 0\,, \tag{11}$$

i.e., the stationary directions of V_n will be the principal axes of the tensor $R_{\alpha\beta}$ in the bivector space, while the stationary curvatures of V_n will be the characteristic values of the secular equation

$$|\, R_{\alpha\beta} - K g_{\alpha\beta}\,| = 0\,. \tag{12}$$

Let (10) holds now for the stationary V^{α}. Since we are interested only in the K satisfying the conditions (9), this K is a continuous function of V^{α} and, hence, it is necessary that the condition

$$R_{\alpha\beta}\, V^{\alpha}V^{\beta} = 0$$

were satisfied. Then one can calculate the value of K for the stationary isotropic direction of V^{α}:

$$K(V^{\alpha}) = \lim_{dV^{\alpha}\to 0} K(V^{\alpha} + dV^{\alpha})\,,$$

assuming the continuity of K as a function of V^α. If, for a given V^α, we denote

$$\varphi = g_{\alpha\beta}\, V^\alpha V^\beta, \qquad \psi = R_{\alpha\beta}\, V^\alpha V^\beta, \tag{13}$$

then for a stationary isotropic V^α

$$K(V^\alpha) = \lim_{dV^\alpha \to 0} \frac{\psi(V^\alpha + dV^\alpha) - \psi(V^\alpha)}{\varphi(V^\alpha + dV^\alpha) - \varphi(V^\alpha)} = \lim \frac{\Sigma_\sigma \frac{\partial}{\partial V^\sigma}\, \psi\, dV^\sigma + \dots}{\Sigma_\sigma \frac{\partial}{\partial V^\sigma}\, \varphi\, dV^\sigma + \dots}\,.$$

As this limit cannot depend on the ways of changing dV^α, then

$$K(V^\alpha) = \frac{\frac{\partial}{\partial V^\sigma}\, \psi}{\frac{\partial}{\partial V^\sigma}\, \varphi} = \frac{R_{\sigma\beta}\, V^\beta}{g_{\sigma\beta}\, V^\beta}\,,$$

so that again we obtain (11).

The determination of stationary curvatures and directions in R_N leads to the study of the pair of the quadratic forms (13). Therefore, the reduction of this pair to canonical form in real space results in a classification for the curvature tensor of V_n at a given point of V_n, as well as in a neighboring plane containing this point, where *the characteristic of the K-matrix*

$$\|\, R_{\alpha\beta} - K g_{\alpha\beta}\, \| \tag{14}$$

remains constant. For each type of the characteristic (14) there is a corresponding field of gravity of a specific type. It is this that determines the sought classification of T_4.

Using real transformations, one can always reduce the matrix $\|\, g_{\alpha\beta}\|$ to the form (6), and it remains to simplify the matrix $\|\, R_{\alpha\beta}\|$ by using real orthogonal transformations.

Theorem 1. *The matrix $\|\, R_{\alpha\beta}\|$ will be symmetrically-double for the orthogonal frame (5).*

For the basic (5) the field equations will take the form

$$\sum_k e_k\, R_{ikjk} = \varkappa\, g_{ij}\,, \qquad e_k = \pm 1,$$

that is, for $i = j$

$$\sum_k e_k\, R_{ikik} = \varkappa\, e_i\,,$$

and for $i \neq j$

$$e_k\, R_{ikjk} + e_l\, R_{iljl} = 0 \qquad (i, j, k, l \neq).$$

Writing these relations with the use of collective indices of the bivector space and taking into account the indexing introduced in § 1, we shall get the following expression for our matrix:

$$\left.\begin{aligned} \|\, R_{\alpha\beta}\,\| = \left\|\begin{array}{c|c} M & N \\ \hline N & -M \end{array}\right\| \\ M = \left\|\begin{matrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{matrix}\right\|, \quad m_{\alpha\beta} = m_{\beta\alpha} \\ N = \left\|\begin{matrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{matrix}\right\|, \quad n_{\alpha\beta} = n_{\beta\alpha} \\ (\alpha, \beta = 1, 2, 3) \end{aligned}\right\}, \tag{15}$$

where $\sum_{i=1}^{3} m_{ii} = \varkappa$ and $\sum_{i=1}^{3} n_{ii} = 0$, due to the Ricci identity, which proves the theorem. Let us note that similar matrices were obtained by V. F. Kagan [3], when studying the group of Lorentz transformations, although he used a condition of orthogonality of these matrices. Under the same assumption of orthogonality, similar matrices were also studied by Ya. S. Dubnov [4] and A. M. Lopshitz [5]. The fact established by the previous theorem takes place for any orthogonal frame and, hence, taking into account that the orthogonal frame has 6 degrees of freedom for $n = 4$, one can expect the possibility of further simplification of the matrix by choosing 6 appropriate rotations.

First let us prove a theorem that would essentially narrow down the number of possible (at first sight) types of the characteristic of the matrix (14).

Theorem 2. *The characteristic of the matrix (14) always consists of two identical parts.*

Let us reduce the matrix (14) to a simpler form by using the so-called elementary transformations, which, as is known, do not change the elementary divisors of a matrix and, therefore, its characteristic. Let us represent this matrix in the following way:

$$\left\|\begin{array}{c|c} m_{\alpha\beta} + K\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline n_{\alpha\beta} & -\, m_{\alpha\beta} - K\delta_{\alpha\beta} \end{array}\right\|,$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. By multiplying the last column by i and adding it to the corresponding first column we shall get the equivalent matrix

$$\left\|\begin{array}{c|c} m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline -\, i\,(m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta}) & -\, m_{\alpha\beta} - K\delta_{\alpha\beta} \end{array}\right\|.$$

By multiplying the first row of the previous matrix by i and adding it to the last row we shall convert the matrix to the form

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline 0 & -m_{\alpha\beta} + i n_{\alpha\beta} - K\delta_{\alpha\beta} \end{array} \right\|.$$

Finally, by multiplying the first column by $\frac{i}{2}$ and adding it to the corresponding last column and making the same operation with the last row, we shall obtain the matrix

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta} & 0 \\ \hline 0 & m_{\alpha\beta} - i n_{\alpha\beta} + K\delta_{\alpha\beta} \end{array} \right\| \equiv \left\| \begin{array}{c|c} P(K) & 0 \\ \hline 0 & \overline{P}(K) \end{array} \right\|, \quad (16)$$

which is equivalent to the K-matrix (14). The task has been reduced to the studying of two three-dimensional matrices $P(K)$ and $\overline{P}(K)$, whose corresponding elements are complex-conjugate. It follows then that the elementary divisors of these two matrices are also complex-conjugate and, hence, their characteristics have the same form. Therefore, the characteristic of our K-matrix consists of two parts repeating each other, so that the theorem holds.

Let us note that the principal directions and invariant bundles of the K-matrix should also be pairwisely complex-conjugate.

Now we can accomplish the classification of the fields of gravity. This classification can be expressed through the following theorem.

Theorem 3. *There exist three and only three types of the fields of gravity.*

The three-dimensional matrix $P(K)$ can have only one of three possible types of characteristic: $[1\ 1\ 1]$, $[2\ 1]$, $[3]$, if we neglect the cases when some of the elementary divisors have the same basis and, thus, some of the numbers in the square brackets should be enclosed in parentheses, e.g., $[(1\,1)\,1]$, $[(2\,1)]$, etc.

The characteristic of $\overline{P}(K)$ will have the same form. Then the characteristics of the K-matrix will be written as following:

$$1)\ [1\overline{1},\ 1\overline{1},\ 1\overline{1}]; \quad 2)\ [2\overline{2},\ 1\overline{1}]; \quad 3)\ [3\ \overline{3}],$$

where the overlined numbers correspond to the power index of the elementary divisor with the basis being complex-conjugate to the basis whose power index is expressed by the previous number.

Each of these types of the gravity fields has to be considered separately; and of a prime importance here is to get the canonical forms of the matrix $\|R_{\alpha\beta}\|$ for each of these types.

§3. The canonical form of the matrix $\|\boldsymbol{R}_{\alpha\beta}\|$. Let us consider the first type with the characteristic $[1\,\bar{1},\,1\bar{1},\,1\bar{1}]$. As in this case the characteristic is of simple type, the tensor $R_{\alpha\beta}$ has 6 non-isotropic, pairwisely orthogonal principal directions [6]. One can show that at a given point of T_4 these directions of the bivector space will give the bivectors of specific structure.

Let us denote the vector components of the real orthogonal frame at a point of T_4 by $\underset{k}{\xi}{}^i$ $(k, i = 1, \ldots, 4)$, denoting for brevity by ξ^{ij}_{kl} the simple bivectors $\underset{[k}{\xi}{}^i \underset{l]}{\xi}{}^j$ $(k \neq l)$ that determine the two-dimensional plane corresponding to the vectors of the frame. In the bivector space, these simple bivectors define 6 non-isotropic, mutually independent and orthogonal coordinate vectors $\underset{\sigma}{\xi}{}^\alpha = \delta^\alpha_\sigma$, so that any vector in R_6 (in particular, the vectors of the principal directions in $R_{\alpha\beta}$) can be represented in terms of these vectors.

Let us show that we can take the vectors

$$W^\alpha = \lambda(\underset{1}{\xi}{}^\alpha \pm i\,\underset{4}{\xi}{}^\alpha) + \mu(\underset{2}{\xi}{}^\alpha \pm i\,\underset{5}{\xi}{}^\alpha) + \nu(\underset{3}{\xi}{}^\alpha \pm i\,\underset{6}{\xi}{}^\alpha) \tag{17}$$

as the vectors of principal directions, which are uniquely defined only in the case when the roots of the secular equation (12) are all distinct.

Indeed, the condition of W^α to define the principal direction of the tensor $R_{\alpha\beta}$ is written as

$$(R_{\alpha\beta} - K g_{\alpha\beta})\,W^\beta = 0\,. \tag{18}$$

But due to the symmetric twoness of the K-matrix this system of six equations can be reduced to three equations

$$(m_{s1} \pm i\,n_{s1} + k)\,\lambda + (m_{s2} \pm i\,n_{s2})\,\mu + (m_{s3} \pm i\,n_{s3})\,\nu = 0\,, \quad s = 1, 2, 3\,.$$

For λ, μ, ν to be the non-zero solutions of this system it is necessary and sufficient that K were the root of one of the equations

$$|\,P(K)| = 0\,, \qquad |\,\overline{P}(K)| = 0\,, \tag{19}$$

i.e., a root of the secular equation (12), which proves the theorem.

At a given point of T_4 the vector W^α (17) of the manifold R_6 corresponds to the bivector of completed rank:

$$W^{ij} = \lambda(\underset{14}{\xi}{}^{ij} \pm i\,\underset{23}{\xi}{}^{ij}) + \mu(\underset{24}{\xi}{}^{ij} \pm i\,\underset{31}{\xi}{}^{ij}) + \nu(\underset{34}{\xi}{}^{ij} \pm i\,\underset{12}{\xi}{}^{ij})\,. \tag{20}$$

One can easily check that, under any (real) orthogonal transformation, W^{ij} grades into a bivector of the same type, with $\lambda, \mu, \nu \longrightarrow \overset{*}{\lambda}, \overset{*}{\mu}, \overset{*}{\nu}$, so that the norm of the bivector remains invariant:

$$\lambda^2 + \mu^2 + \nu^2 = \overset{*}{\lambda}{}^2 + \overset{*}{\mu}{}^2 + \overset{*}{\nu}{}^2 .$$

Let the roots of (12) K $(s = 1, 2, 3)$ correspond to the vectors of the principal direction $\underset{s}{W}{}^{\alpha}$; then, according to the above reasoning, the roots $\underset{s+3}{K}$ should correspond to $\overline{\underset{s}{W}}{}^{\alpha}$, provided the appropriate indexing of the roots.

The root $\underset{1}{K}$ corresponds to the bivector

$$\underset{1}{W}{}^{pq} = \underset{1}{\lambda}(\underset{14}{\xi}{}^{pq} + i\,\underset{23}{\xi}{}^{pq}) + \underset{1}{\mu}(\underset{24}{\xi}{}^{pq} + i\,\underset{31}{\xi}{}^{pq}) + \underset{1}{\nu}(\underset{34}{\xi}{}^{pq} + i\,\underset{12}{\xi}{}^{pq}) ,$$

and the root $\underset{4}{K}$ corresponds to the bivector

$$\underset{4}{W}{}^{pq} = \underset{1}{\overline{\lambda}}(\underset{14}{\xi}{}^{pq} - i\,\underset{23}{\xi}{}^{pq}) + \underset{1}{\overline{\mu}}(\underset{24}{\xi}{}^{pq} - i\,\underset{31}{\xi}{}^{pq}) + \underset{1}{\overline{\nu}}(\underset{34}{\xi}{}^{pq} - i\,\underset{12}{\xi}{}^{pq}) .$$

Let us represent the bivector W^{pq} as a sum of two real bivectors $\underset{1}{V}{}^{pq} + i\,\underset{1}{\overset{*}{V}}{}^{pq}$. Then

$$\underset{4}{W}{}^{pq} = \underset{1}{V}{}^{pq} - i\,\underset{1}{\overset{*}{V}}{}^{pq} .$$

Let

$$\lambda = \underset{1}{a} + i\,\underset{1}{b} , \qquad \mu = \underset{2}{a} + i\,\underset{2}{b} , \qquad \nu = \underset{3}{a} + i\,\underset{3}{b} ,$$

where $\underset{s}{a}$, $\underset{s}{b}$ are real numbers $(s = 1, 2, 3)$; hence

$$\underset{1}{V}{}^{pq} = \underset{1}{a}\,\underset{14}{\xi}{}^{pq} + \underset{2}{a}\,\underset{24}{\xi}{}^{pq} + \underset{3}{a}\,\underset{34}{\xi}{}^{pq} - \underset{1}{b}\,\underset{23}{\xi}{}^{pq} - \underset{2}{b}\,\underset{31}{\xi}{}^{pq} - \underset{3}{b}\,\underset{12}{\xi}{}^{pq} ,$$

$$\underset{1}{\overset{*}{V}}{}^{pq} = \underset{1}{b}\,\underset{14}{\xi}{}^{pq} + \underset{2}{b}\,\underset{24}{\xi}{}^{pq} + \underset{3}{b}\,\underset{34}{\xi}{}^{pq} - \underset{1}{a}\,\underset{23}{\xi}{}^{pq} - \underset{2}{a}\,\underset{31}{\xi}{}^{pq} - \underset{3}{a}\,\underset{12}{\xi}{}^{pq} .$$

Since $\underset{1}{W}{}^{\alpha}$ is not an isotropic vector of R_6, then it can always be regarded as a unit vector

$$g_{\alpha\beta}\,\underset{1}{W}{}^{\alpha}\underset{1}{W}{}^{\beta} = 1 ,$$

which leads us to the conclusion that

$$\sum_{s=1}^{3} \underset{s}{a}\underset{s}{b} = 0 , \tag{21}$$

$$\sum_{s=1}^{3} \underset{s}{b}{}^2 - \underset{s}{a}{}^2 > 0 . \tag{22}$$

Now we can assert the following.

1. The real bivectors $\underset{1}{V}^{pq}$ and $\overset{*}{\underset{1}{V}}{}^{pq}$ are single-foliated. Indeed, by writing down the simplicity condition we shall arrive at (21).
2. They are *0-parallel*. They cannot be be $\frac{2}{2}$-parallel, which would be possible only when the coefficients were proportional at equal $\underset{ij}{\xi}^{pq}$; then they would have to be equal to zero. For example,
$$\frac{\underset{1}{a}}{\underset{1}{b}} = -\frac{\underset{1}{b}}{\underset{1}{a}}\,, \qquad \underset{1}{a}^2 + \underset{1}{b}^2 = 0\,.$$
They cannot be $\frac{1}{2}$-parallel either, as in this case $\underset{1}{W}^{\alpha}$ would be a single-foliated complex bivector; but then by writing the simplicity condition we would arrive at a contradiction with (21) and (22). Therefore, we are left only with the above possibility.
3. These bivectors are $\frac{2}{2}$-perpendicular. For this to be true, it is necessary and sufficient to satisfy the equalities
$$\underset{1}{V}_{is}\overset{*}{\underset{1}{V}}{}^{sj} = 0$$
for any i, j. It is plain to see that these equalities are reduced to (21), so that they are, indeed, satisfied.

Let us consider a simple bivector $\underset{1}{V}^{pq}$. Its norm, according to (22), is
$$g_{\alpha\beta}\underset{1}{V}^{\alpha}\underset{1}{V}^{\beta} = \sum \underset{s}{b}^2 - \underset{s}{a}^2 > 0\,.$$

In the plain of this real bivector, one can always chose two real, orthogonal and non-isotropic vectors η^p, ν^p. Then the norm of our bivector can also be expressed in the form
$$2\,\eta_p\eta^p\,\nu_q\nu^q\,,$$
and, hence, these two vectors are both either *space-like* or *time-like*. Their norms cannot be > 0, because if we took these two real orthogonal vectors as coordinate vectors, we would arrive at a contradiction with the law of inertia of quadratic forms. Therefore, these two vectors have negative norms. Due to this, by re-normalizing them, we can take them as the vectors $\overset{*}{\underset{2}{\xi}}{}^i$, $\overset{*}{\underset{3}{\xi}}{}^i$ of a new real orthogonal frame.

In a similar way, let us define in the plane $\overset{*}{\underset{1}{V}}{}^{pq}$ two orthogonal (mutually and with respect to $\overset{*}{\underset{2}{\xi}}{}^i$, $\overset{*}{\underset{3}{\xi}}{}^i$) vectors, which will be real and non-isotropic but already having the norms of opposite signs, since
$$g_{\alpha\beta}\,\overset{*}{\underset{1}{\nu}}{}^{\alpha}\overset{*}{\underset{1}{\nu}}{}^{\beta} < 0\,.$$

Let us denote these vectors as $\overset{*}{\underset{1}{\xi}}{}^i$ and $\overset{*}{\underset{4}{\xi}}{}^i$. In this coordinate system

$$\overset{*}{\underset{1}{W}}{}^{pq} = \underset{14}{\xi}{}^{pq} + i\,\underset{23}{\xi}{}^{pq}\,,$$
$$\overset{*}{\underset{4}{W}}{}^{pq} = \underset{14}{\xi}{}^{pq} - i\,\underset{23}{\xi}{}^{pq}\,.$$

Let us note that the frame $\{\overset{*}{\xi}\}$ has been chosen up to a rotation in the plane $\{\overset{*}{\underset{2}{\xi}}\overset{*}{\underset{3}{\xi}}\}$ and a Lorentz rotation in the plane $\{\overset{*}{\underset{1}{\xi}}\overset{*}{\underset{4}{\xi}}\}$. Of course, we are interested in the bivectors $\underset{\sigma}{W}{}^{pq}$ only up to a scalar factor.

Now, writing the orthogonality condition for $\overset{*}{\underset{1}{W}}{}^{pq}$ and $\overset{*}{\underset{2}{W}}{}^{pq}$, we find, of course, that the bivector of the second principal direction should have the form

$$\overset{*}{\underset{2}{W}}{}^{pq} = \overset{*}{\underset{2}{\mu}}(\overset{*}{\underset{24}{\xi}}{}^{pq} + i\,\overset{*}{\underset{31}{\xi}}{}^{pq}) + \overset{*}{\underset{2}{\nu}}(\overset{*}{\underset{34}{\xi}}{}^{pq} + i\,\overset{*}{\underset{12}{\xi}}{}^{pq})\,.$$

Let us make use of the above indicated arbitrariness in the choice of the frame and perform the following rotations:

$$\underset{1}{\xi}{}^p = \operatorname{ch}\varphi\,\overset{*}{\underset{1}{\xi}}{}^p + \operatorname{sh}\varphi\,\overset{*}{\underset{4}{\xi}}{}^p\,,$$
$$\underset{4}{\xi}{}^p = \operatorname{sh}\varphi\,\overset{*}{\underset{1}{\xi}}{}^p + \operatorname{ch}\varphi\,\overset{*}{\underset{4}{\xi}}{}^p\,,$$
$$\underset{2}{\xi}{}^p = \cos\psi\,\overset{*}{\underset{2}{\xi}}{}^p + \sin\psi\,\overset{*}{\underset{3}{\xi}}{}^p\,,$$
$$\underset{3}{\xi}{}^p = -\sin\psi\,\overset{*}{\underset{2}{\xi}}{}^p + \cos\psi\,\overset{*}{\underset{3}{\xi}}{}^p\,.$$

After these transformations $\underset{1}{W}$ will have the same form; hence $\underset{2}{W}$ will also be expressed as

$$\underset{2}{\widetilde{W}}{}^{pq} = \underset{2}{\tilde{\mu}}(\underset{24}{\tilde{\xi}}{}^{pq} + i\,\underset{31}{\tilde{\xi}}{}^{pq}) + \underset{2}{\tilde{\nu}}(\underset{34}{\tilde{\xi}}{}^{pq} + i\,\underset{12}{\tilde{\xi}}{}^{pq})\,,$$

where

$$\underset{2}{\tilde{\nu}} = \sin\psi\operatorname{ch}\varphi + p\cos\psi\operatorname{ch}\varphi + q\sin\psi\operatorname{sh}\varphi + \\ + i\,(\cos\psi\operatorname{sh}\varphi + q\cos\psi\operatorname{ch}\varphi - p\sin\psi\operatorname{sh}\varphi)\,,$$

$$p + i\,q = \frac{\overset{*}{\underset{2}{\nu}}}{\overset{*}{\underset{2}{\mu}}}\,,$$

and $\overset{*}{\underset{2}{\mu}}$ can be considered not being equal to zero, otherwise we would be satisfied with the values $\varphi = \psi = 0$. One can find real φ and ψ for any

$\underset{2}{\tilde{\nu}} = 0$. Now the frame is defined uniquely, and, if the orthogonality of $\underset{1}{W}$, $\underset{2}{W}$, $\underset{3}{W}$ is taken into account, the bivectors will have the following form in this frame (up to a scalar factor):

$$\underset{1}{W}^{pq} = \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq}\,,$$
$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}\,,$$
$$\underset{3}{W}^{pq} = \underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq}\,,$$

and, due to the mentioned above complex conjugacy,

$$\underset{4}{W}^{pq} = \overline{\underset{1}{W}}^{\,pq}\,, \qquad \underset{5}{W}^{pq} = \overline{\underset{2}{W}}^{\,pq}\,, \qquad \underset{6}{W}^{pq} = \overline{\underset{3}{W}}^{\,pq}\,.$$

Now, by writing the condition (18) for each of these bivectors and, taking into account that

$$\underset{\alpha}{\xi}^{\sigma} = \delta^{\sigma}_{\alpha}\,,$$

we can easily find

$$m_{ii} = -\underset{i}{\alpha}\,, \quad m_{ij} = 0\,, \quad n_{ii} = -\underset{i}{\beta}\,, \quad n_{ij} = 0\,, \quad (i = 1, 2, 3; \quad i \neq j);$$

and, therefore, *for the first type of T_4 we obtain the following canonical form of the matrix*:

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\underset{1}{\alpha} & & & -\underset{1}{\beta} & & \\ & -\underset{2}{\alpha} & & & -\underset{2}{\beta} & \\ & & -\underset{3}{\alpha} & & & -\underset{3}{\beta} \\ \hline -\underset{1}{\beta} & & & \underset{1}{\alpha} & & \\ & -\underset{2}{\beta} & & & \underset{2}{\alpha} & \\ & & -\underset{3}{\beta} & & & \underset{3}{\alpha} \end{array} \right\|, \tag{23}$$

the real parts of the stationary curvatures being related to each other in the following way:

$$\sum_{1}^{3} \underset{s}{\alpha} = \varkappa\,, \tag{24}$$

whereas the imaginary parts obey the condition

$$\sum_{1}^{3} \underset{s}{\beta} = 0 \tag{25}$$

due to the Ricci identity

$$R_{1423} + R_{1234} + R_{1342} = 0\,.$$

Let us now consider a T_4 with the characteristic of the second type: $[\,21, \overline{21}\,]$. As we have already seen (§ 2), one can use the principal directions and and invariant bundles of the matrices $P(K)$ and $\overline{P}(K)$ for choosing the principal directions and invariant bundles of the K-matrix. It follows that it is sufficient to consider, for example, the matrix $P(K)$ having the characteristic $[\,21\,]$.

With this characteristic, the tensor $P_{\alpha\beta} = -\,m_{\alpha\beta} + i\,n_{\alpha\beta}$ of the three-dimensional space has [6] one non-isotropic principal direction

$$(P_{\alpha\beta} - \underset{1}{K}\, g_{\alpha\beta})\, \underset{1}{W}^{\beta} = 0 \tag{26}$$

and one isotropic principal direction

$$(P_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\, \underset{2}{W}^{\beta} = 0\,, \tag{27}$$

the latter ($\underset{2}{W}$) being orthogonal to $\underset{1}{W}$. Additionally, there exists an isotropic vector $\underset{3}{W}^{\beta}$, orthogonal to $\underset{1}{W}^{\beta}$ and not to $\underset{2}{W}^{\beta}$, which, together with these latter vectors, form an invariant plane $\{\underset{2}{W}, \underset{3}{W}\}$ of the tensor $P_{\alpha\beta}$. This is expressed by

$$(P_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\, \underset{3}{W}^{\beta} = \sigma\, \underset{2}{W}_{\alpha}\,, \tag{28}$$

where σ is an arbitrary nonzero scalar, whose choice is up to us. This arbitrariness is the result of the fact that $\overline{\underset{2}{W}}$, $\overline{\underset{3}{W}}$, being isotropic, can be multiplied by any number without changing their norms.

Any principal direction or bundle of $P_{\alpha\beta}$ will define the corresponding principal directions and bundles of the tensor $R_{\alpha\beta}$; all of them being defined by the bivectors of the type (17).

Let the root $\underset{1}{K}$ corresponds to a simple elementary divisor $(K - \underset{1}{K})$ of the fields of the K-matrix and to a principal direction defined by the bivector $\underset{1}{W}^{\alpha}$. As this bivector is non-isotropic, we can apply to it all the above operations used in the previous case for $\underset{1}{W}^{\alpha}$. Therefore, we can find a real frame, with respect to which

$$\underset{1}{W}^{pq} = \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq}\,.$$

This frame is defined up to a rotation in the plane $\{\overline{\underset{2}{\xi}\underset{3}{\xi}}\}$ and to a Lorentz rotation in the plane $\{\overline{\underset{1}{\xi}}\,\overline{\underset{4}{\xi}}\}$. As the bivectors $\underset{2}{W}^{pq}$ and $\underset{3}{W}^{pq}$

must be orthogonal to $\underset{1}{W}^{pq}$, they have the following form:

$$\underset{2}{W}^{pq} = \underset{2}{\mu}\,(\underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}) + \underset{2}{\nu}\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\,,$$

$$\underset{3}{W}^{pq} = \underset{3}{\mu}\,(\underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}) + \underset{3}{\nu}\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\,.$$

The isotropy condition for these two bivectors results in

$$\underset{2}{\mu}^2 + \underset{2}{\nu}^2 = 0\,, \qquad \underset{3}{\mu}^2 + \underset{3}{\nu}^2 = 0\,,$$

that is,

$$\underset{2}{\nu} = e_1\, i\underset{2}{\mu}\,, \qquad \underset{3}{\nu} = e_2\, i\underset{3}{\mu}\,,$$

where e_1 and e_2 are equal to ± 1. Finally, using the fact that they cannot be orthogonal, we find that $e_1 = -\,e_2$. Therefore, we can put, for example,

$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq} + i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq}),$$

$$\underset{3}{W}^{pq} = \lambda\{\underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq} - i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\},$$

where λ is an arbitrary scalar factor $\neq 0$.

Now we have only to write the conditions similar to (26), (27) and (28) for the tensor $R_{\alpha\beta}$, again, as in the previous case, taking into account that $\underset{\nu}{\xi}^{\alpha} = \delta^{\alpha}_{\nu}$. These conditions will have the form

$$(R_{\alpha\beta} - \underset{1}{K}\, g_{\alpha\beta})\,\underset{1}{W}^{\beta} = 0\,,$$

$$(R_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\,\underset{2}{W}^{\beta} = 0\,,$$

$$(R_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\,\underset{3}{W}^{\beta} = \sigma g_{\alpha\beta}\,\underset{2}{W}^{\beta}\,.$$

The tensor $g_{\alpha\beta}$ is defined by the matrix (6). Assuming here $\alpha = 1, 2, \ldots, 6$, we can readily find that *the matrix* $(R_{\alpha\beta})$ (11) *will be*

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\underset{1}{\alpha} & 0 & 0 & -\underset{1}{\beta} & 0 & 0 \\ 0 & -\underset{2}{\alpha}+\sigma & 0 & 0 & -\underset{2}{\beta} & \sigma \\ 0 & 0 & -\underset{2}{\alpha}-\sigma & 0 & \sigma & -\underset{2}{\beta} \\ \hline -\underset{1}{\beta} & 0 & 0 & \underset{1}{\alpha} & 0 & 0 \\ 0 & -\underset{2}{\beta} & \sigma & 0 & \underset{2}{\alpha}-\sigma & 0 \\ 0 & \sigma & -\underset{2}{\beta} & 0 & 0 & \underset{2}{\alpha}+\sigma \end{array} \right\| , \quad \sigma \neq 0\,. \qquad (29)$$

Here σ can be arbitrary but $\neq 0$. As in the first case, $\underset{s}{\alpha}$ and $\underset{s}{\beta}$ are related to each other through

$$\underset{1}{\alpha} + 2\underset{2}{\alpha} = \varkappa\,, \qquad \underset{1}{\beta} + 2\underset{2}{\beta} = 0\,. \tag{30}$$

The frame is determined up to a rotation in the plane $\{\underset{2}{\overline{\xi}}\,\underset{3}{\overline{\xi}}\}$ and a Lorentz rotation in the plane $\{\underset{1}{\overline{\xi}}\,\underset{4}{\overline{\xi}}\}$.

We have to consider now the third type with the charcteristic $[\,3, \overline{3}\,]$. For this characteristic [6], the tensor $R_{\alpha\beta}$ will have only one principal isotropic direction $\underset{1}{W}^{\beta}$ and, additionally, two more vectors $\underset{2}{W}^{\beta}$ and $\underset{3}{W}^{\beta}$ with the properties

$$\left.\begin{aligned} (R_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta})\underset{1}{W}^{\beta} &= 0 \\ (R_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta})\underset{2}{W}^{\beta} &= \sigma\,\delta_{\alpha\beta}\,\underset{1}{W}^{\beta} \\ (R_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta})\underset{3}{W}^{\beta} &= \tau\,\delta_{\alpha\beta}\,\underset{2}{W}^{\beta} \end{aligned}\right\}, \tag{31}$$

where σ and τ are arbitrary numbers $\neq 0$. The vector $\underset{2}{W}^{\alpha}$ is non-isotropic, whereas $\underset{3}{W}^{\alpha}$ is isotropic. Besides that, $\underset{1}{W}^{\alpha}$ is orthogonal to $\underset{2}{W}^{\alpha}$ and not orthogonal to $\underset{3}{W}^{\alpha}$; while the vector $\underset{2}{W}^{\alpha}$ being orthogonal to $\underset{3}{W}^{\alpha}$.

Since $\underset{2}{W}^{pq}$ is not an isotropic bivector, then, similarly to the previous two cases, we can write this vector as

$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\underset{31}{\xi}^{pq}$$

by choosing an appropriate frame (with two degrees of freedom). Then, by taking into account the above conditions for orthogonality and isotropy, we shall get the following expressions for the bivectors $\underset{1}{W}$ and $\underset{2}{W}$:

$$\begin{aligned} \underset{1}{W}^{pq} &= \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq} + i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\,, \\ \underset{2}{W}^{pq} &= \lambda\{\underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq} - i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\}\,, \end{aligned}$$

where λ is an arbitrary number $\neq 0$. The further study is made following the same scheme as for the previous characteristic types: we should write the conditions (30) for $R_{\alpha\beta}$, fixing the facts that $\underset{1}{W}^{\alpha}$ is the vector of the principal direction (in the bivector space) and that the vectors $\underset{1}{W}^{\alpha}$, $\underset{2}{W}^{\alpha}$, $\underset{3}{W}^{\alpha}$ determine the invariant bundle of the tensor $R_{\alpha\beta}$.

These conditions are as follows:

$$\left.\begin{aligned}(R_{\alpha\beta} - K\,g_{\alpha\beta})\underset{1}{W}^{\beta} &= 0\\ (R_{\alpha\beta} - K\,g_{\alpha\beta})\underset{2}{W}^{\beta} &= \sigma\,g_{\alpha\beta}\underset{1}{W}^{\beta}\\ (R_{\alpha\beta} - K\,g_{\alpha\beta})\underset{3}{W}^{\beta} &= \tau\,g_{\alpha\beta}\underset{2}{W}^{\beta}\end{aligned}\right\}, \tag{32}$$

where σ and τ are non-zero numbers.

Considering that at any given point of T_4 the bivector $\underset{\sigma}{W}^{pq}$ corresponds to the vector $\underset{nt}{W}^{pq} \longrightarrow \underset{\sigma}{W}^{\alpha}$ in a local bivector metric space and taking into account that for the coordinate frame

$$\underset{nt}{\xi}^{pq} \longrightarrow \underset{\sigma}{\xi}^{\alpha} = \delta^{\alpha}_{\sigma},$$

it is not difficult to check that the system of equations (32) is reduced to the following nine independent equations:

$$\begin{aligned}&m_{11} + i n_{11} + i m_{13} - n_{13} = -K\,,\\ &m_{12} + i n_{12} + i m_{23} - n_{23} = 0\,,\\ &m_{13} + i n_{13} + i m_{33} - n_{33} = -\,iK\,,\\ &m_{12} + i n_{12} = -\,\sigma\,,\\ &m_{22} + i n_{22} = -K\,,\\ &m_{23} + i n_{23} = -\,i\sigma\,,\\ &m_{11} + i n_{11} - i m_{13} + n_{13} = -K\,,\\ &m_{12} + i n_{12} - i m_{23} + n_{23} = -\,\tau\,,\\ &m_{13} + i n_{13} - i m_{33} + n_{33} = iK\,,\end{aligned}$$

where $K = \alpha + i\beta$ is one of the two 3-fold roots of the secular equation

$$|\,R_{\alpha\beta} - K\,g_{\alpha\beta}\,| = 0\,,$$

and the numbers σ and τ are arbitrary but not equal to zero. This arbitrariness ensues from the arbitrariness of λ and is due to the isotropy of the vectors $\underset{1}{W}^{\alpha}$, $\underset{3}{W}^{\alpha}$. For instance, one can assume that σ and τ are real numbers.

By solving this system and also taking into account the conditions

$$\sum_{s=1}^{3} e_s m_{ss} = \varkappa\,, \qquad \sum_{s=1}^{3} e_s n_{ss} = 0\,,$$

one can check that $\tau = 2\sigma$, $\beta = 0$, $\alpha = \frac{\varkappa}{3}$, and the matrix $\| R_{\alpha\beta} \|$ takes the following form:

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\frac{\varkappa}{3} & -\sigma & 0 & 0 & 0 & 0 \\ -\sigma & -\frac{\varkappa}{3} & 0 & 0 & 0 & -\sigma \\ 0 & 0 & -\frac{\varkappa}{3} & 0 & -\sigma & 0 \\ \hline 0 & 0 & 0 & \frac{\varkappa}{3} & \sigma & 0 \\ 0 & 0 & -\sigma & \sigma & \frac{\varkappa}{3} & 0 \\ 0 & -\sigma & 0 & 0 & 0 & \frac{\varkappa}{3} \end{array} \right\|, \qquad (33)$$

where σ is an arbitrary non-zero number; the frame is determined up to a rotation in the two-dimensional plane $\{\underset{1}{\xi}\underset{3}{\xi}\}$ and a Lorentz rotation in the plane $\{\underset{2}{\xi}\underset{4}{\xi}\}$.

As the final result, we have the following theorem.

Theorem. *There exist three fundamentally distinct types of gravitational fields:*

The 1st type, with the characteristic of the K-matrix of the simple type $[\,1\,1\,1\,, \overline{1\,1\,1}\,]$, for which a real orthogonal frame is uniquely defined at any point of T_4, and with respect to which the matrix $\| R_{\alpha\beta} \|$ has the form (23) under the conditions (24) and (25).

The 2nd type, with the characteristic of a non-simple type $[\,2\,1\,, \overline{2}\,\overline{1}\,]$, for which the frame is defined having two degrees of freedom, and the matrix $\| R_{\alpha\beta} \|$ has the form (29) under the conditions (30).

The 3rd type has also the characteristic of a non-simple type $[\,3,\ 3\,]$; its frame has two degrees of freedom, and its matrix $\| R_{\alpha\beta} \|$ has the form (33).

Here the overlined numbers in the characteristics denote the power indices of those elementary divisors, whose bases are complex-conjugate to the bases corresponding to the numbers without overlining.

The three indicated types obviously admit some further more detailed classification. For example, one can distinguish the cases of multiple or real roots, as had been already done by the author earlier. This result, which I have obtained in 1950, was first published in 1951 in [1]. There is an ambiguity in the formulation given in that paper. The proof of the theorem from § 2 was also provided by A. P. Norden in 1952 (which was not published), whose starting point was from his study of bi-affine spaces. The proof given here is the third one and it is probably the simplest one.

As for the study carried out in § 3 (i.e., the determination of the canonical form of the matrix $(R_{\alpha\beta})$ for the orthogonal non-holonomic frame), we have to make the following note. At first thought, one might expect to approach this task in the following way: since the characteristic of the matrix $\| R_{\alpha\beta} - K g_{\alpha\beta} \|$ is known, it seems to be possible to write directly the canonical form of this matrix base on the general algebraic theory [6]. However, this cannot be done because the coefficients of admissible linear real transformations can be taken only in the form

$$A^{\alpha'}_{\alpha} = 2A^{[j'j']}_{ij},$$

where $A^{i'}_{i} = \left(\frac{\partial x^{i'}}{\partial x^{i}}\right)_{P}$ are the coefficients of some real orthogonal transformation at a given point P of the manifold T_4. That is, we can only use the transformations belonging to a subgroup of the group of all real orthogonal transformations in a 6-dimensional space.

This fact, which requires the arguments of § 3, is in our case obvious; it is a specific application of a more general theorem proved by G. B. Gurevich [7].

1. Petrov A. Z. On the spaces determining the gravitational fields. *Doklady Akademii Nauk USSR*, 1951, vol. XXXI, 149–152.

2. Landau L. D. and Lifshitz E. M. The Theory of Fields. Nauka Publ., Moscow-Leningrad, 1942, 263–268.

3. Kagan V. F. On Some Systems of Numbers Resulting from the Lorentz Transformation. Moscow State Univ. Press, Moscow, 1926, 1–24.

4. Dubnov Ya. S. On the doubly symmetric orthogonal matrices. In: Kagan V. F. On Some Systems of Numbers Resulting from the Lorentz Transformation. Moscow State Univ. Press, Moscow, 1926, 33–54.

5. Lopshitz A. M. Vector solution of a problem on doubly symmetric matrices. In: *Trans. Rus. Math. Congress 1927 at Moscow*, Moscow-Leningrad, 1928, 186–187.

6. Petrov A. Z. To the theorem on the tensors' principal axes. *Notices of the Kazan Phys. and Math. Soc.*, 1949, vol. 14, 43.

7. Gurevich G. B. On some nonlinear transformations of symmetric tensors and poly-vectors. *Matematicheskii Sbornik* (*New Series*), Steklov Math. Inst., Academy of Sci. USSR, 1950, vol. 26 (68), no. 3, 463–470.

On the Problem of the Existence of Stable Particles in the Metagalaxy

Kyril Stanyukovich

Abstract: In this paper, originally written in 1965, Kyril P. Stanyukovich introduces *planckeons* — fundamental particles, whose characteristics are based on the fundamental mass, length, and time introduced earlier by Max Planck (the Planck mass, the Planck length, and the Planck time). Moses A. Markov defined such particles in 1965 independently from Stanyukovich, and called them *maximons*. Originally published in Russian as: Stanyukovich K. P. K voprosy o suschetvovanii ustoychivykh chastiz v metagalaktike. *Problemy Teorii Gravitazii i Elementarnykh Chastiz*, vol. 1, Atomizdat, Moscow, 1966, 267–279. Translated from the Russian manuscript of 1965 by Dmitri Rabounski, 2008. The translator thanks Andrew K. Stanyukovich, Russia, for permission to reproduce the original version of this paper, and also William C. Daywitt, USA, for assistance.

In our Metagalaxy the following physically reasonable condition, which is obvious and well-verified by observations, is true: the radius of the Metagalaxy, $r_{_M}$, corresponds to its gravitational radius $r_{_{M_g}}$ and its curvature radius a, i.e.

$$r_{_M} = r_{_{M_g}} = \frac{GM_0}{c^2} = \left(\frac{G\delta_{_M}}{c^2}\right)^{-1/2}, \tag{1}$$

where $M_0 \approx \delta_{_M} r_{_M}^3$ is the mass of the Metagalaxy.

The relations in (1) can be compared to the contracted Einstein equation

$$R = \frac{const}{a^2} = -\varkappa T = \frac{8\pi G\delta_{_M}}{c^2}\,, \tag{2}$$

where R is the scalar curvature, $T = -\delta_{_M} c^2$ is the trace of the energy-momentum tensor, and $\delta_{_M} \approx M_0/r_{_M}^3$ is the density of the Metagalaxy. From here it follows that

$$\frac{GM_0}{c^2} = \frac{c^2 r_{_M}^2}{GM_0}\,. \tag{3}$$

This formula is an identity. In other words, the radius of the internal curvature of an object, which equals its gravitational radius, always equals the size L of the object.

Let us try to answer the following question: can the Metagalaxy contain objects which are analogous, in the sense of self-closure or selfcontainment, to the Metagalaxy itself, i.e. objects whose characteristics are

$$L = r_g = a\,, \tag{4}$$

where L is the size of such an object?

J. Oppenheimer and G. Volkoff [1], and also L. D. Landau [2] working independently, showed that, given a star whose mass is larger than $100M_{\odot} = 10^{35}$ g, such a star experiences rapid compression (collapse) so that its radius shrinks to its gravitational radius, or even smaller. It is likely that a reverse process, an anti-collapse process, is also possible that may explain many of the "enigmatic" bulky explosions of star-like objects in the universe.

As M. Planck has shown, the quantities $\hbar$, G, and c (here $\hbar$ is Planck's constant, G is the gravitational constant, c is the velocity of light) provide a base for the construction of the following quantities

$$\left.\begin{aligned} L &= \sqrt{\frac{\hbar G}{c^3}} = 1.6\times10^{-33}\ \text{cm} \\ m_{_L} &= \frac{1}{2}\sqrt{\frac{c\hbar}{G}} = \frac{\hbar}{2cL} = 1.1\times10^{-5}\ \text{g} \\ \tau_{_L} &= \frac{L}{c} = 10^{-43}\ \text{sec} \end{aligned}\right\}. \tag{5}$$

Note that in such a case, according to [3],

$$L = \frac{2Gm_{_L}}{c^2} = r_g\,, \tag{6}$$

where r_g is the gravitational radius specific to the mass $m_{_L}$.

The density of a "particle" whose mass is $m_{_L}$ is [5, Part II]

$$\delta_{_L} = \frac{3m_{_L}}{4\pi L^3} = \frac{3}{4\pi}\,\frac{c^5}{2\hbar G^2} \approx 10^{95}\ \text{g/cm}^3. \tag{7}$$

As a matter of fact, the curvature radius of the internal gravitational field of such a "particle", $a_{_L}$, is L. The scalar curvature is then

$$R = const\times\frac{6}{a_{_L}^2} = -\varkappa T = \frac{8\pi G\delta_{_L}}{c^2} = \frac{8\pi\times3c^3}{2\times4\pi\hbar G} = \frac{3}{L^2}\,, \tag{8}$$

from which we obtain $a_{_L} = L$ (here the $const = 1/2$).

It should be noted that, inside such a particle, we have an "Einstein universe" with variable curvature, or, more precisely, an internal Schwarzschild field [6]. Thus the size of such a quasi-particle is the same as its gravitational radius and its internal curvature radius, and, at the same time, its size satisfies the uncertainty principle.

Being born as a result of random fluctuations of energy, or in the initial stage of expansion of the Friedmann universe, such particles should be stable and neutral to any external radiation (both electromagnetic and gravitational). Such particles, in contrast to unstable geons assumed by Wheeler, should be stable, self-contained Einstein micro-universes. The charge of such a particle is of the order of $e_L = \sqrt{\hbar c} = \sqrt{137}\, e$, the internal field stress being $E \simeq H \simeq \frac{e_L}{L^2} \approx 10^{57}$ Oersteds, while the total energy of such particle corresponds to a rest-mass 10^{-5} g. The "Bohr radius" for such a particle is

$$r_B = \frac{\hbar^2}{e_L^2\, m_L} = \frac{\hbar^2}{137\, e^2\, m_L} \approx L\,.$$

Because the quantities L and m_L are connected to fluctuations of gravitational fields (gravitons), we are allowed to assume the number of such particles to be $N_L = N_g^{1/2} = N_p^{3/4}$, where $N_g = 10^{120}$ and $N_p = 10^{80}$ are the numbers of gravitons and nucleons in the Metagalaxy.

Thus, $N_L = 10^{60}$. In such a case the total mass of these particles is $M_L = m_L N_L = 10^{55}$ g which is the same as the mass of the Metagalaxy. In other words, the energy of these particles is of the same order as the energy of other kinds of matter, as it should be in a homogeneous model of the universe. The number of collisions among these quasi-particles (we will refer to such particles as *planckeons*, in memory of Planck) is determined, within an order of magnitude, by the following formula per unit time per unit of volume

$$n_{col} = \pi r_0^2\, c\, n_L n_p \approx r_0^2\, c N_p^{7/4} a^{-6}, \tag{9}$$

where $n_L = N_L a^{-3}$ and $n_p = N_p a^{-3}$ are the density of planckeons and the density of nucleons respectively, where a is the radius of the Metagalaxy and r_0 is the nucleon radius.

Calculations show that $n_{col} \approx 10^{-40}\,\text{cm}^{-3}\,\text{sec}^{-1}$. The energy radiated in these collisions corresponds to a rest-mass $10^{-45}\,\text{g cm}^{-3}\,\text{sec}^{-1}$ that is the mass necessary for the generation of new nucleons according to the Dirac-Hoyle theory that the law $N_p = T_m^2$ hold (here $T_m = \omega_0\, t_m$ is the dimensionless age, ω_0 is the frequency of strong interactions, t_m is the age of our universe — the Metagalaxy).

Naturally,

$$\triangle m = \frac{\triangle N_p\, m_p}{\triangle t\, a^3} \simeq \frac{T_m\, \omega_0\, m_p}{a^3} \approx 10^{-45}\ \text{g cm}^{-3}\,\text{sec}^{-1}. \tag{10}$$

Formula (9) gives

$$\triangle m = n_m m_{_L} = N_p^{7/4} r_0^2 \, c \, a^{-6} \sqrt{\frac{c\hbar}{G}}\,, \tag{11}$$

because

$$\sqrt{\frac{c\hbar}{G}} = m_p \, T_m^{1/2} = m_p \, N_p^{1/4}.$$

Then, comparing (10) and (11), we obtain

$$N_p^2 \, r_0^2 \, c \, a^{-3} = N_p^{1/2} \omega_0 \,.$$

Therefore, because $r_0 \omega_0 / c \approx 1$, we have

$$N_p^{3/2} = \frac{a^3}{r_0^3}\,, \qquad \frac{a}{r_0} = N_p^{1/2},$$

that is found in nature and so proves the aforementioned claim. Thus we can easily show that $\triangle m = \alpha \, M_0 / a^3$, where $M_0 = N_p \, m_p$ is the mass of the Metagalaxy. Naturally, $\alpha \sim t^{-1} \sim \omega_0 \, T_m^{-1}$; therefore

$$\triangle m = \frac{\omega_0 \, N_p \, m_p}{a^3} = \omega_0 \, N_p^{1/2} m_p \, a^{-3} = \omega_0 \, m_p \, T_m \, a^{-3}$$

that gives equation (10) as a result.

In Hoyle's theory [7] matter is produced from "nothing", a strange assumption at best. In contrast to that assumption, I suggested the hypothesis that the replenishment of the number of particles has its origin in the gravitational background from gravitational transmutations* of "heavy gravitons" [5, Part II]. Now we see that the aforementioned views come together in part. The Hoyle "nothing" is our particles-planckeons ("heavy gravitons"), whose energy is self-contained until the moment when the planckeon, due to interaction with another particle (an elementary particle e.g.), relinquishes its energy, realizing a multiple birth of $\approx 10^{20}$ nucleons. This I assert. Hoyle, however, contradicts the law of conservation of energy, because at a given

$$G \sim T_m^{-1}$$

$m_{_L}$ becomes

$$m_{_L} \sim T_m^{1/2},$$

so $m_{_L}$ is growing with time.

At present, from studies of V. A. Ambarzumian [8] and other modern astronomical observations, it is almost obvious that galaxies have an

*The term "gravitational transmutations" here is intended in the sense according to D. D. Ivanenko.

explosive origin similar to the origin of superstars, quasars, and several other objects in the Metagalaxy. My view on the origin of these objects is [5, Part II] that different kinds of super-dense particles of very small size, which were born and are still being born in interactions within the evolving Metagalaxy, can give birth to superstars and galaxies (meaning that superstars are galaxies in the process of being born). In this process, not only is matter evolving, but also the so-called universal-constants (such as $\hbar$, G, m_p, e) specific to the "universes" where the matter evolves. And now I will justify this view on more solid ground.

Having assumed that

$$G \simeq T_m\,, \qquad \hbar \sim m_p \sim e^2 \sim T_m^{-2},$$
$$c = const, \quad r_0 = const, \quad \omega_0 = const,$$

all conservation laws will be true, and also the relations

$$a = ct_m \sim T_m\,, \qquad \frac{Gm_p^2}{e^2} \simeq T_m^{-1}, \qquad N_p = T_m^2.$$

In such a case, m_L is large for small T_m. Naturally,

$$m_L = \sqrt{\frac{c\hbar}{G}} = \sqrt{\frac{c\hbar_0}{G_0}}\; T_m^{-3/2} = M_0\, T_m^{-3/2}\,. \tag{12}$$

For instance, given $T_m = 10^8$ we obtain $m_L = 10^{44}$ g as the mass of the Metagalaxy. Several of these particles still remain non-interacting, so they replenish the galaxy "reserve". Interactions of these particles with nucleons of that epoch ($m_p = 10^{40}$ g), in large groups, are able to give birth to galaxies. As these particles age, galaxies of smaller size, constellations, and stars of some classes are born. According to the hypothesis suggested by I. D. Novikov [9], super-stars represent the late explosion of a part of the "Friedmann super-dense substance", which was delayed while the main mass of the Metagalaxy evolved. My views [5, Part II], independent of his, are close to his nevertheless, but the mechanism delaying the evolution of the Friedmann substance, as I suggested (the density of this substance coincides with the initial density of the Metagalaxy, $\delta = 10^{95}$ g/cm$^3 = const$), is more specifically developed and reasonable. With this mechanism new developments in the theory of the explosive origin of galaxies and stars are possible [5, Part II].

The likely existence of planckeons suggests that the Metagalaxy itself may be only a "particle" in a complicated structure of a countably-dimensional hierarchy of "particles" in an infinite universe. In other systems similar to our Metagalaxy, other energy stores, light velocities, and particle sizes (on a relative scale of values) are possible. Such systems can be born as a result of interactions (collisions) among "particles", or as a result of fluctuations of other structural systems larger than our Metagalaxy. The "death" of such structural systems can be found in their expansion or compression, or absorption by external sources.

Aside from planckeons, other "self-contained" particles can exist in the Metagalaxy. The main parameters of several classes of such particles are given in the Table.

A few classes of "self-contained" particles can exist between the classes $n=4$ and $n=5$.

In the Table: N is the total number of "elementary" particles whose fluctuations give birth to $N_F = \sqrt{N}$ stable fundamental particles; m, L, and δ are the mass, size, and density of fundamental particles; $\hbar_n/\hbar_4$ is the effective magnitude of the "Planck constants" for these particles (the Table also shows the evolution of $\hbar_n/\hbar_4$ with time for particles of each class); and ν/a^3 is the relative volume "lost" in the self-contained particles.

It is interesting to note from the Table that the probability of large objects (such as galactic clusters and galaxies) emerging from particles whose N is large decreases. In other words, the probabilities of stellar-like objects emerging from particles whose N is small increases. At $n=4$ we have planckeons ($\hbar_4 = \hbar$).

Such stable particles can be called *fundamental particles* and, as a result of interactions (for the most part between particles of neighboring classes), can give birth to different "elementary" quasi-stable particles of stars, i.e. nucleons and leptons. In this process, 10^{-45} g cm^{-3} sec^{-1} of "new" substance is born, on the average, in the Metagalaxy.

M. A. Markov [10] suggests that planckeons ($n=4$ in our Table), referred by him as *maximons*, are quarks. I think however that quarks are particles with $n=5$. I suggest the following scheme of interaction between elementary particles and fundamental particles: fundamental particles can be born due to fluctuations of the fields of elementary particles; then the fundamental particles, in their interaction with the elementary particles and with each other, give birth to other elementary particles.

Because we assume that particles "age" (including the so-called elementary particles), i.e. their energy decreases with time, we should clearly determine how this happens. For particles which are in the quantum ground state, quantum mechanics prohibits both electromagnetic wave radiation and "corpuscular radiation". Quantum mechanics and the quantum field theory assume that such stationary states exist

n	N	$N_F = \sqrt{N}$	m, g	L, cm	δ, g/cm^3	$\hbar_n/\hbar_4$	ν/a^3
1	1	1	10^{55}	$10^{28} \sim T_m$	$10^{-28} \sim T_m^{-3}$	$10^{120} \sim T_m^3$	1
2	$10^{40} \sim T_m$	$10^{20} \sim T_m^{1/2}$	$10^{35} \sim T_m^{-1/2}$	$10^{7} \sim T_m^{1/2}$	$10^{14} \sim T_m^{-2}$	$10^{80} \sim T_m^2$	$10^{-40} \sim T_m^{-1}$
3	$10^{80} \sim T_m^2$	$10^{40} \sim T_m$	$10^{15} \sim T_m^{-1}$	10^{-13}	$10^{-54} \sim T_m^{-1}$	$10^{40} \sim T_m$	$10^{-80} \sim T_m^{-2}$
4	$10^{120} \sim T_m^3$	$10^{60} \sim T_m^{3/2}$	$10^{-5} \sim T_m^{-3/2}$	$10^{-33} \sim T_m^{-1/2}$	10^{95}	1	$10^{-120} \sim T_m^{-3}$
...	......	......	......	......	......	......	......
5	$10^{160} \sim T_m^4$	$10^{80} \sim T_m^2$	$10^{-24} \sim T_m^{-2}$	$10^{-52} \sim T_m^{-1}$	$10^{135} \sim T_m$	$10^{-40} \sim T_m^{-1}$	$10^{-160} \sim T_m^{-4}$

Table: The classes of “self-contained” particles, which are possible in the Metagalaxy.

for a particle in its minimum energy state. This assumption follows from the supposition that a particle can be absolutely isolated (or "shielded") from other particles and fields. Because this is true except for gravitational fields, the modern theories of quantum mechanics and the quantum field theory are theories working in a flat space-time (Minkowski's space), so they don't take into account interactions with the universal gravitational field which, according to the General Theory of Relativity, cannot be "shielded". Experiments of the last decades show that this is true to a measurement precision of at least 10^{-12}. Thus isolating particles from gravitational fields contradicts not only the General Theory of Relativity, but also the experimental evidence.

As an example, a single proton, shielded from all other fields, is a stationary superposition (in the quantum mechanics sense) of the states of a so-called "naked" proton (neutron + meson, etc.; such a proton is also known as "physical proton"). The stationary superposition is spherically symmetric (as it should be for a particle whose spin is 1/2) and, hence, such particles cannot produce radiation; so they remain in their stationary ground states.

If a proton is in the presence of another proton somewhere else in the universe, the impossibility of shielding their gravitational fields destroys the spherically symmetric superpositions of their virtual states due to the tidal forces which perturb their spherical meson shells.

The periodic order of the proton states during the deformation of their spherically symmetric forms results in the braking of the strong stationary states, and leads to the periodic processes of radiation and absorption of the gravitational field energy.

In the case of the gravitational interaction among many moving particles, the perturbation of the surface of each particle is depending on the perturbation or fluctuation of the space metric. The magnitude of such a metric perturbation is known from [1, 2], and is

$$L = \sqrt{\frac{G\hbar}{c^3}} = r_0\, T_m^{-1/2} = 10^{-33}\ \text{cm}. \tag{13}$$

Elementary particles are oscillators whose frequency is of the order of $\omega_0 = c/r_0 = 10^{23}\,\text{sec}^{-1}$. We therefore should consider the probability of various possible quantum transitions of these oscillators, which are due to the action of the fluctuating gravitational fields. Because the perturbations of the fields are small in magnitude, the respective solution of Schrödinger's equation leads to the formula

$$W_{0k} = \frac{\xi_0^{2k}}{2^k k!}\ \mathrm{e}^{-\xi_0^2/2}, \tag{14}$$

where W_{0k} is the transition probability from the stationary ground state to an excited state of level k, and where

$$\xi_0 = \chi_0 \sqrt{\frac{m\omega}{\hbar}}\,, \tag{15}$$

where $\chi_0 = L$, $\omega = \omega_0$, $m = m_p$.

Because

$$\xi_0 = 10^{-33} \sqrt{\frac{10^{-24}\,10^{23}}{10^{-27}}} = 10^{-20},$$

we obtain

$$W_{0k} = \frac{10^{-40k}}{2^k k!}\, \mathrm{e}^{-\frac{1}{2}10^{-4}} = \frac{10^{-40k}}{2^k k!}\,. \tag{16}$$

In the transition to a minimally excited state ($k = 1$), we have

$$W_{0k} = \frac{1}{2}\,10^{-40}\,.$$

The average numerical value $\bar{k} = \frac{\xi_0^2}{2} = \frac{1}{2}\,10^{-40} \approx 10^{-40}$. Therefore the radiation of energy is characterized by the value $\bar{k} = 10^{-40}$, which corresponds, for a nucleon, to the gravitational field energy relative to the energy of the strong interactions. We can also arrive at the same value from the following formula [11]

$$W = \bar{k}\,\frac{\triangle E}{E_0} = \left(\frac{E_g}{E_0}\right)^2. \tag{17}$$

Assuming the "naked" nucleon is not a point-mass, but a continuous particle whose size is $r = L = 10^{-33}$ cm [12], we have

$$E_g = \frac{G m_p^2}{L} = 10^{-20}\, m_p L^2 = 10^{-20}\, E \text{ cm}, \tag{18}$$

with the same result $W \simeq 10^{-40}$ obtained above.

The probability of radiation due to the action of an external field is always larger than the probability of absorption. The remainder of these probabilities, i.e. the probability of excess radiation, is of the order of these probabilities, and is proportional to them.

Because the energy density of such an external gravitational field decreases with time, the relative density of the field energy decreases with time for the time $\triangle\varepsilon_g/\varepsilon_g = 10^{-40} \sim T_m^{-1}$ during a single fluctuation or pulsation of the nucleon. Thus the probability of radiation exceeds the probability of absorption, and leads to a change in the relative energy of the particle.

The method of adiabatic invariants, applied to slow transitions due to adiabatic perturbations, leads naturally to the formula

$$W_{12} = \mathrm{e}^{-2T_m} \int\limits_{t_0}^{t} \omega_{21}(t)\, dt\,, \tag{19}$$

where $\omega_{21} = \frac{E_2 - E_1}{\hbar}$, $t = t_1$ is the current time, t_0 is the initial time, W_{12} is the probability of the particle (system) being in the state characterized by the wave function ψ_2 when $t \to \infty$ under the condition that this particle (system) was in the state ψ_1 as $t \to -\infty$. For the problem at hand, we will assume that $E_1 = E_2 = E_0$ at time $t = t_0$ and that $E_1 = E$ at $t = t_1$.

Let $\omega_{21} = \alpha = \frac{1}{2t}$, and the imaginary part of the complex "time" be $t_1 = t_0 = t = 10^{17}$ sec. In such a case, $E_0 - E_1 = 10^{-44}$ ergs corresponding to a mass $m_g = \frac{E_0 - E_1}{c^2} = 10^{-65}$ g, which is the actual mass of a graviton.

We approximate W_{12} by the equation

$$W_{12} \;=\; \mathrm{e}^{2} \int\limits_{t_1}^{t_0} \alpha\, dt \;=\; \mathrm{e}^{\int_{t_1}^{t_0} \frac{dt}{t}} \;=\; \frac{t_0}{t_1}\,. \tag{20}$$

For $t_1 = t_0 + \triangle t$ we have

$$W_{12} = 1 - \frac{\triangle t}{t_0} = 1 - 10^{-17} \triangle t\,.$$

While a single pulsation of a nucleon takes $\triangle t = 10^{-23}$ sec, we have $W_{12} = 1 - 10^{-40}$. Thus the change of the probability of the nucleon state in a single pulsation is

$$W = 1 - W_{12} = 1 - \frac{t_0}{t} \simeq \frac{\triangle t}{t} = 10^{-40}$$

which corresponds to the above conclusion arrived at in a different way.

These quantum transitions take place only if the frequency of the particle (oscillator) equals the frequency of the external field.

Because the displacement of such an oscillator is $x_0 = F/m\omega^2$, where F is an external force perturbing the oscillator, we arrive at

$$F = m_p \omega_0^2 L = m_p \omega_0^2 r_0 \, \frac{L}{r_0} = \frac{L}{r_0} \times 10^{-10} = 10^{10}. \tag{21}$$

On the other hand, the force of the gravitational field acting on the nucleon (oscillator) is

$$F = m_g^* c\, \omega = m_g c\, \omega \, \frac{m_g^*}{m_g} = 10^{-30} \, \frac{m_g^*}{m_g}\,, \tag{22}$$

where the mass of a quantum of the gravitational field at a distance L from the center of the nucleon is

$$m_g^* = \frac{G m_p^2}{c^2 L}\,, \qquad m_g = \frac{G m_p^2}{c^2 r_0}\,, \tag{23}$$

wherefrom we obtain

$$\frac{m_g^*}{m_g} = \frac{r_0}{L} = 10^{20}, \qquad F = 10^{10},$$

that coincides with the magnitude of F found in equation (21).

Comparing (21) and (22) in their general form, we obtain

$$L^2 = \frac{G m_p}{c\,\omega_0} = \frac{G m_p r_0}{c^2} = r_0\, r_g\,, \tag{24}$$

which is true in general as borne out by experiment.

So, our speculations and calculations show that a particle in a gravitational field (ignoring that field is unrealistic) cannot be in a stationary ground state. The term "stationary state" itself contradicts the covariant laws of the General Theory of Relativity which treats gravitation as a universal disturbance leading to changes in the space metric, i.e. to changes in the geometry of space and any sources located therein. Because a system of bodies interacting through the gravitation field cannot be at rest, the space metric changes with time and forces the bodies to radiate. This radiation is electromagnetic dipole, four-gravitational quadruple radiation.

An opposing view to that taken above would appear to contradict the principles of General Relativity which are well-verified by experiment.

1. Oppenheimer J. and Volkoff G. *Phys. Rev.*, 1939, vol. 55, 374.
2. Landau L. D. and Lifshitz E. M. Statistical Physics. 2nd edition, Nauka, Moscow, 1964.
3. Blokhintzev D. I. *Proc. of the 1st Soviet Conference on Gravitation*, Moscow Univ. Publ., Moscow, 1961.
4. Wheeler J. A. Gravitation, Neutrino, and the Universe. Foreign Literature, Moscow, 1962.
5. Stanyukovich K. P. Gravitational Field and Elementary Particles. Nauka, Moscow, 1965.
6. Landau L. D. and Lifshitz E. M. The Classical Theory of Fields. Phys. Math. Publ. (Fizmatgiz), Moscow, 1960.
7. Hoyle F. *Proc. Phys. Soc.*, 1961, vol. 77, 1.
8. Ambarzumian V. A. On the origin and evolution of galaxies. In: *The Earth and the Universe*, Znanie, Moscow, 1964.

9. Novikov I. D. *Soviet Astron. J.*, 1964, vol. 41, 1075.

10. Markov M. A. Can the gravitational field prove essential for the theory of elementary particles? *Progr. Theor. Phys.*, Supplement Extra Number, 1965.

11. Landau L. D. and Lifshitz E. M. Quantum Mechanics: Non-Relativistic Theory. 2nd edition, Phys. Math. Publ. (Fizmatgiz), Moscow, 1963.

12. Landau L. D. In: *Niels Bohr and the Development of Physics*, ed. by W. Pauli, McGraw-Hill (New York) and Pergamon Press (London), 1955.

On Increasing Entropy in an Infinite Universe

Kyril Stanyukovich

Abstract: In this 1948 presentation Kyril P. Stanyukovich concludes that increasing the entropy of an infinite universe does not lead to a state of equilibrium, but only to a non-cyclic evolution of matter. A very truncated Russian version of this presentation was published in 1949 as: Stanyukovich K. P. O vozrastanii entropii v beskonechnoy vselennoy. *Doklady Akademii Nauk USSR*, 1949, vol. LXIX, no. 6, 793–796. Translated from the Russian manuscript of 1948 by Dmitri Rabounski, 2008. The translator thanks Andrew K. Stanyukovich, Russia, for permission to reproduce the original version of this paper, and also William C. Daywitt, USA, for assistance.

Relativistic thermodynamics shows that a universe does not approach equilibrium by increasing the universe's entropy [1]. In contrast, classical mechanics, statistical mechanics, and thermodynamics come to the opposite conclusion, or they leave the question unanswered.

One often assumes [2] either that unknown physical conditions in the universe lead to a decreasing entropy, or that fluctuations in the infinitude of regions of the universe lead to decreasing entropy as well, that compensate the thermodynamical processes of increasing entropy.

Modern statistical mechanics, which was developed after J. W. Gibbs through the studies of G. D. Birkhoff and A. J. Khinchin [3], considers very large (but finite) sets of particles. As a result, modern statistical mechanics gives no direct answer to the important principal question: is a universe with increasing entropy approaching a state of equilibrium in all its finite regions, or not?

Authors of numerous other studies naturally recognize that entropy increases in most cases of closed and finite systems, while statistical methods are often assumed to apply to an unbounded universe. Nevertheless, even though the infinite universe may be closed as a whole, statistical calculations do not apply to its entirety. In particular, it is wrong to claim that, given an increasing entropy, the universe will automatically approach a state of equilibrium [4]. J. I. Frenkel [5] noted that the entropy of a system, which interacts only minimally with the rest of the universe, increases with time due to the perturbing action of this interaction on the motion of particles in this system. According to Liouville's theorem, a completely isolated system has the property that a given volume $\triangle\Gamma$ of phase space remains unchanged with time. This result follows because the entropy $s \sim \ln \triangle\Gamma$.

Let me now develop my own view on the impossibility of an infinite universe reaching a state of equilibrium.

Split such an infinite universe into a countable (countably infinite) set of finite regions. Clearly, such a splitting is possible.

Obviously each finite region contains a finite number of elementary particles of matter. Because we want to take into account the interaction between these particles and any fields (electromagnetic and gravitational) that may be present, we assume with the quantum theory that a finite region of the universe contains a finite number of quanta. We also assume that the elementary quanta are infinitely small, not in the sense of "energetic points", but in the sense that the countable set of such quanta occupies a finite volume and contains a finite energy.

Therefore, a finite volume of space can contain not only a finite number of elementary particles (including quanta), but also a countable set of them.

In such a case, a countable set of elementary particles inhabits the entire space.

Clearly the set of interactions per a finite interval of time between the particles located in each finite volume of space forms a countable set of interactions if the particles in that set are countable, and forms a finite set if the number of particles is finite. Thus in both cases a countable set of interactions will be realized within the entire infinite space during a finite interval of time. The term "interaction" here means any process in which two particles exchange energy.

Because any infinite interval of time can be split into a countable set of finite intervals, a countable set of interactions can be realized in the entire universe during an infinite interval of time.

Classical statistics, when applied to an infinite universe, has the drawback that it assumes such a universe contains particles of only a single class (an unlikely situation in our Universe). It should also be noted that not all of the theorems of classical statistics are applicable to infinite sets of particles because those theorems only operate on finite sets. So applying these theorems to an infinite set of particles is not correct and can lead to untrustworthy results.

I suggest that, if an infinite universe were inhabited by a countable set of particles of the same class (e.g., like molecules), even in the case where each particle is in the same k energy level allowed to that particle, the universe would evolve to a state of equilibrium after a countable number of interactions between the particles (any and all types of particles are envisioned).

The above is obvious in the case of a finite number of energy levels, because the set of independent distributions of the particles in these levels is of the order of ω^{k-1} (here ω is the number of the particles). In the case of a countable set of levels, the corresponding set of independent distributions of the particles is also a countable set.

Thus we can suppose that, during a finite or infinite interval of time $t \leqslant \infty$, an infinite universe consisting of particles of the same class (excluding their gravitation fields) will arrive at a state of equilibrium. In the case of a countable set of energy levels, the state of equilibrium will also be reached at $t \leqslant \infty$ (this is due in part to the fact that each particle's energy is finite).

Let us introduce, as a postulate, the assumption that a countable set ($\Omega \to \infty$) of classes of different "particles" inhabit an infinite universe, where particles of a class Ω_i can consist of particles of "lower" classes Ω_{i-1}, Ω_{i-2}, ... We can envision such a "particle" as any autonomous structure such as a photon, a molecule, a star, or a stellar system, etc. We can also assume that such an infinite variety of classes of different particles is the result of an interaction between the structure and its fields. Any number of each type of particle can be present in the universe (clearly the number of each type can be infinite). Due to interactions within the countable set of particles of different classes, particles of the same classes and, perhaps, particles of new classes can be born. Given the aforementioned postulate, relations between particles of different classes are inexhaustible as are the results produced by those interactions. Of course, in the interactions of these particles, processes of "association" and "destruction" of other particle types can result. The assumption of strongly one-way processes, however, is not allowed as such an assumption would contradict the experimental evidence. It is enough that a countable set of particles of different classes be present, and that we assume for the particles of each class that the countable set of processes in the class is accompanied by at least one process of the opposite direction.

Considering particles of the same class, the equilibrium state of a system of these particles excludes all other states. The inevitable fluctuations in such a system, however, always lead the system to numerous "states of equilibrium" which differ from each other by a small value. I call such an equilibrium *absolute equilibrium*.

In the case of a countable set of classes of different particles, the term "absolute equilibrium" has no meaning. Naturally, according to the postulate, an infinite universe always contains several non-empty sets of particles of each class (we assume that these are countable sets of particles), and that the entire universe — the set of particles of all classes — is already in a state of equilibrium. We therefore consider the set of particles of a class Ω_i, for instance. Because particles of the lower classes Ω_{i-1}, Ω_{i-2}, ... are elements consisting of particles of the class Ω_i, interactions between particles of the class Ω_i can perturb particles of the lower classes. Therefore interactions between these particles will act on particles of the lower class Ω_{i-1} and also on particles of all other lower classes is such a way that systems of the lower-class particles will never be in an equilibrium state.

Because the order of a class i is unbounded, any structures in the universe can never be in a state of equilibrium. Thus the universe cannot approach a state of equilibrium. So a claim about a state of equilibrium for the entire universe should be looked upon with skepticism.

Clearly, absolute equilibrium can be reached in an infinite universe only if "particles" of different classes, which inhabit this universe, degenerate into "particles" of a single class. As shown above, however, this is not possible. Thus real interactions lead to such states, where substance of the universe experiences permanent evolution.

As interactions between particles of a class Ω_i cause particles of a lower classes to be in a non-equilibrium state, and as the number of particles in each class is variable, the clear result of these interactions will be a set of particles that approaches an equilibrium state. This follows because, as the set of particles reach new states again and again, these states are (more often than not) at a higher level of entropy than the previous states.

Because the order of a class i is unbounded, the result of these interactions leads to a "non-cyclic" evolution of matter that persists indefinitely.

It is interesting to note that the set of all formally imaginable distributions of particles of different classes among their respective energy levels acts as a continuum; so the number of possible sets is effectively inexhaustible.

So finally we arrive at the conclusion that an increasing entropy in the visible part of our Universe is not a factor in causing the Universe to approach its equilibrium state, but is a result of a permanent, non-cyclic, evolution of matter.

Discussion. When I suggested this theory (in the beginning of 1948), the core of which is my thesis that a countable set of molecules of the same class among other classes is always in a state of non-equilibrium, I met with some criticism from I. R. Plotkin. He told me that my conclusion proceeded from the erroneous belief that, given a countable set of particles, the set of independent distributions of the particles among their different states is also a countable set.* Here I would like to answer this criticism in detail.

1. Any infinite universe can be split into a countable set of regions.

2. Each finite region contains a finite number of particles, which is a countable set as well.

3. Thus, the entire space of an infinite universe contains a countable set of particles.

4. Thus, a countable set of interactions between the particles takes place in the entire space during a finite interval of time.

5. Any infinite interval of time can be split into a countable set of finite intervals. So, a countable set of interactions takes place in an infinite universe during infinite interval of time.

6. The set of all possible interactions in a countable set of particles is the set of all sub-sets of the countable set. This set has the power of continuum.

7. A countable set of interactions taking place during even an infinite interval of time cannot exhaust that continuum of interactions which are possible in the set.

8. Suppose an infinite universe is filled with particles of a single class. In such a case the set of all states the particles occupy is $N_n = k^n$ which is a continuum, where k represents a finite number of the energy levels, while $n \to \infty$ is the number of the particles. In the general case, the set of independent distributions of n particles among the energy levels is $N_k = \frac{(n+k-1)!}{n!(k-1)!}$. Having $n \to \infty$ (our case) gives $N_k = \frac{n^{k-1}}{(k-1)!}$, i.e. N_k is a *countable infinity* in our case. Points which characterize the non-equilibrium states in the phase space are distantly separated from the point of "absolute equilibrium" therein. The set of these points is only countable because the set of distributions of the particles among their energy levels is $\frac{n^{k-1}}{(k-1)!}$. Thus, if an infinite universe consists of a single class of particles, such a universe can reach a state of equilibrium only after a countable set of interactions among the particles has taken place, i.e. during an infinite amount of time, while all the rest "bank" of the continuum of the possible interactions was remained unused.

Consider the set $\bar{M}$ of all possible states for the infinite universe. Select the sub-set $\bar{n} \leqslant \bar{M}$ of these states where the universe is in a state

*Later Plotkin has published his criticism in a valuable Soviet journal of physics: Plotkin I. R. *JETP-USSR*, 1950, vol. 20, no. 11, 1051. — Editor's comment. D.R.

of equilibrium. Transitions inside of each "factor-set" $\frac{\bar{M}}{\bar{n}} \neq \bar{n}$ are due to increasing entropy, and are still evolving toward a state of equilibrium. Extract an element $\alpha\bar{n} \in \frac{\bar{M}}{\bar{n}}$ different from $\bar{n}$, i.e. $\alpha\bar{n} \prec \bar{n}$ (here $\prec$ means "much less than"). Assume that, at the moment of time $t = t_0$, the universe is in one of the states of the class $\alpha\bar{n}$. As such the universe will experience the transitions $\alpha\bar{n} \to \beta\bar{n} \to \gamma\bar{n} \to \ldots$ for which $\alpha\bar{n} \prec \beta\bar{n} \prec \gamma\bar{n} \prec \ldots$ We denote $\bar{A}$ as the power of the set of all the transitions experienced by the universe from $t = t_0$ until $t \to \infty$, while $\bar{B}$ denotes the power of the set of all transitions which are necessary for the universe to be in the states of the class $\bar{n}$ (i.e. to be in the state of equilibrium). The universe consisting of particles of the same class is always in a state of non-equilibrium if and only if $\bar{A} < \bar{B}$. However, the opposite condition $\bar{B} < \bar{A}$ is true for the two obvious reasons: 1) given a countable set of particles of the same class, the set of their independent distributions among their energy levels is a countable set; 2) considering heat-conduction or diffusion of a gas in an unbounded space, we conclude that, even if an extremely lopsided distribution of heat exists in the space (where all heat has been condensed into a small region in which the energy density is infinite e.g.), heat eventially becomes equally distributed in the space after an infinite amount of time, so the state of the gas becomes with time only infinitesimally different from the equilibrium state.

9. Imagine an infinite universe filled with a countable set $\Omega \to \infty$ of classes of particles, where each particle of a class Ω_i can contain particles of all lower classes (Ω_{i-1}, Ω_{i-2}, ...). As the assumption of only one-way processes is unacceptable, there are processes of both association and dissociation of the particles. Once a single process appears among a set of exclusively opposite processes in the same class of particles, a countable set of both classes of particles will be generated from the original set some time later.

10. In such a case, not only the set of all possible states, but also the set of all independent dispositions of the different particles among their energy levels, will exist; leading to

$$N_{km} = \prod_{j=1}^{j=m\to\infty} \frac{(n_j + k - 1)!}{n_j(k-1)!} \longrightarrow \left(\frac{n^{k-1}}{(k-1)!}\right)^m \sim 2^m,$$

where $m \to \infty$ is the number of particle classes.

11. Thus, despite increasing entropy in each finite region of the infinite universe, the entire universe containing the countable set of different particle classes is always in a state of non-equilibrium and is unable to reach equilibrium.

12. So, a countable set of particles of the same class reaches the state of equilibrium only through *regular infinity* (i.e. *actual infinity*) after a countable set of interactions among the particles has taken place (see Thesis 8). Therefore Plotkin was wrong when criticized my thesis that a countable set of molecules of the same class among other classes is always in a state of non-equilibrium. He was wrong as well when claiming that a permanent strong non-equilibrium state is specific to a countable set of particles of the same class, if this is the single class of particles in the universe. On the contrary, the universe is able to be in a non-equilibrium state due to the many-level internal structure of the particles which inhabit it.

In conclusion I would thank S. I. Vavilov* for valuable discussions and comments that have made this short paper a better paper. I would also like to thank N. N. Bogoliubov† and O. J. Schmidt‡ who supported me in this discussion. Again, special thank go to S. I. Vavilov, who ordered to publish a truncated version of this presentation in the near issue of the journal of the USSR Academy of Sciences§.

1. Landau L. D. and Lifshitz E. M. The Classical Theory of Fields. Techn. Theor. Literature Publ. (GITTL), Moscow-Leningrad, 1941.
2. Boltzmann L. Notes on the Methodology of Physics. Timiriazev Sci. Res. Inst. Publ., Moscow, 1929.
3. Khinchin A. The Mathematical Foundations of Statistical Mechanics. Joint State Publ. & Head Publ. Techn. Theor. Literature (OGIZ-GITTL), Moscow-Leningrad, 1943.
4. Landau L. D. and Lifshitz E. M. Statistical Physics. Technical Literature Publ. (Gostechizdat), Moscow, 1938.
5. Frenkel J. I. Statistical Physics. Acad. Sci. USSR Publ., Moscow, 1948.

*Sergei Ivanovich Vavilov (1891–1951), a Russian research physicist who was the President of the USSR Academy of Sciences in the years 1945–1951. — Editor's comment. D.R.

†Nikolai Nikolaevich Bogoliubov (1909–1992), a Russian mathematician and theoretical physicist, and a member of the USSR Academy of Sciences. — Editor's comment. D.R.

‡Otto Julius Schmidt (1891–1956), a Russian mathematician, geophysicist, cosmologist, and astronomer, and a member of the USSR Academy of Sciences. — Editor's comment. D.R.

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On the Evolution of the Fundamental Physical Constants

Kyril Stanyukovich

Abstract: This is a presentation held, by Kyril P. Stanyukovich, on May 12, 1971, in Kiev, at the Institute of Theoretical Physics at the seminar on General Relativity maintained by Alexei Z. Petrov. Here Stanyukovich proposes his original theory of evolution of the fundamental physical constants with cosmological time, based on relations between the cosmological and quantum constants. He shows that, given only three experimentally measured fundamental physical quantities G, c, and $\hbar$, and also the scalar curvature R of space, which is changing with time, it is possible to express all rest-frame fundamental constants in terms of the aforementioned four basic parameters. Translated from the Russian manuscript of 1971 by Dmitri Rabounski, 2008. The translator thanks Andrew K. Stanyukovich, Russia, for permission to publish this paper, and also William C. Daywitt, USA, for assistance.

First we introduce the following definition of the gravitational mass, m_g, proceeding from the "linear quantum theory" authored by M. P. Bronstein [1]

$$m_g = \frac{\hbar}{ca}\,, \tag{1}$$

where a is the radius of the Metagalaxy. On the other hand, as one assumes,

$$\frac{m_g}{m_p} = \frac{2Gm_g^2}{\hbar c}\,, \tag{2}$$

where m_p is the mass of a nucleon. Because

$$m_p = \frac{\hbar}{\lambda c}\,, \tag{3}$$

where λ is the Compton wavelength of the nucleon (the "size" of the nucleon), we obtain, on the basis of the formulae (1), (2), and (3), that

$$\frac{m_g}{m_p} = \frac{\lambda}{a} = \frac{2G\hbar}{c^3\lambda^2} = \frac{L^2}{\lambda^2}\,,$$

where $L = \sqrt{2G\hbar/c^3}$ is the Planck length. Thus we obtain the fundamental relaton

$$\lambda^3 = L^2 a = \frac{2G\hbar a}{c^3} = \frac{2G\hbar}{c^2 H}\,, \tag{4}$$

where $H = \frac{c}{a}$ is Hubble's constant.

Let us check the validity of the resulting fundamental relation (4), and how this relation works. Substituting

$$G = 6.7 \times 10^{-8}\ \text{cm}^3/\text{g} \times \text{sec}^2\,, \qquad \hbar = 6.6 \times 10^{-27}\ \text{erg} \times \text{sec}\,,$$

$$c = 3 \times 10^{10}\ \text{cm/sec}\,, \qquad H = 10^{-17}\ \text{sec}^{-1},$$

into (4), we obtain $\lambda^3 \simeq 10^{-38}\,\text{cm}^3$ and $\lambda \simeq 10^{-13}\,\text{cm}$ that meets the real numerical values of these quantities.

We introduce one relation more, namely

$$\frac{2GM_0\, m_g}{\hbar c} = 1\,, \tag{5}$$

where M_0 is the mass of the Metagalaxy. Proceeding from (1) and (3), we have

$$\frac{2GM_0}{c^2} = a = r_{gM}\,, \tag{6}$$

where r_{gM} is the gravitational radius calculated for the entire Metagalaxy (the gravitational radius of the Metagalaxy). In actuality, at the present epoch the size of the Metagalaxy equals its gravitational radius. Therefore the relation (6) is wide used in cosmology.

Thus we verified again the fundamental relation (5). We are going to check it numerically. Because, according to the modern bounds, M_0 in the order of 10^{56} g, and m_g is in the order of 10^{-66} g, we obtain

$$\frac{\frac{8}{3} \times 10^{-7}\, 10^{56}\, 10^{-66}}{3 \times 10^{10}\, 10^{-27}} = \frac{8}{9}\, \frac{10^{-17}}{10^{-17}} \approx 1\,,$$

so the relation (5) has been completely verified. Since, according to the main cosmological assumption, space-time as a whole is homogeneous and isotropic (the assumption of homogeneous time is equivalent to the law of conservation of energy), the relations (5) and (6) should be true not only now, but for all time.

Assume M_0, c, and λ to be constants which remain unchanged with time. This supposition permits the possibility for introducing the main scales of length, time, and mass. In such a case, from (6) and (4), we have

$$G \sim a\,, \qquad G\hbar \sim a^{-1},$$

thus we obtain

$$\hbar \sim a^{-2}.$$

Because $R = \frac{12}{a^2}$ in a homogeneous 4-space, we have

$$G \sim R^{-1/2}, \qquad \hbar \sim R\,.$$

Thus we verified, again, our views according to which several fundamental "constants" are functions of the scalar curvature R in a space-time of variable curvature. In such a case,

$$H \sim R^{1/2}.$$

So, we have already reached a very good result according to which: given three experimentally measured physical quantities G, c, and $\hbar$, and also the scalar curvature of space, R, which changes with time, all rest-frame functions of the fundamental physical quantities can be expressed in terms of these 4 main parameters.

We have already calculated $\lambda = 2\times10^{-13}$ cm. Thus proceeding from (6) we obtain M_0. Since

$$a = \frac{c}{H} = 10^{28} \text{ cm},$$

we obtain

$$M_0 = \frac{a c^2}{2G} = 10^{56} \text{ g}$$

which is exactly the known numerical value 10^{56} g mentioned above. Proceeding from these, we obtain the matter density of the Metagalaxy

$$\rho = \frac{3}{4}\frac{M_0}{\pi a^3} \simeq 10^{-29} \text{ g/cm}^3,$$

then we calculate the mass of a nucleon

$$m_p = \frac{\hbar}{\lambda c} \approx 10^{-24} \text{ g},$$

and also the mass of a gravition

$$m_g = \frac{\hbar}{ca} \approx 10^{-64} \text{ g}.$$

Accordingly we calculate the number of nucleons in the Metagalaxy

$$N_p = \frac{M_0}{m_p} = \frac{a^2}{\lambda^2} = T_m^2 \approx 10^{80}$$

and the number of gravitons in the Metagalaxy

$$N_g = \frac{M_0}{m_g} = \frac{a^3}{\lambda^3} = T_m^3 \approx 10^{120}$$

where $T_m = \frac{a}{\lambda} = \frac{c}{\lambda H} = \frac{\omega}{H}$ is the Dirac dimensionless time.

Now we are going to determine the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}.$$

As L. D. Landau has already shown [2], the electron's charge hidden from observation can be 137 times larger than its observed charge e. This results from the polarization of the vacuum. In [3], assuming the density of dipole charges to be $\delta = \frac{3}{4\pi r^3}$, we obtained that the total charge as

$$e^* = 4\pi e \int_{r_1}^{r_2} \delta\, r^2\, dr = 3\, e \ln \frac{r_2}{r_1}\,, \tag{7}$$

where r_1 is the "minimal radius" of the particle, while r_2 is its "external size". The size r_1 should be understood as the Planck length L, and also the planckeon model of elementary particles should be taken into account.

According to the planckeon model, there in the centre of each particle a planckeon is located — an Einstein micro-universe whose size is $L = \sqrt{2G\hbar/c^3} = 10^{-33}$ cm. A planckeon, due to its own fluctuations, ejects a part of its substance into outer space: this "atmosphere" surrounding planckeons is observed as elementary particles.

Thus we are lead to

$$e^* = 3\, e \ln \frac{\lambda}{L} = e \ln\left(\frac{\lambda}{L}\right)^3 = e \ln \frac{a}{L} = e \ln 10^{60} \simeq 138\, e\,. \tag{8}$$

The efficiency of the charge — the quantity which characteres the interaction between e and e^* — is $e\, e^* = \hbar c$. Thus

$$\frac{1}{\alpha} = \frac{\hbar c}{e^2} = 1 + \ln\left(\frac{\lambda}{L}\right)^3, \tag{9}$$

where $const = 1$ has been introduced for an ultimate case where $\lambda = L$ and $\alpha = 1$. So we obtain

$$e\, e^* = 137\, e^2 = \hbar c = \frac{e^2}{\alpha}\,.$$

From Y. Nambu's empirical formula which characterizes the whole "spectrum" of elementary particles, along the "spectrum" the ratio between the mass m of any elementary particle and the mass of an electron, m_e, is given by the law

$$\frac{m}{m_e} = \frac{2\, n}{\alpha}\,, \tag{10}$$

where n is an integer specific to the particle. Thus we suggest that the "relative particle mass" changes according to logarithm of the space curvature.

Since

$$\left(\frac{\lambda}{L}\right)^3 = \frac{a}{L} = \left(\frac{a}{\lambda}\right)^{3/2},$$

we obtain

$$e^* = 3\,e\ln\frac{\lambda}{L} = \frac{3}{2}\,e\ln\frac{a}{\lambda} = e\ln\frac{a}{L}\,,$$

which allows us the opportunity to interpret $1/\alpha$ as either the logarithm of the probability for any particle of the Metagalaxy to be inside the volume, equal to the volume of this particle, or the entropy of a nucleon, calculated per one particle

$$s = -\,k\ln W = k\ln\left(\frac{a}{\lambda}\right)^3 = \frac{2\,k}{\alpha} = 274\,k\,,$$

where k is Boltzmann's constant.

Because we have derived above how G and $\hbar$ change with the curvature R, we easily calculate

$$m_p = \frac{\hbar}{\lambda c} \sim R\,, \quad m_g = \frac{\hbar}{c\,a} \sim R^{3/2}, \quad L = \sqrt{\frac{2\,G\hbar}{c^3}} \sim R^{-1/4},$$

$$\frac{a}{L} \sim R^{-3/4}, \qquad e^2 \sim \frac{R}{\ln R}\,, \qquad \omega = \frac{c}{\lambda} = const,$$

where ω is the frequency of strong interactions. The frequency of electromagnetic radiation and the radii of the "Bohr orbits" (the first "Bohr orbit", for example) are

$$\omega_\delta = \frac{m_e\,c^2}{\hbar}\left(\frac{e^2}{\hbar c}\right)^2, \qquad r_\delta = \frac{\hbar^2}{m_e\,e^2}\,,$$

where $m_e \simeq 10^{-27}$ g is the mass of the electron which changes logarithmically with time. Thus we obtain (here $ct = a$)

$$\omega_\delta = \frac{m_e\,c^2}{\hbar}\,\frac{1}{\left(1 + 3\ln\frac{\lambda}{L}\right)^2} = \frac{m_e\,c^2}{\hbar}\,\frac{1}{\left(1 + \frac{3}{2}\ln\frac{ct}{\lambda}\right)^2}\,. \tag{11}$$

Let a source of light move away from us with a velocity $v = \frac{r}{t_n}$, and be currently located at a distance r from us. In such a case the observed frequence of the source is determied by the relation

$$\omega = \omega_0\sqrt{\frac{1 - \frac{r}{c\,t_n}}{1 + \frac{r}{c\,t_n}}}\left(\frac{\alpha_0}{\alpha_n}\right)^2,$$

where ω_0 and α_0 are the numerical values of ω and α at the moment of time t_0, while α_n is the numerical value of α at the moment t_n when the light beam was radiated. Here $ct_0 + r = ct_n$, where r is the distance between us and the source of the light as mentioned above.

Since

$$\frac{\alpha_0}{\alpha_n} = \frac{1 + \frac{3}{2}\ln\frac{ct_0}{\lambda}\left(1 + \frac{r}{ct_0}\right)}{1 + \frac{3}{2}\ln\frac{ct_0}{\lambda}}\,,$$

we finally obtain

$$\omega = \omega_0 \sqrt{\frac{1 - \frac{r}{ct_0}}{1 + \frac{r}{ct_0}}} \left(\frac{1 + \frac{3}{2}\ln\frac{ct_0}{\lambda}\left(1 + \frac{r}{ct_0}\right)}{1 + \frac{3}{2}\ln\frac{ct_0}{\lambda}}\right)^2, \tag{12}$$

$$\omega = \omega_0 \sqrt{\frac{1 - \frac{r}{ct_n - r}}{1 + \frac{r}{ct_n - r}}} \left(\frac{1 + \frac{3}{2}\ln\frac{ct_n}{\lambda}}{1 + \frac{3}{2}\ln\frac{ct_n}{\lambda}\left(1 - \frac{r}{ct_n}\right)}\right)^2. \tag{13}$$

Let $r = \alpha ct_0$. Then (12) takes the form

$$\omega = \omega_0 \sqrt{\frac{1-\alpha}{1+\alpha}} \left(1 + \frac{\ln(1+\alpha)}{\frac{2}{3} + \ln\frac{ct_0}{\lambda}}\right)^2. \tag{14}$$

Because $\frac{ct_0}{\lambda} \gg 1$ at the present epoch, the correction to the Doppler effect is infinitesimal; so it can be neglected in the calculation.

On the other hand, at the initial moment of time, when $\frac{ct_0}{\lambda} \simeq 1$, the "ageing effect" was able to have a strong effect on the violet shift in the spectral lines. Most probably, the formulae (12) and (13) should be corrected "logarithmically", because $\frac{m_e}{m_p} \sim \alpha^{1.5}$. In such a case, the exponent in the formula will not be 2, but approximately 3.5 that does not no change the essence of the problem.

Now we calculate the primordial temperature. The initial temperature of the electromagnetic radiation is

$$T_0 = \frac{m_p c^2}{k}\,\frac{\bar{e}^2}{\hbar c}\,, \tag{15}$$

where $\frac{\bar{e}^2}{\hbar c}$ should be in order of $\frac{1}{100}$ (this is because, given at least $t = 1\,\text{sec}$, we have $T_m = 10^{23}$ and $\frac{\bar{e}^2}{\hbar c} = \frac{1}{80}$).

Let us calculate the change of the temperature of electromagnetic radiation with time according to the theory of evolution of the fundamental constants we have suggested here.

Because the total energy radiated by a blackbody per one cm^3 is $\varepsilon = \frac{4\sigma}{c}$, where $\sigma = \frac{\pi^2 k^4}{60\,\hbar^3 c^2}$ is the Stefan–Boltzmann constant, the pressure of the blackbody radiation $p = \frac{1}{3}\varepsilon$ is

$$p = \frac{4\sigma}{3c}\,T^4 = \frac{\pi^2}{45}\left(\frac{k}{\hbar c}\right)^3 k\left(T^0_{EM}\right)^4 . \tag{16}$$

Because $k\,T^0_0 = \hbar\omega$, where $T^0_0 = const$ (an isotermic process) is the temperature of a nucleon, we obtain

$$k \sim \hbar \sim T_m^2\,, \qquad \frac{k}{\hbar} = const\,, \qquad p \sim k\left(T^0_0\right)^4 .$$

Initially the pressure is $p_0 \simeq k\left(T^0_0\right)^4$, so we obtain

$$\frac{p}{p_0} = \frac{k}{k_0}\left(\frac{T^0_{EM}}{T^0_0}\right)^4 = \frac{1}{T_m^2}\left(\frac{T^0_{EM}}{T^0_0}\right)^4 . \tag{17}$$

So far, $\frac{p}{p_0} = T_m^{-3}$ in the isotermic expansion of the Metagalaxy. Therefore, as we showed in [4, Part II, §7],

$$\frac{1}{T_m^2}\left(\frac{T^0_{EM}}{T^0_0}\right)^4 = \frac{1}{T_m^3}\,,$$

hence we obtain

$$\frac{T^0_{EM}}{T^0_0} = \frac{1}{T_m^{1/4}}\,. \tag{18}$$

Finally, proceeding from (15) and (18), we obtain

$$T^0_{EM} = \frac{T^0_0}{T_m^{1/4}} = \frac{m_p c^2}{k\,T_m^{1/4}}\,\frac{\bar{e}^2}{\hbar c} \simeq \frac{10^{-24}\,10^{21}}{3\times 10^{10}\,10^{-16}\,10^2} \simeq 3^\circ\,\mathrm{K}\,,$$

that equals the measured temperature of primordial photons. Because p and the number of primordial photons in one cm^3, n_{EM}, is connected through the obvious relation

$$p = 2\,c\,\hbar N_p n_{EM}^{4/3} \sim T_m^{-3}\,,$$

hence we have

$$n_{EM} \sim T_m^{-3/4}$$

and, because the initial number of primordial photons is

$$n_{EM_0} = \left(\frac{10^{95}\,10^{21}}{2\times 3\times 10^{10}\;\,7\times 10^{-27}\,10^{80}}\right)^{3/4} = 10^{39}\ \mathrm{cm}^{-3},$$

where we used $p_0 = \rho_0 c^2$, $\rho_0 = \frac{3}{4}\frac{M_0}{\pi L^3}$, $M_0 = \frac{L c^2}{2G}$ which are valid at the initially moment of time $(a = L)$, allowing us to use the numerical value $p_0 = \rho_0 c^2 = 10^{95}\,10^{21} = 10^{116}$.

Given the present epoch, we obtain

$$n_{EM} = 10^9 \text{ cm}^{-3}.$$

It is obvious, proceeding from the theory of evolution of the fundamental constants suggested here, that the density of primordial photons should equal the number of gravitons per unit volume.

Naturally, because

$$k_0 T_0^0 = m_{g_0} c^2 = E_0 = M_0 c^2,$$

where E_0 is the total initial energy, and, on the other hand,

$$E_0 = k_0 T_0^0 n_{EM_0} V_0 = k T_{EM}^0 n_{EM} V,$$

we obtain $n_{EM_0} V_0 = 1$. From here, as $N_g = T_m^3$ and $V \sim T_m^3$, we obtain

$$n_{EM_0} = \frac{1}{V_0} = n_g = \frac{N_g}{V} = const\,.$$

Thus it is easy to see that

$$k T_{EM}^0 n_{EM} = \frac{E_0}{V} = p_g\,,$$

where

$$p_g = \frac{E_0}{V} = \frac{2GM_0^2}{V^{4/3}} \simeq 10^{-7} \text{ dynes/cm}^2 = \text{erg/cm}^3.$$

The pressure produced by the intergalactic field equals, within the order of the numerical estimates, the density of the energy of the gravitational fields. Thus the energy of the primordial radiation is found to be related directly to the energy of gravitational fields.

The energy of a primordial particle is

$$E_{PM} = k T_{EM}^0 = 10^{-15} \text{ erg},$$

corresponding to a rest-mass $m_{EM} = 10^{-36}$ g.

The ratio

$$\frac{E_{PM}}{E_p} = \frac{10^{-15}}{10^{-3}} = 10^{-12}$$

corresponds to the ratio specific to forces the weak interactions. This fact cannot be an accident.

It is interesting to note that the relation

$$\frac{q_{c_1}^2}{e^2} = \frac{1}{T_m^{1/4}} = 10^{-10},$$

where $q_{c_1}^2 = 10^{-30}$ characterizes the weak interactions, can also not be an accident. However the reason for this relation is not clear yet.

That fact that the frequency $\approx 10^{12}\,\text{sec}^{-1}$, which is specific to the Lamb shift, corresponds to the energy of the order 10^{-15} erg can also not bound to be an accident.

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1. Bronstein M. P. *JETP-USSR*, 1936, vol. 6, 135.
2. Landau L. D. The quantum field theory. In: *Niels Bohr and the Development of Physics*, ed. by W. Pauli, McGraw-Hill (New York) and Pergamon Press (London), 1955.
3. Stanyukovich K. P. *Trans. Moscow University*, *Phys. and Astron. Series*, 1965, no. 5.
4. Stanyukovich K. P. Gravitational Field and Elementary Particles. Nauka, Moscow, 1965.

Biography of Ole Rømer
(1644–1710)

Ole Rømer. Courtesy Rundetårn, the observatory and museum in Copenhagen.

Ole Christensen Rømer was a Danish astronomer who in 1676 made the first quantitative measurements of the velocity of light. In scientific literature alternative spellings, such as "Roemer", "Römer", and "Romer", are common.

Ole Rømer was born 25 September 1644 in Århus to a merchant and skipper Christen Pedersen and Anna Olufsdatter Storm, daughter of an alderman. Christen Pedersen had taken to using the name Rømer, which means that he was from Rømø, to disambiguate himself from a couple of other people named Christen Pedersen [1]. There are few sources on Ole Rømer until his immatriculation in 1662 at the University of Copenhagen, at which his mentor was Rasmus Bartholin who published his discovery of the double refraction of a light ray by Iceland spar (calcite) in 1668 while Rømer was living in his home. Rømer was given every opportunity to learn mathematics and astronomy using Tycho Brahe's astronomical observations, as Bartholin had been given the task of preparing them for publication [2].

Rømer was employed by the French government: Louis XIV made him teacher for the Dauphin, and he also took part in the construction of the magnificent fountains at Versailles.

In 1681, Rømer returned to Denmark and was appointed professor of astronomy at the University of Copenhagen, and the same year he married Anne Marie Bartholin, the daughter of Rasmus Bartholin. He was active also as an observer, both at the University Observatory at Rundetårn and in his home, using improved instruments of his own construction. Unfortunately, his observations have not survived: they were lost in the great Copenhagen Fire of 1728. However, a former assistant (and later an astronomer in his own right), Peder Horrebow, loyally described and wrote about Rømer's observations.

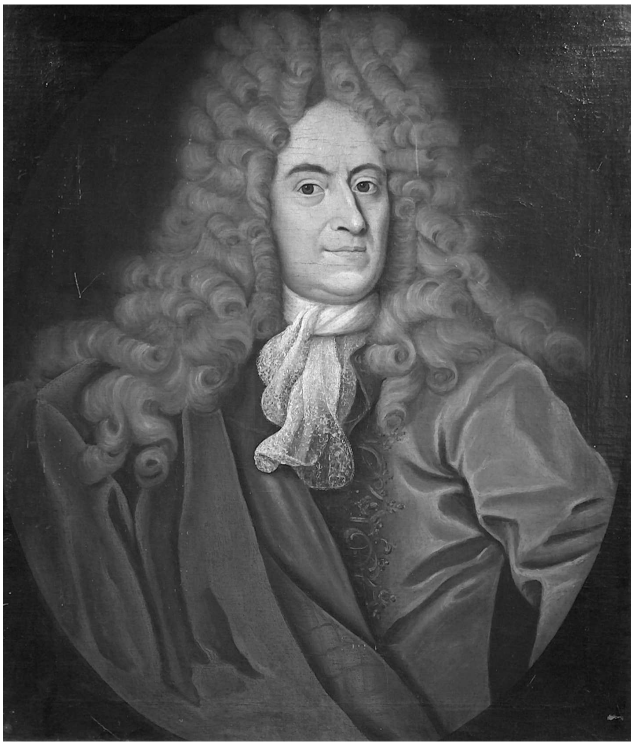
In Rømer's position as royal mathematician, he introduced the first national system for weights and measures in Denmark in May 01, 1683. Initially based on the Rhine foot, a more accurate national standard was adopted in 1698. Later measurements of the standards fabricated for length and volume show an excellent degree of accuracy. His goal was to achieve a definition based on astronomical constants, using a pendulum. This would happen after his death, practicalities making it too inaccurate at the time. Notable is also his definition of the new Danish mile. It was 24,000 Danish feet, which corresponds to 4 minutes of arc latitude, thus making navigation easier. In Norway and Sweden, this 4 minute geographical mile was mainly used at sea (sjømil), up to the beginning of the 20th century.

In 1700, Rømer managed to get the king to introduce the Gregorian calendar in Denmark-Norway — something Tycho Brahe had argued for in vain a hundred years earlier.

Rømer also developed one of the first temperature scales. Fahrenheit visited him in 1708 and improved on the Rømer scale, the result being the familiar Fahrenheit temperature scale still in use today in a few countries.

Rømer also established several schools for marine navigation in many Danish cities.

In 1705, Rømer was made the second Chief of the Copenhagen Police, a position he kept until his death in 1710. As one of his first acts, he fired the entire force, being convinced that the morale was alarm-



The second of two portraits of Rømer painted during his lifetime. Courtesy Rundetårn, Copenhagen.

ingly low. He was the inventor of the first street lights (oil lamps) in Copenhagen, and worked hard to try to control the beggars, poor people, unemployed, and prostitutes of Copenhagen. This was the start of a social reform.

In Copenhagen, Rømer made rules for building new houses, got the city's water supply and sewers back in order, ensured that the city's fire department got new and better equipment, and was the moving force behind the planning and making of new pavement in the streets and on the city squares.

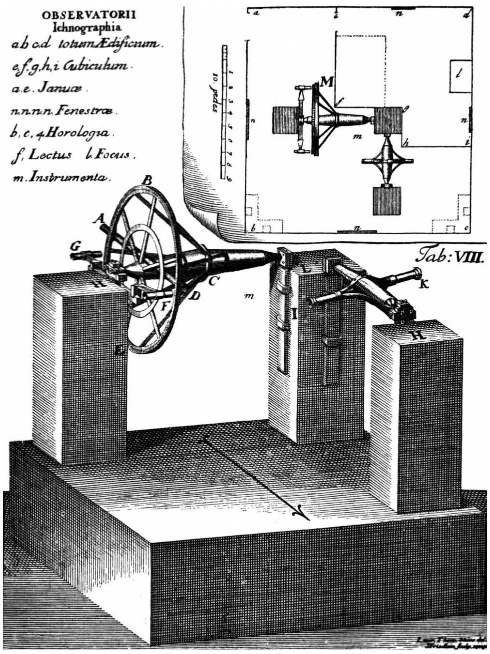
The determination of longitude is a significant practical problem in cartography and navigation. Philip III of Spain offered a prize for a method to determine the longitude of a ship out of sight of land, and Galileo proposed a method of establishing the time of day, and thus longitude, based on the times of the eclipses of the moons of Jupiter, in essence using the Jovian system as a cosmic clock; this method was not significantly improved until accurate mechanical clocks were developed in the eighteenth century. Galileo proposed this method to the Spanish crown (1616–1617) but it proved to be impractical, because of the inaccuracies of Galileo's timetables and the difficulty of observing the eclipses on a ship. However, with refinements the method could be made to work on land.

After studies in Copenhagen, Rømer travelled to the observatory of Uraniborg, then in ruins, on the island of Hven, near Copenhagen, in 1671. Over a period of several months, Jean Picard and Rømer observed about 140 eclipses of Jupiter's moon Io, while in Paris Giovanni Domenico Cassini observed the same eclipses. By comparing the times of the eclipses, the difference in longitude of Paris to Uranienborg was calculated.

Cassini had observed the moons of Jupiter between 1666 and 1668, and discovered discrepancies in his measurements that, at first, he attributed to light having a finite velocity. In 1672 Rømer went to Paris and continued observing the satellites of Jupiter as Cassini's assistant. Rømer added his own observations to Cassini's and observed that times between eclipses (particularly those of Io) got shorter as Earth approached Jupiter, and longer as Earth moved farther away. Cassini published a short paper in August 1675 where he states [3]:

> "This second inequality appears to be due to light taking some time to reach us from the satellite; light seems to take about ten to eleven minutes to cross a distance equal to the half-diameter of the terrestrial orbit."

Oddly, Cassini seems to have abandoned this reasoning, which Rømer adopted and set about buttressing in an irrefutable manner, using a selected number of observations performed by Picard and himself between 1671 and 1677. Rømer presented his results to the French Academy of Sciences, and it was summarized soon after by an anonymous reporter in a short paper, *Démonstration touchant le mouvement de la lumière trouvé par M. Roemer de l'Académie des Sciences*, published on December 7, 1676, in *Journal des Sçavans.* Unfortunately the paper bears the stamp of the reporter failing to understand Rømer's presentation, and as the reporter resorted to cryptic phrasings to hide his lack of understanding, he obfuscated Rømer's reasoning in the process [4]. However only interpretation of the presented numbers makes sense: As forty orbits of Io — each of 42.5 hours — observed as the Earth moves towards Jupiter are in total 22 minutes shorter than forty orbits of Io observed as the Earth moves away from Jupiter, and Rømer



The first Meridian Circle constructed by Rømer in 1704 at his Countryside Observatory near Copenhagen. Courtesy Rundetårn, Copenhagen.

concluded from this that light will travel the distance, which the Earth travels during eighty orbits of Io, in 22 minutes [4]. This makes it possible to calculate the strict result of Rømer's observations: The ratio between the velocity of light of the velocity with which the Earth orbits the Sun, which becomes 80×42.5 hours / 22 minutes $\approx 9{,}300$. In comparison to the result of Rømer's calculation, the modern numerical value is circa $299{,}792\ \mathrm{km} \times \mathrm{sec}^{-1} / 29.8\ \mathrm{km} \times \mathrm{sec}^{-1} \approx 10{,}100$ [5].

Rømer neither calculated this ratio, nor did he give a value for the velocity of light. However, many others calculated a velocity from his data, the first being Christiaan Huygens; after corresponding with Rømer and eliciting more data, Huygens deduced that light travelled $16\frac{2}{3}$ Earth diameters per second, misinterpreting Rømer's value of 22 minutes as the time in which light traverses the diameter of the Earth's orbit [6].

Rømer's view that the velocity of light was finite was not fully accepted until measurements of the so-called aberration of light were made by James Bradley in 1727.

In 1809, again making use of observations of Io, but this time with the benefit of more than a century of increasingly precise observations, the astronomer Jean Baptiste Joseph Delambre reported the time for light to travel from the Sun to the Earth as 8 minutes and 12 seconds. Depending on the value assumed for the astronomical unit, this yields the velocity of light as just a little more than 300,000 kilometres per second.

A plaque at the Observatory of Paris, where the Danish astronomer happened to be working, commemorates what was, in effect, the first measurement of a universal quantity made on this planet.

In addition to inventing the first street lights in Copenhagen, Rømer also invented the Meridian circle, the Altazimuth and the Passage Instrument.

The Ole Rømer Museum is located in the municipality of Høje-Taastrup, Denmark, at the excavated site of Rømer's observatory Observatorium Tusculanum at Vridsløsemagle. The observatory operated until about 1716 when the remaining instruments were moved to Rundetårn in Copenhagen. There is a large collection of ancient and more recent astronomical instruments on display at the museum. Since 2002 this exhibition is a part of the museum Kroppedal at the same location.

Rundetårn (spelled as Rundetaarn), or the Round Tower, is the observatory and museum for astronomical artifacts at the historical centre of Copenhagen, built in 1637–1642. The currently working observatory there was equipped in the 20th century. The author is thankful to Rundetårn, where he maintains the Rømer memorial exhibition and the artifacts, for the permission to use the portraits of Rømer and the lithograph showing his Meridian Circle, in this publication.

Erling Poulsen

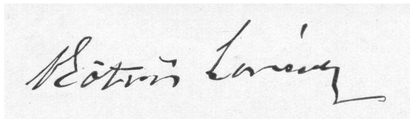
1. Friedrichsen P. and Tortzen C. G. Ole Rømer — Korrespondance og afhandlinger samt et udvalg af dokumenter. C. A. Reitzels Forlag, Copenhagen, 2001, page 16.
2. Friedrichsen P. and Tortzen C. G. *Ibidem*, pages 19–20.
3. Cassini G. D. Cette seconde inégalité paraît venir de ce que la lumière emploie quelques temps à venir du satellite jusqu'à nous, et qu'elle met environ dix à onze minutes à parcourir un espace égal au demi-diamètre de l'orbite terrestre. *Journal des Sçavans*, tome 4, August, 1675.
4. Teuber J. Ole Rømer og den bevaegede Jord — en dansk førsteplads? In: *Ole Rømer — videnskabsmand og samfundstjener*, edited by Friedrichsen P., Henningsen O., Olsen O., Thykier C., and Tortzen C. G., Gads Forlag, Copenhagen, 2004, page 218.
5. Knudsen J. M. and Hjorth P. G. Elements of Newtonian mechanics. 2nd edition, Springer Verlag, Berlin, 1995, page 367.
6. Huygens C. Treatise on light. January 08, 1690. Translated into English by Silvanus P. Thompson, stored at *Project Gutenberg*.

Biography of Loránd Eötvös
(1848–1919)

Loránd Eötvös, 1900. Courtesy the Loránd Eötvös Geophysical Institute of Hungary.

Loránd Eötvös (better known in foreign countries as Roland von Eötvös) is one of the greatest figures of natural sciences in Hungary. He was born in Buda, in Hungary, on July 27th 1848 into an impoverished aristocratic family. His father, Baron József Eötvös, was a novelist, essayist, educator and statesman, whose life and writings were devoted to the creation of modern Hungarian literature and to the establishment of a democratic Hungary. He was a friend of Franz Liszt, the famous pianist and composer. Loránd's mother was Ágnes Rosty, an educated daughter of a well-to-do landowner.

In his younger years, Loránd was educated by private tutors, later he attended the monastic high school of the Piarists where he obtained his final examinations in 1865. In those days it was assumed that boys of aristocratic families who wished to receive higher education had to enter the faculty of law. The law studies, however, failed to satisfy him, therefore he always managed to find time to attend lectures in natural sciences.



Eötvös' signature. Courtesy the Loránd Eötvös Geophys. Institute of Hungary.

Despite the fact that he completed his law studies, his dearest wish was to "study at a university abroad under the guidance of enlightened teachers in order to fully understand the natural forces at work in the scientific field". In 1867 having obtained his father's consent, he took the final decision to follow a career in natural sciences, and to this end entered the University of Heidelberg (Germany). There he became the pupil of famous professors, such as Kirchhoff, Bunsen and Helmholtz. First of all he studied physics, mathematics and chemistry. The following six months he spent at the University of Königsberg, but found the lectures too abstract and returned to Heidelberg. During his university years he kept up a regular correspondence with his father. These letters reveal the depth of understanding and sincerity in the relationship between father and son.

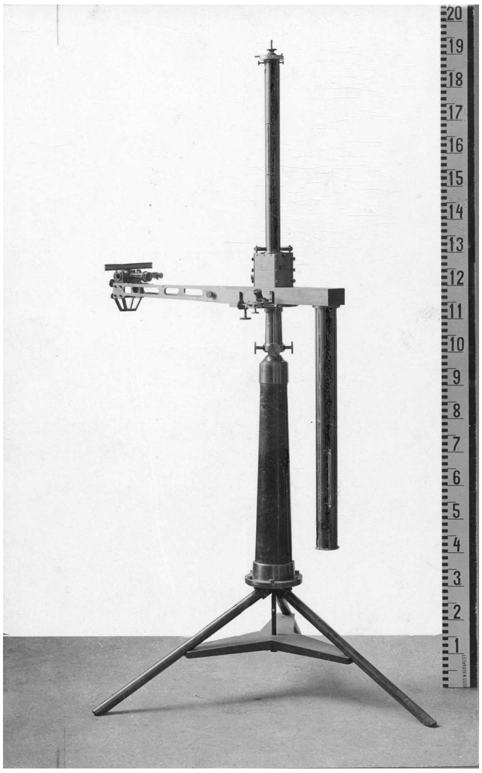
In 1869, the young Eötvös, thirsty for adventure, planned to join Petermann the German geographer on his expedition to the Spitzbergen. At his father's request he gave up the plan to travel and applied all his energy to preparing for his examinations for a doctorate degree, that he absolved "summa cum laude".

Shortly after his return home in February 1871, his father, "the best and truest friend" died. On his death-bed he warned his son once more that his future happiness depended on his devoting himself to science and keeping out of politics.

After his father's death, Eötvös successfully applied for the post of lecturer, advertised by the faculty of theoretical physics at the Pest University, which now bears his name. It was characteristic of the social climate of the time that the majority of the audience attending his inaugural lecture did so because they were curious to see a real baron giving a talk at the university.

After a short period of lecturing, in 1872 he was appointed to the professorship of theoretical physics. In 1874 he was allowed to give lectures in experimental physics and four years later he became professor in this field too. He was then given the task of uniting the departments of experimental and theoretical physics, and was nominated as Director of the newly established Physical Institute.

In 1873 he became Associate Member of the Hungarian Academy of Sciences, then Full Member in 1883, and in 1889 he was elected Presi-

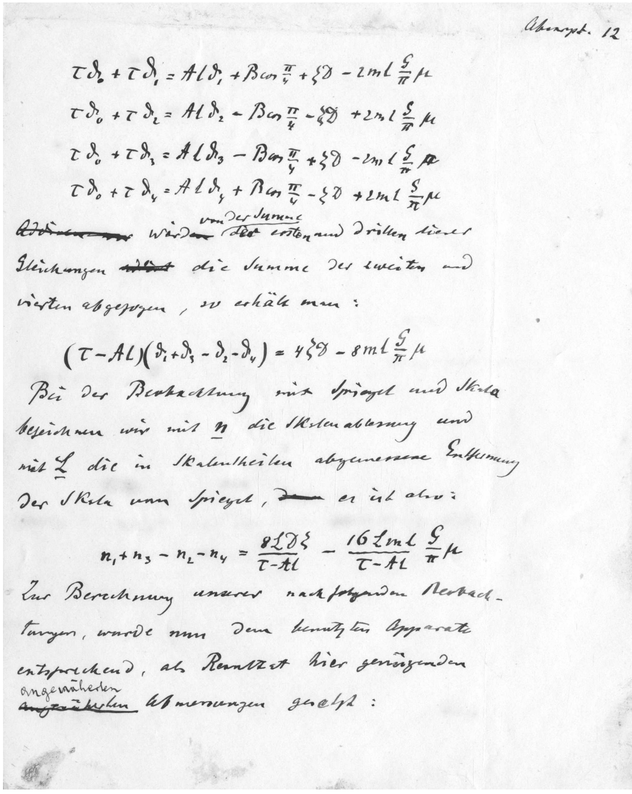


Single torsion balance designed by Eötvös for field work in 1898.
Courtesy the Loránd Eötvös Geophysical Institute of Hungary.

dent. Amongst his offices, he became Minister of religion and education for seven months in 1894. In his inaugural speech as Minister, he addressed the ministerial staff as follows:

> "We must strive, gentlemen, to make the field of public education a true garden of flowers. To achieve this aim we must first create order in the garden, so that every plant has its place. It is also necessary that each one receives the right nourishment, the soil and air that will allow its full development. In short, we have just two things we must do here, to make order and then to help. And gentlemen, I would like us to give more and more assistance and show more tolerance in our regulations."

Eötvös was a modest scientist who shunned the limelight. He disliked noisy ceremonies and did not seek moral or financial reward. In spite of this, he was acclaimed and received awards at home and abroad for



Eötvös' handwriting. A fragment of the manuscript dealing with the law of proportionality of inertia and gravity, submitted to the Univertsity of Göttingen, Germany, and rewarded by the Benecke award in 1909. Courtesy the Loránd Eötvös Geophysical Institute of Hungary.

his scientific work and skill as an organiser. The most important ones included the French Legion of Honour, the Franz Josef award from the Hungarian King, the Saint Sava award from the King of Serbia. He was also elected honorary member of the Prussian Royal Academy of Sciences in Berlin and was given honorary doctorates from the Jagello University in Cracow, and the Norwegian Royal Frederick University in Christiania (now Oslo). In addition to the above, he received several major and minor awards during his lifetime and was elected President or a leading member of various social and scientific societies.

Eötvös was a well-balanced individual. Besides his intensive mental work, he always found time for relaxation and sports. He often went riding and regularly made the eleven-kilometre journey from his home to the university on the horseback. In the summer, he often cycled and indulged in his passion for rock climbing. In the classic time of mountaineering, he ranked among the best. As an enthusiastic photographer, he took hundreds of pictures during his mountaineering excursions. In his latter years, his daughters accompanied him on his expeditions, and also became keen Alpinists. Eötvös' climbing achievements in the southern Tirol made his name so well known that in 1902 one peak of 2837 metres high in the Dolomites (Italy) was named after him Cima di Eötvös (Eötvös summit). In the company of friends he often jokingly said that he was prouder of his mountaineering successes than his discovery of the torsion balance. For many years as President of the Hungarian Touring Society, he achieved a great deal in the popularization of tourism in Hungary.

With the advancing years, he strove to avoid prestigious appointments in order to devote himself entirely to his research. This prompted him to give up his position as President of the Academy in 1905. The last years of his life were clouded by a severe illness, but he continued to lecture at the University as long as it was humanly possible. Until the last moments of his life, he followed torsion balance fieldwork with great interest. In the beginning, he asked his colleagues to inform him of the daily results of their survey by telegram because he was very anxious to know how far the results of the survey supported his theories. He had never been able to tear himself away from his research, even during his summer excursions to the mountains. When on holidays, he always kept up a regular correspondence with his co-workers. He continued his scientific work from his sick bed and sent his last paper to be published only a few days before he died on April 8, 1919.

International scientific life and the whole of Hungarian society mourned his death. Hungary had said farewell to one of the last great representatives of classical physics and to the country's greatest natural scientist. Through his work, however, his name will live forever in the history of physics and geophysics.

As a means of expressing their respect for Eötvös posthumously, the international scientific community named the 1×10^{-9} CGS unit after him, and gave his initial, E, as its symbol.

Eötvös' main scientific achievements

In his scientific research Eötvös was not interested in those topics that were fashionable at that time, and would have brought him immediate public acclaim. He was concerned with capillarity, gravitation and magnetism, phenomena so taken for granted that a superficial observer would fail to see the mysterious powers at work within them. He formulated his *ars poetica* as follows:

> "The true natural scientist ... finds pleasure in research itself and in those results which help to increase the prosperity of Mankind."

It was characteristic of Eötvös' scientific activity that he dealt with all aspects of a problem. He first considered the theoretical base and followed this by designing and constructing the instruments and methodology needed for the experiments. Then came the laboratory and field measurements and, finally, he summarized his conclusions derived from the measurements.

Studies in the field of capillarity The beginnings of Eötvös' scientific career are connected with liquids. He worked out a new way to determine surface tension, which subsequently became known as the reflection method. This method made it possible to determine precisely the surface tension of various liquids. During his experiments, Eötvös found a linear relationship between the molar surface energy of liquids and their temperature. The proportionality factor is constant for all compound liquids independently of their composition. The molar surface energy is equal to the work needed to move one molecule from the inside of the liquid to its surface.

Based on this finding, Eötvös was able to state the following relationship: with increasing temperature, the surface tension of a liquid decreases until, at the critical temperature, it becomes zero. Later this rule was named the Eötvös law and the proportionality constant the Eötvös constant. In case of liquids this constant is as fundamental as the universal gas constant in case of gases.

Eötvös torsion balance Around 1885 his attention turned to gravity and magnetism. Studying the behaviour of the Coulomb balance in the gravity field he invented a modified version of this instrument, which is known in geophysics as the Eötvös torsion balance. This unbelievably sensitive instrument can detect a change of 10^{-12} part/cm of gravity. The instrument has been proved to be suitable for geological exploration and paved the way towards world renown for Eötvös and his balance. In the 1920's and 1930's hundreds of oil fields were discovered throughout the world with the help of Eötvös' ingenious instrument.

Inertial and gravitational mass Eötvös became concerned with the question of the proportionality of the inertial and gravitational mass as early as 1890. In 1908 Eötvös and his colleagues, Jenõ Fekete and Dezsõ Pekár, perfected their measurements to such an extent that they were able to establish that the difference between the inertial and gravitational mass was at the most 1/200,000,000. Their paper on the subject won them the Benecke award at the Göttingen University in 1909.

Eötvös effect While studying Hecker's results who carried out gravity measurements on moving boats on the oceans in the years 1904–1905, Eötvös noticed that no consideration had been given to the vertical component of the Coriolis acceleration developed by the motion of the boat. In his letter to Hecker, he proposed a correction for compensating this effect. The international scientific world recognizes these phenomena as the Eötvös effect and the Eötvös correction, respectively, both having special importance nowadays in the field of sea and air gravimetry.

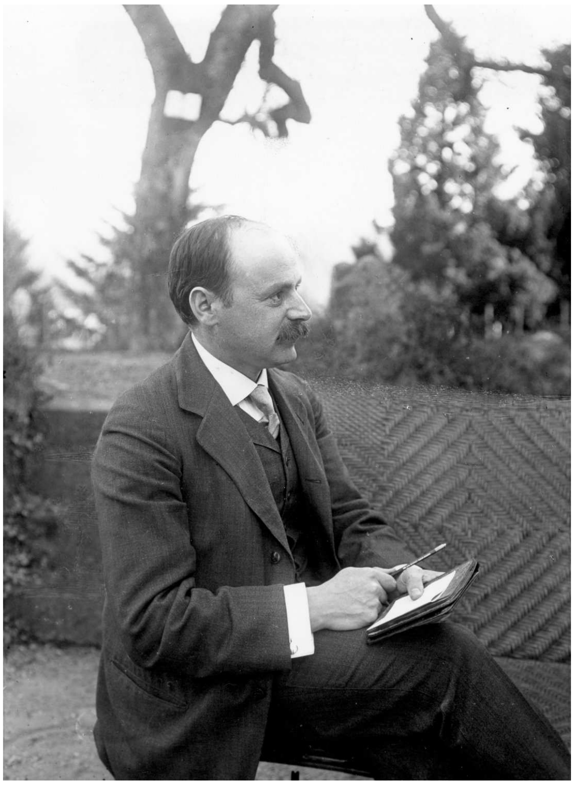
Just to list his further scientific achievements:

- He invented a new method to determine the value of the gravity constant.
- He carried out measurements to study the problem of gravity absorption. He concluded that the gravity absorption of a 5 cm thick lead plate (if there is such phenomenon at all) is less than 5×10^{-10} part of its attraction;
- He constructed a bifilar type gravimeter in 1901, more than a decade earlier as Schweydar did;
- He applied the astatic principle to his gravity compensator to make it so sensitive that he could detect 1 cm variation of the water level of the Danube river in a distance of 100 meter;
- He constructed a magnetic version of his balance and carried out archeomagnetic measurements to determine the inclination of the magnetic field in the past;
- Based on his torsion balance measurements carried out in the Arad region (now in Romania), he elaborated a new method to contour a very detailed geoid map for the region.

Zoltán Szabó

All photo images used in this publication were provided by the Loránd Eötvös Geophysical Institute of Hungary (Eötvös Loránd Geofizikai Intézet, ELGI), where the author of the present biographic article, Zoltán Szabó, maintains the Loránd Eötvös Memorial Museum. The author would like to express his sincere gratitude to László Sõrés and Péter Kovács, the close colleagues at the Institute, for their favour and assistance in preparation of the photo images.

Biography of Karl Schwarzschild
(1873–1916)



Karl Schwarzschild. Courtesy AIP
Emilio Segrè Visual Archives.

The celebrated German astronomer and physicist, Karl Schwarzschild, was born into a wealthy Jewish family in Frankfurt am Main, Germany, on October 9, 1873. He was the oldest of his five brothers and one sister. His mother, Henrietta Sabel, and father, Moses Martin Schwarzschild, were very nice and hearthy persons. His large, extended family was known to cultivate deep interest in art and culture. However, he became the first in the family to pursue a career in science.

Schwarzschild's prodigious talent manifested quite early on while still a student at Frankfurt Gymnasium: at the age of 16, having taught himself some advanced mathematics and studied much of contemporary astronomy, he published his first scientific paper on celestial mechanics.

Karl Schwarzschild writing in his study at Potsdam.
Courtesy AIP Emilio Segrè Visual Archives.

This was soon followed by another paper. Both papers touched upon the orbital theory of binary stars. In his Gymnasium years, he was a close friend of the famous mathematician specializing in number theory, Paul Epstein, with whom there was a sharing of real scientific interests.

He further studied astronomy at the Universities of Strasbourg and Munich, obtaining his doctorate in 1896 for a dissertation on the application of Poincaré's theory of rotating stellar bodies. His supervisor was Hugo von Seeliger whom he often mentioned with much admiration throughout his life.

From 1897 until 1899, he was employed as an assistant at the Kuffner Observatory in the suburb of Vienna called Ottakring. Here, he was engaged in the investigation and measurement of the apparent brightness of stars using photographic plates, from which he produced a formula to calculate the optical density of photographic material. This formula was especially important in dealing with photographic measurements of the intensities of distant, faint astronomical objects.

In the summer of 1899, he became a privatdozent at the University of Munich after submitting his habilitation thesis entitled *Beiträge zur photographischen Photometrie der Gestirne* which dealt with much of the astronomical work he had done at the Kuffner Observatory.

Outdoors. Sitting, second from left, with his family.
Courtesy AIP Emilio Segrè Visual Archives.

It is particularly interesting to note that already in 1900, Schwarzschild already pondered upon the possible non-Euclidean structure of space. His ideas were expounded at the meeting of the German Astronomical Society in Heidelberg that year. In the same year, he published a paper in which he gave a lower limit for the (measurable) radius of the curvature of space as 64 light years (suposing a hyperbolic space) or 1600 light years (an elliptic space). In dealing with solar radiation pressure, he assumed that the tails of comets consisted of spherical particles which acted as light reflectors. Thus he was able to calculate the size of the particles in the tails of the comets. He instinctively knew that radiation pressure had to somehow overcome gravitation, and that the particles did not scatter light. This way, he gave the exact diameters of the particles within the range of 0.07 and 1.5 microns.

From 1901 until 1909 he was an extraordinary professor at the University of Göttingen and also the director of the Observatory there. In Göttingen, he had the opportunity to work with some significant figures inhabiting the University, such as the mathematicians David Hilbert, Felix Klein, and Hermann Minkowski. He studied astrophysical phenomena associated with the energy transport mechanism in a star by means of radiation and produced an important paper on radiative equilibrium within the sun's atmosphere. Following this period, he took up a post of the director at the Astronomical Observatory in Potsdam in 1909, a place which Eddington described for him as "... very congenial..." [1].

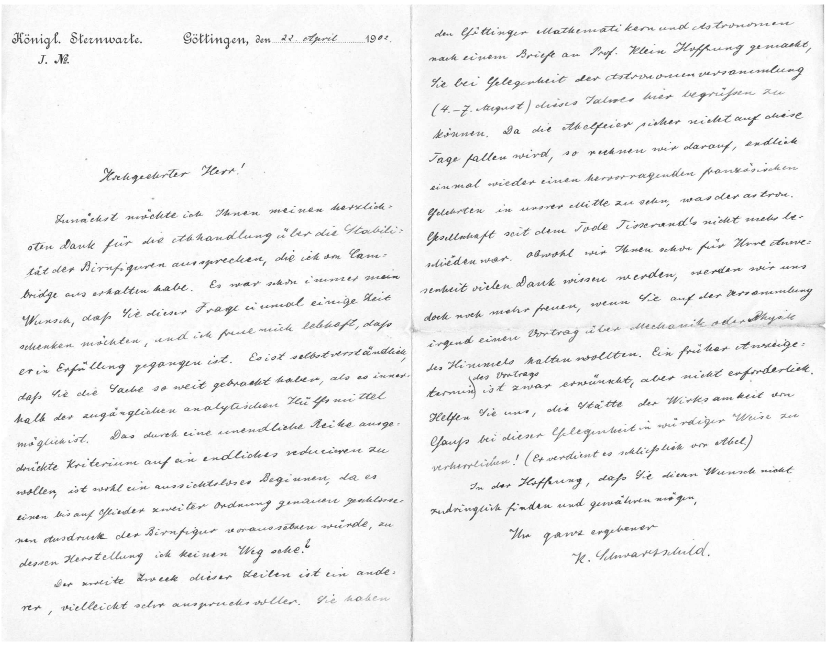
In 1913, Schwarzschild was elected a member of the Prussian Academy of Sciences in Berlin. During his election, he produced a memorable speech which outlined the essence of his attitude towards science [1]:

> "Mathematics, physics, chemistry, astronomy, march in one front. Whichever lags behind is drawn after. Whichever hastens ahead helps on the others. The closest solidarity between astronomy and the whole circle of exact science. ... from this aspect I may count it well that my interest has never been limited to the things beyond the moon, but has followed the threads which spin themselves from there to our sublunar knowledge; I have often been untrue to the heavens. That is an impulse to the universal which was strengthened unwittingly by my teacher Seeliger, and afterwards was further nourished by Felix Klein and the whole scientific circle at Göttingen. There the motto runs that mathematics, physics, and astronomy constitute one knowledge, which, like the Greek culture, is only comprehended as a perfect whole."

Soon, in 1914, Europe was plagued with the outbreak of World War I. Subsequently, he joined the German army as a volunteer despite being over 40 years old. He served on both the Eastern and Western Fronts, eventually rising to the rank of lieutenant in the artillery division. Notably, he served in Belgium where he was put in charge of a local weather station, in France where he produced calculations of missile trajectories, and then in Russia.

While in Russia, despite suffering from a rare and painful skin disease called pemphigus, he managed to write three pivotal papers: two on the exact solutions to Einstein's field equations of the General Theory of Relativity, the new theory of space-time and gravitation, and one on Planck's quantum theory. As it is well-known, his papers on the General Theory of Relativity gave the first exact solutions to Einstein's unimodular field equations of gravitation in the empty space surrounding a spherical mass, a solution which now bears his name, the Schwarzschild metric, which actually involves a slight modification of his original solution. Meanwhile, his paper on quantum theory gave a lucid explanation of the so-called Stark effect.

Upon receiving Schwarzschild's manuscripts, Einstein himself was pleasantly surprised to learn that his non-linear field equations of gravitation did admit exact solutions, despite their "prima facie" complex-


Königl. Sternwarte. Göttingen, den 22. April 1902.
J. №.

Hochgeehrter Herr!

Zunächst möchte ich Ihnen meinen herzlich=
sten Dank für die Abhandlung über die Stabili=
tät der Birnfiguren aussprechen, die ich von Cam=
bridge aus erhalten habe. Es war schon immer mein
Wunsch, daß Sie dieser Frage einmal einige Zeit
schenken möchten, und ich freue mich lebhaft, daß
er in Erfüllung gegangen ist. Es ist selbstverständlich,
daß Sie die Sache so weit gebracht haben, als es inner=
halb der zugänglichen analytischen Hülfsmittel
möglich ist. Das durch eine unendliche Reihe ausge=
drückte Kriterium auf ein endliches reduciren zu
wollen, ist wohl ein aussichtsloses Beginnen, da es
einen bis auf Glieder zweiter Ordnung genauen geschlosse=
nen Ausdruck der Birnfigur voraussetzen würde, zu
dessen Herstellung ich keinen Weg sehe.
Der zweite Zweck dieser Zeilen ist ein ande=
rer, vielleicht sehr anspruchsvoller. Sie haben
den Göttinger Mathematikern und Astronomen
nach einem Briefe an Prof. Klein Hoffnung gemacht,
Sie bei Gelegenheit der Astronomenversammlung
(4.–7. August) dieses Jahres hier begrüßen zu
können. Da die Abelfeier sicher nicht auf diese
Tage fallen wird, so rechnen wir darauf, endlich
einmal wieder einen hervorragenden französischen
Gelehrten in unsrer Mitte zu sehn, was der astron.
Gesellschaft seit dem Tode Tisserand's nicht mehr be=
schieden war. Obwohl wir Ihnen schon für Ihre Anwe=
senheit vielen Dank wissen werden, werden wir uns
doch noch mehr freuen, wenn Sie auf der Versammlung
irgend einen Vortrag über Mechanik oder Physik
des Himmels halten wollten. Ein früher Anzeige=
termin (des Vortrags) ist zwar erwünscht, aber nicht erforderlich.
Helfen Sie uns, die Stätte der Wirksamkeit von
Gauß bei dieser Gelegenheit in würdiger Weise zu
verherrlichen! (Er verdient es schließlich vor Abel)
In der Hoffnung, daß Sie diesen Wunsch nicht
zudringlich finden und gewähren mögen,
Ihr ganz ergebener
K. Schwarzschild.

Brief letter to Henri Poincaré, April 22, 1902. Courtesy Max-Planck-Institut für Wissenschaftsgeschichte, MPIWG Library Collection.

ity, which, according to him, were elegantly shown by Schwarzschild in "...such a simple way..." [2]. Prior to this, Einstein himself was only able to produce an approximate solution, given in his famous 1915 paper on the advance of the perihelion of Mercury. In that paper, Einstein employed a rectilinear coordinate system in order to approximate the gravitational field around a spherically symmetric, static, non-rotating, non-charged mass. Schwarzschild, in contrast to Einstein's initial approach, chose a generalization of the polar coordinate system and was thus able to produce an exact solution in a more elegant manner, a manner somewhat more befitting the splendour and subtlety of the full non-Euclidean nature of Einstein's geometric theory.

In 1916, the elated Einstein famously wrote to Schwarzschild on his newly obtained result [2]:

> "I have read your paper with the utmost interest. I had not expected that one could formulate the exact solution of the problem in such a simple way. I liked very much your mathematical treatment of the subject. Next Thursday I shall present the work to the Academy with a few words of explanation."

Shortly after he sent his last two papers on the General Theory of Relativity to Einstein, Schwarzschild had to succumb to the skin disease he had contracted earlier. The disease, pemphigus, is a rare kind of autimmune blistering skin rash. It is said that for people plagued with this skin rash, the immune system mistakes the cells in the skin as foreign and attacks them, resulting in painful blisters. In Schwarzschild's time there was no known medical treatment or cure for the disease and, after being freed from his military duty to be interned at home in March 1916, he died two months later, on May 11, 1916, at the age of 42.

Schwarzschild died at the height of his scientific achievements. He certainly was a man of wide scientific interests. Apart from his earlier work on astronomy, which included celestial mechanics, observational stellar photometry, optical systems, observational and instrumental astronomy, stellar structure and statistics, comets, and spectroscopy, and his outstanding work in the area of the General Theory of Relativity, he also worked on electrodynamics and geometrical optics (while in Göttingen). He also maintained deep interest in quantum theory.

He married Else Posenbach, the daughter of a professor of surgery at the University of Göttingen, on 22 October 1909. Together, they had three children: Agathe, Martin, and Alfred. His second son, Martin Schwarzschild, followed in his father's footsteps as a professor of astrophysics at Princeton University.

Indranu Suhendro

1. Eddington A. S. Karl Schwarzschild. *Monthly Notices of the Royal Astronomical Society*, 1917, vol. 77, 314–319.

2. Eisenstaedt J. The early interpretation of the Schwarzschild solution. In: *Einstein and the History of General Relativity: Proceedings, 1986 Osgood Hill Conference* (*Einstein Studies*, vol. 1), ed. by D. Howard and J. Stachel, Birkhauser, Boston, 1989, 213–233.

Biography of Abraham Zelmanov
(1913–1987)

A. Zelmanov in the 1940's

Abraham Leonidovich Zelmanov was born on May 15, 1913 in Poltava Gubernya of the Russian Empire. His father was a Judaic religious scientist, a specialist in comments on Torah and Kabbalah. In 1937 Zelmanov completed his education at the Mechanical Mathematical Department of Moscow University. After 1937 he was a research-student at the Sternberg Astronomical Institute in Moscow, where he presented his dissertation in 1944. In 1953 he was arrested for "cosmopolitism" in Stalin's campaign against Jews. However, as soon as Stalin died, Zelmanov was set free, after some months of imprisonment. For several decades Zelmanov and his paralyzed parents lived in a room in a flat shared with neighbours. He took everyday care of his parents, so they lived into old age. Only in the 1970's did he obtain a personal municipal flat. He was married three times. Zelmanov worked on the academic staff of the Sternberg Astronomical Institute all his life, until his death on the winter's day, 2nd of February, 1987.

He was very thin in physique, like an Indian yogi, rather shorter than average, and a very delicate man. From his appearance it was possible to think that his life and thoughts were rather ordinary or uninteresting. However, in acquaintance with him and his scientific discussions in friendly company one formed another opinion about him.

Those were discussions with a great scientist and humanist who reasoned in a very unorthodox way. Sometimes we, people who were with him, thought that we were not speaking with a contemporary scientist of the 20th century, but some famous philosopher from Classical Greece or the Middle Ages. So the themes of those discussions are eternal — the interior and evolution of the Universe, the place of a human being in the Universe, the nature of space and time.

Zelmanov liked to remark that he preferred to make mathematical "instruments" than to use them in practice. Perhaps thereby his main goal in science was the mathematical apparatus of physical observable quantities in the General Theory of Relativity known as the *theory of chronometric invariants*. In developing the apparatus he also created other mathematical methods, namely — *kinemetric invariants* and *monad formalism* (he also referred to monad formalism, the general covariant extension of chronometric invariants, as *orthometric invariants*). Being very demanding of himself, Zelmanov published only a dozen scientific publications during his life, so every publication is a concentrate of his fundamental scientific ideas.

Most of his time was spent in scientific work, but he sometimes gave lectures on the General Theory of Relativity and relativistic cosmology as a science for the geometrical structure of the Universe. Stephen Hawking, a young scientist in the 1960's, attended Zelmanov's seminars on cosmology at the Sternberg Astronomical Institute in Moscow. Zelmanov presented him as a "promising young cosmologist". Hawking read a brief report at one of those seminars. Zelmanov's seminar was visited by also John Wheeler, Kip Thorne, Roger Penrose, and other well-known scientists.

Because Zelmanov made scientific creation the main goal of his life, writing articles was a waste of time to him. However he never regretted time spent on long discussions in friendly company, where he set forth his philosophical concepts on the geometrical structure of the Universe and the process of human evolution. In those discussions he formulated his famous *Anthropic Principle* and the *Infinite Relativity Principle*. He formulated the Anthropic Principle in 1941–1944, many years before the other scientists such as Robert Dicke (1957)* or Brandon Carter (1973)†

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*Dicke R. H. Gravitation without a principle of equivalence. *Reviews of Modern Physics*, 1957, vol. 29, issue 3, 363–376.

†Carter B. Large number coincidences and the anthropic principle in cosmology. In: *Confrontation of Cosmological Theories with Observational Data*. Proceedings of the Symposium (Krakow, Poland, September 10–12, 1973), Dordrecht, D. Reidel Publishing Co., 1974, 291–298.

turned their attention to this problem. Zelmanov, being very demanding of himself, never published it in the scientific journals, meanwhile remaining his formulation of the Anthropic Principle wide known amongst the research stuff and students of the Sternberg Astronomical Institute.

Zelmanov's Anthropic Principle is stated here in his own words, in two versions. The first version sets forth the idea that the law of human evolution is dependent upon fundamental physical constants:

> "Humanity exists at the present time and we observe world constants completely because the constants bear their specific numerical values at this time. When the world constants bore other values humanity did not exist. When the constants change to other values humanity will disappear. That is, humanity can exist only with the specific scale of the numerical values of the cosmological constants. Humanity is only an episode in the life of the Universe. At the present time cosmological conditions are such that humanity develops."

In the second form he argues that any observer depends on the Universe he observes in the same way that the Universe depends on him:

> "The Universe has the interior we observe, because we observe the Universe in this way. It is impossible to divorce the Universe from the observer. The observable Universe depends on the observer and the observer depends on the Universe. If the contemporary physical conditions in the Universe change then the observer is changed. And vice versa, if the observer is changed then he will observe the world in another way, so the Universe he observes will also change. If no observers exist then the observable Universe as well does not exist."

It is probable that by proceeding from his Anthropic Principle, in the years 1941–1944, Zelmanov solved the well-known problem of physical observable quantities in the General Theory of Relativity.

It should be noted that many researchers were working on the theory of observable quantities in the 1940's. For example, Landau and Lifshitz, in their famous *The Classical Theory of Fields*, introduced observable time and the observable three-dimensional interval, similar to those introduced by Zelmanov. But they limited themselves only to this particular case and did not arrive at general mathematical methods to define physical observable quantities in pseudo-Riemannian spaces. It was only Carlo Cattaneo, an Italian mathematician of the Institute of Mathematics, Pisa University, who developed his own approach to the problem, not far removed from Zelmanov's solution. Cattaneo published his results on the theme in 1958 and later. Zelmanov knew those articles, and he highly appreciated Cattaneo's works. Cattaneo also knew of Zelmanov's works, and even cited the theory of chronometric invariants in his last publication.

In 1944 Zelmanov completed his mathematical apparatus for calculating physical observable quantities in four-dimensional pseudo-Riemannian space, in strict solution of that problem. He called the apparatus the *theory of chronometric invariants*.

Solving Einstein's equations with this mathematical apparatus, Zelmanov obtained the total system of all cosmological models (scenarios of the Universe's evolution) which could be possible as derived from the equations. Having this system a base, he developed a classification of all possible models of cosmology which could be theoretically conceivable in the space-time of the General Theory of Relativity. Now, we refer to it as the *Zelmanov classification*. In particular, he had arrived at the possibility that infinitude may be relative. Later, in the 1950's, he enunciated the *Infinite Relativity Principle*:

> "In homogeneous isotropic cosmological models spatial infinity of the Universe depends on our choice of that reference frame from which we observe the Universe (the observer's reference frame). If the three-dimensional space of the Universe, being observed in one reference frame, is infinite, it may be finite in another reference frame. The same is just as well true for the time during which the Universe evolves."

In other words, using purely mathematical methods of the General Theory of Relativity, Zelmanov showed that any observer forms his world-picture from a comparison between his observational results and some standards he has in his laboratory — the standards of different objects and their physical properties. So the "world" we see as a result of our observations depends directly on that set of physical standards we have, so the "visible world" depends directly on our considerations about some objects and phenomena.

The mathematical apparatus of physical observable quantities and those results it gave in relativistic cosmology were the first results of Zelmanov's application of his Anthropic Principle to the General Theory of Relativity. To obtain the results with general covariant methods (standard in General Relativity), where observation results do not depend on the observer's reference properties, would be impossible.

The fact is that Zelmanov published his scientific ideas in only a dozen of very compressed scientific articles with formulae, without essential comments. As a result for more than 60 years Zelmanov's work and the achievements remained known only a close circle of several of his pupils. His book *Chronometric Invariants*, containing his main fundamental studies on the General Theory of Relativity and relativistic cosmology, was written in 1944 and had survived only in manuscript until it has been published in 2006. It is impossible to find a more detailed and systematic description of the theory of chronometric invariants, than there. Even the book *Elements of the General Theory of Relativity*, which Vsevolod Agakov had composed from Zelmanov's lectures and articles, gives a very fragmented account of the mathematical methods that prevents a reader from learning it on his own. The same can be said about Zelmanov's original papers, each no more than a few pages in length. Anyway *Chronometric Invariants* is the best for depth of detail. Sometimes Zelmanov himself said that to use the mathematical methods of chronometric invariants in its full power would be possible only after studying his book of 1944.

Now everyone may read it. I hope that Zelmanov's classical works, in particular the chronometric invariants, the Zelmanov classification, his Anthropic Principle and the Infinite Relativity Principle, will become more widely known and appreciated. May his memory live for ever!

Dmitri Rabounski

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1. Zelmanov A. L. A method for the computation of the orbits of visual binaries. *Astron. J. Soviet Union* (known as *Astronomicheskii Zhurnal*, published in Russian with English summary), 1939, vol. 16, no. 4, 25–30.
2. Zelmanov A. L. On Deformations and the Curvature of Accompanying Space. Ph.D. thesis, preprint of the Sternberg Astronomical Institute, Moscow, 1944 (in Russian).
3. Severny A. B. and Zelmanov A. L. On the origin of big nebulae. *Astronomicheskii Zhurnal*, 1947, vol. 24, no. 6, 372 (in Russian).
4. Zelmanov A. L. On the application of the accompanying co-ordinates in non-relativistic mechanics. *Doklady Academii Nauk USSR*, 1948, vol. 61, no. 6, 993–996 (in Russian).
5. Zelmanov A. L. Chronometric invariants and accompanying frames of reference in the General Theory of Relativity. *Soviet Physics Doklady*, MAIK Nauka/Interperiodica, distributed by Springer, 1956, vol. 1, 227–230 (translated from *Doklady Academii Nauk USSR*, 1956, vol. 107, no. 6, 815–818).
6. Zelmanov A. L. The non-relativistic gravitational paradox and the General Theory of Relativity. *Proc. of the Soviet High School, Physics and Math. Series*, 1958, no. 2, 124.

7. Zelmanov A. L. On the formulation of the problem of the infinity of space in the General Theory of Relativity. *Soviet Physics Doklady*, MAIK Nauka/Interperiodica, distributed by Springer, 1959, vol. 4, 161–164 (translated from *Doklady Academii Nauk USSR*, 1959, vol. 124, no. 5, 1030–1033).
8. Zelmanov A. L. On the relativistic theory of an anisotropic inhomogeneous universe. *Proc. of the 6th Soviet Meeting on Cosmogony*, Acad. Sci. USSR Publ., Moscow, 1959, 144–174 (in Russian).
9. Zelmanov A. L. The deformation of "co-traveling" (concomitant) space in Einstein's gravitational theory. *Soviet Physics Doklady*, MAIK Nauka/Interperiodica, distributed by Springer, 1961, vol. 5, 1309–1312 (translated from *Doklady Academii Nauk USSR*, 1960, vol. 135, no. 6, 1367–1370).
10. Zelmanov A. L. On the deformation of accompanying space. Proc. of the Soviet Conference *Problems in Gravitation*, Tbilisi Univ. Press, Tbilisi, 1965 (in Russian).
11. Zelmanov A. L. and Tverdokhlebova V. A. The problem of energy and momentum of a gravitational-inertial field and chronometric invariants. Proc. of the Soviet Symposium *Novejshie Problemy Gravitatsii*, Moscow, 1968, 76 (in Russian).
12. Zelmanov A. L. The gnoseological problems in cosmology. Proc. of the Soviet Astronomical Symposium *The 500th Anniversary of Nicolaus Copernicus*, Moscow, 1972, 13–42 (in Russian).
13. Zelmanov A. L. On the standards of length and duration. In: *The Methodological Analysis of the Theoretical and Experimental Foundations of the Physics of Gravitation*, Moscow, 1973 (in Russian).
14. Zelmanov A. L. Kinemetric invarants and their relation to chronometric invariants in Einstein's gravitation theory. *Soviet Physics Doklady*, MAIK Nauka/Interperiodica, distributed by Springer, 1973, vol. 18, 231–234 (translated from *Doklady Academii Nauk USSR*, 1973, vol. 209, no. 4, 822–825).
15. Zelmanov A. L. Relationship of the orthometric monad formalism to chronometric and kinemetric invariants. *Soviet Physics Doklady*, MAIK Nauka/Interperiodica, distributed by Springer, 1976, vol. 21, 147–150 (translated from *Doklady Academii Nauk USSR*, 1976, vol. 227, no. 1, 78–81).
16. Zelmanov A. L. The relativity of space and time finiteness-infiniteness for a matter-filled universe. *Soviet Astronomy*, vol. 21, no. 12, Nov.–Dec. 1977, 664–671 (translated from *Astronomicheskii Zhurnal*, vol. 54, no. 12, Nov.–Dec. 1977, 1168–1181).
17. Zelmanov A. L. and Kharbediia L. I. Necessary conditions for relativity of the finiteness and infinity of nonempty space. *Soviet Astronomy*, vol. 22, no. 1, Jan.–Feb. 1978, 105–106 (translated from *Astronomicheskii Zhurnal*, 1978, vol. 55, no. 1, 186–187).
18. Zelmanov A. L. Newtonian cosmological fields and non-relativistic approximations to the relativistic theory. Proc. of the Soviet Conference *Contemporary Theoretical and Experimental Problems of the Theory of Relativity and Gravitation*, Moscow, 1981 (in Russian).
19. Zelmanov A. L. and Khabibov Z. R. Chronometrically invariant variations in the Einstein theory of gravity. *Soviet Physics Doklady*, MAIK Nauka/Interperiodica, distributed by Springer, 1983, vol. 28, 175–178 (translated from *Doklady Academii Nauk USSR*, 1983, vol. 268, no. 6, 1378–1380).

20. Zelmanov A. L. and Agakov V. G. Elements of the General Theory of Relativity. Nauka, Moscow, 1988, 236 pages (in Russian).

21. Zelmanov A. L. Chronometric Invariants: On Deformations and the Curvature of Accompanying Space. Translated into English from the preprint of 1944, American Research Press, Rehoboth (NM), 2006, 232 pages.

22. Zelmanov A. L. Chronometric Invariants: On Deformations and the Curvature of Accompanying Space. The originally edition in Russian, American Research Press, Rehoboth (NM), 2006, 227 pages.

23. Zelmanov A. L. On the relativistic theory of an anisotropic inhomogeneous universe. Translated from the manuscript of 1957, *The Abraham Zelmanov Journal*, 2008, vol. 1, 33–63.

Biography of Alexei Petrov
(1910–1972)

A. Petrov in the 1960's

Alexei Zinovievich Petrov was born on October 28, 1910, in the village of Koshki in Samara Gubernya in the Russian Empire. The eleventh of twelve children, Alexei was ill as a child. His father, Zinovey, was the parish priest of the Russian Orthodox Church and died of tuberculosis when Alexei was only five years old. As a widow, his mother Zoe was without financial resources. She therefore elected to send her two youngest sons, Alexei and Sevir, to live with their aunt, Catherine Petrova, a village school teacher. Catherine adopted both boys, giving them her family name.

In 1926, Alexei was graduated from a normal school at Melekh, a small provincial town. Being a sickly child, he preferred reading to playing games with other boys. He was interested in mathematics, and read many books in this field beyond the scholarly curriculum. In 1932, he entered the Department of Physics and Mathematics of Kazan University, where he became a pupil of Prof. Peter Schirokov. The latter was a prominent investigator in the field of non-Euclidean geometry. Schirokov was highly enthusiastic concerning Einstein's theory of relativity. He supported any achievement in this direction. Amongst Schirokov's students, Alexei built a fine reputation as a powerful and independent scientist, despite his youth. Following many long conversations with Schirokov, Alexei Petrov selected the topic for his PhD Thesis. The resulting manuscript became the basis for famous book Einstein Spaces.

In the beginning of World War II, Alexei Petrov volunteered for military service and became a commander of mortar artillery. In December 1941, he was sent into battle and in August 1943, he was severely wounded. After a prolonged confinement in the hospital, he remained disabled and retired from military service. His health would never return.

Alexei Petrov then took up residence in Kazan, surviving with his wife and son. He continued to study Einstein spaces and was led to the idea of classifying such spaces according to the algebraic structure of the curvature tensor. Today, this is known as the Petrov classification. Petrov published his key papers on this classification in two forms, first as a short communication in the Proceedings of the Academy of Sciences of USSR (1951), and second, as an expanded treatment in the local bulletin of Kazan University (1954).

During the late 1940's through the 1950's, Alexei Petrov was employed as a lecturer. He quickly became a very popular lecturer and a favourite among the student community. He taught at many places within Kazan, including Kazan University where he was elected a professor in 1956.

In 1960, Petrov organized the Faculty of Relativity Theory and Gravitation at Kazan University, and led the Faculty during the next ten years. That year, the first Russian edition of his monograph *Einstein Spaces* was issued. With this publication, Petrov became a well known and admired by theoretical physicists throughout the world. An English translation of his famous book was eventually published in 1969, by Pergamon Press.

Aside from his classification system, Petrov was also interested in the other fields related to Einstein's theory of relativity. He tried to apply the methods of group theory to these problems. Petrov was also interested in gravitational waves. This is because Einstein spaces of the second and the third kind, in the framework of his classification, are related to the fields of gravitational radiation. Eventually, Petrov authored a popular book entitled *Space*, *Time*, *and Matter*, which was the result of his lectures explaining the essence of Einstein's theory of relativity to the general public.

There was a long-term conflict between Petrov and the administration of Kazan University: the Soviet bureaucracy non-allowed independent thinking and behaviour that was very specific to Petrov's individuality. In 1969, this conflict has reached the apogee. As happily Petrov had good friends amongst the fluent scientists at Kiev, the capital city of Ukraine. The friends elected him in 1969 a full member of the Academy of Sciences of the Ukraine at Kiev, and invited to join the academic staff of the Institute of Theoretical Physics. He resolved to leave Kazan immediately and settle in Kiev, but unfortunately these would be the last two years of his life.

Alexei Petrov was consumed by his scientific ideas. The walls of his office were adorned with the portraits of Einstein and Schirokov. Petrov worked for many years without relaxation. It can be said that this train of Petrov's life, in conjunction with his old wounds and generally poor health, slowly killed him. He suffered several heart attacks. However, during each hospitalization, despite intense medical treatment, he would ask his family and friends for a pen and paper. Tragically, he died from the complications of a blood clot on May 09, 1972, at a hospital in Kiev. He was only 61.

Dmitri Rabounski

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This journal is named after Abraham Zelmanov (1913–1987), a prominent scientist working in the General Theory of Relativity and cosmology, whose main goal was the mathematical apparatus for calculation of the physical observable quantities in the General Theory of Relativity (it is also known as the theory of chronometric invariants). He also developed the basics of the theory of an inhomogeneous anisotropic universe, and the classification of all possible models of cosmology which could be theoretically conceivable in the space-time of General Relativity (the Zelmanov classification). He also introduced the Anthropic Principle and the Infinite Relativity Principle in cosmology.

The main idea of this journal is to publish most creative works on relativity produced by the modern authors, and also the legacy of the classics which was unaccessed in English before. This journal therefore is open for submissions containing a valuable result in the General Theory of Relativity, gravitation, cosmology, and also related themes from physics, mathematics, and astronomy.

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