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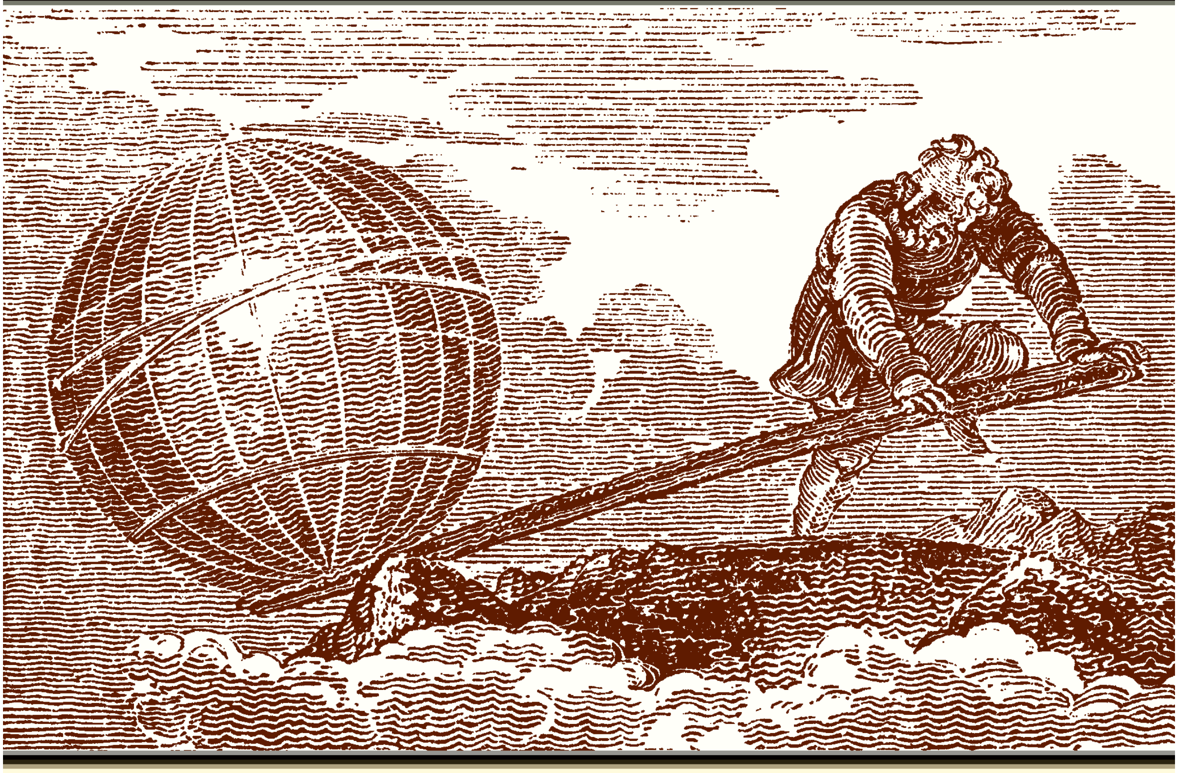

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Cosmological Models in the Generalized Einstein Action

Arbab I. Arbab*

Abstract: We have studied the evolution of the Universe in the generalized Einstein action of the form $R+\beta R^2$, where R is the scalar curvature and $\beta = const.$ We have found exact cosmological solutions that predict the present cosmic acceleration. These models predict an inflationary de-Sitter era occurring in the early Universe. The cosmological constant (Λ) is found to decay with the Hubble constant (H) as, $\Lambda \propto H^4$. In this scenario the cosmological constant varies quadratically with the energy density (ρ), i.e., $\Lambda \propto \rho^2$. Such a variation is found to describe a two-component cosmic fluid in the Universe. One of the components accelerated the Universe in the early era, and the other in the present era. The scale factor of the Universe varies as $a \sim t^n$, $n = 1/2$ in the radiation era. The cosmological constant vanishes when $n = 4/3$ and $n = 1/2$. We have found that the inclusion of the term R^2 mimics a cosmic matter that could substitute the ordinary matter. It is also equivalent to having a bulk viscosity of ordinary cosmology.

Contents:

§1. Introduction. Modified gravity models have been invoked to resolve cosmological and astrophysical problems with observations (see Hawking & Luttrel [1], Whitt [2], Srivastava [3], Srivastava & Sinha [4], Kung [5]). The generalized Einstein action including an additional scalar term R^2 is given (by Kenmoku *et al.*, 1992 [6]; Nojiri & Odintsov, 2005 [7]; Debnath & Paul, 2006 [8]) by

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,(R + 2\Lambda + \beta R^2\ ) + S_{\text{matter}} \qquad (1)$$

where R is Ricci's scalar curvature, Λ is the cosmological constant, g is the negative determinant of the metric tensor $g_{\mu\nu}$, S_{matter} is the

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matter action, and β is a constant. Several authors have studied classical solutions of this action without matter and have concluded that big bang singularity may be avoided (see Kung, 1996 [5]).

In this paper we will study the cosmological implications of this action.

The variation of the metric with respect to $g_{\mu\nu}$ gives

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} + \beta B_{\mu\nu} = -8\pi G T_{\mu\nu}\,, \tag{2}$$

where $T_{\mu\nu}$ is the energy momentum tensor of the cosmic fluid, and

$$B_{\mu\nu} = 2R\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) + 2\left(R_{;\,\mu\nu} - g_{\mu\nu}\Delta R\right)\,, \tag{3}$$

with $R_{;\,\mu\nu} = \nabla_\mu \nabla_\nu R$ and $\Delta R = g^{\mu\nu} R_{;\,\mu\nu}$. For an ideal fluid one has

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}\,, \tag{4}$$

where u_μ, ρ, p are the velocity, density and pressure of the cosmic fluid.

The flat Robertson-Walker line element is given by

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + r^2\sin^2\theta\, d\phi^2)\right]. \tag{5}$$

The time-time and space-space components of (2) give

$$3H^2 - \Lambda - 18\beta\left(6\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2\right) = 8\pi G\rho\,, \tag{6}$$

and

$$-2\dot{H} - 3H^2 + \Lambda + 6\beta\left(2\dddot{H} + 12H\ddot{H} + 18\dot{H}H^2 + 9\dot{H}^2\right) = 8\pi G p\,, \tag{7}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble constant. We have noticed that Debnath and Paul considered a similar action, but with variable G and Λ. They arrive at very similar type of solutions.

§2. Model A.

Now consider the cosmological model when

$$\Lambda = -18\beta\left(6\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2\right) \tag{8}$$

so that (6) is

$$3H^2 = 8\pi G\rho \tag{9}$$

and (7) becomes

$$-2\dot{H} - 3H^2 = 8\pi G\left[p - \frac{3\beta}{2\pi G}\left(\dddot{H} + 3H\ddot{H} + 6\dot{H}^2\right)\right]. \tag{10}$$

A universe with bulk viscosity (η) is obtained by replacing the pressure p by the effective pressure $p-3\eta H$. In this case, one may attribute that the inclusion of the R^2 is equivalent to having a bulk viscosity given by

$$\eta=\frac{\beta}{2\pi G}\left(\frac{\dddot{H}}{H}+3\ddot{H}+6\frac{\dot{H}^2}{H}\right). \tag{11}$$

Notice here that the bulk viscosity is normally parameterized by $\eta=\eta_0\,\rho^s$ $(s,\eta_0=const)$, but here η depends on the rates of universe expansion. Consider a power law expansion of the form

$$a=A\,t^n,\qquad A,n=const \tag{12}$$

so that

$$\eta=\frac{3\beta}{\pi G}\,(2n-1)\,t^{-3}. \tag{13}$$

From (8), the cosmological constant becomes

$$\Lambda=54\;\beta\;n^2(2n-1)\,t^{-4},\qquad n\neq 0\,, \tag{14}$$

and the energy density

$$8\pi G\rho=\frac{3n^2}{t^2}\,. \tag{15}$$

Using (12), the cosmological constant becomes

$$\Lambda=\frac{54\beta\,(2n-1)}{n^2}\,H^4,\qquad n\neq 0\,, \tag{16}$$

where $H=\frac{n}{t}$. Upon using (9), this becomes

$$\Lambda=\frac{6\beta\,(2n-1)(8\pi G)^2}{n^2}\,\rho^2,\qquad n\neq 0\,, \tag{17}$$

i.e., $\Lambda\propto\rho^2$. Substituting (12) into (10), we see that the pressure is given by

$$8\pi Gp=\left[\frac{(2-3n)\,n}{t^2}-\frac{72\beta\,n\,(1-2n)}{t^4}\right]. \tag{18}$$

Using (15), this can be written as

$$p=\left(-1+\frac{2}{3n}\right)\rho-\left(\frac{1-2n}{n^3}\right)N\beta\rho^2,\quad N=64\pi G\,,\quad n\neq 0\,. \tag{19}$$

We know the Van der Waals equation of state is given by

$$p=\frac{\gamma\,\rho}{1-b\gamma}-\alpha\rho^2,\qquad \gamma,b,\alpha=const. \tag{20}$$

Thus, the resulting equation of state of a power law expansion is that due to two-component fluid resembling the van der Waals equation of state. Therefore, introducing a term of R^2 in the Einstein action is like introducing two fluid components in the Universe. We see that one component of the fluid drives the Universe in cosmic acceleration, by making $p<0$, in some period and decelerates it in another period ($p>0$). In the early Universe, when the density was so huge, p was negative if $n<\frac{1}{2}$. During the matter dominated epoch, when the density is very small, $p<0$, if $n>\frac{2}{3}$. Hence, we see that the Universe accelerates for any deviation from the Einstein-de-Sitter expansion.

A. Inflationary Era. We see from (9) when $H=H_0=const$, i.e., $a \propto \exp\,(H_0 t)$, the cosmological constant vanishes, i.e., $\Lambda=0$. From (11) the bulk viscosity also vanishes, i.e., $\eta=0$. We recover the de Sitter solution, $p=-\rho$, as evident from (9) and (10).

For a static universe $n=0$, i.e., $H=0$, and hence, the cosmological constant, the bulk viscosity, the energy density and the pressure vanish, i.e., $\Lambda=\eta=\rho=p=0$. This means that a static universe in this scenario can't exist.

B. Radiation Dominated Era. During the radiation dominated phase, as in the Einstein-de-Sitter model, i.e., $a \propto t^{\frac{1}{2}}$, one has $n=\frac{1}{2}$ so that $\Lambda=\eta=0$, and (19) gives the equation of the state $p=\frac{1}{3}\rho$. Thus the Einstein-de-Sitter model is recovered. In this epoch the cosmological constant vanishes. However, (19) shows that any deviation form $n=\frac{1}{2}$ in the radiation era, viz., $n<\frac{1}{2}$, the second term will be large and negative. Thus, an accelerated expansion of the Universe will be inevitable.

C. Matter Dominated Era. In the matter dominated epoch of Einstein-de-Sitter model one has $n=\frac{2}{3}$. In this case $p=72\pi G\beta\rho^2$. Since ρ is small today, we see that the Universe asymptotically approaches the Einstein-de-Sitter type. However, for any deviation of this expansion law, $n>\frac{2}{3}$ accelerated expansion will be inevitable. In this case $p<0$. So, in the distant future, when $\rho\to 0$, the equation of state reduces to

$$p = \left(-1+\frac{2}{3n}\right)\rho = \omega\,\rho\,, \qquad \omega = -1+\frac{2}{3n}\,. \tag{21}$$

Thus, $n>\frac{2}{3}$ implies $\omega>-1$. We remark here in the distant future, when $n\to\infty$, $p=-\rho$. Hence, the future of our Universe will be a de-Sitter expansion.

§3. Model B. Now, let us define the cosmological constant by

$$\Lambda = -6\beta\left(2\dddot{H}H + 12H\ddot{H} + 18\dot{H}H^2 + 9\dot{H}^2\right), \tag{22}$$

so that (6) and (7) become

$$3H^2 = 8\pi G(\rho + \bar{\rho}), \tag{23}$$

and

$$-2\dot{H} - 3H^2 = 8\pi Gp, \tag{24}$$

where

$$8\pi G\bar{\rho} = -12\beta(\dddot{H} + 3H\ddot{H} + 6\dot{H}^2). \tag{25}$$

A. Inflationary Era. We see that when $H = H_0 = const$, i.e., $a \propto \exp\,(H_0 t)$, $p = -\rho$, $\Lambda = 0$ and $\bar{\rho} = 0$.

B. Radiation and Matter Dominated Eras. Now consider a power law expansion of the scale factor of the form as in (12). We find

$$8\pi Gp = n(2-3n)\,t^{-2}, \tag{26}$$

$$\Lambda = 18\beta n(2n-1)(3n-4)\,t^{-4}, \tag{27}$$

and

$$\rho = \left(\frac{3n}{2-3n}\right)p + N'\left[\frac{2n-1}{n^2(2-3n)^2}\right]p^2, \quad N' = 637\beta G\,, \quad n \neq \frac{2}{3}\,, \tag{28}$$

and

$$8\pi G\bar{\rho} = 72\beta n(1-2n)\,t^{-4}. \tag{29}$$

Equation (28) represents our equation of state for the present cosmology. The cosmological constant here varies as t^{-4}.

The equation of state now reads,

$$p = \omega(t)\,\rho\,, \qquad \omega\,(t) = \left[\frac{3n}{2-3n} + 72\beta\,\frac{(2n-1)}{n(2-3n)}\frac{1}{t^2}\right]^{-1}, \tag{30}$$

where the energy density becomes

$$8\pi G\rho = \frac{3n^2}{t^2}\left[1 + \frac{24\beta(2n-1)}{n}\frac{1}{t^2}\right]. \tag{31}$$

It is evident from (30) that when $n \to \infty$ (i.e., $a \to \infty$), $\omega \to -1$ the Universe becomes vacuum dominated and expands like de-Sitter. It is interesting to note that when $n=\frac{4}{3}$, the cosmological constant vanishes, i.e., $\Lambda=0$. In this case the pressure becomes negative, i.e., $p<0$, and this drives the Universe into an epoch of cosmic acceleration. The deceleration parameter $q=-\frac{\ddot{a}}{aH^2}=-0.25$. Once again, when $n=\frac{1}{2}$, we recover the Einstein-de-Sitter solution, i.e., $a\propto t^{\frac{1}{2}}$ and $p=\frac{1}{3}\rho$ and $\Lambda=0$. For $n=\frac{2}{3}$, $p=0$, $\Lambda=-8\beta t^{-4}$, and $8\pi G\rho=\frac{4}{3t^2}\left(1+\frac{12\beta}{t^2}\right)$. Hence, the Universe approaches the Einstein-de-Sitter solution asymptotically ($t\to\infty$). Notice that (27) and (29) relate the vacuum energy density $\rho_v=\frac{\Lambda}{8\pi G}$ to $\overline{\rho}$ by the equation $\rho_v=(1-\frac{3}{4}n)\overline{\rho}$, so that for $n=\frac{2}{3}$, $\overline{\rho}=2\rho_v$.

§4. Model C. Now consider a cosmological model in which $\Lambda=0$. In this case, (6) and (7) yield

$$3H^2=8\pi G(\rho+\rho') \tag{32}$$

and

$$-2\dot{H}-3H^2=8\pi G(p+p')\,, \tag{33}$$

where

$$\left.\begin{aligned} 8\pi G\rho' &= 18\beta(6\dot{H}H^2+2H\ddot{H}-\dot{H}^2) \\ 8\pi Gp' &= -6\beta(2\dddot{H}+12H\ddot{H}+18\dot{H}H^2+9\dot{H}^2) \end{aligned}\right\}, \tag{34}$$

equations (32) and (33) can be written as

$$3H^2=8\pi G\rho_{\text{eff}} \tag{35}$$

and

$$-2\dot{H}-3H^2=8\pi Gp_{\text{eff}}\,, \tag{36}$$

where

$$\rho_{\text{eff}}=\rho+\rho'\,,\qquad p_{\text{eff}}=p+p'\,. \tag{37}$$

We therefore argue that the inclusion of the term R^2 in the Einstein action induces a fluid in the Universe that has pressure (p') and energy density (ρ'), in addition to the pre-existing matter. This may suggest that our Universe is filled with a fluid with two components; one is *bright* (ρ) and the other is *dark* (ρ') without having a cosmological constant.

A. Inflationary Era. An inflationary solution arises when $H=$ $=H_0=const$ which is solved to give $a\propto\exp(H_0t)$. Equations (35) and (36) give

$$p_{\text{eff}}=-\rho_{\text{eff}}\,. \tag{38}$$

With some scrutiny, one would discover that (34) implies that $p'=$ $=\rho'=0$. Hence, the *dark* component does not contribute to this inflationary era.

B. Radiation Dominated Era. Now consider a power law expansion for the Universe during the radiation dominated era of the form

$$a = Bt^n\,, \qquad n, B = const\,. \tag{39}$$

Substituting this into (34) one gets

$$\rho' = \frac{54\beta n^2(1-2n)}{8\pi G}\,t^{-4}, \tag{40}$$

and

$$p' = \frac{18\beta n(1-2n)(4-3n)}{8\pi G}\,t^{-4}. \tag{41}$$

These two equations are related by the relation

$$p' = \left(-1+\frac{4}{3n}\right)\rho'\,, \qquad n \neq 0\,, \tag{42}$$

which represents the equation of state of the *dark* fluid. It is very interesting to note that, when $n=\frac{1}{2}$, $\rho'=p'=0$. This implies that the *dark* component does not disturb the nucleosynthesis constraints set forth by the Einstein-de-Sitter solution. Notice that when $n\to\infty$, $p'=-\rho'$, so that in the distant future the Universe with or without normal matter will be vacuum dominated. We notice that this *dark* component does not live in a static Universe since it has $p'=\rho'=0$. For a positive energy density, we have (for $\beta>0$) the constraint $n<\frac{1}{2}$. For $\frac{1}{2}<n<\frac{4}{3}$, $\rho'<0$, $p'<0$. When $n=1$, $p'=\frac{1}{3}\overline{\rho}$ so that the dark component mimics the ordinary radiation. The positivity of energy density is recovered if $\beta<0$. It is remarkable to notice that if the matter action is not incorporated in the Einstein action, the inclusion of the quadratic term acts like matter. This type of matter is characterized by its equation of state in (42). Hence, the inclusion of the R^2 mimics the introduction of a new matter in the Universe.

C. Matter Dominated Era. Applying (39)–(41) in (32) and (33), one gets

$$8\pi G\rho = \frac{3n^2}{t^2}\left[1-\frac{18\beta(1-2n)}{t^2}\right], \tag{43}$$

and

$$8\pi G p = \frac{n(2-3n)}{t^2}\left[1-\frac{18\beta(2n-1)(3n-4)}{(2-3n)}\frac{1}{t^2}\right]. \tag{44}$$

Once again, we see from (41) that the pressure of the *dark* fluid vanishes when $n=\frac{4}{3}$ leaving only the *bright* fluid to contribute to the universal pressure. In this case, $p<0$, and this will drive the Universe into a cosmic acceleration era. Substitution of $n=\frac{2}{3}$ in (43) and (44) yields

$$8\pi G\rho = \frac{4}{3t^2}\left(1+\frac{6\beta}{t^2}\right), \qquad 8\pi G p = \frac{8\beta}{t^4}\,. \tag{45}$$

We remark that the Universe approaches the Einstein-de-Sitter asymptotically (when $t\to\infty$, i.e., in the distant future), where $\rho=\frac{1}{6\pi G t^2}$ and $p=0$. However, $\rho_{\rm eff}=\frac{1}{6\pi G t^2}$ and $p_{\rm eff}=0$. Notice that during this era the *dark* component behaves like a stiff matter, with $p'=\rho'$. It is evident that when $\frac{2}{3}<n<\frac{4}{3}$ the Universe enters a period of cosmic acceleration. We may thus argue that the present observed cosmic acceleration happened during this period. Equation (45) looks like having a cosmological constant of the form $\Lambda\propto H^4$ in the Einstein-de-Sitter universe.

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Hubble Redshift due to the Global Non-Holonomity of Space

Dmitri Rabounski

Abstract: In General Relativity, the change in energy of a freely moving photon is given by the scalar equation of the isotropic geodesic equations, which manifests the work produced on a photon being moved along a path. I solved the equation in terms of physical observables (Zelmanov A. L., *Soviet Physics Doklady*, 1956, vol. 1, 227–230) and in the large scale approximation, i.e. with gravitation and deformation neglected, while supposing the isotropic space to be globally non-holonomic (the time lines are non-orthogonal to the spatial section, a condition manifested by the rotation of the space). The solution is $E = E_0 \exp(-\Omega^2 a t/c)$, where Ω is the angular velocity of the space (it meets the Hubble constant $H_0 = c/a = 2.3 \times 10^{-18}$ sec^{-1}), a is the radius of the Universe, $t = r/c$ is the time of the photon's travel. Thus, a photon loses energy with distance due to the work against the field of the space non-holonomity. According to the solution, the redshift should be $z = \exp(H_0 r/c) - 1 \approx H_0 r/c$. This solution explains both the redshift $z = H_0 r/c$ observed at small distances and the non-linearity of the empirical Hubble law due to the exponent (at large r). The ultimate redshift in a non-expanding universe, according to the theory, should be $z = \exp(\pi) - 1 = 22.14$.

Contents:

§1. Hubble redshift in a static universe. In this short presentation, I show how the Hubble law, including its non-linearity with distance, can be deduced directly from the equations of the General Theory of Relativity. The Hubble law I have deduced is present in a non-expanding universe. It is also present, in a slightly different form, in an expanding universe and a compressing universe.

In General Relativity, the change of energy of a freely moving photon should be the solution to the scalar equation of isotropic geodesics, which is also known as the equation of energy and manifests the work produced on the photon being moved along the path. In terms of physically observable quantities — chronometric invariants (Zelmanov, 1944), which are the respective projections of four-dimensional quantities onto the time line and spatial section of a given observer — the isotropic geodesic equations are presented with two projections onto the time line and spatial section, respectively [1–3]

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0\,, \tag{1.1}$$

$$\frac{d(\omega c^i)}{d\tau} - \omega F^i + 2\omega \left(D^i_k + A^{\cdot i}_{k\cdot}\right) c^k + \omega \triangle^i_{nk} c^n c^k = 0\,, \tag{1.2}$$

where ω is the proper frequency of the photon, $d\tau$ is the interval of physically observable time, c^i is the chr.inv.-vector of the observable velocity of light ($c_k c^k = c^2$). The physically observable properties of space are presented with the chr.inv.-vector F_i of the gravitational inertial force, the chr.inv.-tensor A_{ik} of the angular velocity of the rotation of space due to its non-holonomity (the non-orthogonality of the time lines to the spatial section, which is expressed as $g_{0i} \neq 0$, and is manifested as the three-dimensional rotation of space), the chr.inv.-tensor D_{ik} of the deformation of space (shows how space deforms with time), and the chr.inv.-Christoffel symbols $\triangle^i_{nk}$ (indicate the non-uniformity of space). All these three-dimensional quantities bear the property of chronometric invariance (i.e. they are invariant in the spatial section of the observer) and are dependent on the gravitational potential $\mathrm{w} = c^2 (1 - \sqrt{g_{00}})$, on the linear velocity $v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}$ of the rotation of space due to its non-holonomity, and also on the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$, which characterize the time line and spatial section of the observer.

Integration of the scalar equation of isotropic geodesics (the equation of energy) should give a function $E = E(t)$, where $E = \hbar\omega$ is the proper energy of the photon. However, integration of time in a Riemannian space is not a trivial task. This is because the observable interval of time $d\tau = \sqrt{g_{00}}\, dt - \frac{1}{c^2} v_i dx^i$ depends on the gravitational potential w along the path, on the linear velocity v_i of the rotation of space (due to the non-holonomity of it), and on the displacement dx^i of the observer with respect to his coordinate net during the measurement. The result of integration depends on the integration path, so time is not integrable in a general case. We therefore consider the "large scale approximation", where distances are close to the curvature radius of the Universe; so gravitation and deformation are neglected in the space ($g_{00} = 1$ and $D_{ik} = 0$, respectively), and the observer is resting with respect to his coordinate net ($dx^i = 0$). In such a case, integration of time is allowed, and is simply $d\tau = dt$. We also suppose the isotropic space, the "home space" of isotropic (light-like) trajectories and massless light-like particles (e.g. photons), to be globally non-holonomic ($v_i \neq 0$). With these assumptions, the formula for the gravitational inertial force F_i [1–3], losing the gravitational potential which becomes $\mathrm{w} = c^2(1-\sqrt{g_{00}}) = 0$, consists of only the second term

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) \simeq -\frac{\partial v_i}{\partial t}\,, \tag{1.3}$$

which is due to the space non-holonomity. This negative (centrifugal) acceleration, experienced by such a photon in the isotropic space, is the solely factor which is still acting on the energy of the photon in the scalar equation of isotropic geodesics in the framework of the "large scale approximation" in a globally non-holonomic isotropic space. It acts on a photon due to the motion (global rotation) of the isotropic space itself.

It should be noted that, despite the apparent similarity to the centrifugal force of inertia, this factor is not related to the fictitious forces of inertia. The forces of inertia are observed in a rotating coordinate frame, and are due to the transformation of the coordinates and time which include the angular velocity of the coordinate frame (these were considered in 1909 by Max Born, and are known as the Born coordinates). As a result, the space-time metric being written in the Born coordinates gets additional terms in g_{00} and g_{0i}. The additional terms vanish, in common with the forces of inertia produced due to the terms, by the transformation of the coordinates back to another, non-rotating frame. In contrast, the factor we are considering is due to the basic non-holonomity of space, which is only $g_{0i} \neq 0$, and is invariant in the spatial section of the observer (i.e. this is a chronometrically invariant effect), and cannot therefore be removed by the transformation from one coordinate frame to another one in the spatial section.

Of course, one can derive the inertial force effects in General Relativity, when moving to a rotating coordinate frame (the Born coordinates). These will, however, be only the removable (fictitious) effects, observed on the background of the gravitational potential, the non-holonomity, the deformation, the inhomogeneity, and the curvature of space, whose effects cannot be removed by our choice of the coordinate frame in the spatial section of the observer due to the invariance of the effects in the spatial section.

We consider a single photon travelling in the x-direction. In this case, $c^1 = c$, $c^2 = 0$, $c^3 = 0$. With the "large scale approximation" in a globally non-holonomic isotropic space, and assuming the linear velocity of the space rotation to be $v_1 = v_2 = v_3 = v$ and stationary, i.e. $\frac{\partial v}{\partial t} = B = const$, the scalar equation of isotropic geodesics for such a photon takes the form

$$\frac{dE}{dt} = -\frac{B}{c}\,E\,. \tag{1.4}$$

This is a simple uniform differential equation of the 1st order, like $\dot{y} = -\,ky$, so that we have $\frac{dy}{y} = -k\,dt$ or $d(\ln y) = -k\,dt$. It solves as $\ln y = -kt + \ln C$, where C is the integration constant which can be evaluated when the initial conditions of integration ($y = y_0$, $t_0 = 0$) are substituted. Finally, we obtain $y = y_0\, e^{-kt}$. As a result, the scalar equation of isotropic geodesics (the equation of energy), in the "large scale approximation" in the globally non-holonomic space, gives the solution for the photon's energy (frequency) and the redshift $z = \frac{\omega_0 - \omega}{\omega}$ as depending on the distance $r = ct$ travelled from the observer

$$E = E_0\, e^{-kt}, \qquad z = e^{kt} - 1\,, \tag{1.5}$$

such that at small distances of the photon's travel, i.e. with the exponent $e^x = 1 + x + \frac{1}{2}\,x^2 + \ldots \simeq 1 + x$, it takes the form

$$E \simeq E_0\,(1 - kt)\,, \qquad z \simeq kt\,, \tag{1.6}$$

where $k = \frac{1}{c}\,B = \frac{1}{c}\frac{\partial v}{\partial t} = const$. Thus, according to our calculation, which is based on the equations of the General Theory of Relativity, a photon being moved in a non-holonomic space loses its proper energy and frequency due to the work produced by it against the field of the space non-holonomity (or, in other words, the negative work produced by the field on the photon).

We suppose the space (space-time) of our Metagalaxy to be a spherical geometry space, which has a constant curvature and is globally non-holonomic. A constant curvature spherical space, whose metric is sign-definite, is a hypersphere of constant radius (the curvature radius of the space). However, we are considering a four-dimensional spherical space with a sign-alternating metric $(+---)$ or $(-+++)$, which indicates the presence of the special coordinate axis known as time among the four coordinate axes of the space. The sign-alternating metric indicates, in particular, that such a space consists of two subspaces, which are known as the non-isotropic space (the home of non-isotropic trajectories which are the trajectories of mass-bearing particles) and the isotropic space (the home of isotropic trajectories which are the trajectories of massless light-like particles, e.g. photons). We know that, given a point, only one geodesic line can be paved through it in a given direction, and such a unique geodesic line can be either non-isotropic or isotropic (see [4, §6] or [5, §101]). In other words, non-isotropic and isotropic geodesics have no common points. Therefore, the spherical space with the sign-alternating metric we are considering is presented with two concentric hyperspheres — the home of non-isotropic trajectories and that for isotropic ones — which have the same radius of curvature, but are not coinciding with each other.

The constant radius of such a hypersphere manifests that the curvature radius of the space of our Metagalaxy remains unchanged, so the Metagalaxy as a whole does not expand or compress in the framework of this model. Meanwhile, a local volume (a local element of the hypersphere's surface) may experience any stages of the evolution, which could be conceivable in the framework of Zelmanov's theory of a locally inhomogeneous anisotropic universe [2, 3], including the special states of infinite density and infinite rarefraction, if it doesn't change the stationary state of the space (the hypersphere's surface) as a whole.

The non-holonomity of the four-dimensional (non-isotropic or isotropic) space is the basic non-orthogonality of the time lines to the spatial axes on the (non-isotropic or isotropic) hypersphere's surface, and is manifested by its three-dimensional rotation.*

According to the concepts of topology [6, vol. 1], the surface of an $(n+1)$-dimensional sphere is equivalent to the volume of an n-dimensional torus. Thus, the globally non-holonomic spherical space we are considering is representable also with a torus, the home of non-isotropic trajectories and mass-bearing particles, which is coaxial to another torus, the home of isotropic trajectories and massless light-like particles, but is not coinciding with the first.

It is obvious that since the non-holonomity of such a space must be stationary, we can express the acceleration experienced by a photon in the isotropic space due to its non-holonomity, through the angular velocity Ω of the rotation of the isotropic hypersphere and its curvature radius, $a = \frac{c}{H_0}$, which is the same that the curvature radius of our Metagalaxy (H_0 is the Hubble constant). We obtain $\frac{\partial v}{\partial t} = \Omega^2 a = const$. In such a space, the coefficient $k = \frac{1}{c}\frac{\partial v}{\partial t}$ in the solution (1.5) we have

*The non-orthogonality of the time lines to the spatial section is impossible to be in a sign-definite metric space due to the absence of the special coordinate axis known as time. Therefore, all that has been said about holonomic and non-holonomic spaces is valid only for sign-alternating metric spaces such as pseudo-Riemannian spaces (for instance, the four-dimensional pseudo-Riemannian space with the signature $(+---)$ or $(-+++)$, which is the basic space-time of the General Theory of Relativity).

obtained to the scalar equation of isotropic geodesics is

$$k = \frac{1}{c}\,\Omega^2 a = const. \tag{1.7}$$

Then, according to the redshift formula $z \simeq kt$ obtained in the framework of our theory, for the galaxies located at a "small" distance of $r \simeq 630$ Mpc* (the redshift observed on them is $z \simeq 0.16$) we obtain

$$\Omega = \sqrt{\frac{z\,c}{a\,t}} = \sqrt{\frac{z\,c^2}{a\,r}} \simeq 2.4\times10^{-18}\ \text{sec}^{-1}, \tag{1.8}$$

that meets the Hubble constant, which is $H_0 = 72\pm8\times10^5$ cm/sec × Mpc= $= 2.3\pm0.3\times10^{-18}$ sec^{-1} (this is according to the Hubble Space Telescope data, 2001 [7]).

With these we arrive at the following law

$$E = E_0\, e^{-\frac{H_0 r}{c}}, \qquad z = e^{\frac{H_0 r}{c}} - 1\,, \tag{1.9}$$

as a purely theoretical result obtained from our solution to the scalar equation of isotropic geodesics. At small distances of the photon's travel, this law becomes

$$E \simeq E_0\left(1 - \frac{H_0\, r}{c}\right), \qquad z \simeq \frac{H_0\, r}{c}\,. \tag{1.10}$$

As seen, this result provides a complete theoretical ground to the linear Hubble law, empirically obtained by Edwin Hubble for small distances, and also to the non-linearity of the Hubble law observed at large distances close to the size of the Metagalaxy (the non-linearity is explained due to the exponent in our exact solution (1.9), which is becoming a sufficient factor at large r).

Then, proceeding from our solution, we are able to calculate the ultimate redshift, which is allowed in our Universe. It is, according to the exponential law (1.9),

$$z_{\max} = e^{\pi} - 1 = 22.14\,. \tag{1.11}$$

Proceeding from the theoretical considerations presented here, we calculate the linear velocity of the rotation of the isotropic space, which is due to the global non-holonomity of it. It is $\breve{v} = \Omega\, a = H_0\, a = c$, i.e. is equal to the velocity of light. I should note, to avoid misunderstanding, that this linear velocity of rotation is attributed to the isotropic

*1 parsec = 3.0857×10^{18} cm $\simeq 3.1\times10^{18}$ cm.

space, which is the home of isotropic (light-like) trajectories specific to massless light-like particles (e.g. photons). It isn't related to the non-isotropic space of sub-light-speed trajectories, which is the home of mass-bearing particles (e.g. galaxies, stars, planets). In other words, our result doesn't mean that the visible three-dimensional space of cosmic bodies rotates at the velocity of light. The space of galaxies, stars, and planets may be non-holonomic or not, depending on the physical conditions in it.

It is possible to show, by the mathematical methods of orthometric invariants [8] which allow calculation for physically observable quantities in any reference frame of the four-dimensional pseudo-Riemannian space (the basic space-time of the General Theory of Relativity), that the basic non-holonomity of the isotropic space is such that it rotates as a whole with the linear velocity equal to the velocity of light. So, our result concerning the linear velocity of the rotation of the isotropic space meets the basics of geometry of pseudo-Riemannian spaces.

In addition, it should be noted that, according to the theory of chronometric invariants, given the isotropic space rotating at the velocity of light, the observable three-dimensional metric h_{ik} of the space is non-degenerate ($h = \det \| h_{ik} \| \neq 0$). Thus, the four-dimensional metric $g_{\alpha\beta}$ is non-degenerate as well ($g = -h\, g_{00} \neq 0$, where $g = \det \| g_{\alpha\beta} \| \neq 0$). This means that the rotation of the isotropic space at the velocity of light does not lead to a singulary break in it.

§2. The rôle of deformation. The exponential redshift law (1.9) and its linear approximation (1.10) were deduced for a static universe, which does not experience expansion or compression, so its space remains non-deforming. Now, we study how the redshift law does change its formulation in a universe which expands or compresses.

The redshift law (1.9) was obtained as a result of integrating the scalar geodesic equation (1.1). According to the equation, the deformation of space is the second factor which, in addition to the gravitational inertial force, changes energy of a freely moving photon. No other factors are manifested. Space deforms while the universe expands or compresses. Thus, integrating the scalar geodesic equation in a non-static universe, we should take the factor of deformation into account.

The chr.inv.-tensor D_{ik} of the deformation of space is formulated [1–3] as the derivative of the chr.inv.-metric tensor h_{ik} by time

$$D_{ik} = \frac{1}{2} \frac{{}^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\frac{1}{2} \frac{{}^*\partial h^{ik}}{\partial t}\,, \tag{2.1}$$

where the tensor's trace (its physical meaning is the volume deformation of space) is

$$D = h^{ik}D_{ik} = \frac{{}^*\partial \ln\sqrt{h}}{\partial t} = \frac{1}{V}\frac{{}^*\partial V}{\partial t}\,, \tag{2.2}$$

where $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$, $h = \det\|h_{ik}\|$, $dV = \sqrt{h}\,dx^1dx^2dx^3$ is a differential increment of the volume V of space, while the components of the chr.inv.-metric tensor by definition [1–3] are

$$h_{ik} = -\,g_{ik} + \frac{1}{c^2}\,v_i v_k\,, \quad h^{ik} = -\,g^{ik}, \quad h^i_k = -\,g^i_k = \delta^i_k\,. \tag{2.3}$$

We will consider the redshift law in universes of two kinds, according to two simplest types of deformation*.

First, we will consider the redshift law in a *constant deformation* universe. This means that the volume of space undergoes equal relative changes with time†, so the deformation of space remains constant‡

$$D = \frac{1}{V}\frac{{}^*\partial V}{\partial t} = const \quad\Longrightarrow\quad D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = const. \tag{2.4}$$

Deformation of this kind means increase of the linear velocity of the expansion of space in an expanding universe, and decrease of the linear velocity of the compression in a compressing universe. This can be illustrated by calculation of a volume. In the three-dimensional Euclidean space, the volume of a parallelepiped built on the vectors $r^i_{(1)}$, $r^i_{(2)}$, $r^i_{(3)}$ is calculated as $V = \pm\det\|r^i_{(n)}\| = \pm|r^i_{(n)}|$. We obtain the invariant $V^2 = |r^i_{(n)}||r_{(m)i}| = |r^i_{(n)}||h_{ik}\,r^k_{(m)}| = |h_{ik}\,r^i_{(n)}r^k_{(m)}|$. (It should be noted that

*The chr.inv.-quantity D_{ik} takes all changes of the space volume into account. For instance, in a static non-holonomic universe ($v_i \neq 0$), space deforms by its rotation. This is manifested by the derivative from the second term of the chr.inv.-metric tensor h_{ik} (2.3). Meanwhile the coordinate three-dimensional metric g_{ik} changes due to the rotation so that the resulting deformation of space is zero

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = 0 \quad\Longleftrightarrow\quad c^2\frac{\partial g_{ik}}{\partial t} = v_i\frac{\partial v_k}{\partial t} + v_k\frac{\partial v_i}{\partial t}\,.$$

In a static holonomic universe ($v_i = 0$), the condition $D_{ik} = 0$ is realized by the conditions $g_{ik} = const$ and $v_i = 0$.

†I refer to this kind of universes as *homotachydioncotic* (ομοταχυδιογκωτικό). This terms originates in *homotachydioncosis* — ομοταχυδιόγκωσης — volume expansion with a constant speed, from όμο which is the first part of όμοιος (omeos) — the same, ταχύτητα — speed, διόγκωση — volume expansion, while compression can be considered as negative expansion.

‡The stationarity of an invariant metric, such as $g_{\alpha\beta}$ or h_{ik}, leads to the stationarity of its determinant, and vice versa. For instance, in the case under consideration, $h_{ik} = const \Longleftrightarrow h = \det\|h_{ik}\| = const$.

$h_{ik} \equiv -g_{ik}$ in an Euclidean space.) Concerning a differentially small volume, the invariant is $(dV)^2 = |h_{ik}\, dx^i_{(n)} dx^k_{(m)}| = |h_{ik}||dx^i_{(n)}||dx^k_{(m)}| = h\,|dx^i_{(n)}||dx^k_{(m)}|$. Thus, $dV = \sqrt{h}\,|dx^i_{(n)}|$. Expanding this method onto an n-dimensional pseudo-Riemannian space, we obtain $dV = \sqrt{-g}\,|dx^\alpha_{(\nu)}|$. In particular, a three-dimensional differentially small volume in the four-dimensional space-time of General Relativity is $dV = \sqrt{h}\,|dx^i_{(n)}|$, or, if the basic vectors of the parallelepiped meet the spatial coordinate axes, $dV = \sqrt{h}\, dx^1 dx^2 dx^3$.

The volume of a finite space comes with the integration of dV, wherein the differential lengths dx^i, and also the scale of x^i we integrate, do not depend on time (integration with respect to the spatial coordinates is instant). Thus, we obtain

$$D = \frac{{}^*\partial \ln\sqrt{h}}{\partial t} = \frac{1}{\sqrt{h}}\frac{{}^*\partial\sqrt{h}}{\partial t} = \frac{1}{V}\frac{{}^*\partial V}{\partial t} = \gamma\,\frac{1}{a}\frac{{}^*\partial a}{\partial t} = \gamma\,\frac{\mathrm{v}}{a}\,, \qquad (2.5)$$

where $V \sim a^3$ as for any three-dimensional volume, a is the radius of the universe (equal to the curvature radius in a constant curvature space), $\mathrm{v} = \pm|\mathrm{v}|$ is the linear velocity of the expansion or compression of space (positive in an expanding universe and negative in a compressing universe), and γ-factor is a constant numerical coefficient which is specific to the shape of space ($\gamma = 3$ in the homogeneous isotropic models [2,3]). As seen from this formula under $D = const$, in a constant deformation expanding universe, the linear velocity of its expansion increases with the growing radius of space (this means accelerated expansion of the universe). In contrast, in a constant deformation compressing universe, the linear velocity of its compression decreases with the shrinking radius of space (decelerated compression).

Second, we will consider the redshift law in a *constant speed deforming* universe*, i.e. in a universe which expands or compresses with a constant linear velocity $\mathrm{v} = \frac{{}^*\partial a}{\partial t} = const$. In a universe of this kind, the radius of space changes linearly with time $a = a_0 \pm \mathrm{v}\,t$ (here the upper sign is attributed to the expansion of space, while the lower sign characterizes the compression), while the deformation of space (2.5) is

$$D = \gamma\,\frac{1}{a_0 \pm \mathrm{v}\,t}\frac{{}^*\partial a}{\partial t} \simeq \gamma\,\frac{1}{a_0}\left(1 \mp \frac{\mathrm{v}\,t}{a_0}\right)\frac{{}^*\partial a}{\partial t} \simeq \gamma\,\frac{\mathrm{v}}{a_0} \mp \gamma\,\frac{\mathrm{v}^2\,t}{a_0^2}\,, \qquad (2.6)$$

*I refer to this kind of universes as *homotachydiastolic* (ομοταχυδιαστολικός). It's origin is *homotachydiastoli* — ομοταχυδιαστολή — linear expansion with a constant speed, from όμο which is the first part of όμοιος — the same, ταχύτητα — speed, and διαστολή — linear expansion (compression can be considered as negative expansion).

and is a linear function of time: $D = D_0 \mp \mu t$, $\mu = const$. Thus, in a constant speed expanding universe, the deformation decreases with time, while it grows with time in a constant speed compressing universe

$$D \simeq D_0 - \gamma \frac{\mathrm{v}^2 t}{a_0^2} \quad \text{in the case of expansion,} \tag{2.7}$$

$$D \simeq D_0 + \gamma \frac{\mathrm{v}^2 t}{a_0^2} \quad \text{in the case of compression,} \tag{2.8}$$

where $D_0 \simeq \gamma \frac{\mathrm{v}}{a_0}$ is the deformation of space and a_0 is the radius of the universe at the start of measurement.

It should be noted that all that has been said here about the deformation of space is valid to both a finite and an infinite universe. This is because, according to the theory of an inhomogeneous anisotropic universe (Zelmanov, 1944 [2,3]), not only a whole universe can be a subject of evolution, but also any volume element of it, including even differentially small volume elements.

§3. Redshift in a constant deformation universe. In a universe of this kind, $D = const$ and $D_{ik} = const$. We neglect gravitation ($g_{00} = 1$), i.e. the gravitational potential is $\mathrm{w} = c^2 (1 - \sqrt{g_{00}}) = 0$ as in the "large scale approximation". As in our consideration of a static non-holonomic universe, we consider a single photon travelling in the x-direction (in this case, $c^1 = c$, $c^2 = 0$, $c^3 = 0$) and the linear velocity of the space rotation to be $v_1 = v_2 = v_3 = v$. However, v is not stationary in this case. (It is stationary only in a static universe, because it does not change its volume during the rotation.)

We consider the function $v = v(t)$. The relation $\frac{\partial v}{\partial t} = \Omega^2 a$ is obvious in a spherical space. The conservation of angular momentum of the universe therefore means that $\Omega a^2 = const$. These relations lead to $\frac{\partial v}{\partial t} = \frac{\Omega^2 a^4}{a^3} = \frac{\chi}{V}$, where $\chi = \sigma\, \Omega^2 a^4 = const$. Here σ is a constant structural coefficient specific to the shape of space so that the volume of space is expressed as $V = \sigma a^3$ at any stage of the evolution of the universe (we assume that space does not change its shape being homogeneous expanding or compressing). The constant deformation condition $D = \frac{1}{V}\frac{\partial V}{\partial t} = \frac{\partial \ln V}{\partial t} = A = const$ (here $\frac{{}^*\partial}{\partial t} = \frac{\partial}{\partial t}$ because no gravitation) gives $\ln V = At + \ln C$. Thus, $V = V_0\, e^{At}$, where $A = \gamma \frac{\mathrm{v}}{a}$ according to (2.5). Thus we obtain

$$\frac{\partial v}{\partial t} = \frac{\chi}{V_0}\, e^{-At}. \tag{3.1}$$

With these, we adopt the scalar equation of isotropic geodesics (1.1) to a photon travelling in a constant deformation universe. We obtain

$$\frac{dE}{dt} = -\left(\frac{\chi}{cV_0}\, e^{-At} + D_{11}\right) E\,, \tag{3.2}$$

or $\dot{y} = -\,k\,y$, where $k = \frac{\chi}{cV_0}\, e^{-At} + D_{11}$.

This is a simple uniform differential equation of the same kind as the equation of isotropic geodesics (1.4) we deduced for a static (non-deforming) non-holonomic universe with the only difference that being $k = \frac{\chi}{cV_0}\, e^{-At} + D_{11}$. Expanding the contants A and χ, and taking into account that $\Omega = H_0$, $a = \frac{c}{H_0}$, $t = \frac{r}{c}$ (H_0 is the Hubble constant, r is the distance of the photon's travel), and that $D_{11} = D = A$ in the case under consideration, we obtain

$$k = H_0\left(e^{-\gamma\frac{H_0 r \mathrm{v}}{c^2}} + \gamma\,\frac{\mathrm{v}}{c}\right), \tag{3.3}$$

where the linear velocity of the expansion or compression of space is $\mathrm{v} = \pm|\mathrm{v}|$, becoming positive in an expanding universe and negative in a compressing universe.

The equation (3.2) can be solved in the same way as (1.4). The solution will have only k according to the formula (3.3) instead of $k = H_0$ from the solution (1.9) we have obtained in a static (non-deforming) universe.

Thus (3.2) solves as

$$E = E_0\, e^{-\frac{H_0 r}{c}\left(e^{-\gamma\frac{H_0 r|\mathrm{v}|}{c^2}} + \gamma\frac{|\mathrm{v}|}{c}\right)} \quad \text{in an expanding universe,} \tag{3.4}$$

$$E = E_0\, e^{-\frac{H_0 r}{c}\left(e^{\gamma\frac{H_0 r|\mathrm{v}|}{c^2}} - \gamma\frac{|\mathrm{v}|}{c}\right)} \quad \text{in a compressing universe.} \tag{3.5}$$

The redshift in a deforming non-holonomic universe (in the "large scale approximation", where gravitation is neglected) arrives with the sum of two terms. First, the redshift due to the non-holonomity of space, which is resulted from the solution to a photon's scalar equation of motion. Second, the relativistic Doppler redshift, which is an effect of the photon's motion with respect to the observer. Thus, with the obtained solutions (3.4) and (3.5), we obtain the redshift law in a constant deformation non-holonomic universe in the cases of the expansion and compression, respectively

$$z = \left(e^{\frac{H_0 r}{c}\left(e^{-\gamma\frac{H_0 r|\mathrm{v}|}{c^2}} + \gamma\frac{|\mathrm{v}|}{c}\right)} - 1\right) + \left(\frac{1+\frac{|\mathrm{v}|}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1\right), \qquad (3.6)$$

$$z = \left(e^{\frac{H_0 r}{c}\left(e^{\gamma\frac{H_0 r|\mathrm{v}|}{c^2}} - \gamma\frac{|\mathrm{v}|}{c}\right)} - 1\right) + \left(\frac{1-\frac{|\mathrm{v}|}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1\right), \qquad (3.7)$$

where the main goal at sub-relativistic velocities* is due to the first term (a result of the non-holonomity of space), while the numerical value of the second (Doppler-effect) term is much less and consequently plays an auxiliary rôle in the redshift law.

At small distances of the photon's travel and sub-relativistic velocities of the expansion or compression, the redshift law (3.6) and (3.7) takes the linear approximation form

$$z \simeq \frac{H_0 r}{c}\left[1 - \gamma\frac{|\mathrm{v}|}{c}\left(\frac{H_0 r}{c} - 1\right)\right] + \frac{|\mathrm{v}|}{c} \quad \text{in an expanding universe,} \qquad (3.8)$$

$$z \simeq \frac{H_0 r}{c}\left[1 + \gamma\frac{|\mathrm{v}|}{c}\left(\frac{H_0 r}{c} - 1\right)\right] - \frac{|\mathrm{v}|}{c} \quad \text{in a compressing universe.} \qquad (3.9)$$

What is curious in the obtained law is that it will be blueshifted ($z<0$) at only small distances $r \ll a$ in a compressing universe. This is because the first (exponential) term will be positive in any case due to the exponent. At large distances, the first (always positive) term in the law (3.7) is much bigger than the second (Doppler-effect) negative term. For instance, let a photon travel at a distance r equal to the curvature radius of space $a = \frac{c}{H_0} \simeq 1.3\times10^{28}$ cm $\approx 4\times10^{9}$ parsec, while the universe compresses with a linear velocity of 100,000 km/sec. We assume also the shape-factor of space $\gamma = 3$ as for the inhomogeneous isotropic models [2, 3], but this is not principal in the calculation (the numerical value of γ depends weakly from the space of space). In this case, the first term in the redshift law (3.7) is $z_1 = +\,4.6$, while the second term (the relativistic Doppler blueshift) is $z_2 = -\,0.29$. If the universe compresses with a velocity of 10,000 km/sec, for a photon at the same distance $r = a$, we

*It is unbelievable that a universe expands or compresses with a velocity close to the velocity of light. On the other hard, such "ultimate cases" of ultra-relativistic expansion or compression would be interested from purely theoretical viewpoint.

obtain $z_1 = +1.7$ and $z_2 = -0.033$. In contrast, at small distances $r \ll a$, the first term approaches to zero, while the second (Doppler-effect blue-shift) term becomes valuable. For instance, if the universe compresses at 300 km/sec, at a short distance of 10^6 parsec (the Andromeda Galaxy is located at a distance of $\sim 780{,}000$ parsec) we obtain $z_1 = +0.00023$ and $z_2 = -0.001$, so the resulting shift of a photon's frequency at this distance is negative (the photon is definitely blueshifted).

Therefore, I suggest the same name "redshift law" for the obtained law in both expanding universe and compressing universe.

The redshift law (3.6, 3.7) and its linear approximation (3.8, 3.9) were obtained in a constant deformation non-holonomic universe. It is obvious that, in the absence of expansion or compression of space ($\mathrm{v} = 0$), these formulae transform into the redshift law (1.9) and its linear approximation (1.10) as deduced in a static (non-deforming) non-holonomic universe.

§4. Redshift in a constant speed deforming universe. A universe of this kind expands or compresses with a constant linear velocity $\mathrm{v} = const$.

In this case, neglecting gravitation as in the "large scale approximation" ($g_{00} = 1$), and taking the conservation of the angular momentum of the universe ($\Omega a^2 = const$) into account, we obtain

$$\frac{\partial v}{\partial t} = \Omega^2 a = \frac{\Omega^2 a^4}{(a_0 + \mathrm{v}t)^3} \simeq \frac{\Omega^2 a^4}{a_0^3}\left(1 - \gamma\,\frac{\mathrm{v}t}{a_0}\right), \tag{4.1}$$

where $\mathrm{v} = \pm|\mathrm{v}|$ (the positive velocity characterizes the expansion of space, while the sign minus characterizes the compression).

With (4.1) and the formula of deformation with $\mathrm{v} = const$ (2.6), we apply the scalar equation of isotropic geodesics (1.1) to a photon travelling in a constant speed deforming non-holonomic universe. As a result we obtain

$$\frac{dE}{dt} = -\left(\frac{\Omega^2 a^4}{c a_0^3} + \gamma\,\frac{\mathrm{v}}{a_0}\right)E + \gamma\,\frac{\mathrm{v}}{a_0}\left(\frac{\Omega^2 a^4}{c a_0^3} + \frac{\mathrm{v}}{a_0}\right)E\,t\,, \tag{4.2}$$

i.e. a separable first order ordinary differential equation $\dot{y} = -\,ay - byt$, which solves by separation of variables (moving the y terms to one side and the t terms to the other side). Thus, we transform this equation into $\frac{dy}{y} = -\,(a + bt)\,dt$, then obtain $d\ln y = -\big(at + \frac{1}{2}\,bt^2\big) + \ln C$. Finally, we have the solution $y = y_0\, e^{-\left(at + \frac{1}{2}bt^2\right)}$. As a result, the scalar equation of isotropic geodesics (4.2) solves as

$$E = E_0\, e^{-\frac{H_0 r}{c}\left\{1+\gamma\frac{|\mathrm{v}|}{c}-\gamma\frac{H_0 r|\mathrm{v}|}{2c^2}\left(1+\frac{|\mathrm{v}|}{c}\right)\right\}} \quad \text{in an expanding universe,} \tag{4.3}$$

$$E = E_0\, e^{-\frac{H_0 r}{c}\left\{1-\gamma\frac{|\mathrm{v}|}{c}+\gamma\frac{H_0 r|\mathrm{v}|}{2c^2}\left(1-\frac{|\mathrm{v}|}{c}\right)\right\}} \quad \text{in a compressing universe.} \tag{4.4}$$

Accordingly, the redshift law in a constant speed deforming non-holonomic universe is the sum of the redshift proceeded from the solutions to a photon's scalar equation of motion and the relativistic Doppler redshift. With these solutions (4.3) and (4.4) we obtain the redshift law in a constant speed deforming non-holonomic universe, the the cases of expansion and compression, respectively

$$z = \left(e^{\frac{H_0 r}{c}\left\{1+\gamma\frac{|\mathrm{v}|}{c}-\gamma\frac{H_0 r|\mathrm{v}|}{2c^2}\left(1+\frac{|\mathrm{v}|}{c}\right)\right\}} - 1\right) + \left(\frac{1+\frac{|\mathrm{v}|}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1\right), \tag{4.5}$$

$$z = \left(e^{\frac{H_0 r}{c}\left\{1-\gamma\frac{|\mathrm{v}|}{c}+\gamma\frac{H_0 r|\mathrm{v}|}{2c^2}\left(1-\frac{|\mathrm{v}|}{c}\right)\right\}} - 1\right) + \left(\frac{1-\frac{|\mathrm{v}|}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1\right). \tag{4.6}$$

In the case, where the photon travels at a small distance, while space expands or compresses with a sub-relativistic velocity, the redshift law (4.5, 4.6) takes the linear approximation form

$$z \simeq \frac{H_0 r}{c}\left[1-\gamma\frac{|\mathrm{v}|}{c}\left(\frac{H_0 r}{2c}-1\right)\right] + \frac{|\mathrm{v}|}{c} \quad \text{in an expanding universe,} \tag{4.7}$$

$$z \simeq \frac{H_0 r}{c}\left[1+\gamma\frac{|\mathrm{v}|}{c}\left(\frac{H_0 r}{2c}-1\right)\right] - \frac{|\mathrm{v}|}{c} \quad \text{in a compressing universe.} \tag{4.8}$$

As seen, the formulae (4.7, 4.8) differ from the linear form redshift law in a constant deformation universe (3.8, 3.9) by only the numerical multiplier $\frac{1}{2}$ in the brackets of the second term, which is due to the hon-holonomity of space, while the second term (due to the Doppler-effect) remains the same. This means that the redshift in a universe which expands with a constant linear velocity is less that the redshift in a universe whose space expands so that its deformation remains unchanged.

In a constant speed compressing universe, this difference leads to a blueshift (due to the Doppler-effect, manifested by the second term of the redshift law) which is observed at a distance larger than in a constant deformation compressing universe. Then the first term (due to the non-holonomity of space), which is always positive due to the exponent, increases with the distance, so that it exceeds the second (Doppler-effect blueshift) term and the summary shift in a photon's frequency becomes positive: the photon becomes definitely redshifted in a compressing universe.

This tendency is still valid in the exponential redshift law (4.5, 4.6), which takes an account of the large distances and the ultra-relativistic velocity of the expansion or compression.

If no expansion or compression of space ($\mathrm{v} = 0$), these formulae transform into the redshift law (1.9) and its linear approximation (1.10) we have deduced in a static (non-deforming) non-holonomic universe.

§5. Conclusions. To better view of the results obtained in this paper, they have been collected into a Table shown on Page 26. Actually, this is the redshift law and its linear approximation. These have been theoretically deduced in the framework of a globally non-holonomic universe, where the isotropic space (the "home space" of isotropic trajectories and massless light-like particles, e.g. photons) rotates with the velocity of light and at an angular velocity equal to the Hubble constant. In summary, the following results are emphasized:

1. The empirical Hubble law, including its non-linearity at large distances, is completely explained in a static (non-deforming) universe due to the redshift produced by the global non-holonomity of the isotropic space (a photon being moved in a non-holonomic space loses its proper energy/frequency due to the work produced by it against the field of the space non-holonomity).
2. The non-linearity of the Hubble law, observed at large distances close to the curvature radius of space, is explained due to the exponent in the redshift law deduced for a static universe.
3. The ultimate redshift in a static spherical universe, according to the theory, should be $z = \exp(\pi) - 1 = 22.14$.
4. Deformation (expansion or compression) of space results changes in the redshift law. In a deforming universe, it consists of two terms: the first term is due to the non-holonomity of space, while the second term manifests the relativistic Doppler effect observed on a photon due to the rapid expansion or compression of space.
5. In an expanding universe, according to the redshift law, the near objects must be redshifted (on the average) due to the Doppler-

	The Redshift Law	Its Linear Approximation
Static universe	$z = e^{\frac{H_0\, r}{c}} - 1$	$z \simeq \frac{H_0\, r}{c}$
Constant deformation expansion	$z = \left(e^{\frac{H_0\, r}{c}\left(e^{-\gamma\frac{H_0\, r \lvert \mathrm{v} \rvert}{c^2}} + \gamma\frac{\lvert \mathrm{v} \rvert}{c}\right)} - 1 \right) + \left(\frac{1+\frac{\lvert \mathrm{v} \rvert}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1 \right)$	$z \simeq \frac{H_0\, r}{c}\left[1 - \gamma\frac{\lvert \mathrm{v} \rvert}{c}\left(\frac{H_0\, r}{c} - 1\right)\right] + \frac{\lvert \mathrm{v} \rvert}{c}$
Constant deformation compression	$z = \left(e^{\frac{H_0\, r}{c}\left(e^{\gamma\frac{H_0\, r \lvert \mathrm{v} \rvert}{c^2}} - \gamma\frac{\lvert \mathrm{v} \rvert}{c}\right)} - 1 \right) + \left(\frac{1-\frac{\lvert \mathrm{v} \rvert}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1 \right)$	$z \simeq \frac{H_0\, r}{c}\left[1 + \gamma\frac{\lvert \mathrm{v} \rvert}{c}\left(\frac{H_0\, r}{c} - 1\right)\right] - \frac{\lvert \mathrm{v} \rvert}{c}$
Constant speed expansion	$z = \left(e^{\frac{H_0\, r}{c}\left\{1+\gamma\frac{\lvert \mathrm{v} \rvert}{c} - \gamma\frac{H_0\, r \lvert \mathrm{v} \rvert}{2c^2}\left(1+\frac{\lvert \mathrm{v} \rvert}{c}\right)\right\}} - 1 \right) + \left(\frac{1+\frac{\lvert \mathrm{v} \rvert}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1 \right)$	$z \simeq \frac{H_0\, r}{c}\left[1 - \gamma\frac{\lvert \mathrm{v} \rvert}{c}\left(\frac{H_0\, r}{2c} - 1\right)\right] + \frac{\lvert \mathrm{v} \rvert}{c}$
Constant speed compression	$z = \left(e^{\frac{H_0\, r}{c}\left\{1-\gamma\frac{\lvert \mathrm{v} \rvert}{c} + \gamma\frac{H_0\, r \lvert \mathrm{v} \rvert}{2c^2}\left(1-\frac{\lvert \mathrm{v} \rvert}{c}\right)\right\}} - 1 \right) + \left(\frac{1-\frac{\lvert \mathrm{v} \rvert}{c}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} - 1 \right)$	$z \simeq \frac{H_0\, r}{c}\left[1 + \gamma\frac{\lvert \mathrm{v} \rvert}{c}\left(\frac{H_0\, r}{2c} - 1\right)\right] - \frac{\lvert \mathrm{v} \rvert}{c}$

Table 1: The redshift law and its linear approximation, obtained in the framework of a globally non-holonomic universe, where the isotropic space (the "home space" of isotropic trajectories and massless light-like particles, e.g. photons) rotates with the velocity of light and at an angular velocity equal to the Hubble constant.

effect, while the redshift in the spectra of far galaxies must be much larger than the Doppler redshift, and approaching exponential growth at large distances. We however do not observe any systematic redshift on the stars of our Galaxy and the near galaxies (moreover the Andromeda Galaxy is blueshifted). Edwin Hubble had discovered a systematic redshift on only far galaxies. Meanwhile, a systematic Doppler redshift must be observed on the near objects, if our Universe expands. Therefore, the expanding scenario does not seem to properly characterize our Universe.

6. In a compressing universe, according to the redshift law, the near objects must be blueshifted (on the average) due to the Doppler-effect, while far galaxies must be redshifted due to the always positive exponential (growing up with distance) term in the redshift law. Meanwhile, we do not observe any average blueshift on the objects both within our Galaxy or near it (the blueshift of the Andromeda Galaxy can be explained by the relative motion of it toward our Galaxy). Therefore, the compressing scenario also does not seem to properly characterize our Universe.

Consequently, the empirical Hubble law, which is a result of astronomical observations, is completely explained by the theoretical redshift law we have deduced in a static spherical universe, while the expansion or compression of space would lead to the unbelievable changes of the redshift law, never registered in astronomical observations. Therefore, I conclude that we have enough reasons to mean the space of our Universe static as a whole.

On the other hand, this conclusion does not exclude expansion or compression of local volumes of space. According to Zelmanov's theory of an inhomogeneous anisotropic universe [2, 3], a local volume element of a universe can evolve in another way than the universe as a whole. Thus, the local redshift or blueshift anomalies, which differ from the redshift law (1.9, 1.10) we have deduced for a static universe, manifest the fact that the space of our universe, static as a whole, is evolving (expanding or compressing) in its local volume elements.

For instance, supernova explosions lead to the rapid expansion of the surrounding (local) volume of space. An observer near a supernova should register the redshift effect according to the expansion. In contrast, the process of collapse leads to compression of the local space surrounding a collapsing object. Therefore, an observer near a collapsing object should register the blueshift effect which manifests the fact that the surrounding space compresses. Thus, collapsing bodies in the

Universe can be indicated by not only accretion of the near matter onto such a body, but also by the blueshift in the compressing local space of it. Note that, according to the redshift law we have deduced, the blueshift effect of a compressing space is valid at only small distances where the redshift due to the global non-holonomity of the Universe is small. Therefore, in searching for a blueshift effect in a compressing volume (actually, in look for the collapsing bodies in the Universe), we should limit the area of our search by the distance to the Andromeda Galaxy or by a distance which is not much larger.

In this row, bizarre should seem the result of observation produced near a Cepheid, because its local space experiences periodical expansions and compressions, i.e. oscillates, with a short period equal to the period of pulsation of the star itself (days).

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Preface of 2009 to "The Velocity of Light in Uniformly Moving Frame"

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Abstract: This preface gives a brief historical background to my 1958 Stanford Ph.D. thesis, *The Velocity of Light in Uniformly Moving Frames*. As a graduate student at the University of Chicago in the early 1950's. I thought that by modifying the Lorentz Transformation (L.T.) so as to keep a particle's momentum finite at the speed of light, one could solve the divergence problem of QED, and allow for faster-than-light motion. However, after criticisms by eminent physicists, before and after resuming graduate studies at Stanford, this approach was finally abandoned in favor of a truncated version of the L.T. called the Absolute Lorentz Transformation (A.L.T.) that is consistent with Einstein's principle of general covariance, the metric postulate, and experiment.

At this writing, a little over fifty years have elapsed since I began in the autumn of 1957 the line of investigations into special relativity that are presented here in my Stanford thesis, which was completed and submitted to the University in September 1958. Actually, my studies in special relativity had begun about eight years earlier when I was a graduate student at the University of Chicago. My investigations then were along somewhat different lines, and were directed at obtaining a cut-off to cure the logarithmic divergences that occur in quantum electrodynamics, as well to see whether this would also enable particles to travel faster than light. However these investigations led to very complicated expressions mathematically, and after numerous negative comments from distinguished physicists, I put them aside after I had later gone to Stanford to resume my graduate studies in the fall of 1955, following the offer of a graduate fellowship by the physics department chairman, Leonard Schiff, whose textbook on quantum mechanics is well-known [1]. Although I was unaware of it at the time, Schiff had a deep interest in relativity, particularly in the testing of general relativity. I should mention that I left Chicago in the fall of 1952 and came to San Diego for family reasons, where I eventually worked in the space industry for about two and a half years before resuming my graduate studies in physics at Stanford. In the fall of 1956 Sidney Drell, whose work on quantum electrodynamics is well-known, became my thesis adviser, and although while he welcomed the idea of a cut-off, he didn't

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agree with the ideas of my Chicago approach either, hence, as indicated above, I put them aside. and worked on other ideas, such as the then newly-recognized parity and charge conjugation violations in the weak interactions. The following summer I carried out some calculations about photoproduction of neutral pions [2] at the old Mark III linear electron accelerator that was the ancestor of the present two-mile Stanford linear accelerator, commonly known as SLAC, of which Drell became the Vice-Director, but from which he has since retired.

In the fall of 1957, after the pion photo-production calculations were finished, I decided to tackle the Lorentz transformation again, but this time, instead of trying to modify it, I was interested in improving my understanding of the transformation and special relativity more generally, while at the same time retaining the idea I had developed when I was studying at the University of Chicago about there being an ether. This idea had come about as follows. While there I attended many lectures by Enrico Fermi, and in particular, during the winter and spring quarters of 1949, I attended his course on nuclear physics [3]. In these lectures I learnt for the first time about Dirac's idea of space being filled with a sea of negative energy states [4]. Although before coming to Chicago, I had had a course on quantum mechanics as an undergraduate at Harvard, given by Julian Schwinger, it was a non-relativistic course, and I graduated before I could take the course in relativistic quantum mechanics where the Dirac sea and the so-called hole theory of positrons would be discussed. I might add, parenthetically, that it was in a colloquium given late in 1947 by Schwinger concerning his then recent work on quantum electrodynamics that I learnt of the logarithmic divergence of the first-order correction to the mass of the electron. At any rate, Dirac's idea of a sea of negative energy states struck me as supportive of there being an ether, in the sense that space was not empty i.e., it was not a "void". This latter description of space was the view that had emerged from Einstein's famous 1905 work, and that of course was in conflict with the views of Larmor, Lorentz, Poincaré, and indeed nearly all the physicists of Einstein's time. Also, while at Chicago, I learnt of the experiment of Michelson and Gale, that is an optical analogue of the Foucault pendulum experiment, and also the experiment of Sagnac, both of which seemed more easy to interpret in terms of an ether relative to which the Earth was rotating in the first case, or relative to which the Sagnac interferometer was rotating in the second case. This is briefly discussed in the thesis.

To those who have studied only special relativity, my attempt to retain the ether might seem as though I had taken a step backward;

however, while working in San Diego, but continuing to research relativity in my spare time, I found surprising support from the later work of Einstein on the basis of his general theory of relativity. I should note here that I had commenced the study of general relativity on my own while at Harvard using the text by Peter Bergmann [5], and also that by Arthur Eddington [6], while I also attended lectures on tensor analysis by Léon Brillouin [7]. In 1953, I came across a translation of Einstein's [8] inaugural address in Leiden in 1920 where, following the wishes of Lorentz and Ehrenfest, he had been invited to serve as an annual visiting professor, while retaining his primary position in Berlin. In the address he says,

> "From the point of view of the special theory of relativity, the ether hypothesis has certainly been an empty one at first sight... On the other hand, there is an important argument in favour of the ether. To deny the existence of the ether means, in the last analysis, denying all physical properties of empty space. But such a view is inconsistent with the fundamental facts of mechanics".

It is unfortunate that many texts that are used to teach special relativity never reference this important address by Einstein, although a notable exception is the text by Pauli [9], which however is rarely used, so that many students are left only with the view expressed in Einstein's earlier work of 1905 that contributes to the widespread view that space in the absence of bodies and fields is a void. Interestingly, in 1953, as referenced in the thesis, Dirac wrote in support of an ether. For a fairly recent statement in support of an ether, see the article by the particle physicist, Frank Wilczek [10], entitled, *The Persistence of the Ether*.

However, if one does have the ether in the background of one's thinking about the propagation of light through space as a wave, together with the invariance of its speed in each of two uniformly moving frames, one seems to be entertaining contradictory pictures. (Unless the ether is dragged along completely, but such a view had been shown to be untenable.) To be sure, mathematically, it is easy to understand how this invariance arises from the term that depends on space in the Lorentz transformation for the time, and that gives rise to the relativity of simultaneity. The problem is not one of mathematics, but rather one of intuition.

Thus, upon returning to the Lorentz transformation in 1957, I examined how clocks were to be thought of as synchronized according to the Galilean transformation. It seemed to me, then, that the clocks had been synchronized by instantaneous signals, so that if two events were simultaneous in the ether, they were also simultaneous in the frame moving uniformly with respect to it. In other words, when such hypothetical signals were used, simultaneity assumed an invariant, or as I described it in the thesis, an "absolute" character. Although I was unaware of it at that time, Lorentz himself [11] had argued for an absolute or "true" simultaneity. He said, after he had become aware of Einstein's approach to time and simultaneity, that he still favoured the "true" time and the "true" simultaneity and he went on to say,

> ". . . together with this goes that we can imagine velocities of any desired magnitude, e.g., a hundred times greater than the velocity of light (quite apart from whether they actually occur), while according to the relativity principle such velocities are ruled out".

He then describes how such signals, in the limit of infinite velocity, would enable us to know what happened, e.g., on the star Sirius simultaneously with what happened on earth.

Thus the thought underlying the transformation I finally constructed was in accord with Lorentz's ideas on simultaneity. Quite interestingly, the transformation also represented the working out of a question put forward by Poincaré in his 1904 address at the St. Louis Exposition, which, regretfully, I had not read when I wrote my thesis, since if I had been able to refer to it, that would have given further support for the path I was following. In that 1904 address, Poincaré asked [12]:

> "What would happen if one could communicate by non-luminous signals whose velocity of propagation differed from that of light? If, after having adjusted the watches by the optical procedure, one wished to verify the adjustments by the aid of these new signals then would appear divergences which would render evident the common translation of the two stations.And are such signals inconceivable, if we admit with Laplace that universal gravitation is transmitted a million times more rapidly than light?"

In the thesis I assumed that there were signals that propagated instantaneously in the ether, and that these signals could be used to synchronize clocks in the moving frame, and thereby reveal the motion of the frame relative to the ether. However, as indicated in the thesis, I made no assumptions about the physical nature of such signals, so that in this sense, it was a mathematical exercise. On the other hand, unlike the possibility mentioned by Poincaré, I did not regard such hypothetical instantaneous signals as having anything to do with gravity, because I was working in the framework of general relativity in which the gravitational interaction is propagated with the speed of light. I might add that I found out later that Lorentz, in a little known paper [13] in 1900 had showed that the gravitational interaction could take place at the speed of light, in contrast with Laplace's model, without incurring the empirically unobserved effects on the motion of the moon that Laplace had found, essentially because the effects of a finite velocity of interaction could be of second order in $\frac{v}{V}$ rather than of first order, where V is the speed of gravity.

The transformation that I eventually came up with, which is given in Eq. (1.12) of the thesis, I called the Absolute Lorentz Transformation (abbreviated subsequently as A.L.T.), since it had some of the basic properties of the Lorentz transformation: such as the Lorentz contraction and the time dilation, while in keeping with Lorentz's viewpoint that I had inferred from his writings, it kept simultaneity invariant, and consequently it did not keep the one-way velocity of light invariant, although, to be sure, the out-and-back velocity, as with the full Lorentz transformation, remained invariant. If the instantaneous signals were to propagate causally (i.e., not propagate backwards in time), there could be only one frame in which light propagated isotropically with speed c when clocks were synchronized with these signals. This is because, as discussed by Einstein [14], a signal that propagated forward in time and faster-than-light in one Lorentz frame can readily be shown to propagate backwards in time in another Lorentz frame. The idea then was that the above privileged frame would be the ether, and all other frames could be ordered according to their speed relative to this frame, and in this sense, one would return to Newton's idea of there being an absolute velocity of a moving body, a view which Lorentz clearly favoured. On the other hand, in keeping with the principle of relativity, it had to be the case that when measurements were made in the standard way, i.e., with clocks that were synchronized according to the procedure described by Einstein in his 1905 paper, or by slowly moving them apart after they had been synchronized when they were together, the A.L.T. would yield experimental results in agreement with the Lorentz transformation, and would therefore make it impossible to determine one's velocity relative to the ether. Thus the failure to detect this velocity in the numerous efforts that had been made with this purpose would not be a consequence of the fact that such a velocity did not exist, but rather because when measurements were made in the standard way, this velocity always cancelled out, thereby engendering the principle of relativity. Physically, one might say the ether does not exert a drag on bodies, nor bodies on the ether, as maintained by Lorentz, who rejected the interpretation of the Fresnel drag coefficient as an actual dragging of the ether.

In the closing chapter of the thesis, I addressed the issue as to what might give rise to the hypothetical instantaneous signals, and discussed the possibility of faster-than-light particles that are now known as "tachyons". I discussed some of their properties, which I had already investigated to some extent seven years earlier when I was at the University of Chicago. Interesting work on this subject in this time period is due to several authors [15–18], and particularly, G. Feinberg [19], to whom we are indebted for the name, "tachyon", as well as a detailed quantum field theoretical analysis. My own contribution to the quantum field formulation, as indicated in the thesis, was to introduce the idea that the tachyons would be created in the faster-than-light region, thereby avoiding the infinite barrier at the speed of light. I also noted that if they were charged, they would exhibit Cherenkov-like radiation, something I had also been able to show in the Chicago period. For additional references to tachyons see the historical review up to 1969 by Fröman [20]. A later further review on the subject by E. Recami [21] appeared in 1986, in which there is a brief reference to my thesis, in fact, the first reference to it in the literature to my knowledge.

After spending a year at the Niels Bohr Institute of Theoretical Physics (in those days it was known as "Universitetets Institut for Teoretisk Fysik", or "University Institute for Theoretical Physics") in Copenhagen in 1958–1959 as a National Science Foundation post-doctoral fellow, where I had interesting discussions with Christian Møller [22], whose work on relativity had led me to Copenhagen, and then subsequently, on an extension of the N.S.F. fellowship, I spent the following year at the Scuola del Perfezionamento in Fisica Téorica e Nucleare in Naples, of which Eduardo Caianiello was the Director. While there, I began to consider a physically-realizable way of re-interpreting the A.L.T., in view of the obvious experimental absence of faster-than-light signals. The idea, which was briefly mentioned in my introduction to general relativity that I wrote when I was in Naples [23], while giving some informal lectures there in the spring of 1960, is that one can think of the clocks in the moving frame as having been synchronized externally with the clocks in the rest frame, with the latter being any arbitrary Lorentz frame, i.e., an inertial frame in which the clocks have been synchronized so that the one-way speed of light is c in all directions. The following is helpful in visualizing how this external synchronization may be made with existing apparatus, and how the transformation may then be interpreted and experimentally verified.

Imagine, as is customary in pedagogical presentations dealing with special relativity, a railroad station that is taken to be an inertial frame, after suitable corrections, and a line of clocks stretching along the station parallel to the tracks. These clocks have all been synchronized in accordance with special relativity, so that the one-way speed of light is c in the forward and rearward directions, the only direction with which we shall be dealing for simplicity. Now let there be a train travelling through the station with velocity v in the positive x-direction, and on the train assume there is a row of clocks along the length of the train similar to those in the station, but which have not been synchronized. Finally, imagine an electro-mechanical system that enables clocks on the station to transfer their time to the clocks on the train. Assume adjustments have been made so that when the clocks on the station all read zero, the connection is made with the clocks on the train, just once, and this is done so rapidly that the time of exchange can be neglected. After the connection has been made and terminated, the clocks on the train run freely at their own rate. If t denotes the time read by the clocks on the station, and t' the time read by the clocks on the train, then when $t = 0$, $t' = 0$, and hence, under the assumption of a linear relationship, t' is directly proportional to t, i.e., $t' \propto t$, The constant of proportionality follows from special relativity. For example, we know from numerous experiments that have been carried out with relativistic decaying particles, such as the muon, that their lifetimes increase as seen in the lab as they approach the speed of light, and in fact this increase agrees with that predicted by special relativity, so that if T_0 is their lifetime when they are at rest, then when they are travelling with speed v relative to the lab, their lifetime becomes γT_0, where as usual, $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$. This then determines the transformation for the time between the station and the train to be $t' = \gamma^{-1} t$, as given in the thesis. This simple transformation for the time yields the result that if two separated clocks on the station describe an event as simultaneous, $\Delta t = 0$, then clocks on the train will also agree that the events were simultaneous, since $\Delta t' = 0$. This is of course unlike the case for the Lorentz transformation, since $\Delta t_L = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$, and if $\Delta t = 0$, one has that $\Delta t_L = -\gamma \frac{v}{c^2}\Delta x$, which expresses the relativity of simultaneity. As noted in the thesis, the A.L.T. transformation for the time is needed in addition to the Lorentz contraction of the spatial coordinate to guarantee a null-effect in the unequal arm interferometer experiment of Kennedy and Thorndike.

Now, as shown in the thesis, this transformation for the time has the important property that when the clocks on the train that have been synchronized externally are slowly-moved apart, and the one-way speed of light is measured with them, it turns out to be c. Indeed, one finds that the slowly-moved clocks no longer read t' but $t' - \frac{v}{c^2}x'$, which is just the time read by the clocks whose time is described by the Lorentz transformation. Hence, the slowly-moved clocks yield that the one-way speed of light is c.

In keeping with Niels Bohr's idea to look for examples of complementarity outside of the domain of atomic physics [24], it is helpful to recognize that there is a complementarity between the one-way speed of light and simultaneity, which I did not recognize in the thesis, and hence regretably did not discuss with Bohr when I was in Copenhagen. Thus, one can keep the one-way speed of light invariant in transforming between two uniformly moving frames, but then one must relinquish the invariance of simultaneity and let it become relative, as described by the Lorentz transformation, or, one can keep simultaneity invariant, and let the one-way speed of light become relative, as described by the A.L.T. Furthermore, this is fully in keeping with Einstein's principle of general covariance, which enables one to represent the comparison of the two descriptions in mathematical form.

There are two obvious objections to external synchronization: a) the standard special relativistic approach is based on synchronization *within* a given uniformly moving frame; and b) even if external synchronization is allowed, there is no natural frame in space (i.e. no cosmological railroad station) with respect to which such a synchronization could be made.

The reply to a) is that there is no way to prove *empirically* that there is a relativity of simultaneity between two frames in relative motion, unless one is able to compare the measurements of simultaneity in the two frames. Therefore one frame must make a necessarily external contact with the other frame in order that there can be an exchange of information between the two frames. Such an external contact can obviously be also used to make an external synchronization, and hence can be used to keep simultaneity invariant between the two frames.

With respect to b), the view that there is no natural reference frame in space has to be reconsidered because of the discovery of the Cosmic Microwave Background Radiation (CMBR) by Arno Penzias and Robert Wilson in 1965 [25], who were apparently unaware of the earlier theoretical work of George Gamow, who had predicted the existence of such radiation on the basis of his Big Bang model of cosmology, albeit at a different temperature [26]. For an excellent historical review, together with their own important contributions, see the book by his pupils, Ralph Alpher and Robert Herman [27], *Genesis of the Big Bang.*

In later experiments it was found that the radiation is not uniform in all directions, but is warmer in one direction and colder in the opposite direction, so that it exhibits a dipolar structure [27]. This is to be expected if the CMBR defines a rest frame through which our solar system, and hence the Earth is moving. (Note that in view of the magnitude of this velocity of several hundred kilometers/second, the Earth's velocity around the Sun of $\sim$30 km/sec is, to a first approximation, negligible.) Because of the Doppler effect, the radiation temperature is the highest in the direction in which the Earth is moving, and exhibits a typical cosine dependence. The detection of this velocity through the radiation has led Peebles [28] to describe the result as the "new ether drift". To be sure, the measurement is not that of a true ether drift, because if one places the antennae in an electromagnetically sealed laboratory, so that the CMBR cannot penetrate, obviously one will not be able to measure the frame's velocity relative to the radiation, whereas the idea underlying the determination of a true ether drift is that one can make such a measurement in a closed laboratory. Nevertheless, ignoring very small temperature fluctuations and hence anisotropy in the CMBR of order 10^{-5} [29], and assuming the radiation is at rest with respect to the expanding space of the Friedmann, Robertson, Walker, Lemaître cosmological model, then one can synchronize one's clocks in a moving frame with respect to this CMBR frame, and all clocks so synchronized in these frames in uniform motion relative to the CMBR will keep simultaneity invariant with respect to each other, at the expense of not keeping the one-way speed of light invariant.

Since my work is sometimes compared in the literature with that of Herbert Ives, the following comments are in order. When I was at the University of Chicago, I wrote to Ives describing some of my ideas about modifying the Lorentz transformation that would support the idea of an ether, as I was aware from some of his publications that he also strongly supported the idea of an ether. He wrote me back that he preferred his own approach to mine, and kindly sent me a copy of his paper, *The Fitzgerald Contraction*, referenced in the thesis. However, in this work, he objected to the idea, supported by special relativity, of making measurements of the one-way speed of light with clocks that have been synchronized when they are together, and then moved infinitely slowly apart. He apparently had raised this objection earlier to Percy W. Bridgman [30], who pointed out that one can achieve such a measurement by the method of successive approximation and then passing to the limit. I was unaware of Ives' conversation with Bridgman (since Bridgman's book came out after I had written my thesis as well as my Nuovo Cimento article), nor had I noticed in Ives' Fitzgerald article that he rejected synchronization with slowly-moved clocks. However, after I had formulated my transformation, I became concerned as to whether if one had two clocks that had been synchronized together in the moving frame at point A and one of them was slowly moved to point B, and they were used to measure the one-way speed of light, whether one would obtain c as predicted by special relativity. As the work in Chapter 3 of the thesis shows, this is in fact the case, and in Chapter 4, one can use this result in conjunction with other assumptions to derive the A.L.T. A later derivation was published by me in 1994 [31]. Thus, on this issue of slowly-moved clocks, there is a profound disagreement between the approach in my thesis and the work of Ives. Regretably, Ives died in 1953, and consequently I was never able to get comments from him about the A.L.T., but in view of his objections to Bridgman concerning slowly-moved clocks, it is unlikely he would have changed his position. I might add that I sent a revised copy of my thesis, that I had prepared while in Copenhagen, to Bridgman around June of 1959. But he never replied, and tragically, because of a debilitating case of cancer, he ended his life in 1961, although fortunately he was able to complete his *A Sophisticate's Primer of Relativity*, which was edited posthumously with a prologue and epilogue by Adolf Grünbaum, whose earlier article on synchronization is referenced in the thesis. There is a second edition of Bridgman's book [30], edited by Arthur Miller that contains useful information about the chronology of Bridgman's work on his book. As noted in the thesis, Bridgman's operational methodology played a role in the formulation of the A.L.T. I might add that I took his course on advanced thermodynamics during the fall semester of 1947 when I was at Harvard, although relativity was not discussed.

At this point it is appropriate to turn to the idea of alternative synchronization in a given Lorentz frame that was proposed by Hans Reichenbach [32], and which is referred to in the literature as the "conventionality of synchronization". Although I did not mention Reichenbach in the thesis, I became aware of his ideas through the above-mentioned article by Grünbaum cited in the thesis. Reichenbach showed that it is entirely consistent with the special theory of relativity to synchronize clocks so that the one-way velocity of light is not c, by following a different synchronization procedure than that of Einstein. Let two clocks in a uniformly moving frame be an arbitrary distance apart, one at A the other at B. Then if a light signal leaves the clock at A at time t_1 and strikes a mirror at B at time t_2, and then returns to A at time t_3, according to Einstein's synchronization procedure, $t_2 = t_1 + \frac{1}{2}(t_3 - t_1)$, in accordance with the assumption that the speed of light is the same in both directions. However, Reichenbach argued that no contradiction with the other postulates of special relativity arises if instead of the above synchronization, one sets $t_2 = t_1 + \varepsilon(t_3 - t_1)$, with the following restriction that $0 < \varepsilon < 1$, under the assumption that the one-way speed of light is finite in both directions. Since this synchronization procedure of Reichenbach does not describe a coordinate transformation, I did not attempt to deal with it in the thesis beyond what is briefly stated there. However, in recent years, Anderson et al. [33] have given a coordinate representation for Reichenbach synchronization, by means of a linear local time transformation. Thus, let us suppose that after clocks, whose time will be denoted by t_L, have been synchronized either by Einstein's method, or by slowly-moving them, one introduces another set of clocks in the same frame whose time varies along the x-axis according to the linear relation, $t_R = t_L + \frac{k}{c} x_L$, and whose spatial coordinates are the same as the Lorentz observer, i.e., $x_R = x_L$, $y_R = y_L$, $z_R = z_L$ with $-1 < k < 1$. Transformations of this type were called by Lorentz, "local time transformations". Such a linear transformation for the time that also involves the spatial coordinate seems to have been first used in conjunction with the Doppler effect by Voigt [34]. Now let us think of the Reichenbach synchronization having been made on the train travelling uniformly through the above-mentioned railroad station, and compare it with the time read by the clocks that have undergone external synchronization using the A.L.T. Then, from the thesis, one has $t' = t_L + \frac{v}{c^2} x_L$, $x' = x_L$, $y' = y_L$, $z' = z_L$. In other words, the clocks under the A.L.T. are related to the clocks internally synchronized in the moving frame by a local time transformation. Hence, one can always think of a Reichenbach synchronization (as represented by a linear local time transformation following Anderson et al.) as equivalent to an external synchronization for suitable choice of k. It is readily shown that, for the externally synchronized clocks, one has for a signal sent in the direction of the train's motion, i.e., in the positive x-direction, that $t'_2 = t'_1 + \frac{1}{2}\left(1 + \frac{v}{c}\right)(t'_3 - t'_1)$, and that in the reverse direction, one has $t'_2 = t'_1 + \frac{1}{2}\left(1 - \frac{v}{c}\right)(t'_3 - t'_1)$. Hence in the forward direction $\varepsilon = \frac{1}{2}\left(1 + \frac{v}{c}\right)$, and in the rearward direction, $\varepsilon = \frac{1}{2}\left(1 - \frac{v}{c}\right)$, and since $\frac{v}{c} < 1$, Reichenbach's restriction on ε follows. Thus the thesis anticipates to some extent the interesting analysis of Anderson et al. Although it also important to carefully distinguish between external synchronization involving two Lorentz frames, and alternative synchronization within a given Lorentz frame. The effect of external synchronization is to give rise to clocks that exhibit alternative synchronization to the clocks that obey the time transformation of the Lorentz transformation. For a later discussion of Reichenbach's work see Grünbaum [35]. It should also be remarked that from the standpoint of general relativity, and general covariance, it is entirely obvious that one can introduce a Reichenbach synchronization as represented by a linear local time transformation in a given Lorentz frame. However, as discussed in the thesis, the coefficients of the Minkowski line element, i.e. the $g_{\mu\nu}$, no longer remain the same as for a Lorentz transformation. Nevertheless, since they are still constants, it follows that all the Christoffel symbols vanish, and in the absence of external forces one continues to have $\frac{d^2x^\mu}{d\tau^2} = 0$, as for the Minkowski metric, corresponding to the fact that Newton's first law holds, so that such a transformation is inertial.

It is of historical interest that the time dilation that is used in the A.L.T. and that follows from special relativity, as I later found out, was actually first introduced by Larmor [36], who found that he needed it in order to keep the d'Alembertian wave equation invariant. Thus, as some historians of science have observed, one might very well speak of the Lorentz-Larmor transformation. On the other hand, it should be emphasized that Larmor, unlike Lorentz, believed that the speed of gravity if finite at all, vastly exceeded that of light, and hence he did not attribute to the transformation the fundamental significance that Einstein did later on, with his emphasis on the speed of light playing the role of a limiting speed. Also, it should be noted that since the appearance of the 1961 article in which the A.L.T. was first published, there have been published many interesting papers, too numerous to reference here, re-deriving, developing, and applying the transformation.

In 2008 I found [37] further support for the invariance of quantum mechanics under local time transformations, as described in Chapter 11 of the thesis. It was shown that when the standard space and momentum commutation relations are enlarged to a spacetime formulation, they remain invariant under arbitrary linear, non-singular spacetime transformations; while also maintaining the vanishing of the commutator of the time with the Hamiltonian operator, so that time can continue to be treated as a c-number in accordance with quantum mechanics.

Finally, while the thesis was being prepared for publication, I was informed by Dr. Gregory B. Malykin [38] that a literature search had found that Albert Eagle, then a lecturer in mathematics at the University of Manchester, UK, had given the A.L.T. including its inverse in a paper published in 1938, hence twenty years before my thesis. Eagle expressed the view that the transformation could be understood as relating the moving frame to the ether frame with clocks that had been synchronized with instantaneous signals [39], which is entirely similar to the viewpoint expressed in the thesis. (Although, as noted above, I now believe the alternate interpretation of the A.L.T. as associated with external synchronization is physically the more reasonable one.) However, Eagle was under the impression that such instantaneous communication could be accomplished by a rotating spindle, i.e, a mechanical device, and hence that one could use such a device to determine the speed of the frame through the ether. The idea that one could send instantaneous signals with a mechanical system is of course entirely false. See Jammer's recent book on simultaneity [40]. This misunderstanding, together with the fact that the quadratic form describing light propagation is not left invariant under the transformation, apparently led Eagle to a mistaken criticism of Minkowski's spacetime formulation of special relativity, and also to a rejection of the general theory of relativity as well, so that he regretably failed to recognize the A.L.T. is in full conformity with Einstein's principle of general covariance, and the metric postulate, and that the A.L.T. actually constitutes an interesting application of general relativity for the simple case of a linear transformation in flat spacetime, which is, in contrast, the approach taken in my thesis.

I am grateful to Dr. Gregory B. Malykin for valuable scholarly correspondence concerning my thesis, as well as many stimulating questions. I am also indebted to his son, Asst.-Prof. Edward G. Malykin, for his further assistance. I am further indebted to Dr. Dmitri Rabounski for encouraging the publication of the thesis in *The Abraham Zelmanov Journal*, and his considerable assistance in helping to accomplishing this.

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The Velocity of Light in Uniformly Moving Frame

A Dissertation. Stanford University, 1958

Frank Robert Tangherlini

Abstract: This is a dissertation submitted to the Department of Physics and the Committee on Graduate Study of Stanford University in partial fulfillment of the requirements for the degree of Doctor of Philosophy, September 1958. Approved by Sidney D. Drell, thesis advisor, Leonard I. Schiff, Department Chair, and Albert H. Bowker, Dean of the Graduate Division. Note: an abstract was prepared separately and published by University Microfilms, Ann Arbor, Michigan 1959-01456, USA. Briefly, the thesis shows classically and quantum mechanically that a linear transformation can be used in special relativity that keeps simultaneity and the out-and-back speed of light invariant, while the one-way velocity of light varies, depending on the frame's velocity relative to an inertial rest frame.

I certify that I have read this thesis and that in iny opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Sidney D. Drell

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Leonard I. Schiff

Approved for the University Committee on Graduate Study.

Albert H. Bowker, Dean of the Graduate Division

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Contents

Introduction

According to the ideas of special relativity the transformation connecting two uniformly moving frames must be such as to leave the metric tensor an invariant, and given by the diagonal tensor $\eta_{\mu\nu} = (1, -1, -1, -1)$, thus preserving the same value for the speed of light in all uniformly moving frames. On the other hand, from an intuitive standpoint, such a result is quite paradoxical, since one would expect that for two frames in relative motion with speed v, the velocity of light ought to differ in the two frames by quantities of the first order in $\frac{v}{c}$. The absence of such an effect cannot be explained by Lorentz contraction of rods and the slowing down of clocks, since these are second order effects. The answer to the paradox, of course, lies in the fact that in special relativity, one deals with clocks which have been synchronized in a certain manner involving quantities of the first order in $\frac{v}{c}$ so that a cancellation with the above expected effect can occur. Thus the original objection is removed — provided one agrees that this method of synchronization does not contain any assumptions about the propagation of light which involve a *petitio principii*.

It is the purpose of this paper to examine this method of synchronization from the broader mathematical viewpoint of general relativity which, based as it is on general covariance, enables one to envisage more general transformations connecting uniformly moving frames. Indeed we shall consider transformations in which clocks are synchronized with "absolute signals", that is, signals travelling with infinite or arbitrarily large velocity. In our discussion we have not enquired into the dynamics of such signals. For the purpose here, such signals serve merely as a kinematic method for formulating in the framework of general covariance certain types of experiments which are unthinkable in the more restricted framework of special relativity. In the concluding chapter some of the possibilities and difficulties associated with such signals are briefly examined.

Using these signals, one arrives at the view of an absolute rest frame (or ether frame) in which the velocity of light is the same in all directions; but for observers in motion relative to this frame with speed v, the velocity of light is not the same in all directions and differs in different directions to first order by amounts of $\frac{v}{c}$, in agreement with one's intuitive ideas. With absolute signals, it is possible to measure this speed v, and hence to linearly order all frames according to the magnitude of this quantity. On the other hand, measurements made with light signals do not make it possible to measure v. In the Appendix the present status of the absolute frame and Mach's principle in general relativity is reviewed in connection with effects observed in rotating frames.

Some of the basic physical ideas underlying the discussion here are contained in the work of H. E. Ives [1], wherein the view is expressed that the "out" and "back" velocities of light are in general different in uniformly moving frames, and the Lorentz transformation is recast to take this difference into account. The approach in this paper makes it possible to circumvent the unnecessarily cumbersome algebra of his formulation. Recently, in a comprehensive review of the foundations of special relativity Grünbaum [2] has criticized the viewpoint of Ives as being logically inadequate. However, since Grünbaum also observes that other synchronization procedures than the usual one are logically possible, and since an alternative synchronization procedure in general leads to an asymmetry between out and back velocities, the approach given here is mathematically equally valid from either the standpoint of Ives or Grünbaum. Some valuable general remarks on the problems of the one-way velocity of light are to be found in Bridgman [3].

As was already remarked, the mathematical technique that is employed is based on general covariance which permits one to write equations independently of the coordinate system, in contrast with special relativity, where one is restricted to coordinate systems connected by Lorentz transformations. However, while covariance makes it possible to formulate equations independently of the coordinate system, the results obtained by measurement would of course depend on these coordinates if they had direct physical significance in terms of measuring rods and clocks. For example, if one could construct rods and clocks that did not exhibit the Lorentz contraction and time dilatation, one could use these (non-physical) rods and clocks to define a Galilean coordinate system, or set of coordinate systems, in which the velocity of light would not be independent of the motion of the frame. The mathematical framework of general relativity is broad enough to handle measurements made in these arbitrary coordinate systems.

As was pointed out originally by Kretschmann [4], (see also Bridgman [5], Fock [6]) there is therefore a difference between the notion of "relativity" as it is employed in general relativity where it means, from the standpoint of general covariance, a removal of restriction on coordinate systems, and the notion as it is employed in special relativity where it entails a restriction on coordinate systems. As a consequence, from the standpoint of general covariance alone, there is no necessity for two uniformly moving frames to be connected by a Lorentz transformation.

However, the "relativity" of general relativity is to be found really in another equally important assumption [7], namely: in a sufficiently small region of a frame, the propagation of light as measured by (rigid) rods and clocks is such that it is locally describable by the line element of special relativity and more generally, the laws of special relativity hold locally — to a first approximation when such a line element cannot be introduced in the large. In the case of uniformly moving frames, where it is possible to introduce the special relativity line element in the large, it is this assumption which then leads to the Lorentz transformation connecting two such frames. It will be shown in what follows that this latter assumption of general relativity is unnecessarily restrictive on the basis of what is experimentally measured, and can be broadened to permit the use of a line element in which there is an asymmetry in the velocity of propagation of light.

Chapter 1. The Absolute Lorentz Transformation

In the absence of gravitational sources, the field equations of general relativity reduce simply to

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0\,, \qquad \mu,\nu = 0,1,2,3. \tag{1.1}$$

The solutions to (1.1) with $g_{\mu\nu}$ constant are called "Cartesian frames". It is in such frames that we shall work. Since $g_{\mu\nu}$ is by definition a symmetric tensor, the coefficients of the quadratic form $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$, there are ten constants at our disposal, and with no other assumptions, a tenfold infinity of such Cartesian frames. However, because of the symmetry of the $g_{\mu\nu}$ it can always be reduced by real linear transformations to a diagonal matrix with diagonal values given by ± 1, or 0. The case with zero we exclude, since we are interested in working with the full 1+3 dimensionality of the time and space coordinates. By further demanding that the spatial coordinates of the reduced form satisfy the Pythagorean law, the signature of the quadratic form becomes $\pm 1, \pm(1,1,1)$. In order that $ds^2 = 0$, have real solutions corresponding to displacements along the light cone, we finally arrive at the two signatures, $\pm 1, \mp(1,1,1)$, one "time-like", the other, "space-like". For such frames, the determinant of the metric tensor g satisfies the relation $g < 0$. If we adopt the convention that ds^2 should in the limit of small velocities $\frac{dx^i}{dx^0} \approx 0$ (where $i = 1,2,3$) reduce to $(dx^0)^2$, we finally arrive at the canonical time-like metric tensor $\eta_{\mu\nu}$ of special relativity.

However such frames are still too general, for consider a frame which originally had the line element

$$ds^2 = -(d\bar{x}^0)^2 + (d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2 \tag{1.2}$$

by making the transformation, $\bar{x}^0 \to x^1$, $\bar{x}^1 \to x^0$, $\bar{x}^2 \to x^2$, $\bar{x}^3 \to x^3$ it can be brought into the canonical form. Nevertheless the spatial part of the line element originally does not satisfy the Pythagorean law. Frames for which this requirement is not satisfied (to within a spatial coordinate transformation) can be shown in some cases to be moving with velocities greater than that of light. For example, the transformation of [8],

$$\left.\begin{aligned} \bar{x}^1 &= \frac{x^1 - vx^0}{\sqrt{v^2-1}}\,, \qquad & \bar{x}^2 &= x^2 \\ \bar{x}^0 &= \frac{vx^1 - x^0}{\sqrt{v^2-1}}\,, \qquad & \bar{x}^3 &= x^3 \end{aligned}\right\} \tag{1.3}$$

also transforms (1.2) into the canonical form. Since it is not our purpose to consider phenomena in such frames here, it is necessary to restrict the metric tensor $g_{\mu\nu}$ in the following way. Solving for the time Δx^0 for a light signal to propagate through a distance Δx^i one has, setting $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = 0$,

$$\Delta x^0 = -\frac{g_{0i}}{g_{00}}\,\Delta x^i \pm \frac{1}{|g_{00}|}\sqrt{(g_{0i}\,g_{0j} - g_{ij}\,g_{00})\,\Delta x^i \Delta x^j}\ . \tag{1.4}$$

The average out-and-back time for a light signal to propagate is therefore, choosing the positive root in order to make the time delay positive,

$$\frac{1}{2}\left(\Delta x^0_{out} + \Delta x^0_{back}\right) = \frac{1}{|g_{00}|}\sqrt{\gamma_{ij}\Delta x^i \Delta x^j}\ , \tag{1.5}$$

with $\gamma_{ij} \equiv g_{0i}\,g_{0j} - g_{ij}\,g_{00}$. Now unless γ_{ij} is positive definite, there will be directions corresponding to the choice of the Δx^i for which the delay either vanishes or becomes imaginary. Such a situation occurs, for example, in a frame moving faster than light. Moreover, in such a frame, a light signal emitted say from the origin cannot be reflected back to the origin since it cannot overtake the frame. Or again, consider the line element,

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 + (dx^3)^2, \tag{1.6}$$

for which γ_{ij} is not positive definite: light cannot propagate in the cones opening above and below the (x^1, x^2) plane along the x^3 axis. Since, as remarked previously we wish to remain in frames in which light propagates in the customary manner, freely in all directions and with non-zero average delay, and for such frames to provide an alternative description to that given by special relativity, we therefore impose the requirement,

$$\gamma_{ij}: \text{ positive definite}$$

$$\gamma_{11} > 0\,, \qquad \begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} > 0\,, \qquad \begin{vmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{vmatrix} > 0\,. \tag{1.7}$$

This requirement, needless to say, imposes a restriction on allowed coordinate transformations, for example, the transformation, $x^0 \to x^1$, $x^1 \to x^0$, $x^2 \to x^2$, $x^3 \to x^3$, is excluded. In order to arrive at the time-like definition of ds^2, it is necessary to further impose the restriction $g_{00} > 0$; a relation which will then be preserved under all real transformations which leave γ_{ij} positive definite.

By the above assumptions and restrictions we therefore arrive at a multiplicity of frames in which light propagates in the usual manner and such that by a real linear transformation the metric tensor may be brought into the canonical form. Such frames we shall call "Lorentz-reducible" frames. Because these frames are all related to one another by linear transformations, they are easily seen to be in uniform translation (or at rest) with respect to one another. Thus for two such frames

$$dx^\mu = b^\mu_\nu\, dx'^\nu, \tag{1.8}$$

where the b^μ_ν do not depend on the coordinates, then

$$\frac{dx^i}{dx^0} = \frac{b^i_0 + \dfrac{b^i_j\, dx'^j}{dx'^0}}{b^0_0 + \dfrac{b^0_j\, dx'^j}{dx'^0}}\,, \qquad i,j = 1,2,3. \tag{1.9}$$

Hence, if a point in the primed frame is at rest $\frac{dx'^i}{dx'^0} = 0$, its velocity in the unprimed frame is constant and given by

$$\frac{dx^i}{dx^0} = \frac{b^i_0}{b^0_0}\,. \tag{1.10}$$

The above result holds for more general frames than Lorentz-reducible ones (since all that is required is a linear transformation connecting the two frames), so that actually we are dealing with a subset of uniformly translating frames, namely, ones travelling less than the speed of light.

Now it is customary to impose a further restriction on Lorentz-reducible frames, that of special relativity, so that we exclude frames with $g_{\mu\nu}$ not given by $\eta_{\mu\nu}$. For example, we exclude Galilean frames. This exclusion is not demanded by anything in the structure of general covariance, or anything we have done in the above derivation. It is imposed by the hypothesis that all uniformly translating frames are in every sense equivalent, and consequently there should be nothing in the metric tensor which would imply a difference in the propagation of light signals in one frame as distinguished from another.

But if we do not impose this relativity requirement, a variety of other expressions are obtained for the line element, depending upon one's choice of coordinate system. It is our purpose to investigate to what extent some of these alternative line elements are physically permissible, in the sense that they do not violate experimental evidence, taking into account the manner in which the experiments are performed.

Consider a frame with the following expression for the line element,

$$ds^2 = g'_{\mu\nu}\,dx'^{\mu}dx'^{\nu} = dt'^2 - 2v\,dx'dt' - \left(1-v^2\right)dx'^2 - dy'^2 - dz'^2, \tag{1.11}$$

where we introduce units such that $c=1$, also $x'^0 = t'$, $x'^1 = x'$, $x'^2 = y'$, $x'^3 = z'$, and v is a parameter. From the customary standpoint, one would say that this line element represents an improper choice of coordinate system and that one should perform a further coordinate transformation to put the metric tensor in canonical, diagonal form and the observer in a special relativistically admissible coordinate system. But there is more than one way to diagonalize (1.11), each with a different physical significance.

Thus one method to diagonalize (1.11) is to make the coordinate transformation (provided $v<1$),

$$\left.\begin{aligned} x' &= \gamma\,(x - vt)\,, & y' &= y \\ t' &= \frac{1}{\gamma}\,t\,, & z' &= z \end{aligned}\right\} \tag{1.12}$$

with $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$, (1.12) has the inverse

$$\left.\begin{aligned} x &= \frac{1}{\gamma}\,x' + \gamma v t', & y &= y' \\ t &= \gamma t', & z &= z' \end{aligned}\right\}. \tag{1.13}$$

What meaning are we to assign to the transformation (1.12)? We interpret the meaning as follows: 1) the frame with coordinates (t', x',

y', z'), which we shall call S', is in uniform translation with speed v in the x-direction with respect to the frame with coordinates (t, x, y, z), which we shall call S; 2) rods in S' are contracted with respect to those in S by the factor $\frac{1}{\gamma}$, and clocks in S' indicate less elapsed time than those in S, again by the factor $\frac{1}{\gamma}$.

Thus the transformation is in some respects similar to the Lorentz transformation, but clearly the clocks have been synchronized in a different manner, since the time t' in S' only depends on the time t in S, and not on the spatial coordinates.

Moreover, we note that unlike the case with the Lorentz transformation, a measurement of a rod at rest in S, by an observer in S', leads to the conclusion that the rod in S has *expanded* relative to a rod at rest in S', similarly such an observer would say that a clock in S is going *faster* than a clock in S'. One does not have the paradoxical situation of special relativity that both observers say each other's rods have shrunk, or each other's clocks are moving more slowly, rather, one has an absolute relationship. If we regard S as the fundamental frame, then it is the rods in S' which have contracted, so that conversely the rods in S appear expanded with respect to the contracted rods in S', and similarly for clocks. Consider a third frame, S'' in motion with respect to S, and with speed w; clearly, we can state whether S'' is moving faster or slower than S' with respect to S simply by comparing the rates of clocks in S'' and S', since

$$t'' = t' \sqrt{\frac{1-w^2}{1-v^2}}\ . \tag{1.14}$$

In other words, all uniformly moving frames S', S'', etc., may be linearly ordered with respect to S in terms of a parameter v, the speed of the moving frame relative to S, and this ordering is absolute in the sense that observers in the two frames S', S'' by comparing the relative rates of their clocks can assert which is moving faster than the other relative to the frame S — without referring to the frame S — a situation which is not possible in special relativity. Because of this absolute property, we shall refer to (1.12) as the *Absolute Lorentz Transformation* (A.L.T.), and S as the absolute frame.

So far we have not shown that the A.L.T. is actually physically allowable, in the sense that it doesn't violate experimental evidence. In the following chapters we shall show that when measurements are made in the customary manner this is indeed the case.

Let us now observe that instead of diagonalizing the quadratic form by the A.L.T., one might also have chosen to diagonalize it by the transformation,

$$\left.\begin{array}{ll} t_L = t' - vx', & y_L = y' \\ x_L = x', & z_L = z' \end{array}\right\}. \tag{1.15}$$

A point at rest in the primed frame is at rest in the frame S_L and conversely. Thus there is a different physical significance to the two transformations: In the one case (A.L.T), the diagonalized frame is in motion relative to the undiagonalized frame and in the second case, the two frames are at rest relative to one another but there has been a resynchronization of clocks.

Now we note that the transformation above connecting S_L with S', when multiplied by the A.L.T., connecting S' with S, is a Lorentz transformation. Thus since the frame S' may all be ordered with respect to S according to the parameter v, and since to each of these frames S' there is a corresponding Lorentz frame S_L at rest relative to S', it follows the Lorentz frames themselves may be ordered with respect to S. On the other hand, it is clear that unless the observer in S_L has some way of factoring out of the Lorentz transformation the above synchronization of clocks so as to make measurements in S', the ordering with respect to v is lost and one is back to the situation of special relativity.

Chapter 2. Factorization of the Lorentz Transformation

The results of diagonalization obtained above may be stated more elegantly in the following way. Define the three unimodular transformations (we are using "unimodular" in the sense that the determinant is unity),

$$\left.\begin{array}{c} O_1 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad O_2 \equiv \begin{pmatrix} \gamma^{-1} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ O_3 \equiv \begin{pmatrix} 1 & -v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}\right\} \tag{2.1}$$

and the column vectors $X = (t, x, y, z)$, $X' = (t', x', y', z')$, $X_L = (t_L, x_L, y_L, z_L)$, then the A.L.T. may be written

$$X' = O_2 O_1 X \tag{2.2}$$

and the Lorentz transformation

$$X_L = O_3 O_2 O_1 X. \tag{2.3}$$

Thus we have a factorization of the Lorentz transformation into three sub-transformations O_1, O_2, O_3. Such a factorization is meaningless in special relativity (since only transformations which leave the diagonal form invariant are permitted), whereas it is not in general relativity because there is the freedom of considering arbitrary coordinate transformations. Thus, reading from right to left, the transformations say,

O_1X: Make a Galilean transformation from the frame S to a frame moving with velocity v in the x-direction with respect to S;

O_2O_1X: In the new frame, shrink the rods (that are oriented along the x-axis) and slow down the clocks — renormalization of length and time;

$O_3O_2O_1X$: Without changing the state of motion of the frame, resynchronize the clocks.

In addition, because the determinant of each of the transformations is unity, they preserve the four dimensional volume element $dx\,dy\,dz\,dt$ for each of the intermediate steps. Further, since O_1, O_2, O_3 do not commute among one another, the order in which they are performed is significant. For example, if O_2 is performed before O_1, one will pick a frame which does not have velocity v with respect to S, but a velocity $v(1-v^2)$. It is interesting to note that O_1 and O_3 generate subgroups in themselves, since $O_1^2(v) = O_1(2v)$, $O_1^{-1}(v) = O_1(-v)$, $O_1^0(v) = O_1(0) = 1$, the identity, and similarly for O_3, but O_2 does not have this property.

Let us now observe that in the original diagonalization of the line element in S', we might have proceeded by first performing the operation O_2^{-1} which would have brought us without changing the state of motion into the Galilean frame with line element

$$ds^2 = \left(1 - v^2\right) dt_g^2 - 2v\,dt_g\,dx_g - dx_g^2 - dy_g^2 - dz_g^2\,, \tag{2.4}$$

with

$$\left.\begin{aligned} t_g &= \gamma t', \qquad & y_g &= y' \\ x_g &= \frac{1}{\gamma}\,x', \qquad & z_g &= z' \end{aligned}\right\} \tag{2.5}$$

and then proceeded from the Galilean frame S_g to the rest frame S. Note that in the Galilean frame, the velocity of light in the principal directions has the "classical" values,

$$\frac{dx_g}{dt_g} = \pm\,(1 \mp v)\,, \quad \frac{dy_g}{dt_g} = \pm\sqrt{1-v^2}\;, \quad \frac{dz_g}{dt_g} = \pm\sqrt{1-v^2}\;, \tag{2.6}$$

while in the Lorentz frame S_L *at rest* with respect to S_g, the velocity has the values, ± 1. The values in S' we shall discuss in detail in the subsequent chapters.

Chapter 3. Velocity of Light and Synchronization of Clocks under the Absolute Lorentz Transformation

The physical picture presented by A.L.T., then, is that of clocks and rods which have experienced a change in rate and length due to their motion relative to the absolute frame (or ether). (This was the picture used in the era preceding special relativity.) But unlike the situation with the usual Lorentz transformation, we have not further synchronized the clocks in the moving frame by demanding that the velocity of light be the same in all direction as in the absolute frame. Rather, the clocks have been synchronized in the following way: all clocks in both the frame S' and S have been initially synchronized from one clock by a signal travelling with infinite velocity in all directions; upon being synchronized, the clocks keep time at their "natural" rate, the natural rate in the moving frame S' being slower than the natural rate in the rest frame S. This is the physical meaning of the transformation, $t' = \sqrt{1-v^2}\,t$. It is not our purpose here to enquire as to how one might generate such signals, for example, by using the frames previously mentioned which were travelling with $v > 1$. In a later chapter we shall examine the question as to whether such signals violate any fundamental ideas of causality. Consider now, measurements of the velocity of light made by observers in S'. The relative velocity in S' of a point travelling with constant velocity in the x'-direction is given by, upon using the A.L.T.,

$$\frac{dx'}{dt'} = \frac{\dfrac{dx}{dt} - v}{1 - v^2}\,. \tag{3.1}$$

So that since the velocity of light in S is 1, one obtains in the positive and negative directions,

$$\frac{dx'}{dt'} = \frac{1}{1+v}\,,\; -\frac{1}{1-v}\,. \tag{3.2}$$

A result which one obtains directly in the primed frame by setting $ds^2 = g'_{\mu\nu}\,dx'^{\mu}dx'^{\nu} = 0$, and solving for the roots. For the transverse direction one finds

$$\frac{dy'}{dt'} = \pm 1\,, \qquad \frac{dz'}{dt'} = \pm 1\,. \tag{3.3}$$

One might feel that such results violate experience; we shall see that this is not the case because of the way in which measurements are made. Thus consider a measurement of the velocity of light along the x'-axis. One sends a light signal from the origin in S', to a point located at positive distance $\Delta x'$ from the origin and back again. The time required is

$$\left.\begin{aligned} \Delta t'_{out} &= (1+v)\,\Delta x' \\ \Delta t'_{back} &= (1-v)\,\Delta x' \end{aligned}\right\}. \tag{3.4}$$

And hence the average time, which is used in obtaining the velocity of light is

$$\frac{1}{2}\left(\Delta t'_{out} + \Delta t'_{back}\right) = 1\cdot\Delta x', \tag{3.5}$$

so that one obtains the same value as in the unprimed frame, or the Lorentz frame. We see that there is an exact cancellation that comes about due to the fact that the reciprocal of the velocity or "slowness" of the light signal is a linear function of the velocity of the primed frame. For an arbitrary direction, corresponding to displacements, $\Delta x'$, $\Delta y'$, $\Delta z'$, we have, setting $ds^2 = 0$,

$$\Delta t' = v\,\Delta x' + \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}\ . \tag{3.6}$$

On the outward and return paths, $\Delta x'$ changes sign, hence

$$\frac{1}{2}\left(\Delta t'_{out} + \Delta t'_{back}\right) = \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}\ . \tag{3.7}$$

Thus the same average value of the out and back times is obtained in S' as would be obtained by Lorentz observers. We note further that it is the reciprocal of the average slowness which is obtained in a typical out-and-back determination of the "velocity of light".

In sending a light signal to a point $\Delta x'$ from the origin, we see that the delay consists of two parts: $\Delta x'$ and $v\Delta x'$, that is, the delay one uses in special relativity and an additional part associated with the fact that one has synchronized the clocks with absolute signals. Since the extra delay is constant for a given frame, depending only on the location of the clock and the speed v of the frame, one can introduce a new time t_L, given by

$$t_L = t' - vx'. \tag{3.8}$$

So that the delay in sending a light signal becomes

$$\Delta t_{L} = \Delta t' - v\,\Delta x' = (1+v)\,\Delta x' - v\,\Delta x' = 1\cdot\Delta x', \qquad (3.9)$$

the delay assigned in special relativity for the process. Indeed, on introducing the expressions for the A.L.T., (3.8) reduces to

$$t_{L} = \gamma\,(t - vx)\,, \qquad (3.10)$$

or the well known relativistic transformation for time. The transformation (3.8) is our previously described transformation O_3. It is interesting to note historically that the above transformation was first given by Lorentz [9] using Galilean coordinates, under the title, "local time", in order to eliminate first order effects, so that his original transformation was $O_3 O_1$. After he discovered O_2, he still gave the transformation in the form (3.8), instead of the relativistic form. We see therefore that the possibility of introducing the "local time" arises as a consequence of the arbitrariness of the synchronization of separated clocks when there are no absolute signals present. However, one might wonder whether by considering two similar clocks, synchronized at the origin A, and then slowly moving one of the clocks to B, $\Delta x'$ from the origin, and then measuring the velocity of light, one could not perhaps determine $v\Delta x'$, and hence v. This is not possible for the following reason: In terms of a clock located at the origin in the unprimed frame, initially coincident with that of the primed frame, the time of the two clocks at the origin in the primed frame is given by

$$t' = \sqrt{1-v^2}\;t\,. \qquad (3.11)$$

Then, on slowly moving one of the clocks in the primed frame to the point B, one has a change in rate of the clock given by

$$\delta t' = -\frac{v\,\delta v}{\sqrt{1-v^2}}\,t\;. \qquad (3.12)$$

On the other hand, the time t required to move the clock through a distance Δx in the absolute frame is

$$(v+\delta v)\,t = \Delta x = \sqrt{1-v^2}\;\Delta x' + vt\,, \qquad (3.13)$$

or

$$t = \frac{\sqrt{1-v^2}}{\delta v}\,\Delta x', \qquad (3.14)$$

and hence,

$$\delta t' = -\,v\Delta x', \tag{3.15}$$

so that there is an exact cancellation in the limit $\delta v = 0$. We shall use this result later in "deriving" the absolute Lorentz transformation.

In the above we assumed $\delta v \to 0$; for clocks moving with finite velocities, we encounter the following: Since the rate of the clock varies with the velocity with which we move the clock, it is also necessary to know this velocity in order to correct for this change in rate. But how can we measure the velocity of the clock? In order to measure the velocity we need to know the time it left the origin t'_A which we can measure and the time t'_B which it arrived at the point B, which we cannot measure. We can of course send a light signal back to the origin when the clock arrives. But how long did it take the light signal to go from B back to the origin? This is precisely what we were looking for originally! Thus we arrive at the following remarkable and somewhat astonishing result:

> Unless one can synchronize separated clocks absolutely, it is impossible to determine the one-way velocity of an object, since velocity is defined non-locally and one has no way of determining the time of arrival in terms of the time of departure.

Einstein [10], in formulating special relativity attempted to circumvent this difficulty in the following way:

> "We have not defined a common "time" for A and B, for the latter cannot be defined at all unless we establish *by definition* that the "time" required "by light to travel from A to B equals the "time" it requires to travel from B to A".

Such a definition assumes more than is warranted by experiment, since only the out-and-back propagation time of light is measured, or, if measured one-way, the motion of clocks is involved. Eddington [11] considered in detail this problem of the one-way velocity of light and attempted to actually give a "formal proof" that the out and back velocities must be the same. Thus he says:

> "If $v(\theta)$ is the velocity of light in the direction θ, the experimental result is
>
> $$\frac{1}{v(\theta)} + \frac{1}{v(\theta+\pi)} = const = C$$
>
> $$\frac{1}{v'(\theta)} + \frac{1}{v'(\theta+\pi)} = const = C'$$
>
> (v, v' refer to S and S' respectively — our note) for all values of θ. The constancy has been established to about 1 part in 10^{10}.

It is exceedingly unlikely that the first equation would hold unless

$$v\,(\theta) = v\,(\theta + \pi) = const$$

and it is fairly obvious that the existence of the second equation excludes the possibility altogether".

We shall not attempt to discuss his "proof", but merely point out that these "unlikely" results are precisely what the line element associated with the A.L.T. yields. Thus as we have shown, $\Delta t' = v\,\Delta x' + \sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}$, hence introducing $d\sigma' = dx'^2 + dy'^2 + dz'^2$, where $d\sigma'$ is the spatial distance; one may write the above as

$$\frac{\Delta t'}{\Delta \sigma'} = v \cos\theta' + 1\,, \tag{3.16}$$

where $\cos\theta' = \frac{\Delta x'}{\Delta \sigma'}$, and we see that the slowness in the direction θ', and $\theta' + \pi$, satisfy

$$\frac{\Delta t'}{\Delta \sigma'}\,(\theta') + \frac{\Delta t'}{\Delta \sigma'}\,(\theta' + \pi) = 2\,, \tag{3.17}$$

for all θ', and this result is independent of the velocity v of the frame relative to S. The difference in slowness is given by

$$\frac{\Delta t'}{\Delta \sigma'}\,(\theta') - \frac{\Delta t'}{\Delta \sigma'}\,(\theta' + \pi) = 2v \cos\theta', \tag{3.18}$$

which, together with (3.17) summarizes our previous results expressed in terms of the principal directions.

Although for convenience in the above discussion we have chosen v to lie along x, this is clearly not necessary. Thus if the velocity of S' with respect to absolute frame S has components v_i, the line element in S' becomes

$$ds^2 = g'_{\mu\nu}\, dx'^{\mu} dx'^{\nu} = dt'^2 - 2v_i\, dx'^i dt' - dx'^i dx'^j + v_i v_j\, dx'^i dx'^j. \tag{3.19}$$

upon replacing the local time dt_L by $dt' - v_i\, dx'^i$ in the Lorentz line element for the corresponding Lorentz frame. Setting $ds^2 = 0$, the time $\Delta t'$, for light to traverse $\Delta x'^i$ is

$$\Delta t' = v_i\, \Delta x'^i + \Delta\sigma', \tag{3.20}$$

which may be written in the form (3.16) and the above results hold *pari passu*. To avoid confusion it should be noted that the notation "v_i" is not meant in a covariant sense, but as a simplified way of writing quantities which are, mathematically, components of the various coordinate transformations relating S_L, S', S. Thus since S is connected with S' via $dx^\mu = \bar{a}^\mu_\nu dx'^\nu$, and since the v_i are defined as the velocity of S', one has $v_i = \frac{dx^i}{dx^0} = \frac{\bar{a}^i_0}{\bar{a}^0_0}$, since $\bar{a}^0_i = 0$. Also writing $dx^\mu_L = l^\mu_\nu dx'^\nu$, one has $v_i = -l^0_i$. Finally, from (3.19) it follows $g'_{0i} = -v_i$ and we shall also see $g'^{0i} = -v_i$.

Chapter 4. Derivation and Generalization of the Absolute Lorentz Transformation

In the preceding, some of the consequences of the A.L.T. have been examined. Let us now reverse the procedure and undertake to see what postulates are necessary in the framework of general covariance to derive the transformation.

We assume that there exists an absolute (or ether) frame S, and in this frame the propagation of light is governed by (assuming that $R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0$)

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = 0\,, \tag{4.1}$$

so that the time for light to go from A to B is the same as the time from B to A, the "time" being measured by clocks at rest in the absolute frame and synchronized with absolute signals. Such an expression is taken to hold irrespective of the state of motion of the source of the light , an assumption wherein general relativity, special relativity, and the ether theories of light all agree.

We now consider a frame S', with coordinates t', x', y', z' in uniform motion with speed v in the positive x-direction as measured in S, and look for a linear transformation of the form,

$$\left.\begin{aligned} t' &= g_0(v)\, t\,, & y' &= g_2(v)\, y \\ x' &= g_1(v)(x - vt)\,, & z' &= g_3(v)\, z \end{aligned}\right\}. \tag{4.2}$$

The physical interpretation of this transformation is that the rods and clocks in S' have changed their length and rate with respect to those in S, but that the synchronization of clocks in S' with those in S has been without delay. Under the above transformation, the line element becomes

$$\begin{aligned} ds^2 = \frac{\left(1 - v^2\right)}{g_0^2}\left(dt'\right)^2 - \frac{2v}{g_0\, g_1}\, dt' dx' - \\ - \frac{1}{g_1^2}\left(dx'\right)^2 - \frac{1}{g_2^2}\left(dy'\right)^2 - \frac{1}{g_3^2}\left(dz'\right)^2. \end{aligned} \tag{4.3}$$

If we demand that the average slowness (or equivalently, the out-and-back velocity) be the same in all directions — as experiments so far have indicated — we arrive at

$$\frac{g_0}{g_1}\gamma^2 = n(v)\,, \qquad \frac{g_0}{g_2}\gamma = n(v)\,, \qquad \frac{g_0}{g_3}\gamma = n(v)\,, \tag{4.4}$$

where $n(v)$ represents the average slowness in S' and is a positive quantity. It should be remarked that nothing in the above derivation requires that $n(v)$ be unity.

Let us now require that for a slowly moved clock, the shift in setting on being moved from A to B be just such as to compensate the extra delay experienced by light in travelling from A to B, so that a one-way measurement of the velocity of light will give $n(v)$. For a displacement in the positive x' direction by amount $\Delta x'$, this extra delay is given by

$$\left[\frac{g_0}{g_1}\frac{1}{1-v} - n(v)\right]\Delta x'. \tag{4.5}$$

Then by the same argument as in the previous Chapter we are led to the differential equation,

$$\frac{g_0}{g_1}\frac{1}{1-v} - n(v) + \frac{1}{g_1}\frac{dg_0}{dv} = 0\,, \tag{4.6}$$

which becomes, upon substitution from (4.4),

$$\frac{1}{g_0}\frac{dg_0}{dv} = -\frac{v}{1-v^2} \tag{4.7}$$

and hence since $g_0(0) = 1$, because the clock rates are the same when S' is at rest relative to S,

$$g_0(v) = \sqrt{1-v^2}\ . \tag{4.8}$$

The same result would have been obtained had the clock been moved in the negative x' direction. For motion in the transverse direction, the change in setting is zero, since $\delta g_0(v) = -\frac{1}{\sqrt{1-v^2}}\,v_y\,\delta v_y = 0$, since $v_y = 0$. It is therefore necessary and sufficient that $g_0(v) = \sqrt{1-v^2}$ in order that a clock slowly moved in any direction, yield $n(v)$ for a one-way determination of the velocity of light. Using this value for $g_0(v)$ one finds,

$$\left.\begin{aligned} g_1(v) &= \gamma\, n(v)^{-1} \\ g_2(v) &= g_3(v) = n(v)^{-1} \end{aligned}\right\}. \tag{4.9}$$

Thus we are led to a transformation of the form

$$\left.\begin{aligned} t' &= \frac{1}{\gamma}\,t\,, & y' &= n(v)^{-1}y \\ x' &= \gamma n(v)^{-1}(x - vt)\,, & z' &= n(v)^{-1}z \end{aligned}\right\}, \tag{4.10}$$

which we shall refer to as the generalized A.L.T. Under this transformation, the line element becomes,

$$\begin{aligned} ds^2 &= dt'^2 - 2v\,dx'dt'\,n(v) - \\ &\quad - n(v)^2\left(1 - v^2\right)dx'^2 - n(v)^2dy'^2 - n(v)^2dz'^2. \end{aligned} \tag{4.11}$$

If one now assumes that in S', the average slowness must be the same as in S, one has $n(v) = 1$, and the A.L.T. is derived. More generally, if one obtains the same contraction of rods independently of whether the frame was moved in the positive or negative x direction, corresponding to setting $v \to -v$, $n(v) = n(-v)$. Further, for $v = 0$, since the frames coincide, $n(0) = 1$.

One can proceed to define a local time t_L for the generalized transformation (4.10) in the same way as for the A.L.T. Thus set,

$$t_L = t' - n(v)\,v x' \tag{4.12}$$

so that,

$$\Delta t_L = \Delta t' - n(v)\,v\,\Delta x' \tag{4.13}$$

and since the slowness of light from (4.10) is, for the positive and negative x' directions,

$$\frac{\Delta t'}{\Delta x'} = n(v)\,(1 + v)\,,\ -n(v)\,(1 - v)\,, \tag{4.14}$$

it follows

$$\Delta t_L = n(v)\,\Delta x' \tag{4.15}$$

which is the delay desired. One therefore arrives at the transformation connecting t_L with the absolute frame,

$$t_L = \gamma\,(t - vx) \tag{4.16}$$

so that the local time does not actually depend on $n(v)$. Using t_L, the line element (4.11) becomes

$$ds^2 = dt_L^2 - n(v)^2\left(dx'^2 + dy'^2 + dz'^2\right). \tag{4.17}$$

It follows from the derivation of (4.10) and (4.17) that if $n(v)$ were not strictly independent of v, it still would *not* be possible to determine

v by non-absolute measurements of the velocity of light in S' alone. For example, a Michelson-Morley type of experiment in a uniformly moving frame can only lead one to conclude the relations (4.4) when combined with the assumption that the velocity of light is independent of the source.

On the other hand, since for the Earth, v varies as a function of time, an experiment with an interferometer having unequal arms, such as the one of Kennedy-Thorndike [12], would show a periodic shift in the fringe system from one time of the year to the next. Thus with unequal interferometer arms $\Delta x'$, $\Delta y'$, the difference of the average times in the two directions is

$$\Delta T' = n(v)\,(\Delta x' - \Delta y') \tag{4.18}$$

and hence,

$$\delta\Delta T' = 2\left(\frac{dn}{dv^2}\right) v\,\delta v\,(\Delta x' - \Delta y')\,. \tag{4.19}$$

The rotational and orbital motion of the Earth will give rise to periodicities in the term $v\,\delta v$ causing a displacement in the fringe system proportional to $\frac{dn}{dv^2}$. In the theory of their experiment, Kennedy and Thorndike did not consider the possibility that $n(v)$ was not unity and so regarded their measurements in terms of checking the time dilatation, $\Delta t' = \frac{1}{\gamma}\,\Delta t$ and interpreted their data correspondingly. Thus assuming only the Lorentz contraction and independence of the velocity of light on the source, they showed one is led to an expression of the form,

$$\Delta T = \gamma\,(\Delta x' - \Delta y') \approx \left(1 + \frac{1}{2}\,v^2\right)(\Delta x' - \Delta y') \tag{4.20}$$

which can be obtained from (4.18) by setting $n(v)=1$ and $\Delta T'=\frac{1}{\gamma}\,\Delta T$. Hence as v varies,

$$\delta\Delta T = v\,\delta v\,(\Delta x' - \Delta y')\,. \tag{4.21}$$

However, if in fact $\Delta t' = \frac{1}{\gamma}\,\Delta t$, as experiments for example with meson lifetimes indicate, there is still an effect to be expected unless $\frac{dn}{dv^2} = 0$, as derived above. Interpreting their data from this standpoint, the values they quote for an absolute velocity are to be regarded as being the quantity $2v\,\frac{dn}{dv^2}$. They found from an analysis of the diurnal periodicities in the fringe system, $2v\,\frac{dn}{dv^2} = 24 \pm 19$ km/sec and for the annual periodicities $2v\,\frac{dn}{dv^2} = 15 \pm 4$ km/sec. They concluded that because these velocities were so small compared to the velocities of thousands of kilometers per second known to exist among the nebulae, and since moreover the directions of the two velocities differed by 123°, their experiment was to be interpreted as yielding a null result. However the results could mean merely $\frac{dn}{dv^2}$ is small. Since in this paper we are primarily interested in showing that the A.L.T. and the associated line element contain the results of special relativity because of the way in which measurements are made, we shall assume that $\frac{dn}{dv^2}=0$. Under these circumstances, $n(v)=1$ and (4.10) reduces to the A.L.T.

Alternatively, in deriving the A.L.T. we could have proceeded in the following manner: after requiring that the average slowness be independent of direction (which led to equation 4.4), we could have further demanded that it also be independent of the absolute velocity of the frame. Under these circumstances, $n(v)=1$, and (4.4) becomes

$$g_1 = \frac{g_0}{1-v^2}\,, \qquad g_2 = g_3 = \frac{g_0}{\sqrt{1-v^2}}\,. \tag{4.22}$$

The corresponding transformation is

$$\left.\begin{aligned} t' &= g_0 t\,, & y' &= g_0 \gamma y \\ x' &= g_0 \gamma^2 (x - vt)\,, & z' &= g_0 \gamma z \end{aligned}\right\} \tag{4.23}$$

for which the line element takes the form,

$$ds^2 = \frac{1}{(\gamma g_0)^2}\left(dt'^2 - 2v\,dt'dx' - \left(1-v^2\right)dx'^2 - dy'^2 - dz'^2\right). \tag{4.24}$$

For a line element of this form it is clear that no effect is to be expected in either the Michelson-Morley or Kennedy-Thorndike experiment. If one now makes the assumption that the one-way velocity as determined by slowly moved clocks is the same as that yielded by the out-and-back methods, then (4.6) reduces to

$$\frac{1}{g_0}\frac{dg_0}{dv} = -\frac{v}{1-v^2}\,, \tag{4.25}$$

and once again $g_0 = \frac{1}{\gamma}$.

Thus we see, in summary, that the A.L.T. follows uniquely as a consequence of the following postulates:

1. There exists a frame S in which light propagates with a constant velocity, the same in all directions, independently of the motion of the source;
2. In a coordinate frame S', in uniform translation with respect to S, the out-and-back travel time for light is independent of direction, and the velocity of S' with respect to S;

3. The one-way velocity of light as measured with clocks that have been synchronized together and then slowly separated is the same as the value yielded by the out-and-back technique.

In addition, we have employed a synchronization procedure based on the hypothetical "absolute" or "instantaneous" signal, in contrast with the usual relativistic approach based on establishing, by definition, the equality of the out and back times for the propagation of a light signal.

A similar approach to the one above for obtaining the line element in the moving frame has been given by Robertson [13]. However, because he employs the relativistic synchronization procedure before using experiment to restrict the coefficients in the metric of the moving frame, his derivation leads to the ordinary Lorentz transformation and the metric $\eta_{\mu\nu}$, rather than the A.L.T.

Let us now observe that in the above derivation of the A.L.T., nowhere was the assumption made that ds is the proper time, but merely that $ds^2 = 0$ represents the propagation of light. But now setting $\frac{dx^i}{dt} = v_i$ in the line element viewed in the absolute frame, $ds^2 = \left(1 - v^2\right) dt^2$, so that for a clock at rest in S', since $dt'^2 = \left(1 - v^2\right) dt^2$, one has $ds^2 = dt'^2$. Thus the assumption about the property of slowly moved clocks which yielded $g_0 = \frac{1}{\gamma}$, is equivalent to demanding $ds^2 = dt'^2$ for a clock at rest in S', as indeed an examination of the line element (4.3) indicates.

Chapter 5. Velocity of Light in a Moving Refractive Medium and Further Applications Involving Relative Velocity

So far our discussions have pertained only to the velocity of light in the vacuum, and we have seen that in S' the relative velocity of light is different in different directions but unobservable with present techniques so that one cannot measure the velocity v of S'. The questions arise as to whether such a velocity v might be detectable by a) causing the light to pass through a refractive medium at rest in S' and determining whether the out-and-back time is a function of v, b) comparing the time it takes light to travel a distance $\Delta x'$ in the refractive medium to the time it takes to travel the same distance in the vacuum and seeing whether this time difference, varies with v. But we know by, experiment (at least to the approximation $n(v)$ is unity mentioned in the preceding chapter) that there are no effects of the kind a) and b). The problem is therefore to write down a line element for the propagation of light in a refractive medium at rest in S' which exhibits these properties.

Consider the line element,

$$ds^2 = dt'^2 - 2v\,dx'dt' - \left(n^2 - v^2\right)dx'^2 - n^2dy'^2 - n^2dz'^2, \qquad (5.1)$$

which reduces to the vacuum A.L.T. line element for $n=1$, where n is the index of refraction when the refractive medium is at rest in S, the absolute frame (the index of refraction n used here should not be confused with $n(v)$ used in the preceding chapter, although they are somewhat similar in character). The line element (5.1) has the property that under the local time transformation, $t_L = t' - vx'$, it goes into the form

$$ds^2 = dt_L^2 - n^2\left(dx'^2 + dy'^2 + dz'^2\right), \qquad (5.2)$$

which, upon setting $ds^2=0$, yields the same slowness for light in all directions, $\Delta t_L = n\,\Delta\sigma'$, where $\Delta\sigma'$ is as before $\sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}$ and thus (5.1) predicts the results of special relativity.

Setting $ds^2=0$ in (5.1) one finds

$$\Delta t' = v\,\Delta x' + n\,\Delta\sigma\,, \qquad (5.3)$$

and hence the average out-and-back time is

$$\frac{1}{2}\left(\Delta t'_{out} + \Delta t'_{back}\right) = n\,\Delta\sigma \qquad (5.4)$$

so that there are no effects of the kind mentioned in a). The slowness of light in the direction θ', with $\cos\theta' = \frac{\Delta x'}{\Delta\sigma'}$ is,

$$\frac{\Delta t'}{\Delta\sigma'} = n + v\cos\theta'. \qquad (5.5)$$

On the other hand, for a comparison stretch in the vacuum, as was shown previously, $\left(\frac{\Delta t'}{\Delta\sigma'}\right)_{VAC} = 1 + v\cos\theta'$. Hence the time difference per distance $\Delta\sigma'$ is

$$\frac{\Delta t'}{\Delta\sigma'} - \left(\frac{\Delta t'}{\Delta\sigma'}\right)_{VAC} = n - 1 \qquad (5.6)$$

and so there are no effects of the kind mentioned in b).

Let us now enquire as to what the velocity of light through the refractive medium at rest in S' appears to be as measured in the absolute frame. For simplicity we consider the light to be moving in the positive x' direction. Then using the formula for the relative velocity given by the A.L.T., namely, $\frac{dx'}{dt'} = \frac{dx/dt - v}{1-v^2}$, we have

$$\frac{dx}{dt} = \left(1 - v^2\right) \frac{dx'}{dt'} + v \tag{5.7}$$

and hence since $\frac{dx'}{dt'} = \frac{1}{n+v}$ from (5.5), it follows

$$\frac{dx}{dt} = \frac{\frac{1}{n} + v}{1 + \frac{1}{n} v} , \tag{5.8}$$

so that we have the same result as that prescribed by special relativity, but without requiring the slowness of light in the refractive medium in S' to be n.

In the above, for simplicity, we considered only two frames S' and S, but suppose, as in the Fizeau experiment, the refractive medium is in motion relative to the terrestrial frame, which in turn is in motion relative to the absolute frame, what value do we obtain for the refractive index in the terrestrial frame as a function of the relative velocity of the refractive medium? Let the moving refractive medium have velocity v_2 in the positive x direction relative to the absolute frame, and the Earth frame a velocity v_1, then the relative velocity u_1, of the light in the refractive medium with respect to the Earth frame is

$$u_1 = \left(\frac{\frac{1}{n} + v_2}{1 + \frac{1}{n} v_2} - v_1 \right) \frac{1}{1 - v_1^2} , \tag{5.9}$$

and the slowness $\frac{1}{u_1}$. Now in order to measure this slowness one has to know, as remarked previously, the time $\Delta t'$ to traverse a distance $\Delta x'$ which one has no way of measuring without absolute signals. If we employ the special relativistic convention that light travels with unit speed in a comparison vacuum stretch $\Delta x'$, we are actually assigning a slowness $\frac{1}{u_1} - v_1$, to the light in the refractive medium, hence a velocity given by $\frac{u_1}{1-u_1 v_1}$. A simple calculation yields

$$\frac{u_1}{1 - u_1 v_1} = \frac{\frac{1}{n} + v_r}{1 + \frac{1}{n} v_r} , \quad \text{with} \quad v_r = \frac{v_2 - v_1}{1 - v_2 v_1} , \tag{5.10}$$

which is again the relativistic result. On the other hand, measurements made with absolute signals in the Earth frame would give the value u_1. Clearly if $v_1 = 0$, $\frac{u_1}{1-u_1 v_1}$ reduces to u_1 so that the relative velocity with respect to the absolute frame is the same for observers using the A.L.T. or special relativity, since under these circumstances both observers agree that the slowness of light is unity. In order to avoid confusion with the various "relative velocities" that we encounter it will be convenient to use the following terminology:

$$\left.\begin{aligned} &\text{“Galilean relative velocity”:} && \frac{dx_g}{dt_g} = \frac{dx}{dt} - v \\ &\text{“A.L.T. relative velocity”:} && \frac{dx'}{dt'} = \frac{\frac{dx}{dt} - v}{1 - v^2} \\ &\text{“relativistic relative velocity”:} && \frac{dx_L}{dt_L} = \frac{\frac{dx}{dt} - v}{1 - \frac{dx}{dt}v} \end{aligned}\right\} \qquad (5.11)$$

and for the quantity dx/dt, the "velocity relative to the absolute frame" or simply, "absolute velocity".

Another example illustrative of calculating with the A.L.T. is the following: consider twin observers, initially at rest at the origin in the primed frame, and let them be moved with equal and opposite velocities in the positive and negative x' direction, by equal amounts $\Delta x'$; do they have the same age upon arriving at their respective destinations? In special relativity the answer is clearly, yes; however, one might wonder whether the same would be true in the conceptual framework presented here, since the twin that went in the positive x' direction will have a larger velocity relative to the absolute frame than the twin that went in the negative x' direction (except as discussed below) and hence the rate of ageing of the former is greater than the rate of ageing of the latter — and in an absolute sense. However, the key to the discrepancy in the two results lies in the phrase, "equal and opposite velocities". As discussed before, we have with current techniques no way of measuring their velocities; on the other hand, special relativity states that the two twins had equal and opposite velocities, if upon arrival, they each sent back light signals which arrived at the origin simultaneously. Let us calculate with this requirement in the framework of the A.L.T. and see what result is obtained.

If the twins have A.L.T. relative velocities u_-, u_+ to the left and right respectively, the total time elapsed after they have left the origin and the two light signals return is

$$\left(\frac{1}{u_-} + 1 + v\right)\Delta x' = \left(\frac{1}{u_+} + 1 - v\right)\Delta x', \qquad (5.12)$$

since the two signals are required to arrive simultaneously, and the slowness of light is $1+v$ in the positive direction and $1-v$ in the negative direction. Hence, $\frac{1}{u_-}+v=\frac{1}{u_+}-v$. Now the twin that went to the right, required a time $\Delta t'$ in the primed frame given by $\Delta t'=\frac{1}{u_+}\Delta x'$, which in the absolute frame meant a time $\Delta t=\frac{1}{u_+}\Delta x'\gamma$. Hence a clock at rest with the twin indicated a time

$$\Delta t''_+ = \Delta t\sqrt{1-(v_2^+)^2} = \frac{1}{u_+}\,\Delta x'\,\gamma\sqrt{1-(v_2^+)^2}\ , \tag{5.13}$$

where v_2^+ is the absolute velocity of the twin that went to the right. From the relation, $u_+=\frac{v_2^+-v}{1-v^2}$, one has $v_2^+=u_+(1-v^2)+v$ and inserting this in (5.13) and simplifying, there results,

$$\Delta t''_+ = \Delta x'\sqrt{\left(\frac{1}{u_+}-v\right)^2-1}\ , \tag{5.14}$$

and similarly for the twin that went to the left,

$$\Delta t''_- = \Delta x'\sqrt{\left(\frac{1}{u_-}+v\right)^2-1}\ . \tag{5.15}$$

Hence, since $\frac{1}{u_-}+v=\frac{1}{u_+}-v$, the two ages are the same. Moreover we note $\frac{1}{u_+}-v$, $\frac{1}{u_-}-v$ are nothing but the expressions for the reciprocal of the relativistic relative velocities, namely,

$$\frac{1}{u_+}-v=\frac{1-v^2}{v_2^+-v}-v=\frac{1-v_2^+v}{v_2^+-v}\equiv\frac{1}{v_r^+}\ , \tag{5.16}$$

also, remembering $\frac{1}{u_-}$ is treated as a magnitude above,

$$\frac{1}{u_-}+v=\frac{1-v^2}{v-v_2^-}+v=\frac{1-v_2^-v}{v-v_2^-}\equiv\frac{1}{|v_r^-|}\ , \tag{5.17}$$

so that, as one would calculate relativistically,

$$\Delta t''_\pm = \frac{\Delta x'}{v_r^\pm}\sqrt{1-(v_r^\pm)^2}\ , \tag{5.18}$$

where we can omit the absolute magnitude sign of v_r^- treating $\Delta x'$ as negative for motion to the left.

From the above we see that since $\frac{1}{u_+}=\frac{1}{u_-}+2v$, the twin that went to the right actually had less A.L.T. relative velocity than the twin that went to the left. So that an observer using absolute signals would not agree with a relativistic observer that the twins had arrived "simultaneously" at their (respective destinations. Had we therefore made

$u_+ = u_-$, their ages upon arrival would have been different, as may be seen from (5.14) and (5.15) and indeed, as was surmised initially, the twin that went to the right would have been younger when he arrived than-the twin that went to the left.

It should be noted that in the theory presented here there is no twin paradox of the kind in special relativity, since time can be measured in an absolute sense. For a "twin" moved to the right from the origin in S', the rate of ageing is *less* than a twin at the origin.

When the twin returns (provided of course he doesn't return so swiftly that his absolute velocity is greater than v) his rate of ageing is *greater* than the twin at the origin. But the total ageing for the journey is always less than that for the twin who remained at the origin. Thus the out-and-back time $\Delta\bar{t}'$ measured by a twin at the origin is,

$$\Delta\bar{t}' = \frac{1}{u_+}\,\Delta x' + \frac{1}{u_-}\,\Delta x' \tag{5.19}$$

and the total ageing of the twin that travelled out and back is,

$$\Delta\bar{t}'' = \Delta x' \sqrt{\left(\frac{1}{u_+} - v\right)^2 - 1} \; + \; \Delta x' \sqrt{\left(\frac{1}{u_-} + v\right)^2 - 1} \tag{5.20}$$

and using the relations, $\frac{1}{v_r^+} = \frac{1}{u_+} - v$, $\frac{1}{|v_r^-|} = \frac{1}{u_-} + v$, one has

$$\frac{\Delta\bar{t}'}{\Delta x'} = \frac{1}{v_r^+} + \frac{1}{|v_r^-|} \tag{5.21}$$

and

$$\frac{\Delta\bar{t}''}{\Delta x'} = \frac{1}{v_r^+}\sqrt{1 - (v_r^+)^2} + \frac{1}{|v_r^-|}\sqrt{1 - (v_r^-)^2} \tag{5.22}$$

and since all quantities are positive, one always has,

$$\frac{\Delta\bar{t}'}{\Delta x'} > \frac{\Delta\bar{t}''}{\Delta x'} \tag{5.23}$$

a result which is expected from simpler considerations using special relativity. On the other hand, the following result is meaningless in special relativity.

Let there be two pairs of identical clocks in S', one pair at A, call them a_1, and a_2, and the other pair at B, call them b_1, and b_2. And let B be at a positive distance $\Delta x'$ from A. Let both sets of clocks be synchronized with absolute signals at some time $t' = 0$ and then permitted to run at their natural rates. Now let a_2 be moved to the right to

B, and let b_2 be moved to the left to A. Then the above considerations show that while the time indicated by a_2 when it arrives at B will always be *less* than the time indicated by b_1, the time indicated by clock b_2 when it arrives at A will be greater than, equal to, or less than the time indicated by a_1, according to the following scheme,

$$\left.\begin{array}{ll} t_{b_2} > t_{a_1}\colon & \left|\dfrac{-2v}{1-v^2}\right| > u_- > 0 \\ t_{b_2} = t_{a_1}\colon & \left|\dfrac{-2v}{1-v^2}\right| = u_- \\ t_{b_2} < t_{a_1}\colon & \left|\dfrac{-2v}{1-v^2}\right| < u_- \end{array}\right\}. \tag{5.24}$$

Since the elapsed time read by a_1 is $\frac{1}{u_-}\Delta x'$ and the time by b_2 is $\frac{1}{u_-}\Delta x'\gamma\sqrt{1-v_2^2}$, where v_2, is the absolute velocity of b_2 corresponding to u_-, $u_- = \left|\frac{(v_2-v)}{1-v^2}\right|$, and their ratio is $\gamma\sqrt{1-v_2^2}$ which by the above scheme may be adjusted to be equal to, or less than unity.

The relation between u^- and v_r^- which was found in the preceding by essentially a physical argument follows quite simply from the local time transformation: $t_L = t' - vx'$, $x_L^i = x'^i$, since this may be written, upon using $\Delta\sigma_L = \Delta\sigma'$,

$$\frac{\Delta t_L}{\Delta\sigma_L} = \frac{\Delta t'}{\Delta\sigma'} - v\cos\theta', \tag{5.25}$$

and hence for $\theta' = 0$, $\frac{1}{v_r} = \frac{1}{u} - v$. If we take $\frac{\Delta t_L}{\Delta\sigma_L} = 1$, we obtain the expression, for the slowness of light found in (3.16), $\frac{\Delta t'}{\Delta\sigma'} = 1 + v\cos\theta'$.

Chapter 6. Measurements with Signals Travelling with Finite Velocities

As has been shown, it is possible to determine the asymmetries in the propagation of light in S' using absolute signals, but can one measure such asymmetries with signals travelling with merely finite velocities greater than that of light? Before determining the answer to this question, let us note it is possible to define operationally such superlight (or "supervidic") signals without any assumptions about the synchronization of separated clocks. Let there be two similar clocks in S', one at the origin A, and the other $\Delta x'$ from the origin at B. Let a light signal and the signal in question be sent out simultaneously from A, and let their respective times of arrival, t_1' and t_2' be measured at B. Then if

$t'_2 - t'_1 > 0$, the signal in question traveled more slowly than light, and if the time difference is negative, $t'_2 - t'_1 < 0$, the signal traveled faster than light and was a superlight signal. Note that there is nothing in the above definition about the magnitudes of the velocities, but rather a statement about their ordering as to magnitude.

Consider a set of such superlight signals, arriving at B at times t'_2, t'_3, etc., each travelling faster than the other, then

$$|t'_2 - t'_1| < |t'_3 - t'_1| < \dots < |t'_m - t'_1| \,. \tag{6.1}$$

The upper bound of this sequence defines the absolute signal. Moreover there exists such a bound, since the absolute (and largest) delay for light is $(1+v)\,\Delta x'$ in the positive x'-direction, hence $|t'_m - t'_1| \leqslant (1+v)\,\Delta x'$, the equality sign holding for the absolute signal.

Let us now suppose we wish to determine the velocity of the frame S' (say the Earth frame) with respect to the frame S; can this be done with a superlight signaling apparatus? In order to make such a measurement one would do the following: Compare the difference in times of arrival of the light signal and the superlight signal from A to B with the difference in times of arrival from B to A. Thus

$$\left.\begin{aligned} \left(1 + v - \frac{1}{u_+}\right)\Delta x' = \Delta t'_+ \\ \left(1 - v - \frac{1}{u_-}\right)\Delta x' = \Delta t'_- \end{aligned}\right\}, \tag{6.2}$$

where u_+, u_- represent the magnitudes of the A.L.T. relative velocities of the superlight signals in the positive and negative directions. It will be seen one has two equations in three unknowns so that in general there is no solution. However for special cases there are solutions, the simplest situation being $\frac{1}{u_+}, \frac{1}{u_-} \ll v$ so that effectively we are dealing with absolute signals. Or again, if one discovers by experiment that the velocity of the super-light signal is independent of the velocity of the source (as is the case for light signals) and given by $w > 1$, then

$$u_+ = \frac{w - v}{1 - v^2}\,, \qquad u_- = \frac{|-w - v|}{1 - v^2}\,, \tag{6.3}$$

and one has two equations in two unknowns. For the time differences one finds

$$\Delta t'_+ - \Delta t'_- = 2v\left(\frac{w^2 - 1}{w^2 - v^2}\right)\Delta x', \tag{6.4}$$

and for the sum,

$$\Delta t'_+ + \Delta t'_- = 2\left(1 - w\,\frac{1-v^2}{w^2-v^2}\right)\Delta x', \tag{6.5}$$

from which one can determine w and v.

One can for such a signal, formally define a line element ds in the absolute frame given by

$$d\bar{s}^2 = dt^2 - \frac{1}{w^2}\left(dx^2 + dy^2 + dz^2\right), \tag{6.6}$$

which under the A.L.T. becomes in S',

$$d\bar{s}^2 = \frac{1}{w^2}\left[\frac{w^2-v^2}{1-v^2}(dt')^2 - 2v\,dx'dt' - \left(1-v^2\right)dx'^2 - dy'^2 - dz'^2\right], \tag{6.7}$$

and hence setting $d\bar{s}^2 = 0$, the time $\Delta t'$ for such signals to traverse $\Delta x'$, $\Delta y'$, $\Delta z'$ is

$$\Delta t' = \frac{v\left(1-v^2\right)\Delta x'}{w^2-v^2} \pm \pm \frac{1-v^2}{|w^2-v^2|}\sqrt{w^2\left(\Delta x'\right)^2 + \frac{w^2-v^2}{1-v^2}\left(\Delta y'^2 + \Delta z'^2\right)}\,. \tag{6.8}$$

Although the above results were derived assuming $w > 1$, it is interesting to note that they also hold if $w < 1$, except that under these circumstances the slower-than-light signal cannot propagate in certain directions in the primed frame if $w < v$, namely those directions for which

$$w^2\left(\Delta x'\right)^2 + \frac{w^2-v^2}{1-v^2}\left[\left(\Delta y'\right)^2 + \left(\Delta z'\right)^2\right] < 0\,, \qquad \Delta t' < 0 \tag{6.9}$$

since in these directions the delay in sending such a signal is neither real nor positive. The signals are therefore confined to a cone opening in the negative x' direction. On the other hand when $w > v$, all directions are allowed.

Using these slower-than-light signals it would also be possible to detect the absolute motion of the Earth as with superlight signals for which w is constant, employing (6.4) and (6.5) if $w > v$, and if $w < v$, measuring the slope of the cone of preferred directions and $\Delta t'_-$ above — from which it is possible to obtain v and w by a simple calculation. It is interesting to note that if w is zero, the cone shrinks to a line. Physically, this “signal” consists in the identification of a point in the absolute frame which then in S' moves rearward with velocity $\frac{-v}{1-v^2}$. Comparing the delay of such a "signal" with a light signal we find using (6.2) $\Delta t'_- = \left(\frac{1}{v}-1\right)\Delta x'$ and hence v can be found in this case as well.

Thus we see that a sufficient condition for it to be possible to detect the absolute motion of a frame S' is that there be at least one other signal propagating with constant absolute velocity w in the vacuum, with $w \neq 1$.

Chapter 7. Dynamics of a Free Particle

§7.1. Energy-momentum relations in an A.L.T. frame

As was remarked in the Introduction, a fundamental distinction between special relativity and general relativity (from the standpoint of general covariance) is that "invariance" in the former implies a restriction, on coordinate transformations, whereas invariance in the latter is really a tautology. Given any contravariant vector V^μ, and its covariant vector $V_\mu = g_{\mu\nu}V^\nu$, the statement,

$$V_\mu V^\mu = \text{“an invariant”} \tag{7.1}$$

is true independently of what coordinate transformation is made; it is a tautology of general covariance. On the other hand, the statement,

$$V^0V^0 - V^iV^i = \text{“an invariant”} \tag{7.2}$$

is in general not true except for certain transformations, the Lorentz transformations, so that it is a "conditional" invariance relation. This relation in special relativity leads to the result, $p^0_L p^0_L - p^i_L p^i_L = m^2$, where p^μ_L are the momenta in a Lorentz frame. Let us now seek to find the analogous conditional invariance relation when the A.L.T. is employed, and finally, for further comparison, the relation when the Galilean transformation is employed.

Let a particle of mass m, be moving with absolute velocities, $\dot{x}$, $\dot{y}$, $\dot{z}$, the equations of motion are obtained from the variational principle,

$$\delta\int m ds = 0\,, \qquad ds = \sqrt{\eta_{\mu\nu}\, dx^\mu dx^\nu}\ . \tag{7.3}$$

In the primed frame S', under the A.L.T., the variational principle becomes

$$\delta\int m\sqrt{g'_{\mu\nu}\, dx'^\mu dx'^\nu} =$$

$$= \delta\int m\sqrt{1 - 2vu_x - (1-v^2)\,u_x^2 - u_y^2 - u_z^2}\; dt' = 0\,, \tag{7.4}$$

when t' is taken as parameter, and where $u_x = \frac{dx'}{dt'}$, $u_y = \frac{dy'}{dt'}$, $u_z = \frac{dz'}{dt'}$. The Lagrangian is therefore,

$$L = m\sqrt{1 - 2vu_x - (1 - v^2)\, u_x^2 - u_y^2 - u_z^2}\ . \tag{7.5}$$

The covariant momenta are,

$$\left.\begin{aligned} p'_x &= \frac{\partial L}{\partial u_x} = -\,m\Gamma\left[v + \left(1 - v^2\right) u_x\right], \qquad p'_y = -\,m\Gamma u_y \\ p'_0 &= -\,u_i\,\frac{\partial L}{\partial u_i} + L = m\Gamma\,(1 - u_x v)\,, \qquad p'_z = -\,m\Gamma u_z \\ \Gamma &= \frac{m}{L} = \frac{1}{\sqrt{1 - 2vu_x - (1 - v^2)\, u_x^2 - u_y^2 - u_z^2}} \end{aligned}\right\}. \tag{7.6}$$

Alternatively, the expression for p'_0 is more generally obtainable using $p'_0 = \frac{\partial L}{\partial u_0}$, where $u_0 = \frac{dt'}{ds}$, and after differentiation, setting $\frac{dt'}{ds} = 1$, if t' is taken as parameter. The contravariant metric tensor $g'^{\mu\nu}$ is obtained from inverting $g'_{\mu\nu}$ is given by

$$\|\, g'^{\mu\nu}\| = \begin{Vmatrix} 1 - v^2 & -\,v & 0 & 0 \\ -\,v & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{7.7}$$

If the velocity of the frame S' were not along the x axis but in an arbitrary direction v_x, v_y, v_z, $g'^{\mu\nu}$ is obtained from inverting the tensor given in (3.19) and is

$$\|\, g'^{\mu\nu}\| = \begin{Vmatrix} 1 - v^2 & -\,v_x & -\,v_y & -\,v_z \\ -\,v_x & -1 & 0 & 0 \\ -\,v_y & 0 & -1 & 0 \\ -\,v_z & 0 & 0 & -1 \end{Vmatrix}. \tag{7.8}$$

For simplicity we shall continue to restrict our discussion, to motion of S' along the x-axis.

The contravariant momenta are then obtained from (7.6) and (7.7),

$$\left.\begin{aligned} p'^x &= g'^{x0} p'_0 + g'^{xx} p'_x = m\Gamma u_x\,, \qquad p'^y = m\Gamma u_y \\ p'^0 &= g'^{0x} p'_x + g'^{00} p'_0 = m\Gamma\,, \qquad p'^z = m\Gamma u_z \end{aligned}\right\}. \tag{7.9}$$

In order to obtain the momenta in the unprimed frame, it is the contravariant quantities above which are to be transformed via the A.L.T., hence as is the case with the coordinates

$$\left.\begin{aligned} p^x &= \frac{1}{\gamma}\,p'^x + \gamma v p'^0, \qquad & p^y &= p'^y \\ p^0 &= \gamma\,p'^0, & p^z &= p'^z \end{aligned}\right\}. \tag{7.10}$$

Using the transformation properties of the absolute relative velocities,

$$u_x = (\dot{x} - v)\,\gamma^2, \qquad u_y = \gamma\,\dot{y}\,, \qquad u_z = \gamma\,\dot{z}\,, \tag{7.11}$$

one can rewrite Γ as

$$\Gamma = \frac{\bar{\gamma}}{\gamma}\,, \qquad \bar{\gamma} \equiv \frac{1}{\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}}\,. \tag{7.12}$$

Substituting this expression for Γ, together with the expressions for the velocities (7.11) and the momenta (7.10) into the transformation yields

$$p^0 = m\bar{\gamma}\,, \quad p^x = m\bar{\gamma}\,\dot{x}, \quad p^y = m\bar{\gamma}\,\dot{y}, \quad p^z = m\bar{\gamma}\,\dot{z} \tag{7.13}$$

as would have been obtained using the ordinary Lorentz transformation, or as we shall now show, any transformation. The line element, $ds = \sqrt{\eta_{\mu\nu}\,dx^\mu dx^\nu} = \sqrt{\eta_{\mu\nu}\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,d\tau$, where τ is an arbitrary parameter, under the transformation, $dx^\mu = b^\mu_\nu\,dx'^\nu$, becomes

$$ds = \sqrt{\eta_{\mu\nu}\,b^\mu_\lambda b^\nu_\rho\,dx'^\lambda dx'^\rho} = \sqrt{\eta_{\mu\nu}\,b^\mu_\lambda b^\nu_\rho\,\frac{dx'^\lambda}{d\tau}\frac{dx'^\rho}{d\tau}}\;d\tau', \tag{7.14}$$

where τ' is another arbitrary parameter. The momenta (per unit mass) are

$$p'_\rho = \eta_{\mu\nu}\,b^\mu_\lambda b^\nu_\rho\,\Gamma\,\frac{dx'^\lambda}{d\tau'}\,, \qquad p'^\rho = \Gamma\,\frac{dx'^\rho}{d\tau'}\,, \qquad \Gamma \equiv \frac{d\tau'}{ds}\,. \tag{7.15}$$

The coordinate transformation may be written,

$$\frac{dx^\mu}{ds} = b^\mu_\nu\,\frac{dx'^\nu}{ds} = b^\mu_\nu\,\frac{dx'^\nu}{d\tau'}\frac{d\tau'}{ds} \tag{7.16}$$

and if in the unprimed frame $\frac{d\tau}{ds} \equiv \bar{\gamma}$,

$$\bar{\gamma}\,\frac{dx^\mu}{d\tau} = b^\mu_\nu\,\Gamma\,\frac{dx'^\nu}{d\tau'}\,, \tag{7.17}$$

which is the desired result when $d\tau = dx^0 \equiv dt$, $d\tau' = dx'^0 \equiv dt'$.

Let us now observe that in the primed frame, the following relationship holds:

$$(p'_0)^2 - (p'^x)^2 - (p'^y)^2 - (p'^z)^2 = m^2. \tag{7.18}$$

That is, the *covariant* energy and the contravariant spatial momenta satisfy the usual relativistic energy-momentum relationship for a free particle of mass m. Since this invariance relation, does not depend upon v, the absolute velocity of the frame, it is true in all uniformly moving frames for which the A.L.T. holds. It is the conditional invariance relation for which we were seeking that is the analog of (7.2). Moreover one finds after some manipulation that,

$$\left.\begin{aligned} p'^x &= m\gamma_r v_{rx}\,, \qquad & p'^y &= m\gamma_r v_{ry} \\ p'_0 &= m\gamma_r\,, & p'^z &= m\gamma_r v_{rz} \end{aligned}\right\} \tag{7.19}$$

with $\gamma_r \equiv \frac{1}{\sqrt{1-(v_{rx})^2-(v_{ry})^2-(v_{rz})^2}}$, and

$$\left.\begin{aligned} v_{rx} &= \frac{\dot{x}-v}{1-v\dot{x}} = \frac{1}{\frac{1}{u_x}-v} \\ v_{ry} &= \frac{\sqrt{1-v^2}}{1-v\dot{x}}\,\dot{y} = \frac{u_y}{1-u_x v} \\ v_{rz} &= \frac{\sqrt{1-v^2}}{1-v\dot{x}}\,\dot{z} = \frac{u_z}{1-u_x v} \end{aligned}\right\}. \tag{7.20}$$

Thus p'_0, p'^i are to be identified with their relativistic counterparts, p^0_L, p^i_L based on using the ordinary Lorentz transformation. This result may be made more transparent by noting that the transformation connecting p'_0, p'^i to the absolute frame is the Lorentz transformation. Thus, as may be shown,

$$p'_0 = p'^0 - v\,p'^x \tag{7.21}$$

(which follows most easily from recognizing $dt = dt' - v\,dx'$ and making the appropriate identification — alternatively, from suitably reshuffling the terms in the line element or more formally, using $p'_0 = g'_{0\mu}p'^\mu$), and hence substituting for p'^0 in the transformation (7.10), there results,

$$\left.\begin{aligned} p^x &= (p'^x + v\,p'_0)\,\gamma\,, \qquad & p^y &= p'^y \\ p^0 &= (p'_0 + v\,p'^x)\,\gamma\,, & p^z &= p'^z \end{aligned}\right\}. \tag{7.22}$$

The fact that the $(p'_0,\, p'^i)$ are to be identified with what are called the energy and momentum in special relativity, provides us therefore with a very simple way of transcribing the dynamical laws of one theory in terms of the other, and moreover, as we have been showing, when measurements are made in certain ways, they are the *same* laws.

Let us now enquire as to the physical meaning of the momenta complementary to the above, p'^0, p'_i. On using the relations, $\Gamma = \frac{\bar{\gamma}}{\gamma}$, $\dot{x} = (1-v^2)\,u_x + v$ etc., we find,

$$\left.\begin{aligned} p'_x &= -\,m\dot{x}\,\frac{\bar{\gamma}}{\gamma}\,, \qquad & p'_y &= -\,m\dot{y}\,\bar{\gamma} \\ p'^0 &= m\,\frac{\bar{\gamma}}{\gamma}\,, \qquad & p'_x &= -\,m\dot{z}\,\bar{\gamma} \end{aligned}\right\} \tag{7.23}$$

so that apart from the time dilatation factor $\frac{1}{\gamma}$, these quantities are nothing but the *covariant* momenta p_μ as measured in the absolute frame. They are therefore "unobservables" unless measurements are made with absolute signals or the equivalent. Note the square of these momenta satisfy,

$$\gamma^2\left[\left(p'^0\right)^2 - \left(p'_x\right)^2\right] - \left(p'_y\right)^2 - \left(p'_z\right)^2 = m^2, \tag{7.24}$$

which contains explicit reference to the absolute velocity of the frame and is therefore not an invariance relation.

These complementary momenta do not have the same kind of reflection properties that are possessed by the $(p'_0,\, p'^i)$. Thus consider a particle moving in S' along the x' axis, upon colliding elastically with a sufficiently heavy object, the quantities p'_0, p'^x satisfy (the subscripts i and f denoting initial and final states)

$$(p'_0)_f = (p'_0)_i\,, \qquad (p'^x)_f = -\,(p'^x)_i \tag{7.25}$$

the same as for the Lorentz observer. Whereas, for the quantities p'^0, p'_x one has since $p'_x = -p'^x - v\,p'_0$, $p'^0 = p'_0 + v\,p'^x$,

$$\left.\begin{aligned} (p'^0)_f &= (p'_0)_i - v\,(p'^x)_i \neq (p'^0)_i = (p'_0)_i + v\,(p'^x)_i \\ (p'_x)_f &= (p'^x)_i - v\,(p'_0)_i \neq -\,(p'_x)_i = (p'^x)_i + v\,(p'_0)_i \end{aligned}\right\}. \tag{7.26}$$

This result is the analogue of the effect that was discussed in Chapter 5, where we saw that a Lorentz observer says two objects are moving with equal speeds in opposite directions if $|v_r^-| = v_r^+$ whereas the A.L.T. observer using u_+, u_- finds the two objects are in fact travelling in general with different speeds. Despite this lack of symmetry, both species of momenta are conserved in a collision process. For indeed, if one is conserved, so is the other, since they are linearly dependent. Thus if $\sum_i p'^x = \sum_f p'^x$, $\sum_i p'_0 = \sum_f p'_0$, then

$$\left.\begin{aligned} \sum_i p'_x - \sum_f p'_x &= -\Big(\sum_i p'^x - \sum_f p'^x\Big) - v\Big(\sum_i p'_0 - \sum_f p'_0\Big) = 0 \\ \sum_i p'^0 - \sum_f p'^0 &= \sum_i p'_0 - \sum_f p'_0 + v\Big(\sum_i p'^x - \sum_f p'^x\Big) = 0 \end{aligned}\right\}, \tag{7.27}$$

where $\sum_i$, $\sum_f$ represent the summation over the momenta of the particles in the initial and final states.

§7.2. Energy-momentum relations in a Galilean frame

It is of interest to see what the preceding method yields when applied to a Galilean frame. The line element $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ becomes under the Galilean transformation, $t_g = t$, $x_g = x - vt$, $y_g = y$, $z_g = z$, as given earlier in (2.5),

$$ds^2 = \left(1 - v^2\right) dt_g^2 - 2v\, dt_g\, dx_g - dx_g^2 - dy_g^2 - dz_g^2\,,$$

the contravariant metric tensor for the Galilean frame being the covariant metric tensor for the A.L.T. frame and conversely. The momenta are

$$\left.\begin{aligned} p_{gx} &= -m\Gamma_g\left(v + \dot{x}_g\right), & p_{gy} &= -m\Gamma_g\,\dot{y}_g \\ p_{g0} &= m\Gamma_g\left(1 - v\dot{x}_g - v^2\right), & p_{gz} &= -m\Gamma_g\,\dot{z}_g \\ p_g^x &= m\Gamma_g\,\dot{x}_g\,, & p_g^y &= m\Gamma_g\,\dot{y}_g \\ p_g^0 &= m\Gamma_g\,, & p_g^z &= m\Gamma_g\,\dot{z}_g \\ \Gamma_g &= \frac{1}{\sqrt{(1-v^2) - 2v\dot{x}_g - \dot{x}_g^2 - \dot{y}_g^2 - \dot{z}_g^2}} \end{aligned}\right\}. \tag{7.28}$$

Relating the contravariant momenta via the Galilean transformation to their values in the absolute frame by

$$\left.\begin{aligned} p^x &= p_g^x + v p_g^0\,, & p^y &= p_g^y \\ p^0 &= p_g^0\,, & p^z &= p_g^z \end{aligned}\right\} \tag{7.29}$$

one also obtains the expressions for the energy and momenta given by (7.13) as our general arguments showed must be the case.

However, unlike the situation with the A.L.T., these Galilean quantities can not be identified with appropriate relativistic counterparts in the Lorentz frame. Indeed, on using $\dot{x} = \dot{x}_g + v$, etc., and observing, $\Gamma_g = \frac{1}{\sqrt{1-(\dot{x}_g+v)^2-\dot{y}_g^2-\dot{z}_g^2}} = \bar{\gamma}$, we have the following identification,

$$\left.\begin{aligned} p_{gx} &= -m\bar{\gamma}\dot{x} = p_x\,, & p_{gy} &= -m\bar{\gamma}\dot{y} = p_y \\ p_g^0 &= m\bar{\gamma} = p_0\,, & p_{gz} &= -m\bar{\gamma}\dot{z} = p_z \end{aligned}\right\} \tag{7.30}$$

so that these quantities are in fact, the covariant absolute momenta. Thus the Galilean observer states the same invariance relation as the absolute observer but with a change in notation, i.e., $(p_g^0)^2 - (p_{gx})^2 - (p_{gy})^2 - (p_{gz})^2 = m^2$.

Consider now the complementary quantities p_{g0}, p_g^i: we note Γ_g can also be written

$$\Gamma_g = \frac{\sqrt{1-v^2}\;\gamma_r}{1-v\dot{x}} \tag{7.31}$$

and using the expressions for v_{rx}, v_{ry}, v_{rz} we find,

$$\left.\begin{aligned} p_g^x &= \frac{m\gamma_r v_{rx}}{\gamma}\,, & p_g^y &= m\gamma_r v_{ry} \\ p_{g0} &= \frac{m\gamma_r}{\gamma}\,, & p_g^z &= m\gamma_r v_{rz} \end{aligned}\right\} \tag{7.32}$$

so that these quantities are almost, but not quite, the Lorentz momenta. They satisfy,

$$\gamma^2\left[\left(p_{g0}\right)^2 - \left(p_g^x\right)^2\right] - \left(p_g^y\right)^2 - \left(p_g^z\right)^2 = m^2 \tag{7.33}$$

which is not a conditional invariant for arbitrary Galilean observers, depending as it does on v. It is analogous to the expression (7.24).

Chapter 8. Transformation of Maxwell's Equations and Further Applications

§8.1. Transformation of Maxwell's equations

In the absolute frame S, Maxwell's equations may be written

$$\left.\begin{aligned} &\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0 \\ &\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu \end{aligned}\right\}. \tag{8.1}$$

Under a transformation to S', given by

$$dx'^{\mu} = a^{\mu}_{\nu}\, dx^{\nu}, \quad dx^{\nu} = \bar{a}^{\nu}_{\mu}\, dx'^{\mu}, \quad a^{\mu}_{\nu}\, \bar{a}^{\nu}_{\rho} = \delta^{\mu}_{\rho}\,, \tag{8.2}$$

where the a^{μ}_{ν}, $\bar{a}^{\nu}_{\mu}$ are the coefficients of the A.L.T. and its reciprocals, the above equations take the same tensor form,

$$\left.\begin{aligned} &\frac{\partial F'_{\mu\nu}}{\partial x'^{\lambda}} + \frac{\partial F'_{\lambda\mu}}{\partial x'^{\nu}} + \frac{\partial F'_{\nu\lambda}}{\partial x'^{\mu}} = 0 \\ &\frac{\partial F'^{\mu\nu}}{\partial x'^{\nu}} = j'^{\mu} \end{aligned}\right\}. \tag{8.3}$$

The above results of course are not conditional on using the A.L.T.; they are true for any transformation (for non-linear transformations with $F^{\mu\nu}$ and j^{μ} replaced by tensor densities). As remarked in the preceding chapter the above result is simply a tautology of covariance. On introducing the vector potential A'_{μ} defined by $F'_{\mu\nu} = \frac{\partial A'_{\mu}}{\partial x'^{\nu}} - \frac{\partial A'_{\nu}}{\partial x'^{\mu}}$, the second Maxwell equation becomes

$$g'^{\lambda\nu}\frac{\partial^2 A'^{\mu}}{\partial x'^{\lambda}\,\partial x'^{\nu}} = j'^{\mu} \tag{8.4}$$

with the imposition of the gauge condition, $\frac{\partial A'^{\mu}}{\partial x'^{\nu}} = 0$. Since $g'^{\lambda\nu}$ is an explicit function of the absolute velocity of the frame S', it is in this form of the Maxwell's equations that the fundamental difference between the A.L.T. and the Lorentz transformation manifests itself.

An invariant element of charge at rest in the frame S' is given by $\delta e = = j'^{0}\sqrt{-g'}\, dx'dy'dz'$ where g' is the determinant of the metric tensor g', but since A.L.T. is unimodular $-g' = -\eta = 1$ (where η is the determinant of $\eta_{\mu\nu}$). Hence $j'^{0}dx'dy'dz'$ is an invariant of the transformation , and one may write

$$\delta e = j'^{0}dx'dy'dz' = j^{0}\,dxdydz \tag{8.5}$$

which is the same law as for the Lorentz observer. However, for charges in motion in S', the quantity j'^{0} is not what a Lorentz observer would associate with $j^{0}_{L} (= j_{L0})$, as we shall see it is j'_{0}, which for charges at rest in S' is given by $j'_{0} = g'_{0\nu}\, j'^{\nu} = j'^{0}$. The transformation laws for the A'^{μ}, j'^{μ} are, as before, for the momenta,

$$\left.\begin{aligned} &A^{x} = \frac{1}{\gamma}\,A'^{x} + \gamma v A'^{0}, \qquad && A^{y} = A'^{y} \\ &A^{0} = \gamma\, A'^{0}, && A^{z} = A'^{z} \end{aligned}\right\}, \tag{8.6}$$

$$\left.\begin{aligned} j^x &= \frac{1}{\gamma}\, j'^x + \gamma v j'^0, \qquad & j^y &= j'^y \\ j^0 &= \gamma\, j'^0, & j^z &= j'^z \end{aligned}\right\}. \tag{8.7}$$

As before in dealing with the momenta we may write

$$\left.\begin{aligned} A'_0 &= g'_{0\mu} A'^\mu = A'^0 - v A'^x \\ j'_0 &= g'_{0\mu}\, j'^\mu = j'^0 - v j'^x \end{aligned}\right\}. \tag{8.8}$$

And hence the transformation for the potential and current using these mixed quantities becomes that of the Lorentz quantities:

$$\left.\begin{aligned} A^x &= (A'^x + v A'_0)\,\gamma\,, \qquad & A^y &= A'^y \\ A^0 &= (A'_0 + v A'^x)\,\gamma\,, & A^z &= A'^z \end{aligned}\right\} \tag{8.9}$$

and similarly for the currents. Thus $(A'_0,\, A'^i)$, $(j'_0,\, j'^i)$ are to be identified with the Lorentz quantities $(A^0_L,\, A^i_L)$, $(j^0_L,\, j^i_L)$, whereas for example $(j'^0, j'_x)\gamma$ are $(j_0,\, j_x)$, the latter being the quantities measured in the absolute frame — as was the case for the momenta, and is indeed true for all vectors.

Let us now relate the electromagnetic field quantities $F'_{\mu\nu}$, $F'^{\mu\nu}$ to their values in the absolute frame. One has, $F_{\mu\nu} = a^\rho_\mu a^\lambda_\nu F'_{\rho\lambda}$, $F^{\mu\nu} = \bar{a}^\mu_\rho \bar{a}^\nu_\lambda F'^{\rho\lambda}$, which reduce to

$$\left.\begin{aligned} F_{0x} &= F'_{0x}\,, & F^{0x} &= F'^{0x} \\ F_{0y} &= \frac{1}{\gamma}\, F'_{0y} - \gamma v F'_{xy}\,, \qquad & F^{0y} &= \gamma F'^{0y} \\ F_{0z} &= \frac{1}{\gamma}\, F'_{0z} - \gamma v F'_{xz}\,, & F^{0z} &= \gamma F'^{0z} \end{aligned}\right\}, \tag{8.10}$$

$$\left.\begin{aligned} F_{yz} &= F'_{yz}\,, \qquad & F^{yz} &= F'^{yz} \\ F_{xy} &= \gamma F'_{xy}\,, & F^{xy} &= \frac{1}{\gamma}\, F'^{xy} + \gamma v F'^{0y} \\ F_{zx} &= \gamma F'_{zx}\,, & F^{zx} &= \frac{1}{\gamma}\, F'^{zx} + \gamma v F'^{z0} \end{aligned}\right\}. \tag{8.11}$$

In order to compare the quantities $F'_{\mu\nu}$, $F'^{\mu\nu}$ with the values obtained by a Lorentz observer $F_{L\mu\nu}$, $F^{\mu\nu}_L$, we use the transformation O_3 relating Lorentz coordinates to the primed coordinates,

$$dx^\mu_L = \ell^\mu_\nu\, dx'^\nu, \quad dx'^\mu = \bar{\ell}^\mu_\nu\, dx^\nu_L, \quad \ell^\mu_\nu \bar{\ell}^\nu_\lambda = \delta^\mu_\lambda\,. \tag{8.12}$$

as given in Chapter 1. So that $F_{L\mu\nu} = \bar{\ell}^{\rho}_{\mu}\bar{\ell}^{\lambda}_{\nu}F'_{\rho\lambda}$, $F^{\mu\nu}_{L} = \ell^{\mu}_{\rho}\ell^{\nu}_{\lambda}F'^{\rho\lambda}$ one obtains,

$$\left.\begin{aligned} F_{L0x} &= F'_{0x}\,, & F^{0x}_{L} &= F'^{0x} \\ F_{L0y} &= F'_{0y}\,, & F^{0y}_{L} &= F'^{0y} - vF'^{xy} \\ F_{L0z} &= F'_{0z}\,, & F^{0z}_{L} &= F'^{0z} - vF'^{xz} \end{aligned}\right\}, \tag{8.13}$$

$$\left.\begin{aligned} F_{Lyz} &= F'_{yz}\,, & F^{yz}_{L} &= F'^{yz} \\ F_{Lxy} &= F'_{xy} + vF'_{0y}\,, & F^{xy}_{L} &= F'^{xy} \\ F_{Lzx} &= F'_{zx} + vF'_{z0}\,, & F^{zx}_{L} &= F'^{zx} \end{aligned}\right\}. \tag{8.14}$$

Thus the electromagnetic field quantities F'_{0i}, F'^{ij} are what correspond to the electric and magnetic fields as measured by a Lorentz observer at rest with respect to S'. Using the above expressions one can rewrite the transformations from the primed frame to the unprimed frame in the form,

$$\left.\begin{aligned} F_{0x} &= F'_{0x}\,, & F^{yz} &= F'^{yz} \\ F_{0y} &= \left(F'_{0y} - vF'^{xy}\right)\gamma\,, & F^{xy} &= \left(F'^{xy} - vF'_{0y}\right)\gamma \\ F_{0z} &= \left(F'_{0z} - vF'^{xz}\right)\gamma\,, & F^{zx} &= \left(F'^{zx} - vF'_{0z}\right)\gamma \end{aligned}\right\} \tag{8.15}$$

thereby exhibiting explicitly the Lorentz-like behaviour of $(F'_{0i},\ F'^{ij})$. Denoting these quantities then by $\mathcal{E}', \mathcal{H}'$, it follows that $\mathcal{E}'^2 - \mathcal{H}'^2$, $\mathcal{E}'\cdot\mathcal{H}'$ are the conditional invariants under the A.L.T. On the other hand, the quantities $(F'^{0i},\ F'_{ij})$ do not have this property, as may be inferred from the manner in which they are connected with the absolute frame as given above, such a product would contain explicit references to the absolute velocity of the frame.

§8.2. Equations of motion of a charged particle

Let us now consider the equations of motion of a particle interacting with the electromagnetic field, as observed in S'. We shall see that they may be written in a form identical to that seen by a Lorentz observer and for the *same* quantities but with a different label.

The variational principle in the absolute frame is

$$\delta\int mds + eA_{\mu}\dot{x}^{\mu}ds = 0\,, \tag{8.16}$$

(where $\dot{x}^{\mu} = \frac{dx^{\mu}}{ds}$) and under the A.L.T., or indeed any transformation, becomes

$$\delta \int m\,ds + eA'_\mu \dot{x}'^\mu ds = 0\,, \tag{8.17}$$

so that the equations of motion, written in both contra-and covariant form, are

$$\left.\begin{aligned} \frac{d\dot{x}'^\nu}{ds} &= \frac{e}{m}\,\dot{x}'_\mu F'^{\mu\nu} \\ \frac{d\dot{x}'_\nu}{ds} &= \frac{e}{m}\,\dot{x}'^\mu F'_{\mu\nu} \end{aligned}\right\}. \tag{8.18}$$

Consider the equation for the development of the energy,

$$\frac{d\dot{x}'_0}{ds} = \frac{e}{m}\,\dot{x}'^i F'_{i0}\,. \tag{8.19}$$

As we saw $\dot{x}'_0$ has the same value as the Lorentz quantity, $\dot{x}_{L0}$, similarly $\dot{x}'^i = \dot{x}^i_L$, $F'_{i0} = F_{Li0}$ hence, this equation may be written

$$\frac{d\dot{x}_{L0}}{ds} = \frac{e}{m}\,\dot{x}_L F_{Li0} \tag{8.20}$$

and is therefore identical to the corresponding equation as seen by the Lorentz observer. Consider now the equations

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\,\dot{x}'_\mu F'^{\mu i}; \tag{8.21}$$

they may be written

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\,\left(\dot{x}'_0 F'^{0i} + \dot{x}'_j F'^{ji}\right); \tag{8.22}$$

but as we saw in (8.13), $F^{0y}_L = F'^{0y} - vF'^{xy}$ which generalized for motion of the frame S, with velocity v_x, v_y, v_z, becomes $F^{0i}_L = F'^{0i} - v_j F'^{ji}$, but $F^{0i}_L = -F_{L0i} = -F'_{0i}$, hence

$$F'^{0i} = -F'_{0i} + v_j\,F'^{ji}; \tag{8.23}$$

(which may also be derived using $F'_{0i} = g'_{0\mu} g'_{iv} F'^{\mu\nu}$) so that the above equation may be written

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\left(-\dot{x}'_0 F'_{0i} + \left(\dot{x}'_0 v_j + \dot{x}'_j\right) F'^{ji}\right) \tag{8.24}$$

but $\dot{x}'^j = g'^{j\mu}\dot{x}'_\mu = -v\dot{x}'_0 - \dot{x}'_j$, hence

$$\frac{d\dot{x}'^i}{ds} = \frac{e}{m}\left(-\dot{x}_0 F'_{0i} - \dot{x}'^j F'^{ji}\right), \tag{8.25}$$

and now noting $F'_{0i} = -F_L^{0i}$, $\dot{x}'^j = \dot{x}_L^j = -\dot{x}_{Lj}$, the equation is identical to

$$\frac{d\dot{x}'^i_L}{ds} = \frac{e}{m}\,\dot{x}_{L\mu}F_L^{\mu i} \tag{8.26}$$

and our original observation is proved.

The importance of this result is that it means provided an observer in S' makes measurements of velocity using light signals or slowly moving clocks, he always arrives at the same equations of motion as the Lorentz observer at rest with respect to S'; on the other hand, if he makes observations using absolute signals, he can arrive at a second set of equations of motion, namely those given by

$$\left.\begin{aligned} \frac{d\dot{x}'_i}{ds} &= \frac{e}{m}\,\dot{x}'^{\mu}F'_{\mu i} \\ \frac{d\dot{x}'^0}{ds} &= \frac{e}{m}\,\dot{x}'_{\mu}F'^{\mu 0} \end{aligned}\right\} \tag{8.27}$$

which do *not* reduce to the equations of motion as seen by the Lorentz observer. With current techniques, these equations are "unobservables", since they involve knowledge of the absolute velocity of the frame.

Finally, we note that in the presence of an electromagnetic field the conditional invariance relation on the momentum under the A.L.T. (7.18) becomes

$$\left(p'_0 - eA'_0\right)^2 - \left(p'^i - eA'^i\right)^2 = m^2. \tag{8.28}$$

§8.3. Unobservability of a correction to the wave number under the A.L.T.

In the discussions given in previous chapters it was shown that there were no effects to be expected due to the asymmetric propagation of light in S' because of the way in which measurements are made. This was done using the line element and observing the cancellation in the out-and-back slowness. It is also possible to give an analogous discussion from a wave standpoint working with the D'Alembertian equation in S'. One has

$$\left[\left(1-v^2\right)\frac{\partial^2}{\partial t'^2} - 2v\frac{\partial^2}{\partial x'\partial t'} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}\right] A'^i = 0\,, \tag{8.29}$$

and one looks for plane wave solutions of the form, $\exp \pm i\left(k'_\mu x'^\mu\right)$, the k'_μ satisfying

$$\left(1-v^2\right)k'^2_0 - 2vk'_0k'_x - k'^2_x - k'^2_y - k'^2_z = 0\,. \tag{8.30}$$

This expression may be diagonalized by using $k'_x = -v\,k'_0 - k'^x$, so that (upon introducing $k'^y = -k'_y$, $k'^z = -k'_z$), the above reduces to the usual conditional invariance relation,

$$k'^2_0 - \left(k'^x\right)^2 - \left(k'^y\right)^2 - \left(k'^z\right)^2 = 0\,. \qquad (8.31)$$

And hence for the phase we may write

$$k'_0\,t' - \left(vk'_0 + k'^x\right)x' - k'^y\,y' - k'^z\,z'. \qquad (8.32)$$

By introducing the local time $t_L = t' - vx'$, the above expression becomes the usual relativistic one,

$$k'_0\,t_L - k'^x x' - k'^y y' - k'^z z' = k^0_L\,t_L - k^i_L\,x^i_L \qquad (8.33)$$

with the previously noted identity between $(k'_0,\,k'^i)$, and the corresponding Lorentz quantities $(k^0_L,\,k^i_L)$ in the case of particle dynamics.

It is now our purpose to show that even though one has an expression for the phase given by (8.32), that in any typical measurement involving this phase, the quantity vk'_0x' cancels out, so that effectively one is dealing with the Lorentz expression (8.33). The argument is a trivial extension of ones given previously.

Thus consider a typical interference experiment involving two beams of light. They will each propagate from an initial point $P'_1(x'_1, y'_1, z'_1)$, where they were initially in phase, via two different paths, to a final point $P'_2(x'_2, y'_2, z'_2)$ where a phase comparison is to be made. Then we have for their respective phases along the two paths

$$\left.\begin{array}{ll} \text{Path 1:} & \displaystyle\int_{P'_1}^{P'_2} \left(vk'_0 + k'^x\right)dx' + k'^y\,dy' + k'^z\,dz' \\ \text{Path 2:} & \displaystyle\int_{P'_1}^{P'_2} \left(vk'_0 + k'^x\right)dx' + k'^y\,dy' + k'^z\,dz' \end{array}\right\}. \qquad (8.34)$$

And we see that in the phase difference, the term of interest,

$$\int_{P'_1}^{P'_2} vk'_0\,dx' - \int_{P'_1}^{P'_2} vk'_0\,dx' = vk'_0 \oint dx' \qquad (8.35)$$

reduces to a line integral around a closed contour and hence vanishes, leaving the customary relativistic expression for the phase difference.

In the above we have assumed the light paths to be in vacuum, however the interposition of refractive media causes no difficulty. Taking the inverse of the metric tensor given in (5.1), the D'Alembertian equation on these paths becomes

$$\left[\left(1-\frac{v^2}{n^2}\right)\frac{\partial^2}{\partial t'^2}-\frac{2v}{n^2}\frac{\partial^2}{\partial t'\partial x'}-\frac{1}{n^2}\frac{\partial^2}{\partial x'^i\partial x'^i}\right]A'^i=0. \tag{8.36}$$

Proceeding as before, the wave numbers satisfy

$$\left(1-\frac{v^2}{n^2}\right)k_0'^2-\frac{2v}{n^2}\,k_0'k_x'-\frac{1}{n^2}\,(k_i')^2=0\,. \tag{8.37}$$

Using $k'^\mu = g'^{\mu\nu}k'_\nu$, one has $k'_x = -vk'_0 - n^2k'^x$, $k'_y = -n^2k'^y$, $k'_z = -n^2k'^z$, and the above may be written, $k_0'^2 - n^2k'^ik'^i = 0$. Hence nk'^i, rather than k'^i represents the wave number in vacuum. Denoting nk'^i by $\underline{k}'^i$, they satisfy $k_0'^2 - \underline{k}'^i\underline{k}'^i = 0$, and the phase may be written

$$k_0't'-\left(vk_0'+n\underline{k}'^x\right)x'-n\underline{k}'^yy'-n\underline{k}'^zz', \tag{8.38}$$

and as before, $\int vk_0'dx'$ vanishes along a closed path leaving the usual expression. This result may be also derived noting that for a Lorentz observer, the phase in a refractive medium is, $k^0_L t_L - n\underline{k}^i_L x^i_L$, where $\underline{k}^i_L$ are the vacuum wave numbers, hence setting $t_L = t' - vx'$ and making the appropriate correspondences, the result follows.

It is interesting to note that what a Lorentz observer describes to be a plane wave propagating perpendicular to the x' axis (say in the y' direction) with $k^x_L = k'^x = 0$, an A.L.T. observer using absolute signals describes as propagating in a direction tilted with respect to the y' axis and with wave number vk_0' along the x' axis. This follows immediately from the expression for the phase, but it is interesting to give a physical reason for this result. Now a Lorentz observer would declare the relative phase of two portions of a wave front crossing the x' axis to be the same if, as they crossed say at $\frac{-\Delta x'}{2}$, $\frac{\Delta x'}{2}$, they each triggered a device which sent light signals to the origin, one from the left, one from the right, which arrived simultaneously. But as we have seen, the slowness of the light signal propagating from the right to the origin is $(1-v)$, and slowness from the left to the origin is $(1+v)$, hence two signals arriving simultaneously correspond to a time difference of $(1-v)\frac{\Delta x'}{2} - (1+v)\frac{\Delta x'}{2}$, and a phase difference of $k_0'\Big[(1-v)\frac{\Delta x'}{2} - (1+v)\frac{\Delta x'}{2}\Big] = -vk_0'\Delta x'$, as indicated above. Thus for an A.L.T. observer it is necessary for $-k'_x = k'^x + vk'_0$ to vanish to have transverse propagation; one then has, $k'^x = -vk'_0$ also using (8.30) $k'^y = \frac{1}{\gamma}k'_0$, so that a relativistic observer would declare the wave is propagating with direction $\frac{k^x_L}{k^y_L} = \frac{k'^x}{k'^y} = -v\gamma$.

§8.4. Transformation of energy-momentum and angular momentum tensors

Let us now consider the transformation properties of the energy-momentum tensor $T^{\mu\nu}$ which for the electromagnetic field is given by $T^{\mu\nu} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}F_{\lambda\rho}F^{\lambda\rho}g^{\mu\nu}$. However, in the following, for generality, we shall consider $T^{\mu\nu}$ to be an arbitrary energy-momentum tensor that in the absolute frame has the properties of being symmetric and satisfying the conservation law $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0$. Then since these are tensor properties they also hold in the A.L.T. frame,

$$\left.\begin{aligned} T'^{\lambda\mu} &= a^{\lambda}_{\nu}a^{\mu}_{\rho}T^{\nu\rho} = a^{\mu}_{\rho}a^{\lambda}_{\nu}T^{\rho\nu} = T'^{\mu\lambda} \\ \frac{\partial T'^{\lambda\mu}}{\partial x'^{\mu}} &= \bar{a}^{\nu}_{\mu}a^{\lambda}_{\rho}a^{\mu}_{\varsigma}\frac{\partial T^{\rho\varsigma}}{\partial x^{\nu}} = a^{\lambda}_{\rho}\frac{\partial T^{\rho\nu}}{\partial x^{\nu}} = 0 \end{aligned}\right\}. \tag{8.39}$$

It follows from these two properties that angular momentum will be conserved in the A.L.T. frame. Define the generalized angular momentum density $\mathfrak{M}'^{\mu\lambda\nu}$ about the origin to be

$$\mathfrak{M}'^{\mu\lambda\nu} \equiv x'^{\mu}T'^{\lambda\nu} - x'^{\lambda}T'^{\mu\nu}, \tag{8.40}$$

then

$$\frac{\partial \mathfrak{M}'^{\mu\lambda\nu}}{\partial x'^{\nu}} = T'^{\lambda\mu} - T'^{\mu\lambda} = 0\,. \tag{8.41}$$

Since this derivation does not depend on the coordinate system (although we are here only working in Cartesian frames), this conservation law is a typical example of a result which follows from general covariance rather than one which follows from working in the restricted (Lorentz) coordinate frames of special relativity.

The tensors $T'_{\mu\nu}$, $T'^{\mu\nu}$, T'^{μ}_{ν} are related to the tensors in the corresponding Lorentz frame by

$$\left.\begin{aligned} T_{L\mu\nu} &= \bar{\ell}^{\rho}_{\mu}\bar{\ell}^{\lambda}_{\nu}T'_{\rho\lambda} \\ T^{\mu\nu}_{L} &= \ell^{\mu}_{\rho}\ell^{\nu}_{\lambda}T'^{\rho\lambda} \\ T^{\nu}_{L\mu} &= \bar{\ell}^{\lambda}_{\mu}\ell^{\nu}_{\rho}T'^{\rho}_{\lambda} \end{aligned}\right\}. \tag{8.42}$$

Obtaining ℓ^{μ}_{ν}, $\bar{\ell}^{\mu}_{\nu}$, from $dt_{L} = dt' - v_j\,dx'^j$, $dx^j_{L} = dx'^j$, one has

$$\left.\begin{aligned}
T_{L00} &= T'_{00}\\
T_{L0i} &= T'_{0i} + v_i\,T'_{00}\\
T_{Lij} &= T'_{ij} + v_i\,T'_{0j} + v_j\,T'_{0i} + v_i v_j\,T'_{00}\\
T^{00}_{L} &= T'_{00} - 2v_i\,T'^{i0} + v_i v_j\,T'^{ij}\\
T^{0i}_{L} &= T'^{0i} - v_j\,T'^{ji}\\
T^{ij}_{L} &= T'^{ij}\\
T^{0}_{L0} &= T'^{0}_{0} - v_i\,T'^{i}_{0}\\
T^{i}_{L0} &= T'^{i}_{0}\\
T^{0}_{Li} &= T'^{0}_{i} + v_i\,T'^{0}_{0} - v_j\,T'^{j}_{i} - v_i v_j\,T'^{j}_{0}\\
T^{i}_{Lj} &= T'^{i}_{j} + v_j\,T'^{i}_{0}
\end{aligned}\right\}. \tag{8.43}$$

Because of the general relations $T'^{\mu\nu} = g'^{\mu\lambda} g'^{\nu\rho}\,T'_{\lambda\rho}$, $T'^{\mu}_{\nu} = g'^{\mu\lambda}\,T'_{\lambda\nu}$ there are actually only ten linearly independent components to the stress tensor. As can be seen from the above, the ten components which are to be identified with the quantities measured by the Lorentz observer are T'_{00}, T'^{i}_{0}, T'^{ij}; the other twenty-six components $T'_{\mu j}$, $T'^{0\mu}$, T'^{0}_{μ} ,T'^{i}_{j}, except for special cases of symmetry, are unobserveables unless measurements are made with absolute signals. It will be noted that while $T'_{\mu\nu}$, $T'^{\mu\nu}$ are both symmetric tensors T'^{μ}_{ν} does not have this property. In the Lorentz frame one has $T^{0}_{Li} = -T^{i}_{L0}$, $T^{i}_{Lj} = T^{j}_{Li}$ while for T'^{μ}_{ν}, from above

$$T'^{i}_{j} - T'^{j}_{i} = v_i\,T'^{j}_{0} - v_j\,T'^{i}_{0}\,, \tag{8.44}$$

and using $T'^{i}_{0} = g'^{i\lambda}\,T'_{\lambda 0}$, $T'^{0}_{i} = g'^{0\lambda}\,T'_{\lambda i}$,

$$\left.\begin{aligned}
T'^{i}_{0} &= -v_i\,T'_{00} - T'_{i0}\\
T'^{0}_{i} &= \left(1 - v^2\right) T'_{0i} - v_j T'_{ji}
\end{aligned}\right\}. \tag{8.45}$$

In the limiting case $v_i = 0$, the relations that hold on the mixed tensor in the absolute frame and the Lorentz frame follow.

The momentum of the field as measured by the Lorentz observer is $P^{\mu}_{L} = \int T^{\mu}_{L0}\,d^3x_{L}$, and since $d^3x_{L} = d^3x'$, $T^{j}_{L0} = T'^{j}_{0}$, $T^{0}_{L0} = T'_{00}$, this corresponds to the A.L.T. quantities

$$P'_0 = \int T'_{00}\,d^3x', \qquad P'^{i} = \int T'^{i}_{0}\,d^3x' \tag{8.46}$$

in analogy to the correspondence found in the particle case. Consider now, for simplicity, the field to be sufficiently localized (that of a particle) so that the total angular momentum about the origin may be written

$$M'^{ij} = x'^{i} p'^{j} - x'^{j} p'^{i}; \tag{8.47}$$

because of the above correspondence, this is the same as that obtained by the Lorentz observer, that is,

$$M'^{ij} = M_{L}^{ij}\,. \tag{8.48}$$

On the other hand, employing the relations $p'^{i} = -p'_{i} - v_i p_0$, $x'^{i} = -x'_{i} - v_i x'_0$, one obtains

$$M'^{ij} = M'_{ij} - x'_0 \left(v_i p'^{j} - v_j p'^{i}\right) - p'_0 \left(x'^{i} v_j - x'^{j} v_i\right) \tag{8.49}$$

whereas for the Lorentz observer $M_{L}^{ij} = M_{Lij}$.

Thus, in summary, we have shown by explicit calculation that in an A.L.T. frame there always exists a set of tensors which are identical (apart from label) to a corresponding set obtained in the Lorentz frame. Consequently, it is always possible for the A.L.T. observer to write his equations in the same form employing the same quantities as the Lorentz observer. On the other hand, the A.L.T. observer finds there are additional quantities involving the absolute motion of the frame. The fact that the A.L.T. observer finds there are these additional quantities is an expression of the possibility he has for another method of measurement (based on absolute synchronization) not available to the Lorentz observer, the results of which will not in general yield the same value as for the Lorentz observer. This was seen most clearly in connection with the one-way velocity of light. On the other hand, as we also saw in this connection, when the A.L.T. observer performs his measurements in the same way as the Lorentz observer, he obtains the same results — indeed this formed the basis for the derivation of the A.L.T.

Chapter 9. Kinematic Implications of Superlight Signals for Relativistic Causality

The basic idea of causal propagation is that a disturbance from a measurement propagates only forward in time. In order to violate causal propagation it would be necessary to have "signals" that propagate backwards in time. As we shall see, however, this condition is not sufficient due to the different ways in which various observers define time. For brevity, signals that propagate backwards in time will be called "acausal", independently of whether or not they actually lead to a violation of causality.

It will be shown that when there are faster-than-light signals present, they can appear to an infinite class of Lorentz observers as exhibiting this acausal propagation, although for an A.L.T., observer this is not the case. However it will also be shown that if two measurements do not interfere for an A.L.T. observer, they do not interfere for a Lorentz observer, so that the latter cannot use such acausal signals to influence events before they occurred. In the language of general relativity one would say this acausal propagation is due to an improper choice of coordinate system, namely: the use of the Lorentz local time. Indeed one can always in a given frame make any signal propagating forwards in time propagate backwards in time by redefining time, say $T = t - qx$, and choosing q sufficiently large; this will be discussed in detail anon.

Let us now temporarily confine our attention to the absolute frame so that observers using the ordinary Lorentz transformation, the A.L.T. and the Galilean transformation all agree that the velocity of light is the same in all directions and of magnitude unity. Consider two measurements being made on the x axis (for convenience) one at the point x_0 at time t_0, and the other at $x_1 > x_0$ and at time $t_1 > t_0$. Then in order for a light signal emitted at $(x_0,\, t_0)$ to be unable to interfere with the measurement at $(x_1,\, t_1)$, it is necessary that

$$t_1 - t_0 < x_1 - x_0 \tag{9.1}$$

or more generally, $t_1 - t_0 < |\vec{x}_1 - \vec{x}_0|$, that is, the interval must be space-like, to guarantee non-interference of an earlier measurement with a later measurement. This is the "relativistic causality" assumption. But actually it contains two distinct assumptions, namely:

1. There are no signals that travel faster than light forward in time;
2. There are no signals that travel backward in time.

For, if merely 1 were satisfied, a signal violating 2 could leave the later measurement at $(x_1,\, t_1)$ and arrive in time to interfere with the earlier measurement at $(x_0,\, t_0)$.

Let us now suppose that 1 is violated and there are indeed superlight signals available, of constant velocity v_s in the absolute frame S, but that there are no acausal signals present. Under these circumstances, the assumption that the interval be space-like to guarantee non-interference is no longer sufficient. A Lorentz observer, an A.L.T. observer and a Galilean observer at rest in S would all agree that the above requirement is to be replaced by

$$t_1 - t_0 < \frac{1}{v_s}\,(x_1 - x_0)\,, \tag{9.2}$$

and more generally,

$$t_1 - t_0 < \frac{1}{v_s}\,|\vec{x}_1 - \vec{x}_0|\,. \tag{9.3}$$

An interesting situation now arises in the limit $v_s \to \infty$, since the condition for non-interference reduces to the requirement $t_1 - t_0 < 0$, that is the measurement at (x_1) was earlier in time than the measurement at (x_0). This is in contradiction with our initial assumption that $t_1 > t_0$. Indeed, if we allow t_1 to be earlier than t_0, the former measurement could then have influenced the latter measurement, and hence the requirement (9.2) would no longer be applicable in guaranteeing non-interference. In other words we are actually working with a set of inequalities

$$0 < t_1 - t_0 < \frac{1}{v_s}\,(x_1 - x_0)\,. \tag{9.4}$$

The first inequality, to guarantee the measurement at t_1 does not interfere with the one at t_0 (based on assumption 2), and the second inequality to prevent interference of the earlier measurement with the later measurement by putting it outside the superlight cone. Clearly the limiting case $v_s \to \infty$ does not belong to the set.

On the other hand, if instead of the above we employ

$$0 \leqslant t_1 - t_0 \leqslant \frac{1}{v_s}\,(x_1 - x_0)\,, \tag{9.5}$$

we do not arrive at a self-contradictory requirement in the limit. Hence, if we wish to include the absolute signal as a limiting case of causal propagation, the conditions for non-interference of a measurement at t_1 with one at t_0 is that the former measurement be later than or simultaneous with the latter measurement, $t_1 \geqslant t_0$, rather than simply $t_1 > t_0$ as we have been working with above. Since the inclusion of the equality sign may seem paradoxical, it is necessary to include a stipulation that the "effect" or disturbance occur after the arrival of the initiating causal impulse. For example, in a classical case, the effect or disturbance might be a pointer displacement, the cause, a force producing a unit acceleration commencing at time $t = 0$. Then the displacement is $d = \frac{1}{2}t^2$ and there is no displacement until $t > 0$. For simplicity in what follows, v_s will be taken to be finite although arbitrarily large.

With the above qualification on v_s, let us now return to the requirement (9.2) and ask the question, does this requirement guarantee that two measurements which did not interfere in the absolute frame will not interfere for the several observers in the moving frame? Moreover, if a signal propagates causally in the absolute frame, will it propagate causally in the moving frame? We shall consider three cases: Galilean observer, A.L.T. observer, and Lorentz observer.

Case I: Galilean observer. The transformation notation is as before, $x_g = x - vt,\ t_g = t$. The elapsed time $\Delta t(v_s)$ for the signal in the unprimed frame to travel the distance $x_1 - x_0$ is $\frac{x_1 - x_0}{v_s}$; hence the Galilean observer describes the signal as having occupied the same (positive) interval of time

$$\Delta t_g(v_s) = \Delta t(v_s) = \frac{x_1 - x_0}{v_s}\,. \tag{9.6}$$

The time interval between measurements is

$$\Delta t_g(1,0) = \Delta t(1,0) = t_1 - t_0\,. \tag{9.7}$$

Hence by our original assumptions $t_1 - t_0 < \frac{x_1 - x_0}{v_s}$, it follows

$$\Delta t_g(1,0) - \Delta t_g(v_s) < 0\,, \tag{9.8}$$

and the measurements do not interfere; also, the signal is clearly causal, since it traverses the distance, $\Delta x_g(v_s) = (x_1 - x_0) - \frac{v\,(x_1 - x_0)}{v_s} = (x_1 - x_0)\times \times\left(1 - \frac{v}{v_s}\right)$, in a positive interval of time.

Case II: The A.L.T. observer. The above argument is basically unchanged. One has, for the signal,

$$\Delta t'(v_s) = \frac{x_1 - x_0}{v_s\,\gamma} \tag{9.9}$$

and for the time interval between measurements,

$$\Delta t'(1,0) = \frac{t_1 - t_0}{\gamma} \tag{9.10}$$

and hence,

$$\Delta t'(1,0) - \Delta t'(v_s) < 0 \tag{9.11}$$

and the measurements do not interfere. Moreover the signal is causal since the time interval is positive to traverse the distance $\Delta x'(v_s) = = \Delta x_g\,\gamma$.

Case III: The Lorentz observer. Proceeding as above, the time intervals are

$$\Delta t_{L}(v_s) = \left[\frac{x_1 - x_0}{v_s} - v\,(x_1 - x_0)\right]\gamma \tag{9.12}$$

$$\Delta t_{L}(1,0) = \big[(t_1 - t_0) - v\,(x_1 - x_0)\big]\,\gamma \tag{9.13}$$

and once again, $\Delta t_L(1,0) - \Delta t_L(v_s) < 0$.

It will be noted however, that for an infinite class of Lorentz observers it is possible to choose v such that $vv_s > 1$, and the time interval $\Delta t_L(v_s)$ becomes negative. Hence the above class of Lorentz observers declare the superlight signal to have propagated acausally. However if $vv_s > 1$, since $\frac{x_1 - x_0}{t_1 - t_0} > v_s$, $\frac{v(x_1 - x_0)}{t_1 - t_0} > vv_s$, $\Delta t_L(1,0)$ is also negative, and the Lorentz observer asserts that the measurement at x_1 took place *earlier* than the measurement at x_0. Moreover since $\Delta t_L(1,0) < \Delta t_L(v_s)$, he would then say the superlight acausal signal did not propagate backwards in time sufficiently fast to influence the earlier measurement. Thus, insofar as genuinely violating causality is concerned, this "acausal" propagation of the superlight signal is spurious, and arises only because of the method of synchronization employed by the Lorentz observer.

To complete the above discussion, it is necessary to show that when v is chosen such that the later event is mapped onto an earlier event, so that $\Delta t_L(1,0)$ is negative, a superlight signal that was emitted from the event at (x_1, t_1) cannot arrive before the event at (x_0, t_0) as seen in the new Lorentz frame. Denoting by $\Delta t_L(v_s)$ the interval for the signal to propagate from right to left, one has

$$\Delta t_{L}(v_s) = \left[\frac{x_1 - x_0}{v_s} + v\,(x_1 - x_0)\right]\gamma \tag{9.14}$$

so that $\Delta t_L(v_s)$ is positive. Hence it is necessary to show $\Delta t_L(v_s) - \big(-\Delta t_L(1,0)\big) > 0$. Using the expression for $\Delta t_L(1,0)$ given in (9.13) one obtains

$$\Delta t_{L}(v_s) - \big(-\Delta t_L(1,0)\big) = \frac{x_1 - x_0}{v_s} + (t_1 - t_0) > 0 \tag{9.15}$$

and hence the signal arrives later.

Tolman [14] has given a discussion of this problem, but upon showing that a superlight signal can propagate backwards in time for a class of Lorentz observers, he inferred that such signals violate causality. As we have just seen this conclusion is invalid, since it is not sufficient to show merely that such signals propagate backwards in time in the new Lorentz frame, one must also show that the signals interfere with measurements that, in the original frame, occurred before their arrival — which is not the case.

Let us now re-derive the above results working in the moving frame S' alone, thus giving us a more general proof that if two measurements do not interfere for an A.L.T. observer, they do not interfere for a Lorentz observer, although the latter will in general have to admit of acausal propagation to describe the propagation of causal superlight signals.

In the primed frame S', the A.L.T. observer describes the two measurements as having occurred at $(x'_0,\, t'_0)$ and $(x'_1,\, t'_1)$, with $t'_1 > t'_0 > 0$, since the latter measurement is by assumption later. A superlight signal leaves the earlier event and propagates to the point x'_1, with A.L.T. relative velocity u, moreover since the measurements did not interfere

$$t'_1 - t'_0 < \frac{x'_1 - x'_0}{u}\,. \tag{9.16}$$

The time interval for the signal to propagate is,

$$\Delta t'\,(u) = \frac{x'_1 - x'_0}{u}\,. \tag{9.17}$$

The local time interval for the propagation is,

$$\Delta t_{\scriptscriptstyle L}(u) = \Delta t' - v\Delta x' = \frac{x'_1 - x'_0}{u} - v\,(x'_1 - x'_0) \tag{9.18}$$

and the local time interval between measurements is

$$\Delta t_{\scriptscriptstyle L}(1,0) = (t'_1 - t'_0) - v\,(x'_1 - x'_0)\,. \tag{9.19}$$

Hence,

$$\Delta t_{\scriptscriptstyle L}(1,0) - \Delta t_{\scriptscriptstyle L}(u) = (t'_1 - t'_0) - \frac{x'_1 - x'_0}{u} < 0 \tag{9.20}$$

and the measurements did not interfere. It will be noted that the acausality condition is now $vu > 1$, but since $u = \frac{v_s - v}{1 - v^2}$, this is equivalent to $\frac{v_s v - 1}{1 - v^2} + 1 > 1$, and hence $vv_s > 1$, as before.

Alternatively, the above discussion might be carried out using the expressions for relative velocity in the moving frame. The Galilean relative velocity and the A.L.T. relative velocity both transform a superlight signal that was causal in the absolute frame into a causal signal in the moving frame. But the denominator of the relativistic relative velocity, $v_r = \frac{v_s - v}{1 - vv_s}$ changes sign for $vv_s > 1$, which does not mean the signal propagated in a reversed direction in positive time, but as we saw, the signal propagated in a positive direction backwards in time, an interesting example of the ambiguity in velocity. For $vv_s = 1$, the Lorentz observer says the signal propagated with infinite velocity. Consider now the relation between v_r and the A.L.T. relative velocity u developed in preceding chapters. The Lorentz observer has corrected (or phased) his clocks so as to make the slowness of light unity by effectively subtracting $v\Delta x'$, and hence declares a signal of slowness $\frac{1}{u}$ to be of slowness $\frac{1}{v_r} = \frac{1}{u} - v$. For $vv_s < 1$, $\frac{1}{u} > v$, and the relativistic slowness $\frac{1}{v_r}$ is positive, for $v_s = \frac{1}{v}$, u also equals $\frac{1}{v}$ (interestingly enough), and $\frac{1}{v_r}$ vanishes, becoming negative for $v_s > \frac{1}{v}$, and hence $\frac{1}{u} < v$. Thus v_r has the character of a phase velocity in these regions, which can be made to run forward or backward in time by appropriate relative synchronization of clocks corresponding to the choice of Lorentz frame.

Chapter 10. The A.L.T. Line Element under Improper Transformations

As we have seen, given an A.L.T. frame S' with the general line element

$$ds^2 = dt'^2 - 2v_i\, dt' dx'^i - dx'^i dx'^i + v_i v_j\, dx'^i dx'^j$$

it is possible to pass, via the local time transformation, to the corresponding Lorentz frame S_L with line element $ds^2 = dt_L^2 - dx_L^i dx_L^i$. Now the metric tensor $\eta_{\mu\nu}$ of the Lorentz line element is invariant under the improper transformations T: $(t_L \to -t_L,\ x_L^i \to x_L^i)$, P: $(x_L^i \to -x_L^i,\ t_L \to t_L)$ and the problem is to study the behaviour of the A.L.T. line element under similar transformations.

Denoting the improper coordinate transformations by

$$\left.\begin{aligned} T' &: \left(t' \to -t',\ x'^i \to x'^i\right) \\ P' &: \left(x'^i \to -x'^i,\ t' \to t'\right) \end{aligned}\right\}, \tag{10.1}$$

one has that under either T' or P', the line element becomes

$$ds^2 = dt'^2 + 2v_i\, dt' dx'^i - dx'^i dx'^i + v_i v_j\, dx'^i dx'^j \tag{10.2}$$

so that in the reflected coordinate system $g'_{0i} \to -g'_{0i}$, the other components of the metric tensor remaining unchanged. Thus unlike the situation with the Lorentz observer, the metric tensor is not invariant under the improper transformations, the new line element being that for an A.L.T. frame translating with absolute velocity $-v_i$, without reflection of time or space. Let us examine how this lack of invariance would show up in a classical experiment performed in S'.

Consider first an experiment designed to check invariance under T'. An observer in S' sends a light signal from the origin in the direction θ' through a distance $\Delta\sigma'$ to a point A and measures the delay to be $\Delta t' = (1 + v\cos\theta')\,\Delta\sigma'$. Now in reversed time, the signal returned from A back to the origin in the direction $\theta'+\pi$. The expression for slowness in reversed time obtained from (10.2) is $\Delta t' = (1 - v\cos\theta')\,\Delta\sigma'$ and hence for $\theta' \to \theta' + \pi$, the delay is the same as on the outward journey. On the other hand, invariance under time reversal in a given frame means that the motion observed in the time reversed frame is a possible state of motion in the original frame before time reversal. Hence a light signal sent from A to the origin in the original frame should also exhibit the same delay as it did on its outward path, which is of course not the case, the delay being $\Delta t' = (1 + v\cos(\theta' + \pi))\,\Delta\sigma' = (1 - v\cos\theta')\,\Delta\sigma'$. Thus T' is not an invariance operation in the given frame S'.

The physical reason for this lack of invariance is clear: in reversed time not only did the light signal return from A to the origin but the frame itself also reversed its direction of absolute motion, whereas in the above experiment only the direction of motion of the light signal was reversed, not the frame. Thus in order to preserve overall invariance with respect to time reversal it is necessary to go outside the given frame and include the frame travelling with absolute velocity $-v_i$, the only exception being the absolute frame for which $v_i = 0$.

Similarly the parity transformation P': $(x'^i \to -x'^i,\ t' \to t')$ is not an invariance operation in the given frame: a clock slowly moved in the direction θ' does not read the same as a clock moved in the opposite direction $\theta' + \pi$. Once again, to obtain overall invariance with respect to parity one must include the frame travelling in the opposite direction with absolute velocity $-v_i$.

On the other hand, strong reversal, $T'P'$, does represent an invariant transformation, since the two operations have the effect of cancelling the asymmetry produced by the absolute motion so that the metric tensor is left unchanged. Thus unlike the Lorentz line element, for which T, P, TP represent invariant improper coordinate transformations, the A.L.T. line element possesses only one, $T'P'$. However the possibility of performing the local time transformation $t_L = t' - v_i x'^i$ has the effect of restoring the full symmetry of the absolute frame to one in uniform motion.

The lack of invariance under P' in a given A.L.T. frame is of course of an entirely different character than the parity violations observed in the weak interactions. In the former case v_i is a polar vector and the correlated asymmetry in the propagation of light is likewise polar, whereas in the weak interactions one has a correlation between an axial vector and a polar vector. To obtain the latter kind of correlation on the basis of the metrical structure of the line element it would be necessary that g'_{0i} be an axial vector, so that light propagated with different slownesses, for example, parallel and anti-parallel to the "direction" of an axial vector.

Chapter 11. Invariance under the Local Time Transformation

Although it has been shown that for the usual classical-mechanical and electromagnetic type of experiments, the absolute velocity v cancels out in a typical measurement, so that utilization of the A.L.T. does not lead to any contradictions, the question arises as to what are some of the effects to be expected at the quantum level. It will be shown that in a given Lorentz frame, upon resynchronizing the clocks absolutely, by means of the local time transformation, so as to transform to the A.L.T. frame, the Schrödinger state function undergoes a unitary transformation so that the measureables of the two observers are the same. (The method of proof, however, will lead to a result of somewhat greater generality which will be discussed below.) That such a unitary transformation should exist follows on general principles from the fact that as was shown in Chapter 7, the energy-momentum relation for an A.L.T. observer satisfies the conditional in variance relation $p_0'^2 - p'^j p'^j = m^2$, and as was shown in Chapter 8, the equations of motion for an A.L.T. observer can be written in a form involving the same quantities in the same way as for a Lorentz observer.

Consider first the Schrödinger representation in a given Lorentz frame. In this representation, the state vector satisfies the equation, for units in which $\hbar = 1$,

$$i\,\frac{\partial}{\partial t_L}\,\Psi(t_L) = H\,\Psi(t_L)\,, \tag{11.1}$$

where the Hamiltonian H is a time independent Hermitian operator whose relativistic transformation properties will discussed below. Choose a representation for the Ψ's in which H is diagonal and the Ψ's are the energy eigenstates of H, then

$$i\,\frac{\partial}{\partial t_L}\,\Psi_E(t_L) = E\,\Psi_E(t_L)\,. \tag{11.2}$$

Consider now a transformation from the Lorentz frame to the A.L.T. frame, employing $t_L = t' - v_j x'^j$, $x_L^j = x'^j$, then

$$\frac{\partial}{\partial t_L} \rightarrow \frac{\partial}{\partial t'}\,, \qquad \Psi_E(t_L) \rightarrow \Psi_E(t' - v_j x'^j)\,. \tag{11.3}$$

But since the Ψ_E are energy eigenstates, their time dependence is of the form $\exp(-iEt_L)$ and hence under the above time transformation,

$$\Psi_E(t_L) = U\Psi_E(t'), \qquad U \equiv \exp\left(iEv_j x'^j\right). \tag{11.4}$$

Hence the Schrödinger equation becomes

$$i\,\frac{\partial}{\partial t'}\,\Psi_E(t') = U^{-1}EU\Psi_E(t') = E\Psi_E(t') \tag{11.5}$$

and the energy levels and eigenfunctions are the same for the A.L.T. observer as for the Lorentz observer.

It is interesting to note that the proof of (11.5) did not rely on the relativistic properties of H, all that was required was that H be a time independent operator and that there exist stationary solutions of the form: $\Psi_E(t_L) = \exp(-iEt_L)\,\Phi_E$, with $\frac{\partial \Phi_E}{\partial t_L} = 0$, $H\Phi_E = E\Phi_E$. Invariance under the local time transformation is therefore an extremely fundamental property of Schrödinger-type equations which deserves to be further exploited. On the other hand, when we are not dealing with a Lorentz invariant system, upon performing the local time transformation, we are not of course transforming to an A.L.T. frame, since the concepts of Lorentz frame and A.L.T. frame are no longer defined; we are then simply transforming from a frame with coordinates labelled x^μ_L to one with coordinates labelled x'^μ.

So far we have confined our remarks to systems in an eigenstate of energy with H diagonal, when this is not the case, the generalization of U is the displacement operator, which may be formally represented by

$$U = \exp\left(-v_j x'^j\,\frac{\partial}{\partial t'}\right). \tag{11.6}$$

We arrive at such an operator by looking for a generalization of (11.4), namely: an operator for which the following holds

$$\Psi\left(t' - v_j x'^j\right) = U\Psi(t') \tag{11.7}$$

even when Ψ is not eigenstate of energy. Although a great many interesting mathematical questions occur in connection with such an operator, its use here will be justified by showing for the space with which we are working U is unitary. This is done rather simply by first noting that our above analysis has actually given us the unitary "eigenvalues" of U, that is,

$$U\Psi_E(t') = \mathrm{e}^{iEv_j x'^j}\,\Psi_E(t'). \tag{11.8}$$

Now in order to establish unitarity, we must show that U preserves the "lengths" of vectors: $\|U\Psi\| = \|\Psi\|$. But since U clearly leaves invariant the lengths of the orthogonal base vectors Ψ_E, which by completeness span the space, U is unitary and $U^\dagger = U^{-1} = \exp\bigl(v_j\, x'^j \frac{\partial}{\partial t'}\bigr)$.

Thus we have in general that under the above assumptions, under the local time transformation,

$$i\,\frac{\partial}{\partial t'}\,\Psi(t') = U^{-1}HU\,\Psi(t') = H\,\Psi(t')\,. \tag{11.9}$$

We may see more explicitly how U acts by expressing H in the form $H\bigl(\frac{\partial}{\partial x^j_L},\, x^j_L\bigr)$ and noting that under the local time transformation H becomes $H\bigl(\frac{\partial}{\partial x'^j} + v_j\,\frac{\partial}{\partial t'},\, x'^j\bigr)$ but since $\bigl(\frac{\partial}{\partial x'^j} + v_j\,\frac{\partial}{\partial t'}\bigr)^n U\,\Psi = U\bigl(\frac{\partial}{\partial x'^j}\bigr)^n\Psi$, we have, assuming we can expand H in a power series, $H\bigl(\frac{\partial}{\partial x'^j} + v_j\,\frac{\partial}{\partial t'},\, x'^j\bigr)U = U H\bigl(\frac{\partial}{\partial x'^j},\, x'^j\bigr)$ and the result (11.9) follows.

As an illustration of the above in the relativistic case, we consider the Dirac equation $\bigl(-i\gamma^\mu\frac{\partial}{\partial x^\mu_L} + m\bigr)\,\Psi(x^\mu_L) = 0$, since the invariance of the Klein-Gordon equation is immediate. Transforming to the A.L.T. frame under the local time transformation, the Dirac equation becomes

$$\left[-\,i\left(\gamma^0 + \gamma^j v_j\right)\frac{\partial}{\partial t'} - i\,\gamma^j\,\frac{\partial}{\partial x'^j} + m\right]\Psi\left(t' - v_j\,x'^j,\, x'^j\right) = 0\,. \tag{11.10}$$

It will be noted that if we define $\gamma'^0 \equiv \gamma^0 + \gamma^j v_j$, $\gamma'^j \equiv \gamma^j$ they satisfy

$$\left.\begin{aligned} &\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu = 2\,g'^{\mu\nu} \\ &g'^{00} = \left(1 - v^2\right), \quad g'^{j0} = -\,v_j\,, \quad g'^{jk} = -\,\delta_{jk} \end{aligned}\right\} \tag{11.11}$$

as would be the case if we had formulated the equation in the A.L.T. frame directly. Proceeding as above, we set $\Psi(t' - v_j x'^j,\ x'^j) = U\,\Psi(t', x'^j)$, hence since

$$-\,i\gamma^j\,\frac{\partial}{\partial x'^j}\,U\Psi(t', x'^j) = U\left(-\,i\gamma^j\,\frac{\partial}{\partial x'^j} + i\,v_j\,\gamma^j\,\frac{\partial}{\partial t'}\right)\Psi(t', x'^j)\,, \tag{11.12}$$

we obtain finally $\bigl(-i\gamma^\mu\frac{\partial}{\partial x'^\mu} + m\bigr)\Psi(x'^\mu) = 0$.

Let us now observe in connection with the above that after making the unitary transformation, the spatial momenta obtained from $i\,\frac{\partial}{\partial x'^j}$ are *not* p'_j but $-p'^j$ and are therefore the Lorentz observer's covariant momenta p_{Lj}. Before performing the unitary transformation on Ψ for a plane wave state Ψ_E one has

$$p'_j\,\Psi_E = i\,\frac{\partial}{\partial x'^j}\,\Psi_E = -\left(p'^j + v_j E\right)\Psi_E\,. \tag{11.13}$$

The unitary transformation is therefore a method for eliminating the "unobservable" components of the A.L.T. covariant spatial momenta, i.e., $-v_j E$. Alternatively stated, from the standpoint of quantum mechanics, the reason that they are unobservable is that they can be eliminated by a unitary transformation. Thus a breakdown of some one or more of the assumptions we have employed here would be necessary to make v an observable.

The possibility of introducing an absolute time and ether velocity into quantum field theory has been discussed by Dirac [15] in connection with a reformulation of electrodynamics.

Chapter 12. Conclusions

The preceding analysis shows that the experimental results of special relativity may be obtained without imposing the usual requirement that the line element be the same in all uniformly moving frames. Rather, we may employ the A.L.T. line element which leads to an asymmetry in the propagation of light, depending on the absolute velocity of the frame. As we have seen in the various examples presented, this absolute velocity always cancels when measurements are performed in the usual manner. Under these circumstances, from the standpoint of mathematical simplicity, it is advantageous to further introduce the local time transformation, since the results do not depend on the absolute synchronization of separated clocks. Thus the final diagonalization of the line element in the moving frame appears as a convenient but unnecessary step.

On the other hand, one might legitimately raise the question: if this velocity relative to an absolute frame were to always cancel out, would it and the absolute frame have any physical significance? Certainly it would be unsatisfactory to introduce these concepts, together with others employed here such as instantaneous synchronization, superlight signals, etc., in order to justify certain intuitive ideas about the propagation of light, and then show that they play no role in physical phenomena.

At the present time this unsatisfactory situation does seem to exist insofar as uniformly moving frames are concerned; but if we consider phenomena in rotating frames, as is discussed in the Appendix, the situation is somewhat different. As is pointed out there, general relativity does not entail Mach's principle, without which, inertia must be regarded as being relative to space rather than the "fixed stars", and hence, rotation as absolute, rather than merely relative to these fixed stars, indeed, throughout the preceding discussion, we have worked with solutions to $R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0$, that is, a space in which the effect of other matter is vanishingly small. Nevertheless, this did not prevent us from assigning inertia to a particle, and propagation properties to light, which would not have been possible if Mach's principle were contained in the theory. Under these circumstances, one has no choice but to regard the $g_{\mu\nu}$ as representing a description of space-time itself, as manifested in the propagation of light, the behaviour of rods and clocks, and the inertial properties of bodies.

Once this view of the $g_{\mu\nu}$ is accepted, the objections raised above are considerably lessened but not eliminated, since one still has the problem of how uniform motion relative to space is to be measured and the absolute velocity thereby determined. As we have seen, this determination could be made if, in the simplest case, there were signals propagating with arbitrarily large velocities. The well known arguments for excluding such signals are based on the following: Since the energy of a particle increases according to $m\gamma$, it would require infinite energy even to achieve to the speed of light, while beyond the speed of light the energy would become imaginary — both of which are physically untenable requirements. However, while the first objection is certainly valid in classical mechanics, where, to produce a particle travelling faster than light, one would first have to accelerate it through the speed of light, the situation is somewhat different in quantum field theory. For in this case, one can conceive of the possibility of creating, via a collision process, particles (e.g. a pair) that are already in the faster-than-light region. Thus the infinite energy at the speed of light would then divide the spectrum of particles with non-zero mass into two classes: those travelling with $v < 1$, and those with $v > 1$.

It is therefore the second objection, the imaginary energy for $v > 1$, that represents the serious problem. From a classical standpoint, this imaginary energy can be avoided by transferring to the space-like branch of the energy momentum relations as seen in the absolute frame S. Thus we formally define the line element for a particle moving $v > 1$ to be

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2, \tag{12.1}$$

where the coordinates of the particle are measured in S. Then ds^2 is real for $v > 1$, and the energy and momentum satisfy $p^2 - E^2 = m^2$ and are also real. Also setting $ds^2 = 0$, yields, necessarily, the same propagation properties for light in S as the time-like definition. (It should be noted that frequently in the literature (12.1) appears in connection with the usual relativistic theory; however, since $-ds^2$ is employed for the particle variational principle, $\delta \int m\sqrt{-ds^2} = 0$, one is really working with the time-like branch. Similarly in quantum field theory although $g_{\mu\nu} = -\eta_{\mu\nu}$ is sometimes used, this is compensated for by employing $\pm im$ where one would have had m working with the time-like metric.) By employing the transformation (1.3), one can transform to the moving frame bringing the line element into the diagonal form

$$ds^2 = d\bar{t}^2 - d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2, \tag{12.2}$$

or, employing the analogue of the A.L.T., into the non-diagonal form

$$ds^2 = dt'^2 + 2v\,dx'dt' + \left(v^2 - 1\right) dx'^2 + dy'^2 + dz'^2. \tag{12.3}$$

In either case, light cannot propagate freely in all directions as must be the case for a frame travelling with $v > 1$. However, whether one can build a consistent extension to field theory that includes this space-like branch is an open question.

At present, therefore, we can only conclude that the above objections to particles travelling with $v > 1$ do not as yet suffice to exclude such states and further study is necessary. The principal objection, then, that can be raised is that such states have not been experimentally observed — in agreement with the fundamental viewpoint of special relativity. However, this objection can be turned around and used in the construction of a faster-than-light field theory which makes such states difficult to observe. Thus the fact that they have not been observed could be taken to imply some combination of the following: the coupling is weak; the lifetime is short; the threshold for production is high; the particles are neutral. (Charged particles with $v > 1$ would exhibit Cherenkov-like radiation, since space-like energy-momentum relations permit a spontaneous "wake" radiation. Such states need not on this account be eliminated since the threshold might be high, the lifetime short.) Thus the rather extensive possibilities that exist in quantum field theory simply do not permit any firm conclusions about the non-existence of such states to be drawn from present experimental data.

Let us now turn to a question of a different nature: the significance of the transformation O_2. As we have seen, it is this transformation in conjunction with O_1, the Galilean transformation, which gives rise to an extension of the relativity of Newtonian mechanics to include fields propagating with the speed of light. Were it not for O_2, it would not be possible to eliminate the absolute velocity from the line element even after performing a local time transformation. For example, the local time transformation, $dt_\ell = dt_g - dx_g \frac{v}{1-v^2}$, diagonalizes the line element in the Galilean frame but it does not eliminate v; an observer could still detect his motion through space by a Michelson-Morley type of experiment. Thus, while we have made it clear in the derivation of the A.L.T. that O_2 must arise in conjunction with O_1, to make v unobservable in the usual experiments, we have not in turn indicated why this should be the case from the standpoint of some more fundamental dynamical principle. It would perhaps be more satisfactory from the standpoint of logical economy of postulates if it were possible to show that just as O_3 is generated from O_2 and O_1, that O_2 in turn follows from O_1, and as a consequence of the nature of the equations with which we are dealing. It should be stressed that this problem does not exist for the Lorentz observer for whom it is impossible in principle to ever determine v and hence O_2 by measurements made in his frame. But for the A.L.T. observer such measurements are in principle possible, and therefore the contractions and dilatations are something to be explained in the sense that they are for him an observable function of v. On the other hand, as was pointed out when he makes measurements in the same way as the Lorentz observer, v is no longer determinable, and as was proved in Chapter 11, to the extent the usual quantum mechanical principles hold, it is in this manner that he will make measurements. Moreover this rather general invariance that was found under the local time transformation, even when the concepts of Lorentz observer and A.L.T. observer no longer apply, indicate O_3 is in every respect as fundamental to the problem as O_2 and O_1, and so one might rather regard O_2 as being generated as a consequence of O_3 and O_1.

In conclusion then, there are two theoretical problems to be solved to complete the point of view developed here:

1. An extension of quantum field theory to include states with $v > 1$ (or alternatively, a demonstration that there can exist another signal propagating with constant velocity $\neq 1$ in the absolute frame);
2. A further simplification of the postulates underlying the A.L.T.

On the other hand, to lend support to the opposite view that motion relative to the absolute frame has no physical significance, it would be logically necessary to:

a) develop a modification or extension of general relativity that fully incorporates Mach's principle;

b) demonstrate that any extension of field theory along the lines of 1 above would lead to a contradiction with known phenomena.

Appendix. Mach's Principle and the Concept of an Absolute Frame

As the development in the preceding chapters indicates, it is possible to construct a mathematical formalism yielding the same experimental results as special relativity, but in which uniform motion is referred to an absolute frame. Since the treatment was premised on the assumption of absolute signals, for which there is as yet no experimental evidence, the question arises as to whether there are any known physical phenomena which would lend support to the notion of an absolute frame. To find such phenomena it is necessary to go outside the framework of uniformly translating systems and consider other types of motion such as, for example, rotation. Because of the non-inertial character of a rotating frame, an observer located in such a frame can determine he is in rotation, without reference to the "fixed stars", by a variety of mechanical and optical experiments: Foucault pendulum, precessing gyro, the rotating-interferometer of Sagnac [16], the Michelson-Gale experiment [17].

The latter experiment, which may be regarded as the optical analogue of the Foucault pendulum, determines the angular velocity of the Earth by sending two beams of light around a large rectangle (in the actual experiment 2010×1113 feet) in opposite directions, whereupon the beams are made to interfere and a fringe displacement is measured relative to a fringe system produced by sending the beams around a smaller rectangle as a reference. The shift can be calculated very simply from a classical picture in which one takes the Earth as rotating relative to an absolute frame in which the velocity of light is c, so that the velocity of light is different in opposite directions relative to the terrestrial path. Alternatively, one may calculate the effect from the standpoint of general relativity employing the line element

$$ds^2 = \left(1-\frac{\Omega^2 r^2}{c^2}\right)c^2dt^2 - 2r^2\Omega\, d\phi dt - dr^2 - r^2 d\phi - dz^2, \qquad \text{(A.1)}$$

obtained from the usual expression for the line element in cylindrical coordinates by substituting $\phi\rightarrow\phi+\Omega t$, and leaving the other coordinates unchanged. In both cases one finds [18] to first order in Ω (which represents the limits of experimental accuracy) that the time difference is $\frac{1}{c^2}4A\Omega$ for the two beams of light to traverse a figure of area A.

In neither method of calculation is there any reference to the other matter of the universe, the result being a consequence of purely kinematical considerations which, to first order in Ω, do not even involve relativity. Of particular interest is the appearance in the above line element of the cross term $2r^2\Omega\, d\phi dt$, which has the consequence that the time it takes light to go a distance $r\Delta\phi$ is different in opposite directions $\pm\Delta\phi$. The effect of this cross term is therefore entirely analogous to that of the cross term $2v\,dx'dt'$ we encountered in the A.L.T. line element which also gave rise to a difference in the velocity of propagation in different directions. Without absolute signals, however, as we saw, this cross term cancels out in a typical interference measurement, because effectively all that is measured is the average slowness of light which does not involve v. But in a rotating frame one has the opportunity, due to the symmetry involved, to measure the difference of the slowness in opposite directions around the light path and hence obtain the effect of the cross term. Thus, using the A.L.T. expression for the slowness (with $c=1$) $\frac{\Delta t'}{\Delta\sigma} = 1 + v\cos\theta'$, we have the following expression for the time difference for the two light beams traversing a closed circuit in opposite directions $(+,-)$,

$$\int\limits_{(+)} \Delta t' - \int\limits_{(-)} \Delta t' = \int\limits_{(+)} v\cos\theta'\Delta\sigma' - \int\limits_{(-)} v\cos\theta'\Delta\sigma' =$$

$$= \int (\nabla\times\vec{v})\, d\vec{A}' + \int (\nabla\times\vec{v})\, d\vec{A}' = 4\bar{\Omega}A' \tag{A.2}$$

where $\bar{\Omega}$ is the average normal component of $\frac{1}{2}\nabla\times\vec{v}$ over the area. This derivation is of course not rigorous since the expression for the slowness was derived assuming uniform motion; however, one can always consider a series of uniformly moving frames oriented along the light path as having instantaneously the same value of $\vec{v}$ as the point the light is traversing. Since the contractions and dilatations are second order, the use of Stokes' theorem to first order is justified. Thus from the standpoint of the A.L.T., the effect observed in rotation is simply the measurement of the curl of the absolute velocity appearing in the set of A.L.T. line elements instantaneously defining the light path in the rotating frame, and hence Ω is to be regarded as the angular velocity of the terrestrial frame relative to the absolute frame.

Needless to say such an interpretation is inadmissible if one holds to the relativistic viewpoint that motion of a material body has meaning only with respect to other material bodies or reference frames. Under these circumstances it is logically necessary for the relativist to interpret the apparent absolute character of the effects observed in rotating frames (or more generally, non-inertial frames) from the standpoint of Mach's idea [19], as formulated into a principle by Einstein [20]. According to this principle, bodies do not have inertia relative to space but relative to the totality of matter in the universe which not only influences the inertia of a body but somehow produces it. This totality of matter, of which the "fixed stars" constitute the visible and presumably principal component, then determines via an averaging process the fundamental inertial frame (to within an inertial motion: uniform translation, free-fall in a local gravitational field) relative to which rotations and other apparently "absolute" motions are to be referred.

From the standpoint of the relativity of motion and the elimination of non-observable frames, Mach's idea is very attractive; however, it has never been successfully incorporated into a dynamical scheme. Thus in general relativity, as we have seen, a possible solution to the field equations in the absence of sources is $g_{\mu\nu} = \eta_{\mu\nu}$, or $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, and hence the usual Euler-Lagrange equations follow, indicating a single particle can have inertia without other masses being present. The field equations therefore admit of solutions in which inertia is relative to space. In order to satisfy Mach's principle, however, it would be necessary that the field equations have no solutions admitting of inertia in the absence of other matter. It was with the idea of securing this result that Einstein introduced the modification of the field equations involving the famous cosmological constant, an attempt which was later abandoned since, among other reasons, the equations still possessed a solution admitting of inertia in the absence of other matter, de Sitter's "empty" universe [21]. The rotating shell model of Thirring [22], which is sometimes taken as suggesting that general relativity contains Mach's principle, suffers from the difficulties that the "shell" would have to be travelling faster than the speed of light even for the nearest stars, let alone distances as great as the fixed stars, and hence Thirring's solution does not apply. Also the mass of the shell is introduced *ad hoc* into the equations, whereas according to Mach's principle the mass of the shell must arise as a consequence of the interaction.

Thus the situation still remains that when one calculates the effects observed in rotating bodies using general relativity, one is not making a calculation whose physical interpretation is Machian, but rather Newtonian (in the sense of an absolute frame). The view that general relativity in its present form does not entail Mach's principle has been expressed by many authors including Beck [23], Bondi [24] and at the 1955 Jubilee of Relativity at Bern by Heckmann and Robertson and by Pauli [25]. A recent interesting attempt to construct an alternative theory to general relativity by R. H. Dicke [26] can be easily shown to admit a line element of the form $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ even in the absence of matter and therefore likewise does not entail Mach's principle.

On the basis of experimental evidence the simplest assumption that summarizes the facts of rotation seems to be:

> There is a universal frame, embracing the "fixed stars", relative to which it is possible to determine that a material body is in rotation by mechanical and electromagnetic measurements made on the body without reference to the stars. (This frame we shall call the "absolute frame" or simply the "ether".)

The absolute frame as described and defined above is essentially what Newton referred to as "absolute space", except that we have now attributed to it, on the basis of experiment, electromagnetic properties as well as mechanical ones. In so doing, it has been tacitly assumed that the angular velocity one determines by the mechanical experiment (Foucault pendulum) agrees with that determined by the optical experiment (Michelson-Gale experiment), an assumption in accordance with the experimental facts but not *a priori* necessary. The assumption of "universality" is needed to correlate these determinations of Ω with those determined from the observations of the fixed stars. The apparent rotation of the fixed stars is then due simply to the fact that the Earth is rotating in the ether and the stars travelling with velocities less than c relative to the ether, and at these distances $\frac{c}{R\,\Omega} \ll 1$. Their influence on the events (e.g. precession of Foucault pendulum) observed in the Earth frame is, in the absence of Mach's principle, presumably very small and would appear only through their influence on the metric structure of the absolute frame, say via the field equations of general relativity. In addition to the fixed stars, as Eddington [27] points out, more locally gravitating bodies can produce effects simulating a rotation of the coordinate system, however such effects are quite small (a few seconds per century for the Moon's orbit) and do not entail Mach's principle*. However because of these effects, one cannot regard the absolute frame as a rigid structure existing independently of matter as in the Newtonian theory or Lorentz's theory of the ether, but rather as in general relativity, a (space-time) structure capable of being influenced and perturbed by the distribution of matter. Once an absolute frame is admitted on the basis of providing a simple explanation for the rotation experiments,

*Eddington is not talking about the Thirring model, but about the modification of the Moon's orbit based on the analysis due to de Sitter. See pp. 95–99 of the reference, not just p. 99. See also the detailed analysis in Ciufolini & Wheeler's book *Gravitation and Inertia*, 1995, pp. 133–134. — Note by the Author, 2009.

there is no reason for rejecting it on the basis of the Michelson-Morley experiment, etc., since as was shown, the A.L.T. is capable of providing the same results as special relativity without requiring the complete equivalence of uniformly moving frames — the requirement that was primarily responsible for discarding the absolute frame.

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P.S. Note by the Author, 2009:

Since the writing of the thesis fifty-one years ago, with subsequent study, I found that there is an intriguing ambiguity in the Minkowski metric: for suitable choice of the coordinates, in addition to it being seen as an "anti-Machian" metric, it can also be seen as the limiting case of a static, spherical universe with an infinite radius, and an *infinite* amount of mass at *zero* density, together with a *vanishing* cosmological term, and hence Machian. See the following references:

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Tangherlini F. R. Schwarzschild field in n dimensions and the dimensionality of space problem. *Nuovo Cimento*, 1963, vol. 27, 636–651.

Maxwell's Equations and the Absolute Lorentz Transformation

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Abstract: This note supplements Chapter 8 of my thesis that studies Maxwell's equations under the Absolute Lorentz Transformation (A.L.T.), and it compares in greater detail the fields transformed under the A.L.T. with those under the L.T. The general covariance of Maxwell's equations is reviewed, and it is noted that in the case of flat spacetime this includes the A.L.T. The d'Alembertian equation under the A.L.T. is given for the vector potential in the Landau gauge which is shown to be invariant under linear transformations. It is also pointed out that the trajectory of a particle will be the same with the L.T. or the A.L.T., except that the two sets of clocks will record different travel times; although they will agree for a round-trip journey.

This paper is a supplementary background to Chapter 8 of my thesis that will hopefully make it clear that certainly Maxwell's equations hold under the Absolute Lorentz Transformation (A.L.T.) as well as further clarify how the electromagnetic fields transformed under the A.L.T. compare with those transformed under the Lorentz Transformation (L.T.). First of all, it should be kept in mind that, following Einstein's principle of general covariance, when Maxwell's equations are written in generally covariant form they hold in *all* coordinate systems, not just under the L.T. or the A.L.T. Unfortunately, for physicists and engineers only exposed to special relativity, and who therefore think solely in terms of the L.T., this more general result comes as something of a shocker! But of course one has to consider carefully what are the measured quantities when one employs these alternative transformations, and as regards the A.L.T. and the linear local time transformation, this is done in Chapter 8. But before going into this in detail, I wish to review the generally covariant form of Maxwell's equations.

As in special relativity, one introduces a second-rank antisymmetric tensor for the electromagnetic field, $F_{\mu\nu} = -F_{\nu\mu}$, with $\mu, \nu = 0$, 1, 2, 3, and for simplicity, $c = 1$, and further on below I will occasionally set $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. One can readily show that because of the asymmetry, $F_{\mu\nu}$ has only six linearly independent components given by F_{0i} with $(i = 1, 2, 3)$ corresponding to the three components of the electromagnetic field, and to a suitable set of the components of the F_{ij} corresponding to the three components of the magnetic field. Note that different authors have different conventions so that, *e.g.*, F_{0i} might for some correspond to the positive components of the electric field, while for others it might correspond to the negative component. It is also sometimes more convenient to work with the contravariant form of the electromagnetic tensor which is given by $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$, and summation over the repeated indices α, β is understood. The second rank symmetric tensor $g^{\mu\nu}$ is the contravariant form of the metric tensor and is also its inverse, so that $g^{\mu\alpha} g_{\nu\alpha} = \delta^{\mu}_{\nu}$, where the latter is the identity matrix, with $\delta^{\mu}_{\nu} = 0$, $\mu \neq \nu$, and $\delta^{\mu}_{\nu} = 1$, $\mu = \nu$, no sum. One can show that $F^{\mu\nu} = -F^{\nu\mu}$, as is true for the covariant form of the tensor used above. (Note that the term *covariant* is used in two different ways: sometimes it refers to putting equations in tensor form, and sometimes it refers to where the tensorial indices are located, hence with covariant forms, the indices are below, and with contravariant forms, the indices are above, and for second rank tensors or higher, there are mixed forms.) The proof of the asymmetry of $F^{\mu\nu}$ follows from the asymmetry of $F_{\mu\nu}$ and the symmetry of $g^{\mu\nu}$. One has $F^{\nu\mu} = g^{\nu\beta} g^{\mu\alpha} F_{\beta\alpha} = -g^{\nu\beta} g^{\mu\alpha} F_{\alpha\beta} = -F^{\mu\nu}$.

It is shown in textbooks dealing with special relativity and electromagnetism that Maxwell's equations in all Lorentz invariant systems take the following form with partial derivatives replaced by a comma, thus $\frac{\partial f}{\partial x}$ is replaced by $f_{,x}$, so that one has, see (8.1) in the thesis,

$$F_{\mu\nu,\,\lambda} + F_{\lambda\mu,\,\nu} + F_{\nu\lambda,\,\mu} = 0\,, \tag{1}$$

$$F^{\mu\nu}_{\;\;,\,\nu} = j^{\mu}. \tag{2}$$

To put these equations in generally covariant form, one replaces the commas by semicolons that indicate covariant derivatives, so that the above equations become

$$F_{\mu\nu;\,\lambda} + F_{\lambda\mu;\,\nu} + F_{\nu\lambda;\,\mu} = 0\,, \tag{3}$$

$$F^{\mu\nu}_{\;\;;\,\nu} = j^{\mu}\,. \tag{4}$$

Now a remarkable simplification occurs because of the asymmetry of the $F_{\mu\nu}$, and the symmetry of the Christoffel symbols that are involved in the covariant derivatives that are in (3). The first term can be written, $F_{\mu\nu;\,\lambda} = F_{\mu\nu,\,\lambda} - \Gamma^{\alpha}_{\mu\lambda} F_{\alpha\nu} - \Gamma^{\beta}_{\nu\lambda} F_{\mu\beta}$, and similar expressions for the other two covariant derivatives. Well, one finds that all the terms involving the Christoffel symbols cancel, so the covariant derivatives can all be replaced by partial derivatives, or commas, so the equation takes the same form as (1). Although in our work, *all* the Christoffel symbols vanish, since they involve partial derivatives of the metric tensor, and we are working within flat spacetime with Cartesian spacetime coordinates, for which all the metric coefficients are constants, and hence their partial derivatives vanish, nevertheless it is interesting to see that the first equation of Chapter 8 and (1) above holds more generally than in special relativity: *it is true even in general relativity in arbitrary systems of coordinates*. Now let us look at (4), the generally covariant divergence equation. One can show that the covariant derivative takes the form

$$F^{\mu\nu}_{;\,\nu} = F^{\mu\nu}_{,\,\nu} + \Gamma^{\mu}_{\alpha\nu} F^{\alpha\nu} + \Gamma^{\nu}_{\beta\nu} F^{\mu\beta}. \tag{5}$$

Now the second term on the right hand side of (5) vanishes, because $F^{\alpha\nu} = -F^{\nu\alpha}$ and $\Gamma^{\mu}_{\alpha\nu} = \Gamma^{\mu}_{\nu\alpha}$. While for the third term on the right hand side, one can show that $\Gamma^{\nu}_{\beta\nu} = \frac{\partial \ln\sqrt{-g}}{\partial x^{\beta}}$, where g is the determinant of the metric tensor. Hence upon multiplication by $F^{\mu\beta}$ and then replacing β by ν, since it is a "dummy" index, and then combining it with the first term on the right hand side, and substituting in (4), one has

$$\frac{1}{\sqrt{-g}} \frac{\partial\sqrt{-g}\,F^{\mu\nu}}{\partial x^{\nu}} = j^{\mu}. \tag{6}$$

Now it turns out for the A.L.T. and for the L.T., as I point out in the thesis, these transformations are "unimodular" so that their determinant is -1, and hence $\sqrt{-g} = 1$. Thus the second of the two equations labelled (8.1) in the thesis holds not only for the L.T., but for the A.L.T. as well, as given in the second of the two equations labelled (8.3). Also, importantly, because of the asymmetry of the $F^{\mu\nu}$, one readily derives the continuity equation for the current four-vector

$$\frac{\partial^2 F^{\mu\nu}}{\partial x^{\mu}\partial x^{\nu}} = F^{\mu\nu}_{,\,\mu,\,\nu} = j^{\mu}_{,\,\mu} = 0\,. \tag{7}$$

What about the vector potential A_{μ}? In generally covariant form $F_{\mu\nu} = A_{\mu;\,\nu} - A_{\nu;\,\mu}$, and the generally covariant derivatives of the vectors are given by $A_{\mu;\,\nu} = A_{\mu,\,\nu} - \Gamma^{\lambda}_{\mu\nu} A_{\lambda}$ and $A_{\nu;\,\mu} = A_{\nu,\,\mu} - \Gamma^{\lambda}_{\nu\mu} A_{\lambda}$, and since $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\,mu}$, one has that the difference of the generally covariant derivatives reduces to the difference of the partial derivatives, and hence for the A.L.T. as for the L.T., or indeed for all coordinate systems in general, one has remarkably

$$F_{\mu\nu} = A_{\mu,\,\nu} - A_{\nu,\,\mu}\,. \tag{8}$$

And if one substitutes this expression for the $F^{\mu\nu}$ into the first of the two equations in (8.1) of the thesis, and changes the indices in a cyclical fashion, one readily finds that the six resulting terms cancel one another, so that the equation is satisfied. Hence, as pointed out in thesis, if we assume the two equations in (8.1) to hold, say in the rest frame, and then transform them by the A.L.T. to the moving frame, then one has exactly the same form of the equations, but with a prime on the variables, as in (8.3).

On the other hand, when one looks at the d'Alembertian wave equation for the vector potential, for convenience in the contravariant form, A^μ, the difference between the A.L.T. and the L.T. manifests itself. To obtain (8.4), we use $F^{\mu\nu} = g^{\mu\nu} g^{\nu\lambda} F_{\alpha\lambda} = g^{\mu\alpha} g^{\nu\lambda} \left(A_{\alpha,\,\lambda} - A_{\lambda,\,\alpha}\right) = g^{\nu\lambda} A^\mu_{,\lambda} - g^{\mu\alpha} A^\nu_{,\alpha}$. At this point it is of interest to make a brief digression into general relativity. You will notice that I raised the indices on the vector potential in the last two expressions. I could do this because the metric tensor for the A.L.T. as well as the L.T. are constants, and hence their partial derivatives vanish. I could have also done this if we were working with general coordinates for which the components of the metric tensor are not constants, provided the "comma" derivative was replaced by the covariant derivative, i.e., the semicolon derivative, since the covariant derivative of the metric tensor always vanishes. But returning to flat spacetime, and the above results, when one takes the divergence of $F^{\mu\nu}$, one gets two terms: the first term is the d'Alembertian term given in (8.4) of the thesis, and the second term is $-\,g^{\mu\alpha}\frac{\partial}{\partial x^\alpha}\big(\frac{\partial A^\nu}{\partial x^\nu}\big)$. Then (8.4) follows if one sets $\frac{\partial A^\nu}{\partial x^\nu} = 0$; this relation is sometimes called the Landau gauge. Incidentally, you might wonder whether the Landau gauge is invariant under transforming say from the Lorentz frame to the A.L.T. frame. It turns out the gauge is invariant under all linear transformations. Proof: Let the linear coordinate transformation be given as $dx'^\lambda = a^\lambda_\nu\, dx^\nu$, and hence $A^\nu = \bar{a}^\nu_\mu A'^\mu$ with $\bar{a}^\nu_\mu\, a^\lambda_\nu = \delta^\lambda_\mu$. So the matrices are inverse to each other. Then $\frac{\partial A^\nu}{\partial x^\nu} = \bar{a}^\nu_\mu \frac{\partial A'^\mu}{\partial x'^\lambda} \frac{\partial x'^\lambda}{\partial x^\nu} = \bar{a}^\nu_\mu a^\lambda_\nu \frac{\partial A'^\mu}{\partial x'^\lambda} = \frac{\partial A'^\mu}{\partial x'^\mu}$. Hence, if it is the case that $\frac{\partial A^\nu}{\partial x^\nu}$ (which can also be written using the comma notation as $A^\nu_{,\nu}$) is chosen to vanish in one inertial frame, say the ether frame, it vanishes in any other frame connected to it by a linear coordinate transformation. Incidentally you might wonder what the situation is when we work with $A^\nu_{;\nu}$, that is, when we work with the covariant (or semicolon) derivative rather than just the partial (or comma) derivative? Well, from what I have written above one has $A^\nu_{;\nu} = A^\nu_{,\nu} + \Gamma^\nu_{\alpha\nu} A^\alpha = A^\nu_{,\nu} + \frac{\partial \ln\sqrt{-g}}{\partial x^\alpha} A^\alpha$, and upon making use of the fact that α is a dummy index, and can be replaced by ν, the second term upon combining with the comma derivative leads to the following expression for the covariant divergence

$$A^\nu_{;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}\, A^\nu}{\partial x^\nu} = A^\nu_{,\nu}\,, \tag{9}$$

for $\sqrt{-g}=1$ or, more generally, when $\sqrt{-g}$ is a constant.

Now let us suppose we have transformed from the rest frame to the A.L.T. frame which, as above, will continue to be denoted by primes on the coordinates and field quantities, then we arrive at (8.4) in the thesis. However, the contravariant components of the metric tensor, *i.e.*, $g'^{\mu\nu}$ were not given explicitly in (8.4). They are the inverse to the components of $g'^{\mu\nu}$ given in (1.11) of the thesis. One finds with $c=1$ that the non-vanishing components are the following: $g'^{00}=(1-v^2)$, $g'^{01}=g'^{10}=-v$, $g'^{11}=g'^{22}=g'^{33}=-1$. So (8.4) becomes

$$\left[(1-v^2)\frac{\partial^2}{(\partial x'^0)^2}-2v\frac{\partial^2}{\partial x'^0\partial x'^1}-\frac{\partial^2}{(\partial x'^2)^2}-\frac{\partial^2}{(\partial x'^3)^2}\right]A'^{\mu}=j'^{\mu}. \quad (10)$$

This is of course different than the d'Alembertian equation in the corresponding Lorentz transformed frame, which is exactly of the same form as the rest frame. The reason for the difference is that in the A.L.T. frame, the speed of light is not the same in all directions, since the clocks have been synchronized externally so as to keep simultaneity invariant. On the other hand, as pointed out on numerous occasions before, this does not contradict the fact that the out-and-back speed is the same as for the Lorentz observer.

Since the subsequent material in the thesis through (8.7) is self-explanatory, let me go now to the equations given in (8.8). You will note that I have lowered the indices to obtain A'_0, j'_0 in terms of their contravariant expressions. Here we see another difference with the L.T., because $g_{L00}=1$, $g_{L0i}=0$, one has $A_{L0}=A^{L0}$, $j_{L0}=j^{L0}$ in contrast to the relations in (8.8). What is now of interest is that if we work with the mixed components, A'_0, A'^i, the transformation from the rest frame to the primed frame is exactly the same as would be the case for the the Lorentz contravariant components, A^0_L, A^i_L, and similarly for the currents, so that j'_0, j'^i are the same as j^0_L, j^i_L.

In the thesis, I then go on to say that $(j'^0, j'_x)\gamma$ are to be identified with the quantities (j_0, j_x) in the rest frame. It follows from the second equation of (8.7) that we have $j'^0\gamma=j^0$, but since in the rest frame $j^0=j_0$, the results follow for the zeroth component. To show that the above is true for the x-component, we have $j'_x=g'_{x0}j'^0+g'_{xx}j'^x$. Then substituting the values of the metric tensor from (1.11) we have $j'_x=-v\,j'^0-(1-v^2)\,j'^x$, and upon substituting from the first equation in (8.7) for j'^x into the above expression one obtains the following: $j'_x=-v\,j'^0-(1-v^2)(\gamma j^x-\gamma^2 v\,j'^0)=-\frac{1}{\gamma}\,j^x$, and since $j^x=-j_x$, upon multiplying both sides by γ the result follows.

Now let's work out some of the transformations leading to the relations in (8.10). As introduced above, the quantities a^{ρ}_{μ} and $\bar{a}^{\mu}_{\rho}$ are the transformation coefficients and their inverse for the A.L.T., and expressed in terms of partial derivatives, they are given by $a^{\rho}_{\mu} = \frac{\partial x'^{\rho}}{\partial x^{\mu}}$, $\bar{a}^{\mu}_{\rho} = \frac{\partial x^{\mu}}{\partial x'^{\rho}}$. Then, for example, $F_{01} = a^0_0 a^1_1 F'_{01} + a^1_0 a^0_1 F'_{10}$, but since $a^0_1 = \frac{\partial x'^0}{\partial x^1} = 0$, unlike the case for the L.T., one has $F_{01} = \gamma \frac{1}{\gamma} F'_{01} = F'_{01}$ as in the thesis, but with the subscript "1" replaced by the letter "x", and recalling $c = 1$, $x^0 = t$, $x'^0 = t'$. Let us now derive the contravariant expressions given by $F^{01} = \bar{a}^0_0 \bar{a}^1_1 F'^{01} + \bar{a}^0_1 \bar{a}^1_0 F'^{10}$. Once again, the second term on the right hand side vanishes for the A.L.T., and since its inverse is given by $x = \frac{1}{\gamma} x' + v\gamma t'$, $t = \gamma t'$, it follows that $\bar{a}^0_0 = \frac{\partial x^0}{\partial x'^0} = \gamma$ and that $\bar{a}^1_1 = \frac{1}{\gamma}$, hence it follows that $F^{01} = \gamma \frac{1}{\gamma} F'^{01} = F'^{01}$.

Thus we see that the covariant and contravariant forms of the antisymmetric electromagnetic field tensor are invariant under the A.L.T. in the direction of motion! This is also true for the L.T. as shown in the two top relations in (8.13), and as I will prove here explicitly further below. But physically, why is this the case? The argument that I have heard goes as follows. Let us imagine electric charge spread uniformly on an infinite plane metal surface, which we will take to be the yz plane. The electric field is uniform, and given by E_x, and in suitable units is just the surface charge density. Now look at the field in a frame travelling in the x-direction, *i.e.*, normal to the plane. Since the electric charge is conserved, as discussed in the paragraph following (8.4), and since the y and z coordinates are left invariant under both the L.T. and the A.L.T., then the surface charge density is invariant, and hence the electric fields are invariant, which explains physically why the two top equations for the electric fields in (8.13) come out the same for the two transformations. I have only discussed here the electric fields in the direction of motion, and you might find it interesting to work out the case for the electric fields in the y and z directions as given in (8.10).

But before going on to the transformation that links the L.T. fields with the A.L.T. fields, let us look at the transformation for the covariant and contravariant components of the magnetic fields as given in (8.11). Now F_{yz} is the magnetic field in the x-direction in the rest frame, and we see it is the same as in the A.L.T. frame. Mathematically, this comes about because $F_{yz} = a^{\mu}_y a^{\nu}_z F'_{\mu\nu}$, and since $a^{\mu}_y = \delta^{\mu}_y$, $a^{\nu}_z = \delta^{\nu}_z$ from the A.L.T., the result given in the thesis follows. A similar argument holds for F^{yz}. Physically, this means the magnetic field in the direction of motion is invariant under the A.L.T. as it is under the L.T. One can show that this should be the case by an argument similar to the argument for the electric field by thinking in terms of little current loops lying in the yz plane, and using the fact that the y and z coordinates are not changed under either transformation. Once again you may wish to work out the case for the other components of the magnetic field as given in the remainder of the relations in (8.11).

Now let us turn to comparing the fields under the A.L.T. with those under the L.T., and in order to be clear as to the physical circumstance under which the comparison is being made, imagine we are on a train travelling with velocity v in the x-direction with the station taken as the rest frame. On the train there are two sets of clocks: one set has been synchronized internally, either by the Einstein method, or by slowly moving them. According to these clocks, the one-way speed of light is the same in all directions. These are the clocks that obey the Lorentz transformation. The second set of clocks are those associated with the observers using the A.L.T. who have synchronized their clocks with those in the station. As discussed in the thesis, the transformation connecting the A.L.T. with the L.T. is a local time transformation: $t_L = t' - vx'$, $x_L = x'$, $y_L = y'$, $z_L = z'$, which in differential tensorial form as given in (8.12) is written $dx^{\mu}_{L} = \ell^{\mu}_{\nu}\, dx'^{\nu}$, and the inverse transformation is $dx'^{\mu} = \bar{\ell}^{\mu}_{\nu}\, dx^{\nu}_{L}$. Let us work out explicitly $F_{L01} = \bar{\ell}^{\mu}_{0}\,\bar{\ell}^{\nu}_{1} F'_{\mu\nu}$. Now rewrite the local time transformation as $t' = t_L + vx_L$, $x' = x_L$, $y' = y_L$, $z' = z_L$, so that $\bar{\ell}^{\mu}_{0} = \frac{\partial x'^{\mu}}{\partial x^{0}_{L}} = \delta^{\mu}_{0}$, while $\bar{\ell}'^{\nu}_{1} = \frac{\partial x'^{\nu}}{\partial x^{1}_{L}}$ has two nonzero values given by $\bar{\ell}^{0}_{1} = v$, and also $\bar{\ell}^{1}_{1} = 1$. However, because $F'_{\mu\nu}$ is antisymmetric, $F'_{00} = 0$, and therefore the only term that survives corresponds to $F_{L01} = \bar{\ell}^{0}_{0}\,\bar{\ell}^{1}_{1} F'_{01} = F'_{01}$ as given in the top left relation in (8.13). One can of use the same analysis based on the local time transformation to derive the rest of the relations I have given there.

What is very interesting is that we see that the covariant form of the electric field (*i.e.*, with both indices lowered as given in the upper left column of (8.13) for the L.T. is exactly the same as for the corresponding electric field for the A.L.T. This can be summarized in the following way: $F_{L0i} = F'_{0i}$, $i = 1, 2, 3$, or, $i = x, y, z$, as in the thesis. On the other hand, when it comes to the magnetic field, as is clear from the lower right hand column in (8.14), it is the contravariant components describing the magnetic field that are the same for both transformations. It follows that when we use the covariant components of the e-m field tensor for the electric field, and the contravariant components for the magnetic field, *the transformation from the rest frame to the primed frame is exactly the same as for the Lorentz transformation.* We have already shown this is the case for the components in the direction of motion, but now let us look at the transverse components, and specifically, the y-component, since by isotropy, the result will hold for the z-component as well.

So let us go back to (8.10) and look at

$$F_{0y} = \frac{1}{\gamma}\, F'_{0y} - v\gamma\, F'_{xy}\,. \tag{11}$$

We want to rewrite this so that instead of F'_{xy} being present in (11), we have F'^{xy}, and then verify that this relation has the same form as for the Lorentz transformation. We use $F'_{xy} = g'_{x\mu}\, g'_{y\nu} F'^{\mu\nu} = g'_{x0}\, g'_{yy} F'^{0y} + g'_{xx}\, g'_{yy} F'^{xy}$, since all other terms vanish. Note that I have use x, y instead of 1, 2 as indices at this point so as to make it easier to compare with the thesis. Next, substituting values from the A.L.T. metric given in (1.11) one finds

$$F'_{xy} = v\, F'^{0y} + (1 - v^2)\, F'^{xy}. \tag{12}$$

However we see that we have now introduced F'^{0y} which we do not want. So we now use the following relation $F'^{0y} = g'^{0\mu} g'^{y\nu} F'_{\mu\nu} = g'^{00} g'^{yy} F'_{0y} + g'^{0x} g'^{yy} F'_{xy} = -(1-v^2)\, F'_{0y} + v\, F'_{xy}$, which we now substitute in (12) to obtain

$$F'_{xy} = -\,v\,(1 - v^2)\, F'_{0y} + v^2 F'_{xy} + (1 - v^2)\, F'^{xy}. \tag{13}$$

Upon bringing the term $v^2 F'_{xy}$ over to the left hand side and solving, one finds that F'_{xy} can be written as

$$F'_{xy} = -\,v\, F'_{0y} + F'^{xy}. \tag{14}$$

Now substitute (14) in (11), so that we have $F_{0y} = \frac{1}{\gamma}\, F'_{0y} - v\gamma \times (-vF'_{0y} + F'^{xy})$, and rearranging terms, we have $F_{0y} = \gamma\,(\frac{1}{\gamma^2} + v^2)\, F'_{0y} - v\gamma\, F'^{xy}$, and using $\frac{1}{\gamma^2} + v^2 = 1$, we finally have that

$$F_{0y} = \gamma\left(F'_{0y} - v\, F'^{xy}\right), \tag{15}$$

which is exactly the transformation for the corresponding L.T. quantities, *i.e.*, one has that

$$F_{0y} = \gamma\,(F_{L0y} - v\, F^{xy}_{L})\,, \tag{16}$$

which can be obtained directly by employing the L.T. for the covariant components, and noting that for the L.T., unlike the case for the A.L.T., one has

$$F_{Lxy} = g_{Lx\mu}\, g_{Ly\nu} F^{\mu\nu}_{L} = (-1)(-1)\, F^{xy}_{L} = F^{xy}_{L}, \tag{17}$$

since all the off-diagonal terms vanish, and the diagonal terms for the spatial components are equal to -1. And as we have shown, using the local time transformation, $F'_{0y} = F_{L0y}$, and one can show in the same way, $F'^{xy} = F_L^{xy}$, and hence the equivalence of the results for the A.L.T. and the L.T. is established.

It is interesting to note from (15) and (16) that while F'_{0y}, F'^{xy} are equivalent to what the L.T. observer says are the electric and magnetic fields in the x and z directions, respectively, the A.L.T. observer says that in addition, there is another projection of the magnetic field that involves the velocity relative to the rest frame as well as the electric field. Thus from (8.14) one has $F_{Lxy} = F'_{xy} + vF'_{0y}$, and from (8.13) one has $F'_{0y} = F_{L0y}$, so that finally one has

$$F'_{xy} = F_{Lxy} - vF_{L0y}\,. \tag{18}$$

However, unless one has some way of setting up an external synchronization with the rest frame, F'_{xy} is strictly *unobservable*. This is the same situation that exists for the one-way velocity of light: if one has no way of making an external synchronization, one relies on either Einstein synchronization, or that with slowly moved clocks, in which case the speed of light is c in all directions.

I think the remainder of section (8.1) in the thesis is self-explanatory, but here is an additional comment that has bearing on section (8.2) that deals with the equations of motion of a charged particle. Let us suppose in a frame travelling uniformly with speed v in the x-direction relative to the rest frame, one does an experiment, say with electrons in an electromagnetic field, causing them to travel along some path, chosen for simplicity to be in the plane $z_L = z' = 0$. Let us suppose the L.T. observer finds an electron follows a path given by $y_L = f(x_L)$. Then under the local time transformation connecting the A.L.T. with the L.T., we must have $y' = f(x')$, so the electron will travel on the same path for the A.L.T. observer as for the L.T. observer. However, if the electron left the point A at the L.T. time $t_L(A)$, and arrived at the point B at the time $t_L(B)$, these times will in general be *different* for the A.L.T. observer, who will assign them times corresponding to the local time transformation. For convenience, one can assume initially that the A.L.T. observer's clock at A has been seen set to agree with that of the L.T. observer's clock, so that $t'(A) = t_L(A)$. Then, assuming the electron does not travel on a closed path, one has $t'(B) = t_L(B) + vx_L(B)$, so that the two observers assign different travel times to the electron for the *same* path, just as the A.L.T. observer assigns a different travel time for light than the L.T. for the same path, provided it is not closed. For a closed path, they of course agree, since the synchronization of separated clocks is not involved, and the A.L.T. clock and the L.T. clock both keep time at the same rate. This underlies the agreement between the A.L.T. and the L.T. for the Michelson-Morley and Kennedy-Thorndike experiments.

Finally, I would like to remark, as I noted in my 1961 *Supplemento al Nuovo Cimento* article, that the A.L.T. is useful in teaching students about the meaning of general covariance in the simple case when the metric tensor is not diagonal, but its coefficients are all constants, so that all the Christoffel symbols vanish, and the covariant derivatives reduce to ordinary partial derivatives. Also, there is an analogy with quantum mechanics, in that one sees that the A.L.T., because of the different synchronization of clocks from the L.T., splits the degeneracy between covariant and contravariant components which, under the L.T., apart from a possible minus sign, are the same. So in the case of the A.L.T., the student has to confront the different physical interpretation of these now degeneracy-split terms, that would not be the case if one only dealt with the L.T., and this can help to stimulate new insights into special relativity and electromagnetic theory, and possibly even suggest new experiments to be performed.

In conclusion, I would like to thank Dr. Gregory B. Malykin for his interesting questions concerning this Chapter of the thesis, which led to the above comments. I am also grateful to Dr. Dmitri Rabounski for his strong support and helpful suggestions.

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Tangherlini's Dissertation and Its Significance for Physics of the 21th Century

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Abstract: Here we comment on some of the most important results obtained by Frank Robert Tangherlini, in his Dissertation of 1958. We show that the main difference between the Tangherlini transformations and the Lorentz transformations arises from a special method synchronizing two clocks spatially separated in two different inertial reference frames as was suggested by Tangherlini, which is different from Einstein's method of synchronization suggested earlier. It is shown, despite the aforementioned circumstance and also despite the fact that the Tangherlini transformations differ from the Lorentz transformations (in particular, the Tangherlini transformations allow the velocity of light to be anisotropic in a moving inertial frame of reference), the Tangherlini transformations provide adequate explanations to all known well-verified experimental tests of the Special Theory of Relativity. Several possible applications of the Tangherlini transformations could give an explanation to the effects, already predicted by physicists but not yet registered. In particular, once the effects have been experimentally observed (a possible violation of the Lorentz-invariance may be involved), the effects might be more properly described with use of the formalism of the Tangherlini transformations.

During the last seven decades, physicists have discussed kinematic theories which are claimed as alternatives to the Special Theory of Relativity, or are based on transformations of the spatial coordinates and time from one inertial frame into another one which differ from the Lorentz transformations [1]. Meanwhile, despite the fact that several suggested transformations can explain numerous basic experiments of the Special Theory of Relativity, in particular — the Michelson-Morley experiment [2, 3], not one of the suggestions except the Tangherlini transformations [4] and the Sjödin transformations [5], which generalize the former, are able to give a proper explanation to all known experimental tests of the Special Theory of Relativity, in particular — the interference experiments, the measurements of the transverse Doppler effect, and the increased lifetimes of decaying high energy particles. In addition, these alternative transformations can deal with the effects in the rotating frame of the Sagnac experiment [6–10] in which the Einstein

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synchronization procedure cannot be used nor that involving slowly-moved clocks, as was pointed out for the latter case by Tangherlini in a thought experiment described in his general relativity lectures [11], and verified some years later in the well-known Hafele-Keating experiment [12, 13].

It should be noted that, in the framework of most of the aforementioned transformations, due to the specific methods of synchronization of spatially separated clocks, the velocity of light is isotropic only in a single (preferred) inertial reference frame. In the framework of these transformations, the velocity of light still remains isotropic and invariantly constant (equal to c), and independent of the velocity of the source, in agreement with the Special Theory of Relativity, while, in contrast, the value of the transverse Doppler effect predicted according to these transformations (excluding those of Tangherlini and Sjödin) differs from the value predicted by the Special Theory of Relativity, and hence are in disagreement with present experimental evidence. Recall that the constancy of the velocity of light in vacuum in any direction and its independence from the velocity of the radiating source, in an arbitrary inertial frame of reference, are well-verified experimental postulates of the Special Theory of Relativity, which follow directly from the Lorentz transformations [14].

Not one of the transformations alternative to the Lorentz transformations has been so actively discussed in the scientific press as the Tangherlini transformations obtained in 1958 [4]. The discussion has led to a large amount of literature on the subject. The interest to the Tangherlini transformations has been due to the fact that they can be useful in the search for a theoretical explanation of a possible "delicate violation" of the Lorentz-invariance (this is employed sometimes in order to explain several exotic phenomena such as the origin of high-energy cosmic beams, the origin of dark matter and dark energy, several cosmological models, and also quantum gravity models, see [15–24]). Anti-relativists suggest that the Tangherlini transformations can be used as a proof for an absolute (preferred) inertial frame of reference, which is connected with the "luminiferous ether", and as a proof for the violation of Lorentz invariance in physical phenomena. Hence, they declare that the Tangherlini transformations refute the validity of the Special Theory of Relativity. Other researchers emphasize that the Tangherlini transformations are not merely an alternative but an equal replacement for the Lorentz transformations, hence the Special Theory of Relativity is no more than one of many equivalent theories describing physical processes from the viewpoints of observers located in two different inertial frames. Probably, not many of the participants who have discussed the Tangherlini transformations (this discussion started in 1977, after Reza Mansouri and Roman U. Sexl published the paper [25]) have read Tangherlini's original Dissertation of 1958 [4], wherein he deduced the transformations and studied their applications to classical theory and quantum theory. Most scientists know only the later publications of 1961 and 1994 [11, 26], which give the transformations with different deductions and fewer applications. For more than fifty years the Dissertation was able to be accessed, as a manuscript copy, only at the Stanford University Library. Its appearance here in the present issue of *The Abraham Zelmanov Journal* [4] marks its first date of publication. Moreover, it is prefaced by the very interesting discussion written by Frank Robert Tangherlini himself [27], and is accompanied by his fine comment [28] to §8.1 of [11], where he gives an explanation about several properties of the Maxwell electrodynamics on the basis of the transformations.

In his Dissertation [4], Tangherlini uses a special system of units which is specific to several studies on the theory of relativity. In this system, time has the same dimension as length (so that Tangherlini means time t as a regular time multiplied by c, the velocity of light in vacuum), while velocities are dimensionless. In other words, a velocity he uses is a regular velocity divided by c. As a result, the velocity of light in vacuum in this system of units equals 1. This makes the understanding of the obtained result a little complicated. Meanwhile, in the preparation of the manuscript for publication, the Editor of the journal and the Author have decided to keep the original notations and the system of units unchanged for historical reasons. In contrast, in what follows, we will use the regular system of physical units when discussing the Tangherlini transformations and their sequels in physics, which will make their understanding a lot easier.

The main task we are targeting in this paper, which accompanies Tangherlini's Dissertation [4], are comments on the results obtained therein. The biography of Tangherlini and the circumstances of his Dissertation were given in detail in the publications [29, 30]. We therefore limit ourselves to only a brief survey of the events.

Frank Robert Tangherlini was born on March 14, 1924, in Boston (Massachusetts, USA), in the family of a worker. In 1943, during the World War II, he volunteered to be drafted into the U.S. Army, and later volunteered, after arriving in England in 1944 as an infantry replacement, to serve in the 101st Airborne Division as a paratrooper. After parachute training in England, he was dispatched to France, and then fought in Belgium and Germany. He also participated in bloody Battle of the Bulge in the Ardennes, where many of his paratrooper friends were killed in action. In 1946, he returned to the USA and was honorably discharged from the U.S. Army. In 1948, he obtained his BSc (cum laude) from Harvard University, followed by an MSc from the University of Chicago (1952), and a PhD from Stanford University (1959). During 1952–1955 he worked in the space industry in San Diego, and during 1958–1960 he served as a National Science Foundation postoctoral fellow, and from 1960–1994, he worked as a faculty member, or on the staff of different universities and research institutes in the USA and Europe. At the present time he is retired. He lives in San Diego, California, where he is still active in scientific research. The field of his scientific interest is very wide and covers the Special Theory of Relativity, dimensionality of space, relativistic cosmology, Mach's principle, etc. His most important scientific results are the transformations he deduced while working on his Dissertation of 1958. Later, these became widely known as the *Tangherlini transformations* (in 1958 Tangherlini used another, less successful term the *Absolute Lorentz Transformations*). His chief supervisor in this work was Sydney David Drell (b. 1926), the well-known expert in Quantum Electrodynamics who later became a friend of Andrew D. Sakharov. At the initial stage of Tangherlini's work, his actual supervisor was Donald R. Yennie, also well-known for his work on quantum electrodynamics, and who had been appointed his supervisor by Leonard Isaac Schiff (1915–1971) who was the chairman of the Stanford physics department, and was also responsible for Tangherlini receiving a fellowship to continue his graduate studies at Stanford. In June, 1958, Tangherlini reported his results at a colloquium of physics at Stanford University, and then submitted the Dissertation to the Physics Section of the Graduate Division. Positive reviews of the dissertation were submitted by Drell and Schiff. Afterwards, following the receipt of a National Science Foundation postdoctoral fellowship, Tangherlini went to the University Institute for Theoretical Physics in Copenhagen (this Institute was later called Bohr Institute). While abroad, Tangherlini was graduated with a PhD degree from Stanford University in absentia.

The direct and inverse Tangherlini transformations are introduced by means of a special method of synchronization of spatially separated clocks located in two inertial frames of reference, one of which is taken to be the preferred frame in which the one-way speed of light is c, and the other frame is moving relative to it with speed v, in which the speed of light varies with direction and v as discussed below. In this method, the clocks are synchronized by signals travelling with an infinitely high speed. Already in 1898, Henri Poincaré in his paper [31] and his address [32] delivered on September 24, 1904, at the Congress of Arts and Sciences in St. Louis (Missouri, USA), pointed out the significance of faster-than-light synchronization methods in effecting the outcome of the measurements of the velocity of light, and some years earlier, in 1898 [31] had given an interesting discussion of the meaning of simultaneity. This problem was also considered in 1934 by Leonid I. Mandelshtam (1879–1944), in his lectures on the physical grounds of the theory of relativity [33] read in 1933–1934 at Moscow University. Sergey M. Rytov (1908–1996) restored Mandelshtam's lectures after his death, on the basis of the records made by Gabriel S. Gorelik, Maxim A. Divilkovski, Michael A. Leontovich, Z. G. Libin, who heard the lectures, and also on the basis of Mandelshtam's draft notes. Then Rytov published the lectures in 1950 [33] (second edition [34] was printed in 1972). In his lecture, held on March 10, 1934, Mandelshtam was engaged in polemics with anti-relativists on the formulation of the causality principle in the Special Theory of Relativity, and on the problem of the simultaneity of events in different inertial frames of reference. In these polemics, Mandelshtam focused the attention of the listeners on the fact that Reichenbach's method of synchronization does not violate the causality principle*. He said:

> "Thus, the requirement that the causality principle must not be violated in the definition of simultaneity can be simply satisfied ... if there would be a signal travelling at infinite speed, the requirement that the causality principle must be true would give a simple condition universally to all frames [of reference]. ... Thus, it is required to understand that there should not be *such* a faster-than-light signal, which may generate *action*. ... If I make use of a process, which cannot produce action, this does would violate the causality principle. ... Many researchers tried to give such a definition to simultaneity, which they believed does not depend on the possibility of its empirical determination, but rather arises from the supposition that there is an apriori simultaneity".†

Proceeding further, Mandelshtam considered a possibility of synchronization of spatially separated clocks, in different inertial frames of ref-

*Here Mandelshtam obviously considered the method of synchronization of spatially separated clocks suggested by Hans Reichenbach (1891–1953) in [35,36], which is quite different from the synchronization method suggested by Einstein [14].

†Here Mandelshtam attempted to focus the attention of the listeners on the fact that synchronization of clocks by infinite speed signals leads to simultaneity in all inertial frames of reference.

erence, by phase velocity signals (phase velocity may be as faster than light as possible). Such signals transfer neither action nor information. Therefore, if a phase velocity approaches infinity, the causality principle is not violated. Unfortunately, Mandelshtam, in his lectures [33, 34], arrived at the conclusion that this method of synchronization is unable to be realized in practice. This is because he considered the phase velocity of a signal produced by a mechanical device like scissors, namely — the motion of the cross-point of a scissors, which is realized with only a finite phase velocity. Therefore Mandelshtam had not realized the principal step in this research: he had not deduced transformation of the spatial coordinates and time from one inertial reference frame to another one, synchronized by infinite speed signals. It is probable that 1934 was too early a time for this principal step that was made only 24 years later, by Tangherlini.

In this connexion, we should emphasize an interesting and important paper, published by Albert Eagle, the British mathematician who, already in 1938, was extremely close to the Tangherlini transformations. In his paper [37], Eagle considered a method of synchronization of spatially separated clocks by a mechanical shaft, rotating by a clock engine located at its centre. The clocks under synchronization were fixed at the shaft's butt-ends. This method of synchronization was also considered in Eagle's second paper [38]. Albert Eagle, being an obvious anti-relativist*, held the erroneous belief that a mechanical shaft rotates as a perfectly rigid body, so that torsional perturbations would travel instantaneously along it. Meanwhile, as we know, the perturbations do not propagate instantaneously along the shaft, but with the sound velocity specific to the substance of the shaft. This velocity is many orders slower than light†. Eagle targeted his publications [37, 38] as a proof for the reality of an absolute (preferred) reference frame con-

*This fact concerning his personality can be easily concluded from his papers [37, 38] and, especially, from his other paper [39].

†We note that, aside for this simplest reason, synchronization of clocks by a rotating shaft is sensitive to the relativistic change of the figure of the shaft in a resting inertial frame of reference, as was considered by Ives (1882–1953), in [44], in the example of a rotating double Fizeau cogwheel (actually, a double obturator). When Stefan Marinov (1931–1997) performed his single-way measurements of the velocity of light using two mechanically connected systems consisting of rotating mirrors [45], he met a criticism from the side of Simon James Prokhovnik (1920–1994) who pointed out the inconsistency of this method of synchronization [46, 47]. Despite this criticism, Marinov continued measurements based on this synchronization method, but with another mechanical system which was similar to the previous one (it was a modified double obturator called by him the "coupled shutters" system). See [48] and literature referred therein, for Marinov's experiments.

nected with "luminiferous ether". Meanwhile, in spite of his incorrect considerations on the basis of the Newtonian views on absolute simultaneity which were already obsolete in 1938, Eagle [37] arrived at the transformations of the spatial coordinate x and time t associated with an observer's inertial frame of reference to the corresponding space-time coordinates in another inertial reference frame in the x-direction. Eagle's expressions for the transformation of x and t agree with the corresponding expressions of the Tangherlini transformations (1). Direct and inverse transformations for the transverse coordinates y and z were not discussed by Eagle, but he did note that the inverse transformation for the x and t coordinates was not of the same form as the direct transformation, as did Tangherlini.

Unfortunately, Eagle's key paper [37] was ignored by the scientific community from the time commencing in 1938 when it was first published. The sole reference to this paper, which we have found in the scientific literature, appeared in Eagle's second paper [38]*. Why? No simple answer can be given to this question. In any case, whatever be the answer, the real physical sense of the transformations was only achieved 20 years later in Tangherlini's thesis of 1958, in which their derivation and application is discussed clearly and in detail. Dmitri Rabounski, who also discussed Eagle's papers [37, 38], commented this situation as follows [41]:

> "After reading Eagle's paper of 1938, and his following paper, I arrived at the conclusion that Eagle obtained his transformations of the spatial coordinate and time, which particularly meet the Tangherlini transformations, as a result of a formal blindfold of combinations, not a systematical research. Besides, he was mistaken about the obtained result due to his erroneous disbelief in the theory of relativity. His appeal is that the presence of a physical medium fixed to (accompanying) the space as a whole is in contradiction with the theory of relativity. This is absolutely wrong and seems naive. He merely had no clear understanding of the theory of relativity — the geometrical theory of space-time and matter — and how the theory works. The second paper authored by Eagle is a logical continuation of his erroneous views on the theory of relativity, based on the principles of classical physics. According to him, "true synchronization" is synchronization of

*Max Jammer in his book *Concepts of Simultaneity* [40], published in 2006, refers to authors who criticized Eagle's synchronization procedure. Tangherlini refers to Jammer's book in his *Preface of 2009 to "The Velocity of Light in Uniformly Moving Frames"* [27].

> clocks in the Newtonian sense, while all the remaining methods of synchronization leads to non-observable (imaginary) effects, which are unable to be registered in real measurements. In particular, Eagle referred to his first paper of 1938 as an example of how the use of "true synchronization" manifests that all effects of the theory of relativity are non-real, imaginary. Eagle's conclusion is in contradiction to the many decades of experimental verifications of the theory of relativity performed in different experiments with high precision of measurement. In fact, Eagle had no idea about the real physical sense of his formal mathematical deduction. He was in captivity of the views of classical physics, and failed the possibility of all other research methods in physics (the theory of relativity, for instance). This is the same as, given all explanations on the basis of the wave theory of light, denying all the results obtained in the framework of the corpuscular theory".

In continuation of this discussion, Tangherlini wrote recently in his private letter dated June 01, 2009 [42]:

> "With respect to Dr. Rabounski's penetrating criticism, I would add further that Eagle's obvious anti-relativity bias led him to reject the general theory of relativity, and this was most ironic, and indeed tragic, because a key principle in general relativity is Einstein's principle of general covariance which permits arbitrary transformations of the coordinates, and hence permits the transformation Eagle was using when supplemented by the transformation for the transverse y and z coordinates. In fact I would like to take this opportunity to emphasize to you and Dr. Rabounski that I probably would never have undertaken the writing of my thesis were it not for the *logical justification provided by general covariance* for making use of such a transformation when supplemented by the metric postulate as mentioned in the Introduction to my thesis".

Meanwhile, we should pay tribute to Eagle's paper [37], despite the fact that Eagle himself misunderstood the physical sense and meaning of his pioneering result. He was the first person who obtained, in a purely formal way, a part of the common transformations of the spatial coordinates and time which were developed in detail later by Tangherlini, and are known as the Tangherlini transformations.

What is interesting is that, in already 1922, a method of synchronization of spatially separated clocks by means analogous to a locomotive wheel pair was suggested by Carl Axel Fredrik Benedicks, the Swedish physicist who considered a rotating wheel (that can be any space body, the Earth for instance) as an original time-giver [43]:

> "Let us return to the top, which seems to offer a somewhat clearer example of an original time-giver. ... the question arises, how its time-indication may be applied to another process which might occur at a great distance. The simplest example is that where the second process is also an identical, rotary one that is, another identical top, rotating on the same fixed plane. Simultaneity or synchronism is said to prevail if a radius vector of the one is always parallel to a corresponding radius of the other. We ask, in what way can this definition be applied? Evidently, it can be realized in the way used to synchronize two paired wheels of a locomotive; that is, a solid movable connecting rod is pivoted at the end points of two radii, where the length of the rod is equal to the distance separating the two axles. As the two radii have been assumed to be equal, they will during the motion also remain parallel. In principle this will fully define simultaneity, so long as the axes of rotation remain parallel. The first rotating body, A, is the standard which determines time; the second body, B, may act as a *clock* or timepiece, by exactly reproducing A's time. ... We say that *two distant clocks are synchronous, provided that their hands are moving as though their axles were connected by one rigid axle, consisting of an absolutely solid body.* This is the simpler form of synchronizing frequently used, for example, in synchronizing two wagon-wheels belonging to the same axle. ... This definition of synchronism is precise, and has no ambiguity. It is founded only upon the fundamental basis for all measurement of time the accepted unchangeability of the rotation process chosen as standard and upon pure geometry the fundamental basis of which is the existence of the absolutely solid body".

This method of synchronization meets that suggested by Eagle [37, 38]. However, in contrast to Eagle, Benedicks [43] had no idea about respective transformations for the spatial coordinates and time. Also, Benedicks assumed that there are absolutely rigid bodies, i.e. bodies in which signals propagate instantaneously from one end to the other, and hence his proposal encounters the same difficulty as Eagle's proposal, and as was remarked earlier, he seemed unaware that torsional waves, or waves in material bodies more generally, propagate with a finite velocity.

Tangherlini also suggested another method of synchronization of spatially separated clocks, the "external synchronization", which was given in detail later, in his general relativity article [11]. He writes [42]:

> "It was the difficulty I encountered with the failure to find empirical evidence to support faster-than-light signalling that led me to turn to external synchronization, which represents an experimental procedure one can carry out now with existing equipment, that can be used to verify the basic predictions of the transformation empirically, as you discuss. ... Actually *I did not make any hypotheses as to how the instantaneous signals would arise* in my thesis, so my synchronization would include the light spot method, but would be more general. The tachyons (as they are now called) were only mentioned in the concluding Chapter 12 of the thesis as a conceivable way of implementing the instantaneous synchronization, but the argument in the body of the thesis is *independent* of any way of achieving such a synchronization; in a sense, therefore, it is purely mathematical of the type: if we assume X, then the following is the case. ... Actually, I didn't mention external synchronization there, until my later publication *An Introduction to the General Theory of Relativity* (see in *Supplemento al Nuovo Cimento*, 1961, ser. X, vol. 20, 1–86)".

Many physicists believed (and, indeed, still believe) this to be impossible, because superluminal speeds are attributed only to hypothetical faster-than-light particles — tachyons* [49, 50]. In fact, synchronization of this kind can be performed. The easiest case is the one where all clocks of both the moving inertial frame of reference and the resting frame of reference are located along the same line. To perform such

*Tachyons — faster-than-light particles were first coined in the scientific publications on the theory of relativity in the pioneering paper of 1962 [51], authored by Olexa-Myron Bilaniuk, Vijay Deshpande, and George Sudarshan, who worked in Department of Physics and Astronomy, University of Rochester, New York. They pointed out the historical fact that, in pre-relativity times, Thomson, Heaviside, and Sommerfeld had considered particles moving faster than the velocity of light in vacuum. They considered the possibility of such particles in the framework of the Special Theory of Relativity. This term, "tachyon", was introduced into science by Gerald Feinberg (1933–1992) five years later, in 1967 [52], while Feinberg worked at Rockefeller University, New York, the same city as his predecessors. In this background story, many researchers and historians of science missed the fact that Frank Robert Tangherlini was actually the first person who considered the possibility of tachyons and faster-than-light signals, in a very general sense, in the framework of the Special Theory of Relativity, already in 1958. Unfortunately, most papers and books on the history of tachyons do not mention this fact. See [53], for instance. Meanwhile, the most important surveys of this theme such as [50, 54] referred to Tangherlini's goal with respect to this problem. In the last decade [55, 56], the possibility that tachyons had been produced was investigated at CERN.

"instantaneous" synchronization in this case, the "light spot method" should be used, where a light spot travels with a phase velocity which may exceed the velocity of light. This method was suggested by Vitaly L. Ginzburg (b. 1916) in [57], and, in more detail, in the common paper by Boris M. Bolotovskiĭ (b. 1928) and Vitaly L. Ginzburg [58]. In these papers, the motion of a light spot along a screen was considered, where the light spot was due to a light beam produced by a source (searchlight) rotating with an angular velocity Ω. If two points, say A and B, are equally distanced at a very large distance R from the searchlight, the linear velocity v of the light spot on the screen should satisfy the condition $v = R\Omega \gg c$. Of course, a light spot cannot transfer energy/information from A to B (with any velocity, both subluminal and superluminal): photons coming in A never come to B, hence the causality principle is still true, without violation in the experiment. Thus, huge speeds much faster than light are attributed to the light spots produced by the radiation of pulsars [57–60]. More details about synchronization of distant clocks by the light spot method are considered in our works [60–62].

Thus, Frank Robert Tangherlini has deduced transformations for the spatial coordinates and time from one inertial frame of reference to another one in the case where clocks located in both inertial reference frames are synchronized by infinite speed signals in the sense of that which was considered above.

Although not discussed in his Dissertation [4], in his later investigations, Tangherlini also considered another method of synchronization not involving faster-than-light signals, as described in detail in Appendix A of his 1994 paper *Light Travel Times Around a Closed Universe* [26]. This is the so-called "external synchronization", consisting of two steps. First, light signals synchronize clocks which are spatially separated from each other in the same resting ("preferred") inertial frame of reference. Then, these already synchronized clocks of the "preferred" inertial frame are used for synchronization of clocks of moving inertial frames of reference during those moments of time when each of the moving clocks physically meets one of the resting (synchronized) clocks in space. In this method of synchronization, inertial frames of reference are non-equal to each other: the inertial frame where clocks were first synchronized is "preferred" to all remaining (moving) inertial frames.

What is interesting is that Mandelshtam, already in 1934, considered a method of synchronization, which would also give rise to absolute simultaneity in the inertial reference frames K and K', as he described in his lectures [33, 34]:

> "...Suppose we have a frame [of reference], and one set up synchronization in it by a method, for instance — by the Einsteinian method ... Let us also have another frame [of reference]. I could arbitrarily set up synchronization in this other frame [of reference] so that clocks located in it would always show the same time as that displayed by the clocks of the first frame [of reference]".

This is however not the same as Tangherlini's method of external synchronization. Tangherlini, when found Mandelshtam's achievements in the 2000's (his lectures were published only in Russian, so inaccessible to most scientific community), says [42]:

> "This is *not* true for my transformation because (think of the clocks on the train going past the station) after the brief moment of synchronization, the clocks on the train run more slowly than the clocks on the station platform, this is why high energy muons decay significantly more slowly than muons at rest in the laboratory!"

Meanwhile, with use of both methods of synchronization, not all inertial frames of reference are equal: that inertial frame of reference, wherein the first synchronization was performed, becomes preferred to all remaining inertial reference frames. In particular, Mandelshtam wrote [33, 34]:

> "...in this case, we cannot require the relativity principle. ... When Einstein says that the relativity principle takes a place in nature, this means that, if all definitions of age are given equally in any frames [of reference], events [in any reference frames] will be processed equally".

Mandelshtam had not realized any step towards respective transformations in this case. This fact manifests, again, the outstanding thinking and courage Tangherlini showed in scientific research: he worked without looking back on the authoritative persons in science, mostly conservators, so he reached the advanced results which were out of the access for many other scientists.

Consider the direct and inverse Tangherlini transformations

$$\left.\begin{array}{ll} x' = \gamma\,(x - vt)\,, & x = \gamma^{-1} x' + \gamma v t'\,, \\ y' = y, & y = y'\,, \\ z' = z, & z = z'\,, \\ t' = \gamma^{-1}\,t\,, & t = \gamma t'\,, \end{array}\right\} \qquad (1)$$

where x, y, z, t and x', y', z', t' are the spatial coordinates and time in the inertial reference frames K and K' respectively, c is the velocity of light in vacuum, v is the velocity of the reference frame K' with respect to the preferred reference frame K (we assume it to be moving along the x-axis), while $\gamma = 1/\sqrt{1-v^2/c^2}$ is the so-called Lorentz-factor.

Comparatively, the classical Lorentz transformations are

$$\left.\begin{array}{ll} x' = \gamma\,(x - vt)\,, & x = \gamma\,(x' + vt')\,, \\ y' = y\,, & y = y', \\ z' = z\,, & z = z', \\ t' = \gamma\,(t - vx/c^2)\,, & t = \gamma\,(t' + vx'/c^2)\,. \end{array}\right\} \qquad (2)$$

It is obvious that, according to the Tangherlini transformations (1), time t' of a moving inertial frame of reference is slower than time t by the factor γ. That is, the Tangherlini transformations lead to the transverse (relativistic) Doppler effect as in Einstein's Special Theory of Relativity [14]. The direct Tangherlini transformations, which are the first column in formula (1), differ from the direct Lorentz transformations, the first column in (2), by only the transformation of time due to the different methods used in the synchronization of clocks in inertial frames of reference.

Both direct and inverse Lorentz transformations take the same form upon reflecting the velocity, while the direct and inverse Tangherlini transformations are not (see the final part of Tangherlini's comment [28] for detail). Besides, in contrast to the Lorentz transformations, which form a Lie group whose parameter is the velocity, the Tangherlini transformations do not form such a group, since as is clear from (1), the inverse of the Tangherlini transformation does not have the same form as the direct transformation, nor do the product of two such transformations have the same form as the direct transformation. On the other hand, Tangherlini [63] has pointed out that the transformations are members of the group of linear space-time transformations that keep simultaneity invariant, of which the Galilean transformations form a proper Lie subgroup. Also, along with the Lorentz and Galilean transformations, the Tangherlini transformations are also members of the linear unimodular group, i.e., the group of linear space-time transformations with unit determinant, as mentioned in his thesis. The asymmetry of the direct and inverse Tangherlini transformations is connected with the fact that two inertial reference frames K and K' having Galilean (rectangular Cartesian) coordinate frames are equal in the framework of the Lorentz transformations, while in the framework of the Tangherlini transformations the inertial reference frames are non-equal: K still possesses the Galilean (rectangular) coordinate frame because the observer is resting in this frame, while the coordinate frame of K' is non-Galilean (oblique-angled).

Here we should note that the term "Lorentz transformations" was introduced by Henri Poincaré [64, 65]* in 1905, and he also discussed their group properties, as did Einstein in that same year.

Proceeding from the Tangherlini transformations (1), one can obtain a formula connecting the co-linear velocities V and V', measured in the inertial reference frames K and K' respectively, or, similarly, the law of composition of velocities according to the Tangherlini transformations

$$V' = \frac{V - v}{1 - \frac{v^2}{c^2}} \tag{3}$$

as originally obtained by himself in [4]. As visible, this formula is quite different from the law of composition of velocities $V' = \frac{V - v}{1 - vV/c^2}$, which holds according to the Lorentz transformations.

Having the law (3) as a base, Tangherlini [4] has also obtained a formula for the velocity of light in vacuum, measured in a moving inertial reference frame K', and referred to as c'

$$c' = \frac{c}{1 + \frac{v}{c}\cos\theta'}\,, \tag{4}$$

where the angle θ' is counted from the x'-axis in the moving inertial frame K'.

In a common case, where light travels in an optical medium whose refraction coefficient is n, measured in a resting inertial reference frame K, the velocity of light in a moving inertial reference frame K', according to Tangherlini [4] takes the form

$$c' = \frac{c}{n + \frac{v}{c}\cos\theta'}\,. \tag{5}$$

It is clear that formula (4) explains the results obtained in the Michelson-Morley experiment [2, 3] and the Kennedy-Thorndike experiment [67]. This is because, following from (4), the total time of light's travel forward and backward does not depend on the velocity v of the

*Hendrik Antoon Lorentz (1853–1928) passed through a long way, full of many tests, to his understanding of the Special Theory of Relativity. He stopped his research when was at a minor step from the acquisition of the transformations [66].

inertial reference frame K' moving with respect to the preferred inertial reference frame K. Moreover, as was shown in [1], the Tangherlini transformations provide a clear explanation to all interference experiments targeted on checking the Special Theory of Relativity, in particular — the Sagnac experiments [6–10]. And also, as remarked above (see page 121), Tangherlini synchronization can be used to carry out calculations in a rotating frame of reference in which it is not possible to synchronize clocks by the standard methods of Special Relativity.

A simple explanation of the physical sense of the Tangherlini transformation is given by Giancarlo Cavalleri and Carlo Bernasconi [15]. The invariance of the velocity of light and the non-conservation of the simultaneity of events, spatially separated in different inertial frames of reference, are often considered as specific properties of the Special Theory of Relativity. Therefore, Tangherlini [4] formulates his own version of the theory of relativity, where the absolute simultaneity of spatially separated events is allowed, and the velocity of light becomes non-invariant. In the framework of the "standard" Special Theory of Relativity, the velocity of light determined by formula (4) should not be a physical (observed) velocity, but a coordinate velocity. Actually, the conservation/non-conservation of simultaneity and the invariance/non-invariance of the velocity of light depend on the employed method of synchronization of clocks, located in different inertial frames of reference. This fact leads to an infinite number of versions of transformations from one inertial reference frame to another one [5]. In this row, two kinds of transformations — the Lorentz transformations and the Tangherlini transformations — are the limiting cases of the Sjödin transformations [5].

Michael A. Miller, Yuri M. Sorokin, and Nikolai S. Stepanov [68], and then Anatoly Logunov [69] take under consideration an arbitrary linear transformation from Galilean coordinates x, y, z, t of an inertial reference frame to coordinates X, Y, Z, T of a so-called "generalized" inertial reference frame. Such transformations mean non-orthogonal (oblique-angled) coordinate nets X, Y, Z, T [68] described by the metric tensor whose components are constants. As was shown in [69], if we assume different parameters of the components of the metric tensor in the "generalized" inertial frame of reference, different formulae can be obtained for the components of the coordinate velocity of light (V_X, V_Y, V_Z) in the "generalized" frame, and the velocity is anisotropic in a general case, while in contrast, in the Special Theory of Relativity, as is well-known, the speed of light takes the same value for all inertial frames of reference and is isotropic.

Tangherlini [70] was able to show that when the standard canonical commutation relations of Quantum Mechanics in the Schrödinger representation are enlarged to included the energy and time, and one assumes that the energy and momentum transform as a four-vector, these commutation relations are not only invariant under the Lorentz transformations, but under all non-singular linear space-time transformations, which would include the general transformations discussed above.

The main advantage of the Lorentz transformations, in contrast to the other kinds of transformations of the spatial coordinates and time, consists in the Einsteinian method of synchronization of spatially separated clocks that keeps the velocity of light isotropic and constant in transferring from one inertial frame of reference to another one. In "generalized" inertial frames of reference, which are a result of the Tangherlini transformations in particular, no so-called pseudo-forces (the centrifugal force of inertia, or Coriolis' force, for instance) appear as in non-inertial frames of reference where such forces play the rôle equal to the force of gravity. In this connexion, incorrect claims, from the past to the present, can be found in the scientific publications. Thus, Hans C. Ohanian [71] claimed, incorrectly, that the Reichenbach method of synchronization of clocks [35, 36], upon being realized in an inertial frame of reference, should inevitably lead to the formal appearance of the pseudo-forces in the inertial reference frame.

At first, the Tangherlini transformations did not attract much of the attention of scientists. This situation changed after 1977, when an anisotropy in the Cosmic Microwave Background Radiation had been definitely verified in observations on board a U2 sub-stratosphere airplane performed by George Smoot's team [72]*. In fact, this means that the inertial frame of reference connected with the Earth moves in the cosmos with a velocity of about 360 km/sec with respect to a preferred inertial frame of reference, in which the Microwave Background Radiation is "most" isotropic and the common momentum of all masses of the Universe is probably zero. As a result of the experimental success, different suggestions arose to the origin of the observed anisotropy in the Cosmic Microwave Background Radiation as due to the anisotropy of the velocity of light, so the Tangherlini transformations became of interest. The first persons who turned our attention to the Tangherlini transfor-

*The dipole-like anisotropy was first observed in the ground-based observations performed by Edward K. Conklin in 1969 [73], then studied in the balloon observations by Paul S. Henry in 1971 [74] and by Brian E. Corey and David T. Wilkinson in 1976 [75]. The main reason for Smoot's success of 1977 [72], and his fame which followed later, was very certain observations of the anisotropy.

mations as a possibility of explaining the results of the Michelson-Morley experiment [2, 3], following Tangherlini himself*, were Reza Mansouri and Roman U. Sexl [25]: they said in the bibliography to their first paper that the transformation had been considered by Tangherlini. Afterwards, many papers were published, wherein the Tangherlini transformations were employed: see, for instance, [5, 15, 71, 76–91].

There are also numerous papers wherein the Tangherlini transformations were "re-discovered" anew. These are Stefan Marinov's publications of the 1970's [92–94], the paper of 1992 [95] authored by Ernest W. Silvertooth and Cynthia K. Whitney, the papers [96–98] published by Nikolai V. Kupryaev commencing in 1999, and the paper of 2001 [99] by Juri A. Obukhov and Igor I. Zakharchenko. Looking along the scientific literature, we found a note on the absence of priority concerning the earliest of the "re-discovering" papers: Giancarlo Cavalleri and Giancarlo Spinelli [100] commenting on the transformations appearing in Marinov's publications of the 1970's, and claimed by him as his own original achievement, gave the priority to Tangherlini who had actually obtained these already in 1958, although they were not published in a journal until 1961 [11], and it was to this article to which Sexl and Mansouri referred. All the rest of the papers "re-discovering" the Tangherlini transformations were published only much later, commencing in the 1990's, so the absence of priority in those papers was not found somewhere being discussed in the scientific literature.

Interestingly, Frank Robert Tangherlini met Stefan Marinov in person at the *General Relativity 9th Meeting* in Jena, Germany, in 1980. Tangherlini wrote, in his private letter dated October 14, 2006, about how this happened [101]:

> "I met Marinov under a most curious circumstance: he had put up over the doorway of a hall, where many passed through, a poster of about 1/3 meter wide and about 2 meters long in which he criticized me, in artistic calligraphy, for not having followed up on my transformations. I found this very strange behaviour. After all, why didn't he write directly to me, or arrange a meeting at a conference? So I suspected then he was somewhat crazy, although possibly artistically talented. With any crazy person, one shouldn't spend too much time on him except as an example of how people in science, just as in every day life, can go astray".

*In his Nuovo Cimento article of 1961 [11], Tangherlini wrote: "Finally we should note that the usual results of special relativity can obtained from the line element (1.17) and co-ordinate transformation (1.16), as we have already shown for the problem of sending light signals out and back".

In recent years, a second wave of increasing interest in the Tangherlini transformations has risen due to the possibility of a small anisotropy of the velocity of light claimed by the Grenoble group of experimentalists [102, 103] (see also [104–106]). At the present time, there are neither definitely verified experimental facts nor fundamental principles of physics which could require the failure of the Lorentz-invariance in inertial reference frames (see [33, 107], for instance). Meanwhile, physicists are still continuing experimental and theoretical attempts to find violations of Lorentz invariance, and also theoretical grounds to these in the course of interpretation of bizarre physical phenomena such as those in cosmology, quantum gravity, quantum field theory, particle physics, space beam physics and super-high energy physics (see, in particular, [18, 19, 107]). One regularly connects this possible violation, without which CPT-invariance of quantum field theory and the law of charge conservation of classical electrodynamics cannot be violated, with a possible violation of the space-time symmetry due to, say, processes at the Planck (small) scale or due to additional (hypothetical) measurements producing a new vector or tensor field which acts onto physical bodies depending on their velocity and orientation in space (which is different for particles and anti-particles). As a result, theoretical physicists expect various new effects such as a length contraction and time dilation in addition to the Lorentz ones, a variation of the electromagnetic field polarization, a non-zero rest-mass of photons, changes of the masses of decaying particles and of their decay channels, and also many other effects which depend on the motion of the inertial frame of reference wherein the processes occur. The simplest case of theories that violate Lorentz invariance is the so-called Doubly Special Theory of Relativity (see [20, 108–111], for instance), wherein elementary particles cannot be accelerated up to a velocity exceeding the velocity of light, nor can they acquire an energy exceeding a fixed numerical value specific to each particle (the so-called Planck energy). The aforementioned vector or tensor field has no direct connexion to the gravitational field. Whether such possible violations of Lorentz invariance would lead to changes of the gravitational field of a moving body, or to changes of the properties of a black hole that are not predicted by the General Theory of Relativity are issues for further research.

Putting aside gravitation, it is useful to study various consequences of the violation of Lorentz invariance. In particular, as already discussed above, the possibility of introducing alternative methods of synchronizations in a given inertial frame. In this regard, it is important to keep in mind that whether clocks are synchronized according to the Einstein procedure, or externally, one has not changed inertial frames, but effectively one has merely introduced another set of clocks in the same inertial frame. In addition to the Tangherlini synchronization which has already been described, still another form of synchronization has been suggested by Torgny Sjödin in his paper of 1979 [5] in which clocks are synchronized etc. The discussion in Chapter 6 of Tangherlini's thesis, entitled *Measurements with Signals Travelling with Finite Velocities*, to some extent anticipates Sjödin's considerations, in that Tangherlini considers the possibility of other signals propagating with constant speeds greater than or less than the speed of light relative to the rest frame.*

Even if digressing from gravitation and other effects of the General Theory of Relativity, different scenarios of the violation of the Lorentz-invariance are useful to be studied on the basis of not only the Lorentz frames of reference (their preference is due to the Einsteinian method of synchronizations of clocks, where the out and back travel times for light are equal), but also on the basis of other inertial frames in which there has been an alternative synchronization of clocks. In particular, the time dilation is to be considered/described by the use of a frame of reference whose clocks are synchronized by infinite speed signals (in practice — a respective light spot). Such reference frames were studied by Tangherlini, when he compared descriptions of physical processes obtained in such a reference frame to the well-known Lorentz description. A larger class of *alternatively synchronized inertial frames*, where clocks are synchronized by signals travelling with a finite speed which can exceed the velocity of light, was suggested later by Torgny Sjödin in his paper of 1979 [5].

The Tangherlini transformations and also the Sjödin transformations which generalize them gave rise to a substantial discussion a quarter century ago, and then found respective places in the Special Theory of Relativity. Despite the fact that the Tangherlini and Sjödin transformations can yield the same results as the Special Theory of Relativity, these transformations are more complicated than the Lorentz transformation since they don't leave the speed of light invariant. However, physicists will probably turn to these transformations each time when there is

*In this concern, Tangherlini writes [42]: "... changing synchronization *does not change the inertial frame*. Think of it this way. You have a train moving with constant velocity relative to the railroad station. You may synchronize clocks according to Einstein on the train, or according to my method, which is related by a local time transformation to the Einstein synchronization, or to that of Reichenbach, or to that of Sjödin, but that doesn't change the uniform motion of the train. It is only when one considers transformations from, say, the station to the train, or vice versa that one has changed inertial frames".

even the smallest chance that they are encountering Lorentz-invariance violating effects in their experiments.

The anisotropy of the coordinate velocity of light c' (3), measured in a moving inertial frame of reference K', is the price one has to pay to keep simultaneity unchanged between all inertial frames of reference. Note: *within a given inertial frame*, there is agreement everywhere in *that* frame as to when two events are simultaneous, after the clocks have all been synchronized, say, by the Einstein method, and in this sense *simultaneity is absolute within a given frame*. It is whether simultaneity within one frame agrees with simultaneity within another frame that the problem of relative simultaneity arises.

Because the Tangherlini transformations are linear, Maxwell's equations are invariant with respect to the transformations. Meanwhile, as shown by Tangherlini [4], in a moving inertial reference frame K' an effective "optical medium" appears which makes the velocity of light different in the forward and backward directions, with respect to the motion of K'. Hence, the Tangherlini transformations in common with the Lorentz transformations can provide adequate description of physical processes in a moving inertial frame of reference, but the Lorentz transformations are more useful in this deal because they keep the velocity of light constant and isotropic in all inertial frames of reference.

Finally, it should be mentioned that Tangherlini in his thesis used the fact that since Maxwell's equations can be written in generally covariant form, they obviously hold under his transformations as well as for the Lorentz transformations. However, because his transformations are linear and unimodular, as are the Lorentz and Galilean transformations, and also include the Lorentz contraction and time dilation, which the Galilean transformations do not, he found that despite the difference with the Lorentz Transformation as to synchronization, a set of tensorial expressions for the electromagnetic fields could be extracted that were exactly the same as for the Lorentz transformation, so that the equations of motion of a charged particle, when written in term of proper velocity and proper acceleration (i.e., derivatives taken with respect to proper time) could be written so as to take the *same form* as for the Lorentz transformation, and, importantly do not involve the velocity of the moving frame relative to the rest frame. He points out in the thesis that there exist a second set of equations of motion which do not reduce to the equations of motion, as seen by the observer in the moving frame who uses the Lorentz transformation, that explicitly involve the velocity of the moving frame relative to the rest frame, but that in the absence of any way to synchronize the clocks in the moving frame with the rest frame, these equations of motion are unobservables. He also points out that the d'Alembertian operator is not invariant under his transformation and that this is a consequence of the fact that the one-way velocity of light in the moving frame has not remained invariant in the moving frame as is the case for the Lorentz transformation, but that in the absence of the possibility of synchronization with the rest frame, this anisotropy is unobservable, in agreement with observation.

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The EGR Theory: An Extended Formulation of General Relativity

Patrick Marquet*

Abstract: This is an extended formulation of General Relativity based on the existence of an additional segment curvature, due to the non-vanishing covariant derivative of the metric tensor. The resulting enlarged manifold allows for a permanent "free" field to exist next to the usual phenomenological energy-momentum tensor. This field may provide plausible explanation to further unanswered pending issues.

Contents:

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Notations:

To completely appreciate this article, it is imperative to define some notations employed.

Indices. Throughout this paper, we adopt the Einstein summation convention whereby a repeated index implies summation over all values of this index:

4-tensor or 4-vector: small Latin indices $a, b, \ldots = 1, 2, 3, 4$;

3-tensor or 3-vector: small Greek indices $\alpha, \beta, \ldots = 1, 2, 3$;

4-volume element: d^4x;

3-volume element: d^3x.

Signature of space-time metric:

Hyperbolic $(+---)$ unless otherwise specified.

Operations:

Scalar function: $U(x^a)$;

Ordinary derivative: $\partial_a U$;

Covariant derivative in GR: ∇_a;

Covariant derivative in EGR: D_a or $'$, (alternatively).

Tensors:

Symmetrization: $A_{(ab)} = \frac{1}{2}!\,(A_{ab} + A_{ba})$;

Anti-symmetrization: $A_{[ab]} = \frac{1}{2}!\,(A_{ab} - A_{ba})$;

Kronecker symbol: $\delta_{ab} = (+1 \;\text{ if }\; a = b; \;\; 0 \;\text{ if }\; a \neq b)$;

Levi-Civita tensor: ϵ_{abcd} (where $\epsilon^{1234} = 0$).

Three-dimensional vectorial quantities:

$\boldsymbol{P} = P_\alpha\,.$

Dedication:

The author dedicates this paper to the memory
of André Lichnérowicz (1915–1998).

As a young student in Theoretical Physics at the Faculty of Sciences-Jussieu-Paris, I enthusiastically began to follow the lectures on GR analysis given by Prof. André Lichnérowicz at the prestigious Collège de France where he held Chaire of Physique Mathématique since 1952.

He was himself a student of Georges Darmois, a pioneer in differential geometry, as well as Elie Cartan, gently nicknamed "Papa Cartan" because of his permanent availability to his students.

"Lichné" had certainly inherited those virtues: intelligent, broad-minded, curious and always accessible to "fresh men" as well as to post doctorate students. He generously arranged for me to have permanent access to the restricted library of the Institut Henri Poincaré, rue Curie, where I spent daylong research readings. This is indeed the evidence of a great humanist and a far reaching thinker internationally acknowledged.

It was then this autumn of 1985, when, already retired, he received me in his apartments, rue Paul Appel in South Paris, where we discussed the current status of Physics, although he had rather wished to remain a mathematician. (P. A. M. Dirac was in his opinion the greatest mind next to Einstein.)

In my last visit to him in 1995, I showed him a collection of my paper drafts including a new theory on a possible extension of General Relativity: he warmly encouraged me to proceed and kindly gave me several of his own text books, which I now modestly consider as a legacy.

As a dedication to his memory, I am honored to present here this paper.

Patrick Marquet

Introduction

As early as 1915, Einstein's General Theory of Relativity (GR) has successfully generalised Newton's original equations wherefrom most of the cosmological observations have been accurately described (to a certain extent).

As a possible doorway to further analysis, I would like to present here a new approach of the concept of gravity by considering a "free" gravity-like field which is assumed to be present and "localizable" throughout our Universe. Like the usual gravitational field classically resulting from the mass, this specific field interacts with matter and this coupling actually accounts for the known gravitational mass.

In this paper, our basic idea rests upon following observation. In the framework of classical physics, electrodynamics is described by means of two tensors:

- A pure electromagnetic field tensor described by Maxwell's tensor;
- A massive tensor which constitutes the charged particle.

Interaction of both quantities results in a conserved global momentum vector. Proceeding, in perfect analogy with the above, we suggest that gravitation also be described by two tensors:

- One tensor inherent to a pure field;
- The other tensor generalized only relative to the particle's mass.

By doing so however, we come across a major difficulty. Classical electrodynamics takes place in either an Euclidean space or on a Riemannian manifold. A straightforward gravitational analogy is not admissible, for whatever be the gravity field, it defines the space-time structure, which in turn will affect the matter field coupling.

Our line of attack consists of assigning to the macroscopic energy-momentum tensors a "dominant Riemannian" characteristic which is embedded in a more global geometry. In the framework of this scheme, the Riemannian physics would then just appear as a large scale approximation characterizing the elementary masses and energies, thus never conflicting with the known results of GR. On the very small scale, however, the non-Riemannian geometry is no longer negligible and its properties should be taken into consideration.

To achieve such a construction, we develop an antisymmetric extended (torsion-free) General Relativity (while keeping the four space-time dimensions), by ruling out the restrictive metric condition $\nabla g_{ab} = 0$, thus introducing a new connection built from the non-vanishing covariant derivative of the metric tensor.

The resulting enlarged manifold displays here an extra curvature called the *segment curvature*.

With this preparation, we can derive a generalized Einstein tensor denoted here the *EGR tensor* (i.e. the Extended GR tensor), which implies the existence of the so-called *EGR field equations* very close to the classical case.

In the absence of energy (e.g. mass), the EGR field equations however, do not reduce to the Riemannian source free equations: they actually always retain a "remnant-like" energy-momentum field tensor, which can be regarded as a vacuum "background" displaying a non-vanishing low level Riemannian part and non-Riemannian part due to the covariant derivative of the metric tensor.

As a conceptual gift, with this new theory, one no longer requires a "vanishing" (symmetric) gravitational energy-momentum pseudo-tensor [1] attributed to the mass and whose physical meaning has always remained unclear. In this sense, the EGR "residual" (true) field tensor is just a continuation of this pseudo-tensor when escaping a massive body. As well, its deep antisymmetric nature arises naturally from the theory.

It clearly confirms Einstein's early choice (as well as Dirac), and thus simply avoids the confusing controversy between the two versions. Last but not the least, the cosmological constant term $g_{ab}\lambda$, which is initially discarded in the text, automatically reappears under the form of a (small) term $g_{ab}J^2$ where J^2 is the square of a slightly varying four-vector fundamentally related to the extra segment curvature.

In our opinion, J^2, which prevails among other terms on the right hand side of the EGR field equations, has been "erroneously" approximated to the famous constant λ, thus misleading, since the complete structure of the EGR equations has been ignored.

Chapter 1. Gravitational Field: The Classical Theory

§1.1. The GR fundamental equations

Typically, the source-free field equations are non-linear equations of propagation which must contain derivatives of g_{ab} up to 2.

So, we consider the action

$$S = \int L_{\mathrm{E}} \sqrt{-g}\, d^4x\,, \qquad \det \| g_{ab} \| = g\,, \tag{1.1}$$

which must be stationary when the metric tensor is varied and where the Lagrangian L_{E} and its density $\mathcal{L}_{\mathrm{E}}$ are expressed with the Christoffel symbols as in the classical Einsteinian theory (Riemannian geometry)

$$\mathcal{L}_{\rm E} = \sqrt{-g}\, L_{\rm E} = \sqrt{-g}\, g^{ab} \left(\left\{ {}^{\,e}_{ab} \right\} \left\{ {}^{\,d}_{de} \right\} + \left\{ {}^{\,d}_{ae} \right\} \left\{ {}^{\,e}_{bd} \right\} \right) \tag{1.2}$$

being derived from the contracted curvature tensor (Ricci's tensor)

$$R_{bc} = \partial_a \left\{ {}^{\,a}_{bc} \right\} - \partial_c \left\{ {}^{\,a}_{ba} \right\} + \left\{ {}^{\,d}_{bc} \right\} \left\{ {}^{\,a}_{da} \right\} - \left\{ {}^{\,d}_{ba} \right\} \left\{ {}^{\,a}_{dc} \right\} . \tag{1.3}$$

Thus one infers the source-free field equations

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab} R = 0\,. \tag{1.4}$$

The Einstein tensor G_{ab} is a symmetric second-rank tensor, which is a function of only g_{ab} and their first and second derivatives. We have thus ten equations in (1.4) with partial derivatives which are not mutually independent.

There exists only 6 independent conditions, since the space-time coordinates can be subjected to an arbitrary transformation allowing us to choose four out of the ten generalizations of the metric tensor g_{ab}.

In order for the four conservation identities resulting from (1.4)

$$\nabla_a G^a_b = 0 \tag{1.5}$$

to be satisfied along with the previous conditions, Elie Cartan showed that the tensor G_{ab} should have the following form

$$G_{ab} = k \left[R_{ab} - \frac{1}{2}\, g_{ab} \left(R - 2\lambda \right) \right], \qquad k = const, \tag{1.6}$$

where λ is known as the cosmological constant, and it will be discarded here. When a source (matter) is present, we obtain ten non-linear equations

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab} R = \varkappa\, T_{ab}\,, \tag{1.7}$$

which show that masses and space-time are not mutually independent. Also, here

$$\varkappa = -\frac{8\pi\mathfrak{G}}{c^4} \tag{1.8}$$

is Einstein's constant and $\mathfrak{G}$ is Newton's constant.

The (massive) energy-momentum tensor here is given by

$$T_{ab} = \rho c^2\, u_a u_b\,, \tag{1.9}$$

where ρ is the matter density.

The fundamental equation (1.8) generalizes the Poisson equation, which is clearly valid in Newtonian physics when the macroscopic velocities are slow compared to the light velocity c.

§1.2. Energy-momentum pseudo-tensor density

The condition $\nabla_a G^a_b = 0$ implies

$$\nabla_a T^a_b = 0$$

or

$$\partial_a \Im^a_b = 0 \tag{1.10}$$

with the tensor density

$$\Im^a_b = \sqrt{-g}\; T^a_b \,.$$

However, inspection shows that

$$\partial_a \Im^a_b = \frac{1}{2}\,\Im^{cd}\,\partial_b\, g_{cd}$$

and the condition (1.10) is thus never satisfied in a general coordinate system.

The classical theory requires that the total four-momentum of matter and its gravitational field

$$P^a = \frac{1}{c}\int (T^{ab} + t^{ab})\,\sqrt{-g}\; dS_b$$

should be conserved.

We thus have to introduce a tensor density

$$\mathcal{T}_{ab} = \sqrt{-g}\; t_{ab}$$

such that

$$\partial_a (\Im^a_b + \mathcal{T}^a_b) = 0 \tag{1.11}$$

with the explicit form

$$\mathcal{T}^c_d = \frac{1}{2\varkappa}\left[\frac{(\partial_d\,\mathcal{G}^{ab})\,\partial\mathcal{L}_{\rm E}}{\partial\,(\partial_c\,\mathcal{G}^{ab})} - \delta^c_d\,\mathcal{L}_{\rm E}\right], \tag{1.12}$$

where

$$\mathcal{G}^{ab} = \sqrt{-g}\; g^{ab}$$

is the *metric tensor density*, constructed from the fundamental metric tensor g_{ab}.

The quantities $\mathcal{T}^{ab}$ are called *pseudo-tensor densities* of Landau-Lifshitz, for they can be transformed away by a suitable choice of the reference frame. The densities $\mathcal{T}^{ab}$ are just formed with the Christoffel symbols, themselves becoming a generalization of a true tensor only with respect to linear coordinate transformations. This is why the classical theory stipulates that the gravitational energy, which is attributed to masses, is *not localizable* and therefore *cannot be engineered*.

Chapter 2. The Basics of the EGR Theory

§2.1. Extended Riemannian geometry

§2.1.1. Structure of the extended manifold

Consider the generalization $\mathcal{R}^{e\cdot\cdot\cdot}_{\cdot acd}$ having the same form as the Riemann curvature tensor $R^{e\cdot\cdot\cdot}_{\cdot acd}$, but constructed on other connection coefficients

$$\mathcal{R}^{e\cdot\cdot\cdot}_{\cdot acd} = \partial_d \Gamma^e_{ac} - \partial_c \Gamma^e_{ad} + \Gamma^e_{ac}\Gamma^k_{kd} - \Gamma^k_{ad}\Gamma^e_{kc}\,. \tag{2.1}$$

On a manifold M referred to a natural basis, e_a, it is known that the connection coefficients Γ^c_{ab} can be decomposed as follows

$$\Gamma^c_{ab} = \left\{{c \atop ab}\right\} + K^c_{ab} + (\Gamma^c_{ab})_s\,, \tag{2.2}$$

where $\left\{{c \atop ab}\right\}$ are the conventional Christoffel symbols of the second kind, used in General Relativity, and (see Tonnelat [1, p. 30–32] for detail)

$$K^c_{ab} = \frac{1}{2}\, g^{ce}\left(T_{[ae],b} + T_{[be],a} + T_{[ab],e}\right) \tag{2.3}$$

is referred to as the *contorsion tensor*, which includes the torsion tensor* $T^c_{[ba]} = \frac{1}{2}(\Gamma^c_{ba} - \Gamma^c_{ab})$. The quantity

$$(\Gamma^c_{ab})_s = \frac{1}{2}\, g^{ce}\left(\mathrm{D}_b\, g_{ae} + \mathrm{D}_a\, g_{be} - \mathrm{D}_e\, g_{ab}\right) \tag{2.4}$$

is the so-called *segment connection*, which is formed with the covariant derivatives of the metric tensor

$$\mathrm{D}_c\, g_{ab} = \partial_c\, g_{ab} - \Gamma_{ac,b} - \Gamma_{bc,a}\,. \tag{2.5}$$

This last connection characterizes a particular property of the manifold M, which is related to a specific type of curvature called the "segment curvature".

In a dual basis θ defined on M, to any parallel-transported vector along a closed path can be associated:

- A rotation curvature

$$\Omega^a_b = -\frac{1}{2}\, \mathcal{R}^{a\cdot\cdot\cdot}_{\cdot bcd}\, \theta^c \wedge \theta^d; \tag{2.6}$$

- A torsion

$$\Omega^a = \frac{1}{2}\, T^a_{cd}\, \theta^c \wedge \theta^d; \tag{2.7}$$

- A segment curvature

$$\Omega = -\frac{1}{2}\, \mathcal{R}^{a\cdot\cdot\cdot}_{\cdot acd}\, \theta^c \wedge \theta^d. \tag{2.8}$$

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*Somewhere in the scientific literature, the torsion tensor is used in the other form $T^c_{[ba]} = \Gamma^c_{ba} - \Gamma^c_{ab}$ that does not matter in the present case.

§2.1.2. Modified action principle

As we know, the classical General Relativity is constructed from the Riemannian action $S = \int L_{\mathrm{E}} \sqrt{-g}\, d^4x$ (1.1), which is varied with respect to g_{ab}. The derived source free equations are

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab} R = 0\,.$$

In this case, the geometry is Riemannian, i.e.

$$T^{c}_{[ab]} = 0\,,$$

$$\mathrm{D}\, g_{ab} = \nabla g_{ab} = 0\,.$$

At this stage, our way of generalizing the GR theory is legitimized by the following remarks:

- The symmetry of the Einstein tensor is not sufficiently natural. Indeed, when derived from the relativistic theory, the canonical energy-momentum tensor is always antisymmetric, as in the electromagnetic field with a source

  $$\theta^{bc} = \frac{1}{4}\, g^{bc} F_{de} F^{de} - F^{ba} \partial^{c} A_a + g^{bc} j_a A^a$$

  and, in order to fit in the field equations, this tensor has to afterwards be symmetrized;
- The condition $\mathrm{D}\, g_{ab} \neq 0$ is more general than the restrictive Riemannian condition

  $$\nabla g_{ab} = 0\,;$$
- Moreover, we deem that the torsion tensor T^{c}_{ab} resorts more to an artificial mathematical property and does not offer a full physical and clear meaning. We therefore postulate a torsion-free manifold with 40 general symmetrical connection coefficients

  $$\Gamma^{a}_{bc} = \left\{ {a \atop bc} \right\} + (\Gamma^{a}_{bc})_J\,, \tag{2.9}$$

  where the latter connection is not necessary (2.4).

The Riemannian manifold should nevertheless be recovered when $\mathrm{D}\, g_{ab} = 0$. To begin with, we follow here the basic ideas of Einstein: instead of the potentials g_{ab}, we consider 40 connection coefficients (2.9) as the "field" variables.

In this context, the generalization of the Ricci tensor formed with Γ^{a}_{bc} is still expressed by

$$R_{bc} = \partial_a \Gamma^{a}_{bc} - \partial_c \Gamma^{a}_{ba} + \Gamma^{d}_{bc} \Gamma^{a}_{da} - \Gamma^{d}_{ba} \Gamma^{a}_{dc}\,. \tag{2.10}$$

§2.1.3. Eulerian equations

We consider the tensor density

$$\mathfrak{R}^{ab} = R^{ab}\sqrt{-g}\ , \tag{2.11}$$

from which we construct the invariant density

$$\mathcal{H} = \mathfrak{R}^{ab} R_{ab} \tag{2.12}$$

with

$$\mathfrak{R}^{ab} = \frac{\partial \mathcal{H}}{\partial R_{ab}}\,.$$

The least action principle is then

$$\delta S = \int \delta\mathcal{H}\, d^4x = 0\,. \tag{2.13}$$

For a variation $\delta\Gamma^a_{bc}$, we obtain

$$\delta S = \int \left[\left(\frac{\partial\mathcal{H}}{\partial\Gamma^a_{bc}}\right)\delta\Gamma^a_{bc} + \left(\frac{\partial\mathcal{H}}{\partial(\partial_e\,\Gamma^a_{bc})}\right)\delta\left(\partial_e\,\Gamma^a_{bc}\right)\right] d^4x = 0\,. \tag{2.14}$$

The variation of $\mathcal{H}$ is also expressed by

$$\delta\int \mathfrak{R}^{bc} R_{bc}\, d^4x = \int\left[\frac{\mathfrak{R}^{bc}\partial R_{bc}\,\delta\Gamma^a_{de}}{\partial\Gamma^a_{de}} + \frac{\mathfrak{R}^{bc}\partial R_{bc}\,\delta(\partial_k\Gamma^a_{de})}{\partial\left(\partial_k\Gamma^a_{de}\right)}\right] d^4x$$

and, integrating by parts, we obtain

$$\delta\int\left[\frac{\mathfrak{R}^{bc}\,\partial R_{bc}}{\partial\Gamma^a_{de}} - \partial_k\left(\frac{\mathfrak{R}^{bc}\,\partial R_{bc}}{\partial(\partial_k\Gamma^a_{de})}\right)\right]\delta\Gamma^a_{de} +$$

$$+\int\partial_k\left[\mathfrak{R}^{bc}\,\frac{\partial R_{bc}\,\delta\Gamma^a_{de}}{\partial(\partial_k\Gamma^a_{de})}\right] d^4x = 0\,. \tag{2.15}$$

If the variations $\delta\Gamma^a_{de}$ are zero on the integration boundary, the last divergence integral has no contribution.

The condition (2.15) reduces to

$$\delta\int\mathfrak{R}^{ab} R_{ab}\, d^4x = \int\left(Q^{bc}_a\,\delta\Gamma^a_{bc}\right) d^4x = 0 \tag{2.16}$$

with

$$Q^{bc}_a = \mathfrak{R}^{de}\,\frac{\partial R_{de}}{\partial\Gamma^a_{bc}} - \partial_k\left[\mathfrak{R}^{de}\,\frac{\partial R_{de}}{\partial\left(\partial_k\Gamma^a_{bc}\right)}\right]. \tag{2.17}$$

The stationary principle for the symmetric Γ^a_{bc} leads to the Eulerian equations

$$Q^{(bc)}_a = \frac{1}{2}\left(Q^{bc}_a + Q^{cb}_a\right) = 0\,. \tag{2.18}$$

From the expression (2.10), we derive the derivatives

$$\frac{\partial R_{dk}}{\partial\left(\partial_e \Gamma^a_{bc}\right)} = \delta^e_m \,\delta^b_d \,\delta^c_k \,\delta^m_a - \delta^e_k \,\delta^b_d \,\delta^c_m \,\delta^m_a \tag{2.19}$$

and

$$\frac{\partial R_{dk}}{\partial \Gamma^a_{bc}} = \delta^n_a \,\delta^b_d \delta^c_k \Gamma^m_{nm} + \delta^m_a \,\delta^b_n \delta^c_k \Gamma^n_{dm} - \\ - \delta^n_a \,\delta^b_d \,\delta^c_m \Gamma^m_{nk} - \delta^m_a \,\delta^b_n \,\delta^c_k \Gamma^n_{dm} \,. \tag{2.20}$$

Now substituting these into (2.17) yields

$$-\,Q^{bc}_a = \partial_a \mathfrak{R}^{bc} - \delta^c_a \partial_e \mathfrak{R}^{be} - \mathfrak{R}^{bc} \Gamma^m_{am} - \delta^k_a \mathfrak{R}^{dc} \Gamma^b_{dk} + \\ + \mathfrak{R}^{bk} \Gamma^c_{ak} + \mathfrak{R}^{kc} \Gamma^b_{ka} = (\mathfrak{R}^{bc})_{\prime,a} - \delta^c_a \,(\mathfrak{R}^{be})_{\prime,e} \tag{2.21}$$

with

$$(\mathfrak{R}^{bc})_{\prime,a} = \partial_a \mathfrak{R}^{bc} + \Gamma^b_{ea} \mathfrak{R}^{ec} + \Gamma^c_{ea} \mathfrak{R}^{eb} - \Gamma^e_{ae} \mathfrak{R}^{bc} \tag{2.22}$$

where $\prime,$ are the covariant derivatives constructed with the global Γ^a_{bc} defined in (2.9).

The condition (2.18) explicitly yields

$$(\mathfrak{R}^{bc} + \mathfrak{R}^{cb})_{\prime,a} - \delta^c_a \,(\mathfrak{R}^{be})_{\prime,e} - \delta^b_a \,(\mathfrak{R}^{ce})_{\prime,e} = 0 \,. \tag{2.23}$$

§2.2. Connection coefficients

In order to determine the exact form of the connection, we first decompose $\mathfrak{R}^{bc}$ into the metric density $\mathcal{G}^{bc} = \sqrt{-g}\, g^{bc}$ and two parts $\mathcal{E}^{bc} + \mathcal{A}^{bc}$, where $\mathcal{A}^{bc}$ is antisymmetric

$$\mathfrak{R}^{bc} = (\mathcal{G}^{bc} + \mathcal{E}^{bc}) + \mathcal{A}^{bc}. \tag{2.24}$$

The two-term quantity in brackets represents the Riemann-Ricci tensor density

$$R^{bc} \sqrt{-g} = \mathcal{R}^{bc} = \mathcal{G}^{bc} + \mathcal{E}^{bc} \tag{2.25}$$

so that when $\mathcal{A}^{bc} = 0$, (2.24) reduces, as it should be, to (2.25).

Consistency of our theory leads to impose the following constraint

$$(\mathcal{E}^{bc})_{\prime,b} = 0 \,. \tag{2.26}$$

So forth we set

$$\mathcal{J}^b = (\mathcal{A}^{ba})_{\prime,a} = \partial_a \mathcal{A}^{ba} \tag{2.27}$$

(due to the antisymmetry of $\mathcal{A}^{ab}$) with

$$\mathcal{J}^a = \sqrt{-g}\; J^a,$$

where the four-vector J^a will play a central role.

We now aim to check whether the condition $(\mathcal{G}^{bc})_{\prime,c} = 0$ reinstates a Riemannian connection whereby the curvature tensor R_{ab} (2.10) would reduce to the Riemann-Ricci tensor R_{ab}.

By contracting (2.23) on c and a, and taking into account (2.26), one finds

$$(\mathcal{G}^{bc})_{\prime,a} = -\frac{5}{3}\,\mathcal{J}^b. \qquad (2.28)$$

If inserting (2.28) into (2.23), the conditions (2.18) eventually read

$$(\mathcal{G}^{bc})_{\prime,a} = -\frac{1}{3}\left(\delta^b_a\,\mathcal{J}^c + \delta^c_a\,\mathcal{J}^b\right). \qquad (2.29)$$

Dividing by $\sqrt{-g}$, we obtain

$$\partial_a\,g^{bc} + g^{bc}\,\partial_a \ln\sqrt{-g} + \Gamma^b_{ea}\,g^{ec} + \Gamma^c_{ea}\,g^{be} - \Gamma^e_{ea}\,g^{bc} = \\ = -\frac{1}{3}\left(\delta^b_a\,J^c + \delta^c_a\,J^b\right) \qquad (2.30)$$

and multiplying through by g_{bc}, having $g_{ba}\,g^{ca} = \delta^c_b$ taken into account as well as

$$dg = g\,g^{bc}\,dg_{bc} = -\,g\,g_{bc}\,dg^{bc},$$

we infer

$$\Gamma^e_{ae} = \partial_a \ln\sqrt{-g} + \frac{1}{3}\,J_a\,. \qquad (2.31)$$

Substituting this last relation into (2.30) and multiplying it by $g_{bd}\,g_{kc}$ (after noting that $dg_{ed} = -\,g_{ec}\,g_{bd}\,dg^{bc}$), we eventually find

$$\partial_a\,g_{bc} - \Gamma^k_{ba}\,g_{kc} - \Gamma^k_{ca}\,g_{bk} = \frac{1}{3}\left(J_c\,g_{ab} + J_b\,g_{ac} - J_a\,g_{bc}\right) = \mathrm{D}_a\,g_{bc}\,. \qquad (2.32)$$

Interchanging the indices a and b, then a and c, we obtain two more equations of type (2.32), which could be virtually denoted by $(2.32)'$ and $(2.32)''$. From the linear combination $(2.32)' + (2.32)'' - (2.32)$, we eventually get the explicit form of the global connection

$$\Gamma^d_{ab} = \left\{{d \atop ab}\right\} + (\Gamma^d_{ab})_J = \left\{{d \atop ab}\right\} + \frac{1}{6}\left(\delta^d_a\,J_b + \delta^d_b\,J_a - 3g_{ab}\,J^d\right). \qquad (2.33)$$

Our last equation (2.33) shows that when $J_a = 0$, we have $\mathrm{D}_a\,g_{bc} = 0$ and thus

$$(\mathcal{G}^{ab})_{\prime,b} = 0\,. \qquad (2.34)$$

From (2.31), the condition $J_a = 0$ implies $(\Gamma^b_{ae})_J = 0$, so we see that in the case, the generalized curvature tensor R_{ab} (2.10) reduces to the Riemann-Ricci tensor R_{ab}.

Chapter 3. The EGR Field Equations

§3.1. EGR curvature tensors

§3.1.1. The fourth-rank curvature tensor

From the connection

$$\Gamma^d_{ab} = \left\{{d \atop ab}\right\} + (\Gamma^d_{ab})_J = \left\{{d \atop ab}\right\} + \frac{1}{6}\left(\delta^d_a J_b + \delta^d_b J_a - 3 g_{ab} J^d\right) \tag{3.1}$$

the EGR curvature tensor can be derived

$$\mathsf{R}^{a\cdots}_{\cdot bcd} = R^{a\cdots}_{\cdot bcd} + \nabla_d \Gamma^a_{bc} - \nabla_c \Gamma^a_{bd} + \Gamma^k_{bc}\Gamma^a_{kd} - \Gamma^k_{bd}\Gamma^a_{kc}\,. \tag{3.2}$$

Inspection shows that the following relations hold

$$(\mathsf{R}^{e\cdots}_{\cdot dab})_{\prime,k} + (\mathsf{R}^{e\cdots}_{\cdot dka})_{\prime,b} + (\mathsf{R}^{e\cdots}_{\cdot dbk})_{\prime,a} = 0\,, \tag{3.3}$$

$$\mathsf{R}^{e\cdots}_{\cdot dab} + \mathsf{R}^{e\cdots}_{\cdot bda} + \mathsf{R}^{e\cdots}_{\cdot abd} = 0\,. \tag{3.4}$$

Let us now contract

$$g_{ce}\,\mathsf{R}^{e\cdots}_{\cdot dab} = \mathsf{R}_{cdab}\,, \tag{3.5}$$

we then note that, from $\nabla_a(\Gamma^e_{db})_J$,

$$g_{ce}\nabla_a\left[(\Gamma^e_{bk})_J\,\delta^k_m\,\delta^m_d\right] = g_{cd}\nabla_a(\Gamma^e_{be})_J$$

and the curvature tensor (3.5) now reads

$$\mathsf{R}_{cdab} = R_{cdab} + g_{ce}\nabla_b(\Gamma^e_{da})_J - \frac{1}{2}\,g_{ce}\left[\nabla_a(\Gamma^e_{db})_J + \nabla_d(\Gamma^e_{ab})_J\right] + $$
$$+ g_{ce}\left[(\Gamma^e_{kb})_J(\Gamma^k_{da})_J - (\Gamma^e_{ka})_J(\Gamma^k_{db})_J\right] + g_{cd}\left[\partial_a(\Gamma^e_{be})_J - \partial_b(\Gamma^e_{ae})_J\right]. \tag{3.6}$$

With the definition (3.1) we have

$$(\Gamma^d_{ad})_J = \frac{1}{3}\,J_a \tag{3.7}$$

and

$$\partial_a(\Gamma^d_{bd})_J - \partial_b(\Gamma^d_{ad})_J = \frac{1}{3}\,J_{ab} \tag{3.8}$$

with

$$J_{ab} = \partial_a J_b - \partial_b J_a\,. \tag{3.9}$$

§3.1.2. The EGR second-rank tensor

The relation (3.6) eventually leads to the contracted tensor

$$\mathsf{R}^{d\cdots}_{\cdot abd} = \mathsf{R}_{ab} = R_{ab} + \nabla_d(\Gamma^d_{ab})_J - \nabla_b(\Gamma^d_{ad})_J + $$
$$+ (\Gamma^k_{ab})_J(\Gamma^d_{kd})_J - (\Gamma^k_{ae})_J(\Gamma^e_{kb})_J\,, \tag{3.10}$$

we then have once more the splitting

$$\mathsf{R}_{ab} = \mathsf{R}_{(ab)} + \mathsf{R}_{[ab]} \tag{3.11}$$

with

$$\mathsf{R}_{(ab)} = R_{ab} + \nabla_d \,(\Gamma^d_{ab})_J - \frac{1}{2}\left[\nabla_b \,(\Gamma^d_{ad})_J + \nabla_a \,(\Gamma^d_{bd})_J\right] + \\ + (\Gamma^k_{ab})_J \,(\Gamma^d_{kd})_J - (\Gamma^k_{ae})_J \,(\Gamma^e_{kb})_J \tag{3.12}$$

and

$$\mathsf{R}_{[ab]} = \frac{1}{2}\left[\partial_a \,(\Gamma^d_{bd})_J - \partial_b \,(\Gamma^d_{ad})_J\right] \tag{3.13}$$

that is

$$\mathsf{R}_{(ab)} = R_{ab} - \frac{1}{2}\left(g_{ab} \nabla_d J^d + \frac{1}{3} \, J_a \, J_b\right), \tag{3.14}$$

$$\mathsf{R}_{[ab]} = \frac{1}{6}\,(\partial_a J_b - \partial_b J_a)\,. \tag{3.15}$$

§3.1.3. The EGR curvature scalar

Applying $\mathsf{R} = g^{da} \mathsf{R}_{da}$, we have

$$\mathsf{R} = R - \nabla_e\left[g^{da}(\Gamma^e_{da})_J\right] - \nabla_e\left[g^{dc}(\Gamma^e_{dc})_J\right] - \\ - g^{da}\left[(\Gamma^e_{da})_J \,(\Gamma^c_{ce})_J - (\Gamma^k_{de})_J \,(\Gamma^e_{ka})_J\right] \tag{3.16}$$

or

$$\mathsf{R} = R - \frac{1}{3}\left(\nabla_e J^e + \frac{1}{2}\, J^2\right). \tag{3.17}$$

§3.2. The EGR Einstein tensor

Unlike the Riemann curvature tensor, the EGR curvature tensor is no longer antisymmetric on the indices pair ca

$$\mathsf{R}_{cabk} + \mathsf{R}_{acbk} = \frac{2}{3}\, g_{ca} J_{bk} \tag{3.18}$$

or

$$\mathsf{R}^{ca\cdot\cdot}_{\cdot\cdot bk} + \mathsf{R}^{ac\cdot\cdot}_{\cdot\cdot bk} = \frac{2}{3}\, g^{ca} J_{bk}\,. \tag{3.19}$$

Lifting the indices d in the equation (3.3) and contracting on d and k as well as on b and e, we obtain

$$(\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ab})_{\prime,k} + (\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ka})_{\prime,b} + (\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot bk})_{\prime,a} = 0 \tag{3.20}$$

then we replace $\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ab}$ by its value from (3.19). We eventually find

$$(\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot bk})_{\prime,a} + 2\,(\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ab})_{\prime,k} + \frac{2}{3}\, g^{bk}\,(J_{ka})_{\prime,b} = 0 \tag{3.21}$$

that is

$$\left(\mathsf{R}^k_a - \frac{1}{2}\,\delta^k_a\,\mathsf{R}\right)_{\prime,\,k} = -\frac{1}{3}\,(J^k_a)_{\prime,\,k} \tag{3.22}$$

which is just the generalized conservation law for the EGR tensor $\mathsf{G}_{da} = (G_{da})_{\mathrm{EGR}}$ (here we substitute $d=k$)

$$\mathsf{G}_{da} = \mathsf{R}_{(da)} - \frac{1}{2}\left(g_{da}\,\mathsf{R} - \frac{2}{3}\,J_{da}\right). \tag{3.23}$$

The latter will be called here the *EGR Einstein Tensor.* It obviously reduces to the "Riemannian" Einstein tensor

$$G_{da} = R_{da} - \frac{1}{2}\,g_{da}\,R = 0$$

in the framework of the classical GR field equations.

The equations (3.23) are a transcription of the tensor density EGR field equations

$$\mathfrak{R}^{da} + \mathcal{B}^{da} = 0\,, \tag{3.24}$$

whose conservation law is

$$\left(\mathfrak{R}^b_a + \mathcal{B}^b_a\right)_{\prime,\,b} = 0\,.$$

In the strong (ideal) Riemannian regime $\mathcal{J}^a = 0$, thus

$$(\mathfrak{R}^b_a)_{\prime,\,b} = \nabla_b\,\mathcal{R}^b_a = 0\,,$$

$$\partial_b\,\mathcal{R}^b_a - \left\{ {c \atop ba} \right\} \mathcal{R}^b_c = 0\,,$$

or

$$\partial_b\,\mathcal{R}^b_a - \frac{1}{2}\,\mathcal{R}^{cb}\,\partial_a\,g_{cb} = 0$$

eventually

$$\nabla_b\left(R^b_a - \frac{1}{2}\,\delta^b_a\,R\right) = 0\,,$$

which is just the conserved Einstein tensor G_{ab} as inferred from the Bianchi identities in the classical GR

$$G_{ab} = R_{ab} - \frac{1}{2}\,g_{ab}\,R\,.$$

It is now easy to derive the expression of the tensor B_{da} corresponding to $\mathcal{B}^{da}$

$$B_{da} = -\frac{1}{2}\left(\frac{3}{2}\,g_{da}\,\nabla_e\,J^e + \frac{1}{3}\,J_d\,J_a - \frac{1}{6}\,g_{da}\,J^2 + \frac{2}{3}\,J_{da}\right). \tag{3.25}$$

By doing so, we note that $\frac{1}{6}\,g_{da}J^2$ is only the term in the bracket which carries J^2, and which prevails over the others as a candidate to generalize $g_{da}\,\lambda$ (the cosmological term). The term $\frac{1}{6}\,g_{da}\,J^2$ is reminiscent of the classical $g_{da}\,\lambda$, where λ was long regarded as a mere constant in the usual Riemannian theories.

§3.3. The persistent field

§3.3.1. The EGR field equations

In the framework of the EGR theory, our universe is completely described by $\mathcal{G}_{ab} = (G_{ab})_{\text{EGR}}$.

In the classical GR, the source-free field equations are

$$G_{ab} = 0\,, \tag{3.26}$$

but according to our basic postulate, the latter "Riemannian" equation is merely a particular case in the framework of the global EGR geometry. Therefore, in the absence of the macroscopic energy term, there should always remain a faint energy tensor described by the extra curvature.

The classical vacuum equations (3.26) should be replaced by the following EGR field equations

$$\mathfrak{R}^{ab} + \mathcal{B}^{ab} = \varkappa\,(\Im^{ab})_{\text{field}}\,. \tag{3.27}$$

When matter or ponderomotive energy is present, we simply write

$$\Im^{ab} = (\Im^{ab})_{\text{Riem}} + (\Im^{ab})_{\text{field}}\,, \tag{3.28}$$

which has a certain analogy with the "Riemannian" electrodynamics, where there exists a massive tensor for a conductor, and an interacting electromagnetic energy-momentum tensor.

In the immediate neighbourhood of a mass, the Riemannian geometry represented by $(\Im^{ab})_{\text{Riem}}$ becomes increasingly dominant inside the global one, and $(\Im^{ab})_{\text{field}}$ coincides with the gravitational pseudo-tensor density classically attributed to the mass.

§3.3.2. The persistent energy-momentum tensor

By considering the tensor density $\mathcal{T}_{ab} = \sqrt{-g}\,t_{ab}$ (see Page 154),

$$\mathcal{T}^{c}_{d} = \frac{1}{2\varkappa}\left[(\partial_d\,\mathcal{G}^{ab})\,\frac{\partial\mathcal{L}_{\text{E}}}{\partial\,(\partial_c\,\mathcal{G}^{ab})} - \delta^{c}_{d}\,\mathcal{L}_{\text{E}}\right] \tag{3.29}$$

one can express the tensor density $(\Im^{ab})_{\text{field}}$, which can be determined through the usual canonical equations

$$(\Im^{a}_{\cdot\,b})_{\text{field}} = \frac{1}{2\varkappa}\left[\mathcal{H}\,\delta^{a}_{b} - (\partial_b\,\Gamma^{e}_{dk})\,\frac{\partial\mathcal{H}}{\partial\,(\partial_a\,\Gamma^{e}_{dk})}\right]. \tag{3.30}$$

It has a tensor counterpart $(T_{ab})_{\rm field}$ which is written as

$$\sqrt{-g}\;(T^{ab})_{\rm field} = (\Im^{ab})_{\rm field}\,. \tag{3.31}$$

As expected from B_{ab}, we can easily check that this tensor is antisymmetric on the indices a and b.

In accordance with (3.28), we now suggest that the massive tensor density $(\Im^{ab})_{\rm Riem}$ still be given by

$$(T_{ab})_{\rm Riem} = \rho\, u_a u_b\,, \tag{3.32}$$

where ρ is the density of the (neutral) massive fluid.

The conservation law (1.10) then corresponds to

$$(\Im^{\cdot b}_{a\cdot})_{\prime,b} = \left[(\Im^{\cdot b}_{a\cdot})_{{\rm field}\,\prime,b} + (\Im^{\cdot b}_{a\cdot})_{{\rm Riem}\,\prime,b}\right] = 0\,, \tag{3.33}$$

$$(\Im^{\cdot b}_{a\cdot})_{{\rm mass}\,\prime,b} = \partial_b(\Im^{\cdot b}_{a\cdot})_{\rm Riem} - \left[\left\{{c \atop ba}\right\}(\Im^{\cdot b}_{c\cdot})_{\rm Riem} + (\Gamma^c_{ba})_J\,(\Im^{\cdot b}_{c\cdot})_{\rm Riem}\right] = \\ = \rho\sqrt{-g}\,\frac{{\rm D}u_a}{d\tau}\,.$$

In the Riemannian regime, $\mathcal{J}^a = 0$ (which is an ideal case) and we should have

$$\nabla_b\,\Im^{\cdot b}_{a\cdot} = 0\,, \qquad \rho\sqrt{-g}\,\frac{\nabla u_a}{d\tau} = \partial_b\,(\Im^{\cdot b}_{a\cdot})_{\rm Riem} - \left\{{c \atop ba}\right\}(\Im^{\cdot b}_{c\cdot})_{\rm Riem}$$

that is

$$\rho\,\frac{\nabla u_a}{d\tau} = \frac{1}{\sqrt{-g}}\,\partial_b\,(\Im^{\cdot b}_{a\cdot})_{\rm Riem} - \left\{{c \atop ba}\right\}(T^b_c)_{\rm Riem} = \nabla_b\,T^b_a$$

in accordance with the classical result inferred from the definition of the massive tensor

$$T^{ab} = \rho u^a u^b.$$

Strictly speaking, the four-velocity u^a should be slightly modified since the Universe is characterized here by two forms:

- The quadratic form
$$ds^2 = g_{ab}\,dx^a dx^b;$$
- The linear form
$$dJ = f(J_b)\,dx^b.$$

A reasonable choice for $(u^a)_{\rm EGR}$ can be

$$(u^a)_{\rm EGR} = \frac{dx^a}{\sqrt{ds^2 + dJ}}\,. \tag{3.34}$$

§3.3.3. Interaction with matter

Reinstating the light velocity c, we now consider the immediate vicinity of a massive body. We write the total energy-momentum four-vector (attributed to field and mass)

$$P^a = \frac{1}{c}\int \left[(T^{ab})_{\text{field}} + (T^{ab})_{\text{Riem}}\right]\sqrt{-g}\; dS_b \tag{3.35}$$

across any given hypersurface.

In the framework of the immediate neighbourhood of the mass, the field tensor $(T_{ab})_{\text{field}}$ is replaced by the tensor t_{ab} which coincides with the classical gravitational energy-momentum pseudo-tensor.

In this case, the total energy-momentum four-vector reduces to the "Riemannian" result

$$P^a = \frac{1}{c}\int \left[t^{ab} + (T^{ab})_{\text{Riem}}\right]\sqrt{-g}\; dS_b\,. \tag{3.36}$$

Consider then the "contact" situation for which $(T_{ab})_{\text{Riem}}$, when integrated over the volume ϑ of the mass, gives the contribution

$$m_0c^2 = \int \left[(T^1_1)_{\text{Riem}} + (T^2_2)_{\text{Riem}} + (T^3_3)_{\text{Riem}} - (T^4_4)_{\text{Riem}}\right]\sqrt{-g}\; d\vartheta \tag{3.37}$$

into the total energy-momentum four-vector (3.35).

On the other hand, the "Riemannian" static field equations result in the follows

$$R^4_4 = \frac{8\pi\mathfrak{G}}{c^4}\left[(T^4_4)_{\text{Riem}} - \frac{1}{2}(T)_{\text{Riem}}\right] =$$

$$= \frac{4\pi\mathfrak{G}}{c^4}\left[(T^4_4)_{\text{Riem}} - (T^1_1)_{\text{Riem}} - (T^2_2)_{\text{Riem}} - (T^3_3)_{\text{Riem}}\right], \tag{3.38}$$

where we have first established that classically

$$\int R^4_4\sqrt{-g}\; d\vartheta = -\,4\pi\mathfrak{G}\,\frac{P^4}{c^3}\,, \qquad P^4 = m_0c\,, \tag{3.39}$$

$$P^a = \frac{1}{c}\int \left[(\Im^{ab})_{\text{field}} + (\Im^{ab})_{\text{Riem}}\right] dS_b \tag{3.40}$$

across any given hypersurface.

At a large distance from a source, $(\Im^{ab})_{\text{Riem}} \to 0$, thus

$$P^a \approx \frac{1}{c}\int (\Im^{ab})_{\text{field}}\, dS_b\,. \tag{3.41}$$

Chapter 4. Concluding Remarks

As a temporary conclusion, we would like to consider the foregoing theoretical elements in the light of the long discussed "MOND" paradigm and its developments. Let us first recall some relevant history.

§4.1. The MOND formulation

Newtonian gravitational theory, when applied to describe acceleration of stars and gas as estimated from Doppler velocities, does not fit with the Newtonian field generated by the visible matter. This is known as the "missing mass" problem, which has led astrophysicists to invoke some sort of dark energy or exotic matter while it has actually never been detected.

In the meanwhile, some scientists have turned to a possible new law of gravity which would be more appropriate in predicting the observed anomalies. In the beginning of the 1980's, the astronomer Mordehai Milgrom [2] restated Newton's second law with the following scheme

$$\mu\left(\frac{a}{a_0}\right)\boldsymbol{a} = -\,\partial_\alpha \Phi_{\mathrm{N}} \qquad (4.1)$$

where a is the generic acceleration, Φ_{N} is the Newtonian potential of the visible matter, $\boldsymbol{a} = a_\alpha$ is the three-dimensional acceleration vector, and $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ is the three-dimensional spatial differential operator. Milgrom termed the acceleration scale a_0, where the function μ satisfies

$$\mu\,\frac{a}{a_0} = 1 \text{ for } a \gg a_0\,, \quad \text{and} \quad \mu\,\frac{a}{a_0} = \frac{a}{a_0} \text{ for } a \ll a_0 \qquad (4.2)$$

with an estimate numerical value of

$$a_0 \approx 10^{-8}\,\mathrm{cm/s}^2.$$

In the limit of low accelerations, Newton's second law should be quadratic and approach the following form (in the direction of the radial coordinate r) in the presence of a gravitational potential

$$\frac{a}{a_0}\,a_r = \partial_r \Phi\,.$$

For a point mass m the attractive potential at r is

$$V^4 = \mathfrak{G}\,m\,a_0\,,$$

which describes a flat rotation curve.

This is the *MOND paradigm*, the "modified Newtonian dynamics", which so far predicts most of the observed anomalies.

§4.2. Non-relativistic reformulation of MOND

A first Lagrangian (Riemannian) density was found to be

$$L = \frac{a_0^2}{8\pi\mathfrak{G}'}\, f\left(\frac{(\partial_\alpha\Phi)^2}{a_0^2}\right) - \rho\,\Phi\,, \tag{4.3}$$

where $(\partial_\alpha\Phi)^2 = \partial_\alpha\Phi\,\partial^\alpha\Phi$. This leads to the following gravitational field equation

$$\partial_\alpha\left[\mu\left(\frac{\sqrt{(\partial_\mu\Phi)^2}}{a_0}\right)\partial^\alpha\Phi\right] = 4\pi\mathfrak{G}'\rho\,, \tag{4.4}$$

where

$$\mu\left(\sqrt{y}\right) = \frac{df(y)}{dy}\,,$$

assuming

$$f(y) = \begin{cases} y & \text{for} \quad y \gg 1 \\ \frac{2}{3}\,y^{2/3} & \text{for} \quad y \ll 1 \end{cases}$$

and $\mathfrak{G}'$ is a constant which reduces to Newton's gravitational constant $\mathfrak{G}$ in the classical regime Φ_{N}. Inspection shows that, when the usual form of the generic acceleration is applied

$$\boldsymbol{a} = -\,\partial_\alpha\Phi\,, \tag{4.5}$$

the solution corresponds to (4.1).

The Lagrangian density (4.3) is "aquadratic", therefore the theory is known as the *AQUAL theory.*

§4.3. A theory of Te-Ve-S

First Relativistic AQUAL. It has been suggested to consider a physical metric $(g_{ab})'$ conformal to a "primitive" Einstein metric g_{ab} according to

$$(g_{ab})' = e^{2\Psi} g_{ab}\,, \tag{4.6}$$

where Ψ is a real scalar field.

The action of a particle of mass m_0 is expressed as

$$S = -\,m_0\int e^{\Psi}\sqrt{-\,g_{ab}\,dx^a dx^b}\;. \tag{4.7}$$

For slow motion in a quasi-static situation with nearly flat metric and in a weak field Ψ,

$$e^{\Psi}\sqrt{-\,g_{ab}\,dx^a dx^b} \approx \left(1 + \Phi_{\text{N}} + \Psi - \frac{v^2}{2}\right)dt\,,$$

where

$$\Phi_{\text{N}} = -\frac{g_{44}+1}{2} \tag{4.8}$$

is the known tensor form of the Newtonian potential induced by the mass density ρ as inferred from the linearized Einstein equations, while v is the velocity with respect to the Minkowski metric $\eta_{ab} = g_{ab} - h_{ab}$.

The particle's Lagrangian is thus

$$m_0\left(\frac{v^2}{2} - \Phi_{\text{N}} - \Psi\right), \tag{4.9}$$

which leads to the equation of motion

$$\boldsymbol{a} \approx -\,\partial_\alpha\left(\Phi_{\text{N}} + \Psi\right). \tag{4.10}$$

Whenever

$$|\partial_\alpha \Psi| \gg |\partial_\alpha \Phi|\,,$$

so (4.10) reduces to (4.1). Thus we obtain the MOND-like dynamics, and also

$$|\partial_\alpha \Psi| \ll a_0\,.$$

In the regime where $|\partial_\alpha \Psi| \gg a_0$, $\mu \approx 1$, $f(y) \approx y$, the quantity Ψ reduces to Φ_{N}.

To keep the particles' acceleration Newtonian, the measurable Newtonian gravitational constant $\mathfrak{G}$ is twice to the bare constant $\mathfrak{G}'$ introduced in (4.4).

However Bekenstein [3] pointed out the setbacks of the relativistic AQUAL: it turns out that the Ψ-waves can propagate faster than light due to the conformal transformation of the physical null cone, and therefore the contribution of Ψ should be kept to a minimum. The last assumption is quite contradicting to the actual galaxies and clusters, which are observed to deflect light stronger than the visible mass.

Disformal related metrics

a) Field $\nabla_a \Psi$. The light deflection problem can be cured by discarding the relation (4.6). It is then suggested to replace the conformal relation by a "disformal" generalized

$$(g_{ab})' = e^{-2\Psi}\left(\mathcal{A}\, g_{ab} + \mathcal{B} L^2\, \nabla_a \Psi\, \nabla_b \Psi\right) \tag{4.11}$$

with $\mathcal{A}$ and $\mathcal{B}$ functions of the invariant $g^{ab}\, \nabla_a \Psi\, \nabla_b \Psi$, and $L = \frac{1}{a_0}$. This allows to deflect light via the term $\nabla_a \Psi\, \nabla_b \Psi$ of the physical metric.

Here the causality is fully maintained, but it yields smaller light deflection instead of enhancing it.

b) Field $\mathcal{U}^a$. In February 2008, Jacob D. Bekenstein [3] suggested a possible relativistic generalization of the MOND paradigm. This is known as the *Tensor-Vector-Scalar content theory* (in short, Te-Ve-S [3]), which introduces, next to the metric tensor g_{ab}, a timelike four-vector field $\mathcal{U}^a$ and a scalar field ϕ. This vector is normalized so that

$$g^{ab}\,\mathcal{U}_a\mathcal{U}_b = -1\,. \tag{4.12}$$

The physical (real) metric here is obtained by stretching the Einstein metric in the space-time directions orthogonal to $\mathcal{U}^a = g^{ab}\mathcal{U}_b$, by a factor $e^{-2\phi}$, while shrinking it by the same factor in the direction parallel to $\mathcal{U}^a$ according to

$$(g_{ab})' = e^{-2\phi}\,(g_{ab} + \mathcal{U}_a\mathcal{U}_b) - e^{2\phi}\mathcal{U}_a\mathcal{U}_b\,. \tag{4.13}$$

When a specific matter content is present, with a density ρ, the *physical* velocity $(u_a)'$ of the matter, normalised with respect to $(g_{ab})'$, is taken to be collinear with $\mathcal{U}_a$

$$(u_a)' = e^{\phi}\,\mathcal{U}_a\,,$$

from which it follows that

$$(g_{ab})' + (u_a)'(u_b)' = e^{-2\phi}\,(g_{ab} + \mathcal{U}_a\mathcal{U}_b)\;. \tag{4.14}$$

With these elements, Bekenstein's MOND relativistic theory successfully provides a suitable explanation for mass discrepancy (hypothetical dark matter), and also for several cosmological anomalies without conflicting with GR.

§4.4. Matching the relativistic MOND formulation

Let us consider again the contracted EGR tensors

$$\mathsf{R}_{(ab)} = R_{ab} - \frac{1}{2}\left(g_{ab}\,\nabla_d\,J^d + \frac{1}{3}\,J_aJ_b\right), \tag{4.15}$$

$$\mathsf{R}_{[ab]} = \frac{1}{6}\,(\partial_a\,J_b - \partial_b\,J_a)\,, \tag{4.16}$$

where the time components reduce to

$$\mathsf{R}_{(44)} = R_{44} - \frac{1}{2}\left(g_{44}\,\nabla_d\,J^d + \frac{1}{3}\,J_4J_4\right), \tag{4.17}$$

$$\mathsf{R}_{[44]} = 0\,, \tag{4.18}$$

(we note that although, obviously, $\mathsf{R}_{[44]} = 0$ and $J_4 \neq 0$).

For low velocities and weak fields, the quasi-Euclidean approximation holds and $\nabla_d J^d$ is negligible with respect to $\frac{1}{3} J_4 J_4$

$$R_{(44)} = R_{44} - \frac{1}{2}\left(\frac{1}{3} J_4 J_4\right), \tag{4.19}$$

whereas in the classical Newtonian theory

$$R_{44} = \frac{1}{c^2} \frac{\partial^2 \Phi_{\mathrm{N}}}{\partial x^\alpha \partial x^\alpha}\,.$$

Upon the linear approximation, the quantity

$$B_{44} = -\frac{1}{6} J_4 J_4 \tag{4.20}$$

can be identified with the Laplacian of the scalar field Ψ, i.e. with the quantity $-\Delta\Psi$, and we find back the conclusions inferred from the conclusions of the AQUAL model, without recurring to the conformal metric,

$$\boldsymbol{a} \approx -\,\partial_\alpha \left(\Phi_{\mathrm{N}} + \Psi\right). \tag{4.21}$$

Causality is therefore respected since no hypothesis is formulated on the light cone structure. As a result, we see that there is no need to introduce the specific (real) metric

$$(g_{ab})' = e^{-2\phi}\left(g_{ab} + \mathcal{U}_a \mathcal{U}_b\right) - e^{2\phi}\,\mathcal{U}_a \mathcal{U}_b\,. \tag{4.22}$$

This purely theoretical approach does not take into account the order of magnitude of the extra curvature which describes the residual field.

Because of this, it may not fit in the relativistic MOND formulation. However we just want to focus our attention onto the fact that the new outlook made possible here by the EGR theory.

Indeed, as we will see in the forthcoming papers, the existence of a persistent field, which is viable through only the EGR theory, provides a sound consistency in other known theories.

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Charged Particle in the Extended Formulation of General Relativity

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Abstract: In the recently presented extended formulation of General Relativity (the EGR theory), a "persistent field" expressed by a gravity-like energy-momentum tensor has been suggested. Due to the non-Riemannian curvature manifested by the theory, this field tensor is a true entity unlike Einstein's pseudo-tensor. Here this tensor is considered in the case of a charged particle in a gravitational field. In the "gravitational radiation damping", the usual relativistic treatment leads to a mass renormalization process. In the framework of the presented theory, this renormalization is not longer required.

Contents:

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Notations:

To completely appreciate this article, it is imperative to define some notations employed.

Indices. Throughout this paper, we adopt the Einstein summation convention whereby a repeated index implies summation over all values of this index:

4-tensor or 4-vector: small Latin indices $a, b, \ldots = 1, 2, 3, 4$;

3-tensor or 3-vector: small Greek indices $\alpha, \beta, \ldots = 1, 2, 3$;

4-volume element: d^4x;

3-volume element: d^3x.

Signature of space-time metric:

Hyperbolic $(+---)$ unless otherwise specified.

Operations:

Scalar function: $U(x^a)$;

Ordinary derivative: $\partial_a U$;

Covariant derivative in GR: ∇_a;

Covariant derivative in EGR: D_a or $'$, (alternatively).

Tensors:

Symmetrization: $A_{(ab)} = \frac{1}{2}\,!\,(A_{ab} + A_{ba})$;

Anti-symmetrization: $A_{[ab]} = \frac{1}{2}\,!\,(A_{ab} - A_{ba})$;

Kronecker symbol: $\delta_{ab} = (+1 \;\text{ if }\; a = b; \;\; 0 \;\text{ if }\; a \neq b)$;

Levi-Civita tensor: ϵ_{abcd} (where $\epsilon^{1234} = 0$).

Three-dimensional vectorial quantities:

$\boldsymbol{P} = P_\alpha\,.$

Introduction

As a follow up to my recent paper *The EGR Theory: An Extended Formulation of General Relativity* [1], we now turn to the consequences of this field contribution to an accelerated charged particle.

We recall the classical concept: In an electrostatic situation, the energy of a charged particle such as the electron, is $\frac{eV}{2}$, where V is the scalar potential of the field generated by the charge e. However, the Special Theory of Relativity tells us that any elementary particle is assumed to be a point mass or charge (non-elastic body), thus implying that at its "centre" $R=0$, where $V=\frac{e}{R}$ must become infinite. As a result, the proper energy (i.e. the proper mass of the electron) would also become infinite, which is physically irrelevant.

The usual way to overcome this difficulty leads to an implicit kind of external negative "mass" which compensates for the divergent one: this is accepted as the "renormalisation" process.

The free field predicted by the EGR theory is introduced in in the form of a "gravitational" energy-momentum tensor density $(\Im^{ab})_{\text{field}}$ next to the mass tensor density $(\Im^{ab})_{\text{mass}}$, which is the "continuation" of the classical energy-momentum pseudo-tensor so far associated with matter. In the framework of my understanding, this extra field, linked with the space-time segment curvature, naturally allows us to avoid the renormalisation requirement, providing the general electrodynamics with a clear and consistent explanation. The EGR Universe is entirely described by two curvatures. Accordingly, the present theory implicitly involves the EGR Ricci tensor R_{ab} rather than the Ricci tensor G_{ab}.

We begin this paper by recalling that, according to the Special Theory of Relativity, an accelerated electron will radiate and produce a reactive damping force in addition to the mechanical inertia force [2]. In the framework of the classical representation of the General Theory of Relativity (we will refer to it as GR), a charged particle does not suffer a reactive damping as long as its absolute acceleration is uniform. We may then expect that this particle actually radiates when deflected by a gravitational field i.e. when a kind of "Bremßtrahlung" effect takes a place; however, it has been shown that a more subtle phenomenon occurs. As has been pointed out by De Witt and Brehme [3], a plane or spherical sharp pulse of light when propagating in a curved 4-dimensional hyperbolic manifold, gradually develops a "tail" which is responsible for this electrogravitic "Bremßtrahlung". This "thinning out" of the elementary waves appears as an extra term in the relativistic equation of a moving charge [2].

Chapter 1. Relativistic Electrodynamics

§1.1. Electromagnetic radiation: variable fields

Variable potentials. Here, the charges are assumed to be located inside a volume element $d\vartheta$ where the variable charge density is $\mu(t)$.

Inside this volume, the scalar electrostatic potential V, which is derived from the electric field E, is

$$E = -\operatorname{grad} V. \tag{1.1}$$

Maxwell's second group of equations states that the variable field produced by arbitrary moving charges obeys the equation

$$\partial_b F^{ab} = -\frac{4\pi}{c}\, j^a, \tag{1.2}$$

$$j^a = \mu\,\frac{dx^a}{dt}\,, \tag{1.3}$$

where j^a is the four-vector density of charge μ. By setting the Lorentz gauge, $\partial_a A^a = 0$, we realise that

$$\frac{\partial^2 A^a}{\partial x^b\,\partial x_b} = \frac{4\pi}{c}\, j^a, \tag{1.4}$$

which can be decomposed into two equations

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\,\boldsymbol{j} \tag{1.5}$$

and

$$\Delta V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 4\pi\mu\,. \tag{1.6}$$

If de is the variable charge in a given volume element $d\vartheta$, the charge density is

$$\mu = de\,(t)\,\delta R\,, \tag{1.7}$$

where δ is the Dirac function which will be analysed in the next section, while R is the distance from the origin of the coordinates, a unique point at which δR is not zero.

Retarded potentials. For an arbitrary charge distribution $\mu(x^a)$, we write

$$de = \mu\, d\vartheta\,.$$

For a volume ϑ we have

$$\mu = e\,\delta R\,.$$

In this case, equation (1.4) can be reduced to a plane wave equation whose solution is of the type

$$V = f\left(t - \frac{R}{c}\right).$$

This represents the progression of the potential V along R, however, with some retarded amplitude measured at the time t. This retarded amplitude results from the signal velocity limited by the light velocity c. Adding V_0 and A_0 to the solutions of equations (1.4) and (1.5), we have

$$\boldsymbol{A} = \frac{1}{c}\int \frac{\left(\boldsymbol{j}_{t-R/c}\right) d\vartheta}{R_a} + \boldsymbol{A}_0\,, \tag{1.8}$$

$$V = \int \frac{\left(\mu_{t-R/c}\right) d\vartheta}{R_a} + V_0\,. \tag{1.9}$$

If $R_a(t) = r_a - (r_a)_0$ is the distance to an electron e observed in $P(x_a)$ at t, the state of motion of the charge at an earlier time t' is determined by the equation

$$t' = t - \frac{R(t')}{c}\,. \tag{1.10}$$

In the resting frame at t', the field at $P(t)$ is simply given by the Coulomb potential

$$V = \frac{e}{c}\,(t - t') \quad \text{since} \quad \boldsymbol{A} = 0\,. \tag{1.11}$$

In a four-dimensional situation, in any arbitrary frame, we find the potential in the form

$$A^a = e\,\frac{u^a}{R_b}\,u^b, \tag{1.12}$$

which is the well-known expression of the Liénard-Wiechert potential, where

$$R_a = \left[\,c\,(t - t')\,,\; r_a - r'_a\,\right].$$

§1.2. Electromagnetic radiation: radiative damping

General coordinate system. On a general metric manifold, the dynamical equations for the electron of mass m_0 and charge e, in an electromagnetic field, are

$$m_0\frac{\mathrm{D}u^a}{ds} = \frac{e}{c}\,F^{ab}u_b\,, \tag{1.13}$$

where $u^a = \frac{dx^a}{ds}$ is the four-velocity and the Maxwell tensor F_{ab} is

$$F_{ab} = \mathrm{D}_a A_b - \mathrm{D}_b A_a\,. \tag{1.14}$$

Here, the electromagnetic field's energy-momentum tensor is

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F^{c\cdot}_{\cdot a} + \frac{1}{4}\,g_{ab}\,F_{ck}F^{ck}\right). \tag{1.15}$$

In a general coordinate reference frame, we assume the following dynamical equations for a particle at z_a

$$\dot{z}^a = \frac{\mathrm{D}z^a}{d\tau} = u^b\,\mathrm{D}_b\,u^a, \tag{1.16}$$

$$\ddot{z}^a = \frac{\mathrm{D}\dot{z}^a}{d\tau} = \frac{d\dot{z}^a}{d\tau} + \Gamma^a_{bd}\,\dot{z}^b\,\dot{z}^d, \tag{1.17}$$

$$\dddot{z}^a = \frac{d\ddot{z}^a}{d\tau} + \Gamma^a_{bd}\,\ddot{z}^b\,\dot{z}^d, \tag{1.18}$$

where τ is the proper time of the particle*.

Three-dimensional radiative damping. An arbitrary distribution of charges with the velocities slow to c does not substantially vary during the time $\frac{R}{c}$. Therefore we expand $\mu_{t-R/c}$ and $\boldsymbol{j}_{t-R/c}$ into series of $\frac{R}{c}$.

Up to third order, we find for the scalar potential

$$V = -\frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\mu\,d\vartheta\,. \tag{1.19}$$

Since the vector potential $\boldsymbol{A}$ already contains a term in $\frac{1}{c}$, we can restrict the expansion to second order. We take

$$\boldsymbol{A} = -\frac{1}{c^2}\,\partial_t f\int \boldsymbol{j}\,d\vartheta\,,$$

then follow with the transformations

$$\boldsymbol{A}' = \boldsymbol{A} + \operatorname{grad} f \quad \text{and} \quad V' = V - \frac{1}{c}\,\partial_t f\,.$$

Then we choose the function f so that the scalar potential V vanished, i.e.

$$f = -\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int R^2\mu\,d\vartheta\,.$$

Hence

$$\boldsymbol{A}' = -\frac{1}{c^2}\,\partial_t\int \boldsymbol{j}\,d\vartheta - \frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\,\operatorname{grad}\int R^2\mu\,d\vartheta\,.$$

*The connection coefficients (Christoffel symbols) are here assumed general for keeping the theory compatible with the EGR theory, we denote Γ^a_{bd} instead of the conventional Christoffel symbols $\left\{{a \atop bd}\right\}$ of General Relativity. See Page 178.

Thus, we arrive at the formula

$$\boldsymbol{A} = -\frac{1}{c^2}\,\partial_t\!\int \boldsymbol{j}\,d\vartheta - \frac{1}{3c^2}\frac{\partial^2}{\partial t^2}\int \boldsymbol{R}\mu d\vartheta\,. \tag{1.20}$$

The first-order terms of the field equation exhibit an additional force exerted on the charge. This force depends on the time derivative of the charge's acceleration. This force, resulting from a higher approximation, is called the *Lorentzian damping force*

$$F_\alpha = \frac{2}{3}\frac{e^2}{c^3}\,\dddot{z}_\alpha\,. \tag{1.21}$$

The equation of motion of the electron without external fields and solely subjected to (1.21), is due to the action of the charge itself

$$m_0\,\ddot{z}_\alpha = \frac{2}{3}\frac{e^2}{c^3}\,\dddot{z}_\alpha\,. \tag{1.22}$$

Ultrarelativistic case. In the Special Theory of Relativity, the equations of motion for the electron should be written

$$m_0\,\frac{du^a}{ds} = \frac{e}{c}\,F^{ab}u_b + f^a. \tag{1.23}$$

For the state of low velocity of the electron, the relation (1.23) should reduce to the expression (1.22). This condition is satisfied when

$$f^a = \frac{2}{3}\frac{e^2}{c}\left(\frac{d^2u^a}{ds^2} - u^a u^b\,\frac{d^2u_b}{ds^2}\right). \tag{1.24}$$

The second term in the brackets is chosen so as to satisfy the physical condition $f^a u_a = 0$, and so (1.24) can be written equally as

$$f^a = \frac{2}{3}\frac{e^2}{c^3}\left(\dddot{z}^a - \frac{1}{c^2}\,\dot{z}^a\,\ddot{z}^2\right). \tag{1.25}$$

Chapter 2. Trajectory of a Charged Particle in a Gravitational Field

§2.1. Brief reminder of the EGR theory

Free gravity field. In the EGR theory, the field equations

$$G_{da} = R_{da} - \frac{1}{2}\,g_{da}R$$

are generalized to

$$\mathsf{G}_{da} = \mathsf{R}_{da} - \frac{1}{2}\left(g_{da}\,\mathsf{R} - \frac{2}{3}\,J_{da}\right), \tag{2.1}$$

where $G_{da} = (G_{da})_{\text{EGR}}$, The antisymmetric Ricci tensor $R_{da} = (R_{da})_{\text{EGR}}$ is constructed with the general connection

$$\Gamma = \{\,\} + (\Gamma)_J \tag{2.2}$$

with the latter coefficients $(\Gamma)_J$, additional to the conventional Christoffel symbols $\{\,\}$, depending on the extra "segment curvature" through the 4-vector J according to

$$\mathrm{D}\, g_{ab} = \frac{1}{3}\left(g_{ac}\, J_b + g_{cb}\, J_a - g_{ab}\, J_c\right) dx^c.$$

The new generalized field equations are written down as

$$\mathcal{R}^{ab} + \mathcal{F}^{ab} = \varkappa \left[(\Im^{ab})_{\text{mass}} + (\Im^{ab})_{\text{field}}\right] , \tag{2.3}$$

where the $(\Im^{ab})_{\text{field}}$ represents the "energy-momentum" free field tensor density which is persistent even in the source-free EGR field equations.

Having defined the Lagrange density $\mathcal{H} = \mathfrak{R}^{ab} R_{ab}$ with

$$\mathfrak{R}^{ab} = \frac{\partial \mathcal{H}}{\partial R_{ab}}\,,$$

the free field density is inferred from the canonical equations

$$(\Im^a_b)_{\text{field}} = \frac{1}{2\varkappa}\left[\mathcal{H}\,\delta^a_b - \partial_b \Gamma^e_{dk}\, \frac{\partial \mathcal{H}}{\partial\left(\partial_a \Gamma^e_{dk}\right)}\right] \tag{2.4}$$

(we have decomposed the curvature tensor density $\mathfrak{R}^{bc} = \sqrt{-g}\, R^{bc}$ into a symmetric part $\mathcal{G}^{bc}$ and an antisymmetric part $\mathcal{A}^{bc}$),

$$\mathfrak{R}^{bc} = \mathcal{G}^{bc} + \mathcal{A}^{bc} \quad \text{with} \quad \mathcal{G}^{bc} = \mathcal{R}^{bc} + \mathcal{E}^{bc}$$

so that

$$(\mathcal{E}^{bc})_{\prime,c} = 0 \quad \text{and} \quad \mathcal{J}^a = (\mathcal{A}^{ba})_{\prime,a} = \partial_a \mathcal{A}^{ba}$$

(due to the antisymmetry of $\mathcal{A}^{ba}$), we have a set of

$$\mathcal{J}^a = \sqrt{-g}\, J^a,$$

$$\mathcal{G}^{ab} = \sqrt{-g}\; g^{ab},$$

$$\mathcal{R}^{bc} = \sqrt{-g}\; R^{bc},$$

$$(\mathcal{G}^{bc})_{\prime,c} = -\frac{5}{3}\,\mathcal{J}^b,$$

$$(\mathcal{G}^{bc})_{\prime,a} = -\frac{1}{3}\,\delta^b_a \mathcal{J}^c + \delta^c_a \mathcal{J}^b,$$

where $'$, is the covariant derivative formed with Γ in (2.2). In fact, within the framework of my theory, the field equations (2.3) *always* have their second term, which corresponds to the free field tensor density

$$\mathcal{R}^{ab} + \mathcal{F}^{ab} = \varkappa\, (\Im^{ab})_{\text{field}}\,. \tag{2.5}$$

Thus, in the EGR theory, in the neighbourhood of matter, the mass density $(\Im^{ab})_{\text{mass}}$ increasingly dominates over the free field density $(\Im^{ab})_{\text{field}}$. This is the quasi-Riemannian regime of the classical theory.

Four-momentum vector of the free field. In tensor notation, we write the global four-energy momentum vector for the field and mass as

$$P^a = \frac{1}{c}\int \left[\, (T^{ab})_{\text{field}} + (T^{ab})_{\text{mass}} \right] \sqrt{-g}\, dS_b$$

across any hypersurface. Inspection shows that the pseudo-tensor $\Im^{ab}$ is a true tensor quantity lending support to the theory of a free field (which is merely the natural extension of the Riemannian gravitational field), for which the quantity is classically attributed to the mass. When integration is performed on the volume ϑ containing this mass, the tensor field $(T_{ab})_{\text{field}}$ vanishes inside the matter, thus only the time component of the four-momentum vector remains (i.e. we are in the Riemannian regime)

$$P^4 = m_0 c = \frac{1}{c}\int \left[-(T^a_a)_{\text{mass}}\right] \sqrt{-g}\, d\vartheta$$

or

$$m_0 c^2 = \int \left[(T^1_1)_{\text{mass}} + (T^2_2)_{\text{mass}} + (T^3_3)_{\text{mass}} - (T^4_4)_{\text{mass}}\right] \sqrt{-g}\, d\vartheta$$

that is the total mass of the given corpuscle (particle). If distantly located from the source, $(\Im^{ab})_{\text{mass}} \to 0$ and

$$P^a \approx \frac{1}{c}\int (\Im^{ab})_{\text{field}}\, dS_b\,. \tag{2.6}$$

§2.2. Gravitational influence

§2.2.1. Dirac bi-tensors

Dirac's distribution function. We now consider the *delta function* introduced by P. A. M. Dirac [4]

$$\delta(x' - x)\,, \tag{2.7}$$

which is known as the Dirac distribution function

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0\,,\ \delta(0) = \infty\,, \tag{2.8}$$

hence

$$\int_{-\infty}^{+\infty} dx = 1\,. \tag{2.9}$$

If $f(x)$ is a continuous function at the point $x = 0$, we have

$$\int_{-\infty}^{+\infty} \delta(x)\, f(x)\, dx = f(0) \tag{2.10}$$

under a more general form

$$\int \delta(x-a)\, f(x)\, dx = f(a)\,, \tag{2.11}$$

where the integration domain contains the point $x = a$, and $f(x)$ is continuous at the point $x = a$.

We write (2.11) as

$$\langle \delta(x, x'),\ f(x') \rangle = f(x)\,. \tag{2.12}$$

The notation

$$\delta(x, x') \tag{2.13}$$

is called the *Dirac bi-scalar*. It will be generalized in the next section.

Displacement bi-tensors. On a differential manifold V_n, we are going to consider a point x' located in the neighbourhood of another point x. Along the geodesic connecting x' to x, we define a "displacement" which represents a "canonical isomorphism" (basis-independent) of the tangent space T_x at x on the manifold, into the tangent space $\mathrm{T}_{x'}$ at x'. The free bases $e_a(x)$ and $e_c(x')$ are attributed to the neighbourhoods.

The relevant isomorphism therefore defines a "bi-tensor", which we call a *displacement tensor*, and denote as

$$\mathsf{t}_a^{c'}\,, \tag{2.14}$$

hence

$$g_{ab}\, \mathsf{t}^a_{c'} \mathsf{t}^b_{d'} = g_{c'd'}\,. \tag{2.15}$$

Here we have

$$\mathsf{t}_{ac'} = g_{c'd'}\, \mathsf{t}_a^{d'} = g_{ab}\, \mathsf{t}^b_{c'}\,, \quad (\mathsf{t}_a^{d'} \mathsf{t}^b_{d'} = \mathsf{t}^d_{a'} \mathsf{t}_d^{b'} = \delta^b_a)\,, \tag{2.16}$$

$$\mathsf{t} = \det\| \mathsf{t}_{ac'}(x)\| = \sqrt{\det\| g_{ab}(x)\|}\,\sqrt{\det\| g_{c'd'}(x)\|}\,, \tag{2.17}$$

the particular case $x = x'$ implies

$$\mathsf{t}_a^{c'}(x, x' = x) = \delta_a^{c'} \quad \text{or} \quad \mathsf{t}_{ac'}(x, x' = x) = g_{ac'}\,, \tag{2.18}$$

$$\nabla_a\, \mathsf{t}_c^{d'}(x, x' = x) = \nabla_{c'}\, \mathsf{t}_a^{d'}(x, x' = x) = 0\,. \tag{2.19}$$

If V_n is an Euclidean space of the given signature (e.g. Minkowskian), we simply have

$$\mathfrak{t}_{ac'} = e_a \, e_{c'} \,. \tag{2.20}$$

We choose the space-time signature, as earlier, to be

$$g_{ab} = \text{diag}\,(+---)\,, \tag{2.21}$$

so the determinant is $g = \det \| g_{ab} \| < 0$, while $\sqrt{-g} > 0$.

§2.2.2. The Feynman propagator (reminder)

The Pauli-Jordan propagator (reminder). In the quantized field technique, the commutation function of the scalar field is introduced. It satisfies

$$\mathcal{D}(x) = \mathcal{D}^{+}(x) + \mathcal{D}^{-}(x) \tag{2.22}$$

with

$$\mathcal{D}^{+}(x) = -\mathcal{D}^{-}(-x) = \\ = \frac{1}{(2\pi)^3}\, i \int \left[\exp\,(i P x)\right] \delta(P^2 - m_0^2)\, \theta(P^4)\, d^3P\,. \tag{2.23}$$

This commutation function or the *Pauli-Jordan propagator* is explicitly written

$$\mathcal{D}(x, x') = \frac{1}{(2\pi)^3}\, i \int \left[\exp\,(i P x)\right] \epsilon(P^4)\, \delta(P^2 - m_0^2)\, d^3P\,, \tag{2.24}$$

where

$$\epsilon(P^4) = \theta(P^4) - \theta(-P^4)$$

is the "sign function"

$$\epsilon = +1 \quad \text{for} \quad P^4 > 0\,,$$

$$\epsilon = -1 \quad \text{for} \quad P^4 < 0\,.$$

The upper indices + and − indicate, respectively, the positive or negative energy parts contributed into the complete commutator $\mathcal{D}$, which corresponds to the future and the past, and whose boundaries are the characteristic hyperboloids in the Minkowski representation.

The Green function. The Jordan-Pauli commutation relation is an odd function

$$\mathcal{D}(x, x') = -\,\mathcal{D}(x', x)\,. \tag{2.25}$$

This (scalar) propagator is Lorentz invariant. It satisfies the homogeneous Klein-Gordon equation

$$\left(-\,\partial^a \partial_a - m_0^2\right) \mathcal{D}(x) = 0\,. \tag{2.26}$$

We then define the *Green function* of the scalar field by the equation

$$(-\,\partial^a\partial_a - m_0^2)\,\mathbb{G}(x,x') = -\,\delta^{(4)}(x,x')\,, \tag{2.27}$$

and then, passing to the momentum representation

$$\mathbb{G}(x,x') = \frac{1}{(2\pi)^4}\int \big[\exp\big(i\,P(x,x')\big)\big]\,\mathbb{G}(P)\,d^4P\,, \tag{2.28}$$

we obtain, for $\mathbb{G}(P)$, the expression

$$\mathbb{G}(P) = \frac{1}{m_0^2 - P^2}\,. \tag{2.29}$$

Writting the denominator as

$$P^2 = (P^4)^2 - (\mathcal{P}^2 + m_0^2)\,, \tag{2.30}$$

we see that for a given $\mathcal{P}^2$, the time component P_4 has two poles

$$P_4 = \pm E\,, \tag{2.31}$$

where the total energy of the particle is

$$E = \sqrt{P^2 + m_0^2}\,. \tag{2.32}$$

In order to remove this ambiguity when integrating (2.29) over d^4P, the Feynman contour rules should be used to circumvent the poles. First we consider the "retarded" Green function defined by the condition

$$\mathbb{G}^-(x,x') = 0 \quad \text{for} \quad x^4 - x'^4 < 0\,. \tag{2.33}$$

We then remark that the function (2.29) is not substantially modified, if multiplied by $\exp\big[-\,\epsilon(x^4 - x'^4)\big]$, where $\epsilon > 0$,

$$\mathbb{G}^-\exp\big[-\,\epsilon(x^4 - x'^4)\big] = \mathbb{G}_\epsilon\,, \tag{2.34}$$

and it can thus be represented by

$$\mathbb{G}^-(x,x') = \lim_{\epsilon\to 0}\mathbb{G}_\epsilon\,, \tag{2.35}$$

and $\mathbb{G}_\epsilon$ is defined by (2.31), hence satisfies

$$\big[\Delta - (\partial_t + \epsilon)^2 - m_0^2\big]\,\mathbb{G}_\epsilon = -\,\delta(x,x')\,. \tag{2.36}$$

In the momentum representation, when $\epsilon \to 0$, we have

$$\mathbb{G}_\epsilon = \frac{1}{m_0^2 - (P^4 - i\,\epsilon)^2 + \mathcal{P}^2} \longrightarrow \frac{1}{m_0^2 - P^2 - 2i\,\epsilon\,\mathcal{P}^4}\,, \tag{2.37}$$

and (2.31) takes the form

$$\mathbb{G}^{-}(x,x') = \frac{1}{(2\pi)^4}\int \frac{\exp\left[iP(x,x')\right]}{m_0^2 - P^2 + 2i\epsilon P^4}\, d^4P\,. \tag{2.38}$$

The same effect can be achieved if one integrates along the real axis by shifting the poles by an infinitesimal mass of the particle in the complex plane.

In the same way, the advanced Green function defined by

$$\mathbb{G}^{+}(x,x') = 0\,, \quad \text{for} \quad x^4 - x'^4 > 0\,, \tag{2.39}$$

which satisfies (3.36), is of the form

$$\mathbb{G}^{+}(x,x') = \frac{1}{(2\pi)^4}\int \frac{\exp(iPx)}{m_0^2 - P^2 + 2i\epsilon P^4}\, d^4P\,. \tag{2.40}$$

The integral

$$\mathbb{G}(x) = \frac{1}{(2\pi)^4}\int \frac{\exp(iPx)}{m_0^2 - P^2}\, d^4P \tag{2.41}$$

can be taken over the principal value, upon being separated into real and imaginary parts

$$\frac{1}{x + i\epsilon P^4} = \frac{P}{x} - i\pi\,\delta(x)\,\epsilon(P^4)\,, \tag{2.42}$$

and we obtain

$$\frac{P}{(2\pi)^4}\int \frac{\exp(iPx)}{m_0^2 - P^2}\, d^4P = \frac{1}{2}\,\epsilon(x^4)\,\mathbb{G}(x)\,. \tag{2.43}$$

Local bi-tensors on a four-dimensional manifold. Let us recall that the simplest example of a bi-tensor is the product of two local vectors taken at the different space-time points x and x'

$$A^k(x) \quad \text{and} \quad B_a(x')\,, \tag{2.44}$$

$$C_a^k(x,x') = A^k(x)\,B_a(x')\,. \tag{2.45}$$

We shall here adopt De Witt's convention that indices taken from the Latin characters $a \dots k$ are always to be associated with the point x' (denoted, from now, by z), while indices taken from k to y, are always associated with the point x.

The transformation law for the bi-tensor (2.45) is given by

$$C'^k_a = \left(\frac{\partial x'^k}{\partial x_m}\right)\left(\frac{\partial z^b}{\partial z'^a}\right) C_b^m\,. \tag{2.46}$$

The Dirac bi-scalar (2.13) extended to the Minkowski space is

$$\delta^{(4)}(x,z) = \delta(x^4-z^4)\,\delta(x^1-z^1)\,\delta(x^2-z^2)\,\delta(x^3-z^3) = \\ = \delta^{(4)}(z,x)\,, \qquad (2.47)$$

and is also called the *bi-density.*

We also define the geodesic interval bi-scalar $s\,(z,x)$ by the invariant

$$g^{km}\,\delta_k s\,\delta_m s = g^{ab}\,\partial_a s\,\partial_b s = \pm 1 \qquad (2.48)$$

with

$$\lim_{x\to z} s = 0\,.$$

§2.2.3. Trajectory of a charged particle

World tube. Let us consider a particle describing a world line whose point coordinate will always be denoted by z. We construct a small sphere surrounding the particle. The energy-momentum flow will be determined across the surface. In the course of time, such a sphere generates a hypersurface called a world tube.

We begin by introducing, at the point z on the world line of the particle, three unit vectors orthogonal to each other and to the world line itself

$$n^a_\alpha\, n_{\beta a} = \delta_{\beta\alpha}\,, \qquad n_{\alpha a}\,\dot{z}^a = 0\,. \qquad (2.49)$$

We next introduce a set of direction cosines ς satisfying

$$\varsigma_\alpha\,\varsigma_\alpha = 1 \qquad (2.50)$$

in terms of which we can specify the direction, relative to n^a_α, of an arbitrary unit vector perpendicular to the world line at z.

Then, in the direction of this arbitrary vector, we construct a geodesic from z extending throughout a fixed distance ξ to a point x of the "tube wall". The coordinates at the point z depend on the direction cosines ς_α and on the proper time τ at this point, which is explicitly expressed at the tube wall by the function $x^k(\varsigma,\tau)$.

Let us set up a bi-scalar σ related to the distance ξ as

$$\sigma = \frac{1}{2}\,\xi^2,$$

whence

$$\partial_a\sigma = -\,\xi\, n_{\alpha a}\,\varsigma^\alpha\,, \qquad (\partial_a\sigma)\,\dot{z}^a = 0\,. \qquad (2.51)$$

A pair of independent variations $\delta_1\varsigma_\alpha$, $\delta_2\varsigma_\alpha$ in the direction cosines defines an element $d\Omega$ of solid angle by the relation

$$\varsigma_\alpha d\Omega = \epsilon_{\alpha\beta\gamma}\,\delta_1\varsigma^\beta\,\delta_2\varsigma^\gamma\,,$$

or in virtue of (2.51), we have

$$\partial_q \partial_a \sigma \, \delta_3 x^q = \varsigma_a \, d\xi \,,$$

or

$$\delta_3 x^q = -(D^{qa})^{-1} \varsigma_a \, d\xi \,, \qquad D_{qa} = -\, \partial_q \partial_a \sigma \,.$$

We define a "tube section" as

$$dS_q = \epsilon_{qruw} \, \delta_1 x^r \, \delta_2 x^u \, \delta_3 x^w$$

and with $\Delta = -\mathfrak{t}^{-1} \det \| - D_{qa} \|$, where $\mathfrak{t}$ is the determinant (2.17), we obtain

$$dS_q = -\frac{1}{c\sqrt{-g}} \, \Delta^{-1} \, D_{qa} \, \dot{z}^a \, \xi^2 \, d\xi \, d\Omega \,. \tag{2.52}$$

§2.2.4. Dynamical equations for a particle

The conserved energy-momentum tensor. Let L denotes the surface of the world tube limited by two sections of hypersurfaces S_1 and S_2 corresponding to two proper times τ_1 and τ_2 (with $\tau_1 < \tau_2$).

We choose the integration volume d^4x as a portion of the tube, in order to express an integral conservation condition for the energy-momentum bi-tensor density $\Im^{qr}$.

However, one cannot integrate the divergence of $\Im^{qr}$ over the four-volume (at x) d^4x, to replace the volume integral by an integral over the hypersurface S_r containing z, since Gauss' theorem is not longer applicable for a bi-tensor.

There is nevertheless a natural procedure to overcome this difficulty by introducing the displacement bi-tensor $\mathfrak{t}_q^a$ in order to refer to the contributions into the integral

$$I^a = \int (\mathfrak{t}_q^a \, \partial_r \Im^{qr}) \, d^4 r \tag{2.53}$$

at the point x back to some fixed point z.

The latter integral becomes a local four-vector at z where

- x_a corresponds to x_q, and
- x'_a corresponds to z_a.

Let us then consider the integral over S_1, S_2 and the volume ϑ, the conservation condition for $\Im_{ab}$ is then written down as

$$\frac{1}{c} \int (\mathfrak{t}_q^a \, \partial_r \Im^{qr}) \, d^4 x = 0 \,, \tag{2.54}$$

Integrating by parts

$$\frac{1}{c}\left(\int_L + \int_{S_1} + \int_{S_2}\right) \mathfrak{t}^a_q \Im^{qr}\, dS_r - \frac{1}{c}\int_\vartheta (\partial_r \mathfrak{t}^a_q)\Im^{qr}\, d^4x = 0 \tag{2.55}$$

with zero contribution of the last integral, and considering the replacement

$$\int_L \longrightarrow \int_{\tau_1}^{\tau_2}\int_{4\pi}\,, \tag{2.56}$$

we can write (2.55) in the limit $\xi \to 0$, while taking (2.52) into account,

$$\lim_{\xi\to 0}\,\frac{1}{c}\int_{\tau_1}^{\tau_2}\!\!\int_{4\pi} \mathfrak{t}^a_q \Im^{qr} dS_r + m_0\left[\,\mathfrak{t}^a_{b'}\big(z(\tau'), z(\tau)\big)\,\dot{z}^{b'}(\tau')\right]_{\tau=\tau_1}^{\tau'=\tau_2} -$$

$$- m_0\int_{\tau_1}^{\tau_2}\partial_{r'}\mathfrak{t}^a_{b'}\big(z(\tau'), z(\tau)\big)\,\dot{z}^{b'}\dot{z}^{r'}(\tau')\, d\tau' = 0\,. \tag{2.57}$$

The next step is to let τ_1 and τ_2 both approach τ, and denoting their infinitesimal separation in the limit by $d\tau$, we express the relation (2.57) as follows

$$m_0\,\ddot{z}_a\, d\tau = -\lim_{\xi\to 0}\,\frac{1}{c}\int \mathfrak{t}^a_q \Im^{qr} dS_r\,. \tag{2.58}$$

The geodesic principle is obviously given by

$$m_0\,\ddot{z}_a = 0\,. \tag{2.59}$$

In the framework of the Euclidean approximation, when the particle's trajectory is taken along x, the latter equation reduces to

$$m_0\,\frac{d^2x}{d\tau^2} = 0\,. \tag{2.60}$$

Chapter 3. Gravitational Damping

§3.1. Green functions on a curved manifold

§3.1.1. Scalar Green functions

Elementary solutions of J. Hadamard. In a non-Euclidean space, the second derivatives of any vector or tensor are not equivalent

$$(\mathrm{D}_e\mathrm{D}_k - \mathrm{D}_k\mathrm{D}_e)\,A^{h\dots}_{d\dots} = -\,R^{h\,\cdots}_{\cdot iek}\,A^{i\dots}_{d\dots} - 2\Gamma^i_{ke}\mathrm{D}_i A^{h\dots}_{d\dots} + R^{\,i\,\cdots}_{\cdot\, dke}\,A^{h\dots}_{i\dots}\,. \tag{3.1}$$

From the identities, the equations

$$-\,\mathrm{D}_e F^{de} = j^d \tag{3.2}$$

read

$$- \sqrt{-g}\; g^{dh} g^{ek} \mathrm{D}_e \left(\mathrm{D}_h A_k - \mathrm{D}_k A_h\right) =$$
$$= \sqrt{-g}\; g^{ek} \mathrm{D}_e \mathrm{D}_k A^d - \sqrt{-g}\; \mathrm{D}^d \mathrm{D}_k A^k - \sqrt{-g}\; R^{cd} A_c = -\, j^d, \quad (3.3)$$

and by fixing a gauge

$$\mathrm{D}_k A^k = 0\,, \quad (3.4)$$

we have

$$\sqrt{-g}\; \left(g^{ek} \mathrm{D}_e \mathrm{D}_k A^d - R^{dh} A_h\right) = -\, j^d. \quad (3.5)$$

Consider then the vector wave equation

$$g^{ek} \mathrm{D}_e \mathrm{D}_k A^d - R^{dh} A_h = 0\,. \quad (3.6)$$

Following Hadamard, we shall try to find so called "elementary solutions" corresponding to Green functions. We can then infer the particular solutions of (3.5).

The Feynman propagator. We first consider here the scalar wave equation on a four-dimensional manifold

$$g^{dh} \partial_d \partial_h A = 0\,. \quad (3.7)$$

Here, we find the elementary solution which is a bi-scalar having the form

$$\mathbb{G}^{(1)} = \frac{1}{(2\pi)^2} \left(\frac{\mathfrak{u}}{\xi} + \mathfrak{b} \ln |\xi| + \mathfrak{w}\right), \quad (3.8)$$

where $\mathfrak{u}$, $\mathfrak{b}$, $\mathfrak{w}$ are bi-scalars satisfying the normalization condition

$$\lim_{x \to z} \mathfrak{u} = 1\,. \quad (3.9)$$

After some algebra, we show the validity of the equation

$$\mathfrak{u}^{-1} \partial_d \mathfrak{u} = \frac{1}{2}\, \Delta^{-1} \partial_d \Delta\,, \quad (3.10)$$

which, with the boundary condition (3.9), has the unique solution

$$\mathfrak{u} = \sqrt{\Delta}\,. \quad (3.11)$$

Eventually we arrive at

$$\lim_{x \to z} \mathfrak{b} = \frac{1}{12}\, \mathbb{G}\,. \quad (3.12)$$

Separating the full Green function $\mathbb{G}^{\mathrm{F}}$ into real and imaginary parts

$$\mathbb{G}^{\mathrm{F}} = \mathbb{G}^{(1)} - 2i\, \mathbb{G}\,, \quad (3.13)$$

where

$$\mathbb{G}^{\mathrm{F}} = \frac{1}{(2\pi)^2}\left[\frac{\sqrt{\Delta}}{(\xi + i0)} + \mathfrak{b}\ln(\xi + i0) + \mathfrak{w}\right] \tag{3.14}$$

is identified with the Feynman propagator.

The formula (2.42) becomes

$$\frac{1}{\xi + i0} = \frac{\mathfrak{p}}{\xi - \pi i\,\delta(\xi)} \tag{3.15}$$

and

$$\ln\,(\xi + i0) = \ln|\xi| + \pi i\,\epsilon(-\xi) \tag{3.16}$$

with the sign function such that

$$\epsilon(\xi) = 0 \quad \text{for} \quad \xi < 0$$

and

$$\epsilon(\xi) = 1 \quad \text{for} \quad \xi > 0\,.$$

The scalar Green function corresponding to the bi-scalar $\mathfrak{b}$ can be computed as

$$\mathbb{G} = \frac{1}{8\pi}\left[\sqrt{\Delta}\;\delta(\xi) - \mathfrak{b}\,\epsilon(-\xi)\right]. \tag{3.17}$$

§3.1.2. Vector Green functions

Hadamard solutions. Consider now the wave equation

$$g^{hk}\,\mathrm{D}_h\mathrm{D}_k A_d + R_{d\cdot}^{\cdot h} A_h = 0\,. \tag{3.18}$$

The procedure is entirely analogous to the above, thus we introduce the elementary solution of the form

$$\mathbb{G}^{(1)}_{qa} = \frac{1}{(2\pi)^2}\left[\frac{\mathfrak{u}_{qa}}{\xi + \mathfrak{b}_{qa}}\ln|\xi| + \mathfrak{w}_{qa}\right], \tag{3.19}$$

where the functions $\mathfrak{u}_{qa}$, $\mathfrak{b}_{qa}$ and $\mathfrak{w}_{qa}$ are now bi-vectors. Normalization for $\mathfrak{u}_{qa}$ leads to

$$\lim_{x\to z}\mathfrak{u}_{qa}(x,z) = g_{qa}(z)$$

and, after some algebra, we find

$$\mathfrak{u}_{qa} = \sqrt{\Delta}\;\mathfrak{t}_{qa}\,. \tag{3.20}$$

Making use of the extension, for the bi-vector $\mathfrak{b}_{qa}$,

$$\mathfrak{b}_{qa} = \mathfrak{t}_{qa}\left(1 - \frac{1}{12}\,R^{be}\,\partial_b\sigma\,\partial_e\sigma + O(s^2)\right) \tag{3.21}$$

at the limit

$$\lim_{x \to z} \mathfrak{b}_{qa} = -\frac{1}{2}\, \mathfrak{t}_q^b \left(R_{ab} - \frac{1}{6}\, g_{ab} R \right) . \tag{3.22}$$

The presence of the determinant (Δ symbol) in (3.20) reveals the singular behaviour of the elementary waves originating from the point z: this represents actually the so-called "thinning out" of these waves due to the induced curvature.

The Feynman propagator. The full propagator is of the form

$$\mathbb{G}^{\text{F}}_{qa} = \mathbb{G}^{(1)}_{qa} - 2i\,\mathbb{G} \tag{3.23}$$

that is

$$\mathbb{G}^{\text{F}}_{qa} = \frac{1}{(2\pi)^2} \left[\sqrt{\Delta}\, \frac{\mathfrak{t}_{qa}}{(\xi + i0) + \mathfrak{b}_{qa}} \ln(\xi + i0) + \mathfrak{w}_{qa} \right] . \tag{3.24}$$

Advanced or retarded Green functions. We set

$$\mathbb{G}^{\pm}_{qa} = \int \sqrt{\mathfrak{t}}\ \mathbb{G}^{\pm}_{qr'}\, \mathfrak{t}_a^{r'} \delta^{(4)} d^4x'. \tag{3.25}$$

The quantities $\mathbb{G}^{\pm}_{qa}$ correspond to advanced and retarded portions of the Green functions $\mathbb{G}_{qa}$, whose components depend on two distinct points x and z: they define a bi-vector.

If we consider an arbitrary space-like hypersurface $S(x)$ containing x, we regard "actions" as retarded when the source z^a lies to the past of S, and advanced when the source z^a lies to the future of S. The "symmetric" Green function is then

$$\mathbb{G}_{qa} = \frac{1}{8\pi} \left[\sqrt{\Delta}\ \mathfrak{t}_{qa}\, \delta\xi - \mathfrak{b}_{qa}\, \epsilon\,(-\xi) \right] , \tag{3.26}$$

where the functions $\mathbb{G}$ can, just as in the flat-space case, be separated into advanced and retarded parts

$$\mathbb{G}_{qa} = \frac{1}{2} \left(\mathbb{G}^{-}_{qa} + \mathbb{G}^{+}_{qa} \right) \tag{3.27}$$

with

$$\mathbb{G}^{-}_{qa} = 2\,\epsilon(S,\, z)\, \mathbb{G}_{qa}(x, z)\,, \tag{3.28}$$

$$\mathbb{G}^{+}_{qa} = 2\,\epsilon(z,\, S)\, \mathbb{G}_{qa}(x, z)\,, \tag{3.29}$$

$$\epsilon \left[S(x),\, z \right] = 1 - \epsilon \left[z,\, S(x) \right] = 1\,,$$

when z lies to the past of S, and vanishes when it lies to the future.

§3.2. Dynamical equations for the electron

§3.2.1. Tensor density of the electromagnetic field

Energy-momentum field global tensor density. On an arbitrary manifold, the approximated Lagrangian for a particle of mass m_0 is

$$L_m = -\,m_0 c^2 \int \sqrt{-\,g_{ab}\,\dot{z}^a\,\dot{z}^b}\;\delta^{(4)}\,d\tau\,. \qquad (3.30)$$

The inferred massive tensor density of this particle with respect to the proper time τ following the geodesic $z(\tau)$ is

$$M^{qr} = m_0 c \int \sqrt{\mathfrak{t}}\;\mathfrak{t}^q_a\,\mathfrak{t}^r_b\,\dot{z}^a\,\dot{z}^b\,\delta^{(4)}\,d\tau\,. \qquad (3.31)$$

For an electron interacting with an electromagnetic field, the Lagrangian density becomes

$$L = -\,m_0 c^2 \int \sqrt{-\,g_{ab}\,\dot{z}^a\,\dot{z}^b}\;\delta^{(4)}\,d\tau\,+ \\ +\,e \int A_a\,\dot{z}^a\,\delta^{(4)}\,d\tau - \frac{1}{16\pi}\sqrt{-g}\;F_{qr}F^{qr}. \qquad (3.32)$$

The current vector density expressed with the charge density (1.7) can be determined from the four-velocity $\dot{z}^a$ at the point z, by parallel displacement along the geodesic extending z to x

$$j^q = e \int \sqrt{\mathfrak{t}}\;\mathfrak{t}^q_a\,\dot{z}^a\,\delta^{(4)}\,d\tau\,. \qquad (3.33)$$

The form of this density justifies the form of the second term in (3.32), which corresponds to the classical electron-field interaction, $eA_q j^q$. Application of the least action principle to

$$S = \frac{1}{c}\int L\,d^4x \qquad (3.34)$$

yields the dynamical equations

$$m_0\,\ddot{z}^a = \frac{e}{c}\,F^{a\cdot}_{\cdot b}\,\dot{z}^b, \qquad (3.35)$$

$$\sqrt{-g}\;\partial_r F^{qr} = \frac{4\pi}{c}\,j^q = \partial_r F^{qr}. \qquad (3.36)$$

Given the current density j^q, the tensor $F^{a\cdot}_{\cdot b}$ appearing on the right hand side of (3.32) is divergent, and this leads to the well-known difficulty that the electron's proper mass m_0 is infinite, which must thereby be renormalized.

For reasons which will become clear later, we hereby proceed to consider the tensor density of the whole system as

$$\Im^{ab} = (\Im^{ab})_{\text{mass}} + (\Im^{ab})_{\text{field}} + (\Im^{ab})_{\text{elec}} \,. \tag{3.37}$$

Advanced or retarded potentials. According to Quantum Electrodynamics, the particular solutions of equation (3.5) are the retarded and/or advanced potentials

$$A^{-}_{q}(x) = \frac{4\pi}{c} \int \mathbb{G}^{-}_{qr}(x, x')\, j^{r'}(x')\, d^4x', \tag{3.38}$$

$$A^{+}_{q}(x) = \frac{4\pi}{c} \int \mathbb{G}^{+}_{qr}(x, x')\, j^{r'}(x')\, d^4x'. \tag{3.39}$$

Substituting the expressions of j^q in the previous equations, we obtain

$$A^{\pm}_{q} = 4\pi e \int_{-\infty}^{+\infty} \mathbb{G}^{\pm}_{qa}\, \dot{z}^a\, d\tau = \\ = \pm\, e \int_{\tau_s}^{\pm\infty} \left[\mathfrak{u}_{qa}\, \delta\xi - \mathfrak{b}_{qa}\, \epsilon\,(-\xi)\right] \dot{z}^a\, d\tau\,, \tag{3.40}$$

where $\tau_{\rm S}$ is the value of the proper time at the point of intersection of the world line of the particle with an arbitrary hypersurface $S(x)$ containing x.

Defining the advanced and retarded proper time of the particle relative to the point x, $\tau_{\pm}$, we obtain the advanced and retarded potentials as

$$A^{\pm}_{q} = \mp\, e \left[\sqrt{\Delta}\;\mathfrak{t}_{qa}\, \dot{z}^a \left(\dot{z}^b\, \partial_b\, \xi\right)^{-1}\right]_{\tau=\tau^{\pm}} \mp\, e \int_{\tau^{\pm}}^{\infty} \mathfrak{b}_{qa}\, \dot{z}^a\, d\tau\,. \tag{3.41}$$

These are the covariant Liénard-Wiechert potentials. For flat space-time these potentials obviously reduce to the form (1.12).

Retarded and advanced fields. From the potentials defined above, we define the corresponding proper fields

$$F^{\pm}_{qr} = \partial_q\, A^{\pm}_{r} - \partial_r\, A^{\pm}_{q}. \tag{3.42}$$

The total field can be expressed in the alternative forms

$$F_{qr} = (F_{qr})^{\text{in}} + F^{-}_{qr} = (F_{qr})^{\text{out}} + F^{+}_{qr}\,, \tag{3.43}$$

where *in* and *out* mean the incoming field and the outgoing field, respectively.

Defining the average field

$$\langle F_{qr}\rangle = \frac{1}{2}\big(F^{-}_{qr} + F^{+}_{qr}\big)\,, \tag{3.44}$$

we write the total field in terms of the average free non-radiative field

$$F_{qr} = \langle (F_{qr})^{\text{free}}\rangle + \langle F_{qr}\rangle \tag{3.45}$$

with

$$\langle (F_{qr})^{\text{free}}\rangle = \frac{1}{2}\left[(F_{qr})^{\text{in}} + (F_{qr})^{\text{out}}\right]. \tag{3.46}$$

The field strengths can be explicitly written as

$$\begin{aligned} F^{\pm}_{qr} = \mp\, e\,\Big\{&(\mathfrak{u}_{ra}\partial_q\xi - \mathfrak{u}_{qa}\partial_r\xi)\,\dot z^a\,(\dot z^b\dot z^e\partial_b\partial_e\xi + \ddot z^b\partial_b\xi)(\dot z^d\partial_d\xi)^{-3} - \\ &- \big[\partial_b(\mathfrak{u}_{ra}\partial_q\xi - \mathfrak{u}_{qa}\partial_r\xi)\,\dot z^a\dot z^b + (\mathfrak{u}_{ra}\partial_q\xi - \mathfrak{u}_{qa}\partial_r\xi)\,\ddot z^a\big](\dot z^e\partial_e\xi)^{-2} + \\ &+ (\partial_q\mathfrak{u}_{ra} - \partial_r\mathfrak{u}_{qa} + \mathfrak{b}_{ra}\partial_q\xi - \mathfrak{b}_{qa}\partial_r\xi)\,\dot z^a\,(\partial_b\xi\,\dot z^b)^{-1}\Big\}_{\tau=\tau^{\pm}} \mp \\ &\mp e\int_{\tau\pm}^{\pm\infty}\mathfrak{f}_{qra}\,\dot z^a\,d\tau + O(\xi)\,, \end{aligned} \tag{3.47}$$

where

$$\mathfrak{f}_{qra} = (\partial_q\mathfrak{b}_{ra} - \partial_r\mathfrak{b}_{qa})\,. \tag{3.48}$$

§3.2.2. Global damping

Energy-momentum tensor density. We consider the energy-momentum tensor density $\Im^{ab} = (\Im^{ab})_{\text{mass}} + (\Im^{ab})_{\text{field}} + (\Im^{ab})_{\text{elec}}$ (3.37) of the system at the point x. Thus (here O vanishes when $\xi\to 0$),

$$\begin{aligned} \frac{1}{c}\,\mathfrak{t}^a_q\,\Im^{qr} dS_r = \frac{1}{4\pi c}\sqrt{-g}\,\Big[&\mathfrak{t}^a_q\big(\langle F^{q\cdot}_{\cdot s}\rangle\langle F^{rs}\rangle + \langle (F^{q\cdot}_{\cdot s})^{\text{free}}\rangle\langle F^{rs}\rangle + \\ &+ \langle F^{q\cdot}_{\cdot s}\rangle\langle (F^{rs})^{\text{free}}\rangle\big)\,dS_r - \left(\frac{1}{4}\langle F_{st}\rangle\langle F^{st}\rangle + \frac{1}{2}\langle (F_{st})^{\text{free}}\rangle\langle F^{st}\rangle\right)\times \\ &\times\,\mathfrak{t}^{qa}\,dS_q\Big] + \frac{1}{c}\,\mathfrak{t}^a_q\,(\Im^{qr})_{\text{field}} + O(\xi)\,. \end{aligned} \tag{3.49}$$

Avoiding the mass renormalization. Reverting to the result inferred in the present theory, we integrate (3.49) according to

$$\begin{aligned} \frac{1}{c}\int_{4\pi}\mathfrak{t}^a_q\Im^{qr} dS_r = \Big[&\frac{e^2}{2\epsilon c^2}\,\ddot z^a - \frac{e^2}{c}\,\dot z^b\int_{-\infty}^{+\infty}\mathfrak{f}^{a\cdot\cdot}_{\cdot be'}\,\dot z^{e'}\tau'\,d\tau' - \\ &- \frac{e}{c}\,\langle (F^{a\cdot}_{\cdot b})^{\text{free}}\rangle\,\dot z^b\Big]\,d\tau + \frac{1}{c}\int_{4\pi}\mathfrak{t}^a_q\,(\Im^{qr})_{\text{field}}\,dS_r + O(\xi)\,, \end{aligned} \tag{3.50}$$

where $\ddot{z}^a$ is a function of the electromagnetic field.

In this equation, we must get rid of the term $\frac{e^2}{2\xi c^2}\,\ddot{z}^a$ which is divergent. This term has the same kinematical structure as the mass term in (2.58). Therefore, we renormalize the mass as follows

$$m = m_0 + \lim_{\xi \to 0} \frac{e^2}{2\,\xi\, c^2} \tag{3.51}$$

and (2.58) reads now

$$m\,\ddot{z}^a = \frac{e}{c}\,\langle (F^{a\cdot}_{\cdot b})^{\text{free}} \rangle\, \dot{z}^b + \frac{e^2}{2\,c}\,\dot{z}^b \int_{-\infty}^{+\infty} \mathfrak{f}^{a\cdot\cdot}_{\cdot\, be'}\, \dot{z}^{e'}\, d\tau'. \tag{3.52}$$

By setting

$$\ddot{z}^a\,\frac{e^2}{2\,\xi\, c^2}\, d\tau = -\frac{1}{c}\int_{4\pi} \mathfrak{t}^a_q\, (\Im^{qr})_{\text{field}}\, dS_r\,, \tag{3.53}$$

we remark that the renormalization is no longer required, which gives a better physical consistency to the present theory. This particular circumstance tends to lend support to the existence of free gravitational fields predicted by the EGR theory.

In the absence of charge, we obtain the well-known inertia law

$$m_0\,\ddot{z}^a = 0\,. \tag{3.54}$$

As outlined by De Witt and Brehme [3], for purposes of application to physically set boundary conditions, it is more appropriate to deal with the "incoming" field $(F_{ab})^{\text{in}}$

$$m_0\,\ddot{z}^a = \frac{e}{c}\,(F^{a\cdot}_{\cdot b})^{\text{in}}\,\dot{z}^b + \frac{2}{3}\,\frac{e^2}{c^3}\left(\dddot{z}^a - \frac{\dot{z}^a\,\dot{z}^2}{c^2}\right) + \\ + \frac{e^2}{c}\,\dot{z}^b \int_{-\infty}^{r} \mathfrak{f}^{a\cdot\cdot}_{\cdot\, be'}\,\dot{z}^{e'}\,d\tau\,. \tag{3.55}$$

On the right hand side, one recognizes the first two terms of the relativistic equation (1.23), bearing in mind that the derivatives are covariant here, while keeping the proper mass m_0 on the left hand side.

The third term determined by $\mathfrak{b}_{aq}$ of the Green function is the "tail" due to the space-time curvature and radiation damping occurs even when $(F^{a\cdot}_{\cdot b})^{\text{in}}$ vanishes.

Concluding remarks

Let us stress some important points about the "tail" term:

- It is spanned by R_{ab}, which are built with the general connection coefficients (2.2): $\Gamma = \{\ \} + (\Gamma)_J$ as defined by the EGR theory;

- We thus have implicitly assumed that the elementary solutions of Hadamard and subsequent relations hold within the suggested extension of the GR theory.

Upon this assumption, it is clearly shown that, with the introduction of the related persistent free field, one no longer requires a negative external mass, thus avoiding an unphysical "pathology" found in the Riemannian theory.

In the Euclidean approximation, the third term (3.55) vanishes anyway and the formula (1.23) is recovered.

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On the Physical Nature of the Wave Function: A New Approach through the EGR Theory

Patrick Marquet*

Abstract: In this paper, we first recall the quantum theory of Louis de Broglie which attempted to give the usual wave function a real and truly physical nature, and which is closely associated to a massive particle. The resulting Double Solution Theory is then interpreted in terms of a physical fluid described in the framework of the Extended General Relativity Theory (EGR theory) approximation. This approach may provide an explanation to the problem arising from the "hidden" medium as set forth by the initial theory.

Contents:

Notations:

4-tensor or 4-vector: small Latin indices $a, b, \ldots = 1, 2, 3, 4$;

3-tensor or 3-vector: small Greek indices $\alpha, \beta, \ldots = 1, 2, 3$;

Ordinary derivative: $\partial_a U$;

Covariant derivative: ∇_a in GR, and D_a in EGR;

Kronecker symbol: $\delta_{ab} = (+1 \text{ if } a = b; \; 0 \text{ if } a \neq b)$;

Three-dimensional vectorial quantities: $\boldsymbol{P} = P_\alpha$.

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Introduction

The original wave function first discovered by Louis de Broglie [1] in his famous Wave Mechanics Theory is always acknowledged as a statistical entity. Its physical meaning was almost totally denied in all subsequent quantum field developments despite the Davisson and Germer experiment, which actually detected the wave through diffraction of electrons by a nickel crystal lattice.

This problem dates back from the Solvay Symposium of 1927 in Brussels, when most of the physicists decided to adopt the so-called Copenhagen School Concept of considering quantum physics on pure statistical grounds. Throughout the remainder of his life, de Broglie yet could not believe observable physical phenomena to only follow from abstract mathematical wave functions.

In the late 1960's, he improved his first theory called the *double solution interpretation of Quantum Mechanics*, which describes a particle as closely related to its physical wave and constantly in phase with it. The theory is extremely simple and elegant, but to remain consistent, it requires two constraints:

- The guided particle should permanently exchange energy and momentum from an external (unknown) medium which he named "hidden thermostat";
- In addition, the considered particle should also undergo small energetic random perturbations.

In the past decades, many interesting theories have been provided for explaining the nature of this "sub-quantum" medium, which is assumed to exchange energy and momentum at the quantum level.

In this paper, we suggest to identify this "energy background" with the persistent field of the EGR theory. The hydrodynamic interpretation for the particle's probability density as depicted by de Broglie, is here given with physical consistency.

Chapter 1. Interpretation of Wave Mechanics by Means of Louis de Broglie's Theory of Double Solution

§1.1. The reasons for implementing the theory

For almost a century, the wave-particle duality first discovered by Einstein, in his theory of light quanta, has been the basis of present day Quantum Physics. As an essential contribution, the wave mechanics theory of Louis de Broglie has successfully extended this duality to all known particles. Shortly after, de Broglie further developed the Double Solution Theory based on two striking observations.

In the framework of the Special Theory of Relativity, it is noticed that the frequency ν_0 of a plane monochromatic wave is transformed as

$$\nu_c = \nu_0 \sqrt{1-\beta^2}\,,$$

while a clock's frequency ν_0 is transformed according to

$$\nu_c = \frac{\nu_0}{\sqrt{1-\beta^2}}$$

with the phase velocity

$$\widetilde{\mathrm{v}} = \frac{c}{\beta} = \frac{c^2}{\mathrm{v}}\,.$$

It is noticed that the four-vector defined by the gradient of the plane monochromatic wave can be linked to the energy-momentum four-vector of a particle by introducing Planck's constant h, thus writing

$$W = h\nu\,, \qquad \lambda = \frac{h}{p}\,, \tag{1.1}$$

where p is the particle's momentum and λ is its wavelength.

If the particle is considered as containing the rest energy

$$M_0\, c^2 = h\nu_0\,,$$

we may compare it to a small clock of a frequency ν_0 so that when moving with a velocity $\mathrm{v} = \beta c$, its frequency, different from that of the wave, is then

$$\nu_c = \nu_0 \sqrt{1-\beta^2}\,.$$

It can be further shown that the particle has an internal vibration which is constantly in phase with the vibration of the surrounding wave.

In the spirit of the theory, the wave is regarded as a physical entity having a very small amplitude which cannot be arbitrarily normed, and which is distinct from the wave ψ having a statistical significance in the usual quantum mechanical formalism.

Let us call ϑ the physical wave which is connected to the wave ψ by the relation $\psi = C\vartheta$, where C is a normalizing factor. The wave ψ has then the nature of a subjective probability representation formulated by means of the objective wave ϑ. Therefore, the wave mechanics is complemented by the Double Solution Theory, so ψ and ϑ are two solutions of the same equation.

If the complete solution of the equation representing the wave ϑ, or, if preferred, the wave ψ (since both waves are equivalent according to $\psi = C\vartheta$) is written as

$$\vartheta = a(x,y,z,t)\, \exp\left[\frac{i}{\hbar}\,\phi(x,y,z,t)\right], \qquad \hbar = \frac{h}{2\pi}, \tag{1.2}$$

where a and ϕ are real functions, the energy W and momentum vector $\boldsymbol{p}$ of the particle localized at point x, y, z at time t are given by

$$W = \partial_t \phi, \qquad \boldsymbol{p} = -\,\mathrm{grad}\,\phi, \tag{1.3}$$

which in the case of a plane monochromatic wave, where one has

$$\phi = h\left[\nu t - \frac{(\alpha x + \beta y + \gamma z)}{\lambda}\right],$$

yields equation (1.1) for W and $\boldsymbol{p}$.

§1.2. The guidance formula and the quantum potential

Taking Schrödinger's equation for the scalar wave phase ϑ in the external potential U, we get

$$\partial_t \vartheta = \frac{\hbar}{2im}\,\Delta\vartheta + \frac{i}{\hbar}\,U\vartheta. \tag{1.4}$$

This complex equation suggests that ϑ should be represented by two real functions linked by two real equations, leading to

$$\vartheta = a\,\exp\frac{i\phi}{\hbar}, \tag{1.5}$$

where a (the wave's amplitude) and ϕ (its phase) are both real values. Taking this value into equation (1.4), we arrive at two important equations

$$\partial_t \phi - U - \frac{1}{2m}\,(\mathrm{grad}\,\phi)^2 = \frac{\hbar}{2m}\,\frac{\Delta a}{a}, \tag{1.6}$$

$$\partial_t a^2 - \frac{1}{m}\,\mathrm{div}\big(a^2\,\mathrm{grad}\,\phi\big) = 0. \tag{1.7}$$

If terms involving Planck's constant $\hbar$ in equation (1.6) are neglected (which amounts to disregarding quanta), and if we set $\phi = S$, this equation becomes

$$\partial_t S - U = \frac{1}{2m}\,(\mathrm{grad}\,S)^2. \tag{1.8}$$

As S is the Jacobi function, equation (1.8) is the Jacobi equation of Classical Mechanics. Only the term containing $\hbar^2$ is responsible for the particle's motion being different from the classical motion.

It can be interpreted as another potential $\mathcal{Q}$, distinct from the classical potential U,

$$\mathcal{Q} = -\frac{\hbar^2}{2m}\frac{\Delta a}{a}\,. \tag{1.9}$$

One has thus a variable proper mass

$$M_0 = m_0 + \frac{\mathcal{Q}_0}{c^2}\,,$$

where, in the particle's rest frame, $\mathcal{Q}_0$ is a positive or negative variation of the rest mass, and it represents the "quantum potential" which causes the wave function's amplitude to vary.

By analogy with the classical formulae $\partial_t S = E$ and $\boldsymbol{p} = -\,\mathrm{grad}\, S$, with E and $\boldsymbol{p}$ being the classical energy and momentum vector, one may write

$$\partial_t \phi = E\,, \qquad \boldsymbol{p} = -\mathrm{grad}\,\phi\,. \tag{1.10}$$

As in non-relativistic mechanics, where $\boldsymbol{p}$ is expressed as a function of velocity by the relation $\boldsymbol{p} = m\mathbf{v}$, one eventually finds the following result

$$\mathbf{v} = \frac{\boldsymbol{p}}{m} = -\frac{1}{m}\,\mathrm{grad}\,\phi\,, \tag{1.11}$$

which is the *guidance formula*; it gives the particle's velocity, at point x, y, z and time t as a function of the local phase variation at this point. Inspection shows that relativistic dynamics applied to the variable proper mass M_0 eventually leads to the following result

$$W = \frac{M_0 c^2}{\sqrt{1-\beta^2}} = M_0 c^2\sqrt{1-\beta^2} + \frac{M_0 v^2}{\sqrt{1-\beta^2}}\,, \tag{1.12}$$

known as the *Planck-Laue formula*.

In the relativistic form of the theory, equation (1.4) is replaced by the Klein-Gordon equation applied to the wave function ϑ

$$\Box\vartheta - \frac{2i}{\hbar}\frac{eV}{c^2}\,\partial_t\vartheta + \frac{2i}{\hbar}\frac{e}{c}\sum_{xyz} A_x\,\partial_x\vartheta + {} + \frac{1}{\hbar^2}\left[m_0^2 c^2 - \frac{e^2}{c^2}\left(V^2 - \boldsymbol{A}^2\right)\right]\vartheta = 0\,, \tag{1.13}$$

where the particle's charge e is acted upon by an electromagnetic field with a scalar potential $V(x,\,y,\,z,\,t)$ and a vector potential $\boldsymbol{A}(x,\,y,\,z,\,t)$. Note that the d'Alembertian, as usual, is

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,.$$

Inserting equation (1.5) into equation (1.13) yields the generalized Jacobi equation (1.14) as well as another continuity equation (1.15)

$$\frac{1}{c^2}\left(\partial_t\phi - eV\right)^2 - \sum_{xyz}\left(\partial_t\phi + \frac{e}{c}A_x\right)^2 = m_0^2c^2 + \hbar^2\frac{\Box a}{a} = M_0^2c^2, \quad (1.14)$$

$$\frac{1}{c^2}\left(\partial_t\phi - eV\right)\partial_t a - \sum_{xyz}\left(\partial_x\phi + \frac{e}{c}A_x\right)\partial_x a + \frac{a}{2}\,\Box\phi = 0\,, \qquad (1.15)$$

where on the right hand side of (1.14) we have introduced a variable proper mass M_0 defined by

$$M_0 = \sqrt{m_0^2 + \frac{\hbar^2}{c^2}\frac{\Box a}{a}}\;. \qquad (1.16)$$

Neglecting the terms in $\hbar^2$, equation (1.14) leads to

$$\frac{1}{c^2}\left(\partial_t S - eV\right)^2 - \sum_{xyz}\left(\partial_x S + eA_x\right)^2 = m_0^2c^2, \qquad (1.17)$$

which is the Jacobi equation for a charged particle moving in an electromagnetic field with scalar and vector potentials V and $\boldsymbol{A}$, and considered in the framework of relativistic mechanics without quanta.

Keeping the terms in $\hbar^2$ and considering the proper mass M_0 as defined in equation (1.16), one gets

$$\frac{M_0c^2}{\sqrt{1-\beta^2}} = \partial_t\phi - eV, \qquad \frac{M_0\mathbf{v}}{\sqrt{1-\beta^2}} = -\left(\mathrm{grad}\,\phi + e\boldsymbol{A}\right), \qquad (1.18)$$

thus, with $\beta = \frac{\mathrm{v}}{c}$, we find the relativistic guidance formula

$$\mathbf{v} = -\frac{c^2\left(\mathrm{grad}\,\phi + e\boldsymbol{A}\right)}{\partial_t\phi - eV}\,. \qquad (1.19)$$

For the Newtonian approximation with $\boldsymbol{A}=0$ and $\partial_t\phi - eV \cong m_0c^2$, equation (1.11) is found again.

Here, the quantum force results from the variation of M_0c^2 as the particle moves. For a monochromatic wave, the quantum potential is constantly zero, and one simply has

$$\mathcal{Q} = M_0c^2 - m_0c^2. \qquad (1.20)$$

In the non-relativistic approximation, $c\to\infty$ and $\Box a \cong -\Delta a$. Therefore we obtain

$$\mathcal{Q} = \sqrt{\frac{m_0^2c^4 + c^2\hbar^2\,\Box a}{a}} \cong -\frac{\hbar^2}{2m_0}\frac{\Delta a}{a}\,. \qquad (1.21)$$

§1.3. Particles with internal vibration and the hidden thermodynamics

The idea of considering the particle as a small clock is of central importance here. Let us look at the self-energy M_0c^2 as the hidden heat content of a particle. One easily conceives that such a small clock has (in its proper system) an internal periodic energy of agitation which does not contribute to the momentum of the whole. This energy is similar to that of a heat-containing body in a thermal equilibrium. Let Q_0 be the heat content of the particle in its own (resting) frame, and viewed in a frame where it has a velocity βc. The contained heat in the second frame will be

$$Q = Q_0 \sqrt{1-\beta^2} = M_0\, c^2 \sqrt{1-\beta^2} = h\,\nu_0 \sqrt{1-\beta^2}\;. \tag{1.22}$$

The particle thus appears at the same time as being a small clock of a frequency

$$\nu = \nu_0 \sqrt{1-\beta^2}$$

and a small reservoir of a heat

$$Q = Q_0 \sqrt{1-\beta^2}$$

moving with the velocity βc.

If ϕ is the phase of the wave $a \exp \frac{i\phi}{\hbar}$, where a and ϕ are real, the guidance theory states that

$$\partial_t\, \phi = \frac{M_0\, c^2}{\sqrt{1-\beta^2}}\,, \qquad -\operatorname{grad} \phi = \frac{M_0\, \boldsymbol{v}}{\sqrt{1-\beta^2}}\,, \tag{1.23}$$

so the Planck-Laue equation can be written as

$$Q = M_0\, c^2 \sqrt{1-\beta^2} = \frac{M_0\, c^2}{\sqrt{1-\beta^2}} - \boldsymbol{vp}\,. \tag{1.24}$$

Combining (1.23) and (1.24) results in

$$M_0\, c^2 \sqrt{1-\beta^2} = \partial_t\, \phi + \mathbf{v} \operatorname{grad} \phi = \frac{d\phi}{dt}\,.$$

Since the particle is regarded as a clock of a proper frequency $\frac{M_0 c^2}{h}$, the phase of its internal vibration expressed by $a_i \exp \frac{i\phi_i}{\hbar}$, with a_i and ϕ_i real values, will be

$$\phi_i = h\,\nu_0 \sqrt{1-\beta^2}\; t = M_0\, c^2 \sqrt{1-\beta^2}\; t\,,$$

thus

$$d\,(\phi_i - \phi) = 0\,. \tag{1.25}$$

The fundamental result agrees with the assumption according to which a particle, as it moves in its own wave, remains constantly in phase with it.

§1.4. The equations of continuity

The equations of continuity are (1.7) and (1.15). First we revert to equation (1.7)

$$\partial_t a^2 - \frac{1}{m}\,\mathrm{div}\big(a^2\,\mathrm{grad}\,\phi\big) = 0\,.$$

Making use of the guidance law (1.11) and setting $\rho = Ka^2$, where K is a constant, equation (1.7) becomes

$$\partial_t \rho + \mathrm{div}\,(\rho\boldsymbol{v}) = 0\,. \tag{1.26}$$

In hydrodynamics, this equation represents the continuity equation. The quantity $\rho d\tau$ is the number of the fluid's molecules in the volume element $d\tau$ moved with the velocity $\boldsymbol{v}$.

We denote by $\frac{D}{dt}$ the derivative taken along the direction of motion of the molecules. The expression

$$\frac{D\,(\rho\,d\tau)}{dt} = 0$$

then expresses the conservation of the fluid.

With a normalization factor, $\rho\,d\tau = a^2\,(x,\,y,\,z,\,t)\,d\tau$ is here assumed to be the probability of finding a single particle at time t in the volume element $d\tau$, at x, y, z.

As the statistical wave ψ solution of the linear equation is purely virtual, it can be defined as everywhere proportional to the real wave ϑ, and so we may set $\psi = C\,\vartheta$, where C is the normalization factor chosen so as to satisfy

$$\int |\psi|^2\,d\tau = 1\,.$$

Chapter 2. The Random Perturbation in the Framework of the EGR Theory

§2.1. The physical requirement

With the simple hydrodynamic picture (1.26), and with the constant K taken to be 1, it is assumed that $a^2\,(x^a) = \rho$ multiplied by $d\tau$ gives, with a normalizing factor, the probability of finding a single particle in the volume element $d\tau$, which is the absolute value of the statistical wave $\int \left|\psi^2\right| d\tau$.

This hydrodynamic model is however not adequate by itself for it contains nothing to describe the actual location of the particle: by examining a simple quantized state of a hydrogen atom, inspection shows that the guidance formula for the electron gives $\boldsymbol{v}=0$, which makes equation (1.26) irrelevant.

We may however circumvent this difficulty by considering a random perturbation of Brownian character superimposed onto the guided motion. In that case, the particle's regular motion obeying the guidance law should be subjected to a slight random influence of hidden origin, so as to switch from one guided trajectory to another.

The "main" trajectory would then appear as a "mean-valued motion".

Such a concept was brought forward by Bohm and Vigier [1] who referred to this invisible "thermostat" as the "sub-quantum medium". Referring to the same interpretation of the continuity equation (1.26), they showed that when random fluctuations would perturb the density ρ, a systematic tendency must exist for fluid elements to move to certain regions in such a way as to maintain the stability of the mean density ρ.

This tendency may find its origin in a kind of pressure which tends to correct the deviation.

A good example is a gas in a gravitational field in which the pressures automatically adjust themselves to maintain a local mean density close to

$$\rho = \rho_0 \exp\left(-\frac{mgz}{\mathfrak{K}T}\right),$$

where g is the acceleration of the force of gravity, and $\mathfrak{K}$ is Boltzmann's constant.

It should be stressed however, that the suggested medium does not serve as a universal reference system.

§2.2. The EGR picture

The velocity of light c will be taken here to be equal to 1.

When $V=0$ and $\boldsymbol{A}=0$, in a Riemannian situation, the relativistic continuity equation (1.15) may be conveniently generalized as

$$\left(g^{bc}\,\partial_b\,\phi\right)\partial_c\,a + \frac{1}{2}\,a\,\Box_{\text{Riem}}\,\phi = 0\,, \tag{2.1}$$

where $\Box_{\text{Riem}} = g^{bc}\,\nabla_b\nabla_c$.

In the framework of the Riemannian relativistic hydrodynamics, the classical continuity equation (1.26) for a neutral mass density ρ is

$$\nabla_a\left(\rho u^a\right) = 0\,, \tag{2.2}$$

where the unit velocity $u^a = \frac{dx^a}{ds}$ obeys

$$g_{ab}\, u^a u^b = g^{ab}\, u_a u_b = 1\,.$$

If we define the "guidance lines" by the differential equations (in the absence of external potentials)

$$g^{bc}\, \partial_c\, \phi = u^b,$$

where u^b generalizes the three-spatial guidance velocity $\boldsymbol{v}$ defined by the equation (1.19), which characterizes the flow lines of the fluid of proper density ρ

$$\mathbf{v} = v^\beta = \frac{u^\beta}{u^4}\,. \tag{2.3}$$

To maintain the form given by (1.26) it is easy to see that we must set

$$\rho = a^2 u^4 \tag{2.4}$$

and, taking into account (1.19), we obtain

$$\rho = a^2 \left(-\,\partial_t\, \phi\right)\,, \tag{2.5}$$

with the space-time signature $(-+++)$.

To apply the generalized equation (2.2), we must start from the tensor

$$T_{ab} = a^2\, u_a u_b \tag{2.6}$$

and the equations

$$u^a\, \nabla_a\, u_b = 0\,, \tag{2.7}$$

which are a differential systems satisfied by the flow lines, which expresses that those lines are geodesics of the Riemannian metric ds^2.

Following now the above example of a pressurized gas, we consider a neutral perfect fluid whose well-known tensor is

$$T_{ab} = \left(a^2 + P\right) u_a u_b - P\, g_{ab} \tag{2.8}$$

with a prescribed equation of state $a^2 = f(P)$.

Equation (2.7) takes the form in a holonomic frame

$$\dot{u}_b = h_{ab}\, \partial^a U\,, \tag{2.9}$$

(here $h_{ab} = g_{ab} - u_a u_b$ is the projecting tensor) with

$$U = \int_{P_1}^{P_2} \frac{1}{a^2 + P}\, dP\,.$$

The quantity $\dot{u}_b$ represent the acceleration of the flow lines satisfying the differential system (2.9). Those flow lines are everywhere tangent to the four-unit vector u_c.

The differential system (2.9) is also written as

$$u^a \nabla_a u_b - \partial_a (U h^a_b) = 0 \,. \qquad (2.10)$$

In this case, the continuity equation becomes now

$$\nabla_b (a^2 u^b) - a^2 u^b \partial_b U = 0 \,. \qquad (2.11)$$

By doing so, our final aim is to put in evidence a "perturbed" density $\tilde{a}^2$, while keeping the usual classical form

$$\nabla_b (\tilde{a}^2 u^b) = 0 \,. \qquad (2.12)$$

A rigorous demonstration of Lichnérowicz [3] states, concerning the aforementioned flow lines, that:

> "... the flow lines satisfying the differential system* $\dot{u}_b = h_{ab}\, \partial^a U$ are geodesics of the metric
>
> $$ds^{2\prime} = e^{2U} ds^2$$
>
> conformal to the Einstein metric ds^2."

In other words, the presence of an internal pressure P readily induces a conformal factor (here e^{2U}), which is referred to as the *fluid index*.

Let us now introduce ∇' as the covariant derivative operator of the conformal metric $ds^{2\prime}$. We also define the "current vector" C^a of the considered fluid, whose components are

$$C^a = e^U u^a.$$

The current vector C' of $ds^{2\prime}$ has covariant components defined by

$$C'_a = C_a \,,$$

so as to remain unitary in the new metric

$$g^{ab\prime}\, C'_a C'_b = e^{-2U} g^{ab} (e^U u_a)(e^U u_b) = 1 \,.$$

The contravariant components of the vector C' are

$$C^{a\prime} = g^{ab\prime}\, C_b = e^{-U} u^a.$$

---

*The equation $\dot{u}_b = h_{ab}\, \partial^a U$ quoted by Lichnérowicz is given by formula (2.9) of this paper. — P.M.

Inspection shows that these flow lines are geodesics of $ds^{2\prime}$, according to

$$C^{a\prime}\,\nabla'_a\,C^{b\prime}=0\,, \tag{2.13}$$

which are fully equivalent to equations (2.10).

Likewise, the continuity equation (2.11)

$$\nabla_a\left(a^2\,u^a\right)-a^2\,u^a\,\partial_a U=0$$

must coincide with the one describing the fluid density $\tilde{a}^2$

$$\nabla'_a\left(\tilde{a}^2\,C^{a\prime}\right)=0\,, \tag{2.14}$$

which amounts to the recognition that the perturbation exerted on the fluid density $\tilde{a}^2$, i.e. the pressure P, is implicitly described by the conformal derivative ∇'

$$\left\{{}^{\,d}_{ab}\right\}'=\left\{{}^{\,d}_{ab}\right\}+\left(\delta^d_b\,\partial_a U+\delta^d_a\partial_b U-g_{ab}\,\partial^d U\right).$$

However, a conformal metric is not suitable for describing the physical influence of an external medium which is defined in the initial ds^2.

This model has nevertheless an interesting virtue: following the same pattern, we will see that it is possible to build a plausible representation in the framework of the EGR theory.

§2.3. The influence of the metric fluctuations

In the extended formulation of General Relativity, the EGR theory [4], we may establish a continuity equation analogous to (2.14)

$$\mathrm{D}_a\left(\tilde{a}^2\,u^a_{\mathrm{EGR}}\right)=0\,, \tag{2.15}$$

where the four-velocity u^a_{EGR} has the form defined in the EGR theory [4].

We suggest the following interpretation. The fluctuating density $\tilde{a}^2$ is related to the general connection defined in the EGR theory

$$\Gamma^d_{ab}=\left\{{}^{\,d}_{ab}\right\}+\left(\Gamma^d_{ab}\right)_J=\left\{{}^{\,d}_{ab}\right\}+\frac{1}{6}\left(\delta^d_a\,J_b+\delta^d_b\,J_a-3\,g_{ab}\,J^d\right).$$

Unlike the conformal metric, which does not present a physical significance, the EGR theory provides a consistent scheme which enables to consider a one-to-one influence from an external medium manifestly represented by the "residual field" T^{ab}_{field}.

The "approximated" Riemannian continuity equation defined in the metric ds^2, which generalizes (2.12), should be written

$$\nabla_a\left(\langle\tilde{a}^2\rangle\,u^a\right)=0\,. \tag{2.16}$$

The density a^2 in the Riemannian continuity equation would then appear as a *mean value* of the "instant" fluctuating density $\tilde{a}^2$ of the fluid, which actually obeys the equation (2.15) on the very small scale.

According to the EGR postulate, like for elementary particles, the Double Solution Theory is always considered in the framework of the dominant Riemannian geometry.

Bearing this in mind, remember that the wave ϑ is a *physical entity*, and so is the amplitude a, therefore the relativistic hydrodynamics applied here is fully legitimate.

Within the EGR theory, the Riemannian part of the "residual field" at its lowest level (but not vanishing) supplies the energy background ("thermostat") required by de Broglie's theory, and the small random disturbances are directly related to the covariant fluctuations of the metric through the non-Riemannian part of the persistent field.

Concluding remarks

As a concluding remark, let us stress that in this study we have made use of non-linear considerations, as we should in General Relativity, in accordance with de Broglie's ideas.

Francis Fer [5] has constructed a non-linear equation corresponding to the equation (1.13), and showed that in this general case, the relativistic continuity equation (2.1) defined in a Minkowski space remains unaffected. This remarkable result lends support to the aforementioned interpretation based on the EGR theory.

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On the Speed of Rotation of the Isotropic Space: Insight into the Redshift Problem

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The presentation delivered at the Fall 2009 Meeting of the Ohio Section of the APS, held on October 9–10, 2009, in Delaware, Ohio

Abstract: This study applies the mathematical method of chronometric invariants, which are physically observable quantities in the four-dimensional space-time (Zelmanov A. L., *Soviet Physics Doklady*, 1956, vol. 1, 227–230). The isotropic region of the space-time is considered (it is known as the isotropic space). This is the home of massless light-like particles (e.g. photons). It is shown that the isotropic space rotates with a linear velocity equal to the velocity of light. The rotation slows in the presence of gravitation. Even under the simplified conditions of Special Relativity, the isotropic space still rotates with the velocity of light. A manifestation of this effect is the observed Hubble redshift explained as energy loss of photons with distance, for work against the non-holonomity (rotation) field of the isotropic space wherein they travel (Rabounski D. *The Abraham Zelmanov Journal*, 2009, vol. 2, 11–28). It is shown that the light-speed rotation of the isotropic space has a purely geometrical origin due to the space-time metric, where time is presented as the fourth coordinate, expressed through the velocity of light.

Contents:

This presentation is dedicated to Hermann Minkowski (1864–1909), on the 100th anniversary of his publication of "Raum und Zeit".

§1. Foreword. When I presented *Hubble Redshift due to the Global Non-Holonomity of Space** [1], the scientific community asked: Why do you believe that the isotropic space (the home of photons) rotates with

*The presentation was also delivered, in two parts, at Meetings of the American Physical Society, held in Spring, 2009 [2, 3]. A brief account of the study was preliminary published in [4].

the velocity of light, and what are its foundations in the basic space-time geometry?

Naturally, this question is not trivial, and cannot be answered in brief. I therefore decided to provide the answer, in detail, in this special presentation.

This problem will be considered in the framework of both General Relativity and Special Relativity. In both cases, it will be observed that the sign-alternating structure of the space-time metric, where time is presented as the fourth coordinate $x^0 = ct$, expressed through the velocity of light, is solely responsible reason for the ligh-speed rotation of the isotropic space. Now, I have to offer all the explanations to the attention of readers.

§2. A short explanation of the isotropic space. First of all we need to give a short explanation of the isotropic space and of its origin in the geometric structure of space-time.

The basic space-time of the General Theory of Relativity is a four-dimensional pseudo-Riemannian space, with the signature $(-+++)$ or $(+---)$. This is one member of the family of Riemannian spaces, the metric spaces where the square of distance between any two infinitely close points is set up by the square form $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. This form is invariant along all the space (that also is specific to Riemannian spaces). Due to invariance of the metric, the length of any n-dimensional vector Q^α, being transferred in parallel to itself in a Riemannian space of n-dimensions, remains unchanged: $Q_\alpha Q^\alpha = g_{\alpha\beta}\, Q^\alpha Q^\beta = const$. This is known as *Levi-Civita parallel transfer*, due to Tullio Levi-Civita, and is specific to Riemannian spaces. The kind metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = inv$ is referred to as a *Riemannian metric*, in memory of Bernhard Riemann who introduced it in the 1850's. The prefix "pseudo" means a class of Riemannian spaces, where the metric is sign-alternating. In this case, algebraically, the diagonal components $g_{\alpha\alpha}$ of the fundamental metric tensor $g_{\alpha\beta}$ do not bear the same sign. Geometrically, this means that two types of coordinate axes are present in the space: the axes of real coordinates (the "plus" sign in the diagonal components) and the axes where coordinates are imaginary (the "minus" sign). Pseudo-Riemannian spaces were introduced in 1908 by Hermann Minkowski, who first considered a particular case of these, having four dimensions, wherein one axis is imaginary and three other axes are real, or, alternatively, one axis is real while the other three are imaginary*. So, the

*In a general case, pseudo-Riemannian spaces can have any number of dimensions, with any combinations of positive and negative signs in the signature.

signature of the space metric is $(-+++)$ or $(+---)$, respectively. Thus, Minkowski emphasized time as a coordinate axis $x^0 = ct$, which is segregate from three axes of the spatial spread. Historically, he studied a highly simplified case, where the space metric can be reduced, by transformation of the coordinates, to a simplest diagonal form, where $g_{\alpha\alpha} = \{-1, +1, +1, +1\}$ or $\{+1, -1, -1, -1\}$, and the non-diagonal components of $g_{\alpha\beta}$ are zero. This is the basic space-time of the Special Theory of Relativity. In this case, the Riemann-Christoffel curvature tensor is zero, so the space is non-curved as can be illustrated by a pile of flat spatial sections (three-dimensional spaces) threaded up onto the time axis. This simplified case of the four-dimensional pseudo-Riemannian space is known as *Minkowski's space*. This, however, differs from a four-dimensional pseudo-Euclidean space, which also is non-curved, but all spatial coordinates are homogeneous therein (the unit coordinate marks are uniformly distributed along the coordinate axes). In contrast, the spatial coordinates can be inhomogeneous in Minkowski's space, producing some forces therein.

The four-dimensional pseudo-Riemannian space is not a "monolite" single spread as a sign-definite metric space. Due to its sign-alternating metric, it is presented with two segregate spreads:

a) The non-isotropic space (space-time), where the time interval and the spatial interval always differ from each other. As such, $ds^2 \neq 0$ and any world-vector's length is $Q_\alpha Q^\alpha = const \neq 0$ in the space. Thus, this is the home for mass-bearing particles (such a particle, being characterized with the world-vector $P^\alpha = m_0 \frac{dx^\alpha}{ds}$, has a non-zero rest mass $P_\alpha P^\alpha = m_0 = const \neq 0$).

b) The isotropic space (space-time), where the time interval and the spatial interval have the same length. As such, the space-time interval is always zero ($ds^2 = 0$). Any world-vector of the isotropic space has zero length ($Q_\alpha Q^\alpha = const = 0$). The isotropic space particles are characterized with the world-vector $P^\alpha = \frac{m}{c}\frac{dx^\alpha}{d\tau}$, expressed through the relativistic mass m and the observable time interval $d\tau$. They have zero rest mass ($P_\alpha P^\alpha = const = 0$), but non-zero relativistic mass and energy according to $E = mc^2$. All isotropic space particles move at the velocity of light. Thus, these are massless light-like particles, e.g. photons.

This terminology, "non-isotropic" and "isotropic", does not seem to be very successful when being applied to the space-time regions. This is because, for a physicist, the terms mean something different than in the geometry of pseudo-Riemannian spaces. A physicist, when hearing that something (space or medium) is non-isotropic, thinks about the presence of a preferred direction in it. Conversely, the absence of a preferred direction is regularly understood as isotropy. Relativists and mathematicians refer to a space region as isotropic if $ds^2 = 0$ therein, so the length of any world-vector is zero: the vector is equally targeting all four-dimensional directions. On the other hand, spatial vectors of the isotropic space, having one dimension lesser than the four-dimensional space itself, are not world-vectors therein. The vectors have surely non-zero lengths, and target their specific spatial direction. That is, the term "isotropic" is attributed to the four-dimensional space (space-time) of photons, but is unrelated to the three-dimensional space where photons travel (it can be isotropic or anisotropic, depending on the particular physical conditions in it).

I, and the relativists in general, adhere to this terminology, because it is well accepted in the scientific literature on the space-time geometry and the theory of relativity.

§3. The light-speed rotation. We are going to consider the isotropic space from the viewpoint of a regular observer, whose home is the non-isotropic space filled with mass-bearing particles. Thus, his reference body is a rigid physical body over which a real (deformed) coordinate net is spanned, and real clocks are located on its surface. To find physical quantities, registered by the observer, we should project the four-dimensional quantities onto the time line and coordinate net of his body of reference. This problem was resolved, in 1944, by Abraham Zelmanov. His mathematical apparatus of chronometric invariants [5–7] targets physically observable quantities for a regular observer at rest with respect to his body of reference.

In particular, the theory introduces the chronometrically invariant (physically observable) intervals of time and the spatial coordinates as the projections of the interval of the four-dimensional coordinates dx^α onto the observer's time line and the spatial section. The observable spatial coordinates meet the regular three-dimensional coordinates, while the physically observable time interval

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i = \sqrt{g_{00}}\, dt - \frac{1}{c^2}\, v_i\, dx^i \tag{3.1}$$

depends on the gravitational potential $\mathrm{w} = c^2 (1 - \sqrt{g_{00}})$ and the linear velocity of rotation of the observer's three-dimensional space

$$v_i = -\, c\, \frac{g_{0i}}{\sqrt{g_{00}}}\,. \tag{3.2}$$

The chronometrically invariant metric tensor

$$h_{ik} = -\,g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}} = -\,g_{ik} + \frac{1}{c^2}\,v_i\,v_k\,, \tag{3.3}$$

obtained as the spatial projection of the fundamental metric tensor $g_{\alpha\beta}$, gives the chronometrically invariant (observable) spatial interval

$$d\sigma^2 = h_{ik}\,dx^i dx^k. \tag{3.4}$$

Due to these formulae, the space-time interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ is expressed through the observable time interval and the observable spatial interval as

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \tag{3.5}$$

that is true in the space-time of General Relativity, because the observable quantities, $d\tau$ and $d\sigma$, take all components of the fundamental metric tensor $g_{\alpha\beta}$ into account. This is in contrast to the analogous formula of Special Relativity, $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$, which assumes that only the diagonal terms of $g_{\alpha\beta}$ are non-zero, and are units.

Now, I show how rotation of the isotropic space can be easily found with use of the mathematical method of physically observable quantities (chronometric invariants).

Two physical conditions specific to the isotropic space,

$$ds^2 = 0\,, \qquad c^2 d\tau^2 = d\sigma^2 \neq 0\,, \tag{3.6}$$

were highlighted in §2*. These conditions set that the time spread and the spatial spread meet each other everywhere in the isotropic space.

Time and regular three-dimensional space can meet each other in terms of the linear velocity of rotation of the space, according to the definition of the velocity (3.2).

This can be visualized by introducing a locally geodesic frame of reference in the point of observation (where the observer is located). The main advantage of such a reference frame is that it is the same, within infinitesimal vicinities of the point of observation, for all other regions of the space (space-time)†.

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*The second condition, $c^2 d\tau^2 = d\sigma^2 \neq 0$, is stronger. This is because the first, $ds^2 = c^2 d\tau^2 - d\sigma^2 = 0$, includes also the fully degenerate case $c^2 d\tau^2 = d\sigma^2 = 0$, which means something out of the isotropic space due to the full degeneration of the observable time durations and the observable lengths.

†Locally geodesic coordinates and reference frames rise from Riemann's pioneering studies, and are much explained in the scientific literature. For instance, see §7 of Petrov's *Einstein Spaces* [8].

Within infinitesimal vicinities of any point of such a reference frame the fundamental metric tensor is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\alpha\beta}}{\partial\tilde{x}^\mu \partial\tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \dots \,, \tag{3.7}$$

i.e. its components $\tilde{g}_{\alpha\beta}$ at a point, located in the vicinities, are different from $g_{\alpha\beta}$ at the point of observation to within only the higher order terms, which can be neglected. Therefore, at any point of a locally geodesic reference frame the fundamental metric tensor can be considered constant, so its first derivatives (the Christoffel symbols) and the second derivatives (the space curvature) are zero.

As a matter of fact, within infinitesimal vicinities of any point located in a Riemannian space, a locally geodesic reference frame can be set up. At the same time, at any point of this locally geodesic reference frame, a tangential flat Euclidean space can be set up so that this reference frame, being locally geodesic for the Riemannian space, is the global geodesic for that tangential flat space.

The fundamental metric tensor of a flat Euclidean space is constant, so the values of the tensor $\tilde{g}_{\alpha\beta}$, taken in the vicinities of a point of the Riemannian space, converge to the values of the tensor $g_{\alpha\beta}$ in the flat space tangential at this point. Therefore, we can build a system of basis vectors $\vec{e}_{(\alpha)}$, which are located along the coordinate axes in this flat space, and tangential to curved coordinate lines of the Riemannian space in the point of observation.

It should be noted that, in a general case, real coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other. So the lengths of the basis vectors may sometimes be very different from unity.

We denote a four-dimensional vector of infinitesimal displacement by $d\vec{r} = \{dx^0, dx^1, dx^2, dx^3\}$. So $d\vec{r} = \vec{e}_{(\alpha)}dx^\alpha$, where components of the basis vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e^0_{(0)}, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e^1_{(1)}, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e^2_{(2)}, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e^3_{(3)}\}$. The scalar product of the vector $d\vec{r}$ with itself is $d\vec{r}d\vec{r} = ds^2$. On the other hand, $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. As a result we arrive at the formula

$$g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)} \cos\left(x^\alpha; x^\beta\right), \tag{3.8}$$

which shows how components of the fundamental metric tensor of the observer's space depend on the lengths of the basis vectors (tangential to his real coordinate axes, inhomogeneous and curved), and on the angle between them.

In particular, formula (3.8) gives

$$g_{00} = e^2_{(0)}\,, \tag{3.9}$$

$$g_{0i} = e_{(0)} e_{(i)} \cos\left(x^0; x^i\right), \tag{3.10}$$

$$g_{ik} = e_{(i)} e_{(k)} \cos\left(x^i; x^k\right). \tag{3.11}$$

Finally, applying these to the definitions of v_i (3.2) and h_{ik} (3.3), we derive how these depend on the lengths of the basis vectors $\vec{e}_{(0)}$ and $\vec{e}_{(i)}$ (tangential to the real coordinate axes, inhomogeneous, and curved), and on the angle between them. That is

$$v_i = -\,c\,e_{(i)} \cos\left(x^0; x^i\right), \tag{3.12}$$

$$h_{ik} = e_{(i)} e_{(k)} \left[\cos\left(x^0; x^i\right) \cos\left(x^0; x^k\right) - \cos\left(x^i; x^k\right)\right]. \tag{3.13}$$

Consider these equations under the isotropic space condition, $c^2 d\tau^2 = d\sigma^2 \neq 0$. According to this condition, time and regular three-dimensional space meet each other. Geometrically, this means that the time basis vector $\vec{e}_{(0)}$ meets all three spatial basis vectors $\vec{e}_{(i)}$ (this fact does not mean, however, that the spatial basis vectors coincide, because the time basis vector is the same for all the spatial frame). In other words, $\cos\left(x^0; x^k\right) = \pm 1$ everywhere in the isotropic space. Also, in observing a photon, only its direction of motion (direction of travelling light) is counted, and $e_{(0)} = e_{(i)}$ along it (according to the isotropic space condition). This can be expressed through the gravitational potential $\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right)$, because $e_{(0)} = \sqrt{g_{00}}$ in a general case (3.9). Finally, in the isotropic space, we have

$$\cos\left(x^0; x^k\right) = \pm 1\,, \qquad e_{(i)} = e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{3.14}$$

and, hence,

$$v_i = \mp \sqrt{g_{00}}\; c_i = \mp \left(1 - \frac{\mathrm{w}}{c^2}\right) c_i\,, \tag{3.15}$$

$$h_{ik} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 \left[1 - \cos\left(x^i; x^k\right)\right], \tag{3.16}$$

where c^i is the chronometrically invariant three-dimensional vector of the physically observable velocity of light, $c_i c^i = h_{ik}\, c^i c^k = c^2$.

According to the formula derived (3.15), we immediately come to the following conclusion:

The isotropic space rotates, at each its point and in each direction where a photon travels, with a linear velocity equal to the velocity of light. This fundamental rotation can slow down relative to the light speed in the presence of gravitation.

§4. Consequences of the light-speed rotation. Now, we investigate two sequels of the light-speed rotation of the isotropic space.

First consequence. Consider physically observable time $d\tau$ (3.1), which is dependent on the linear velocity of rotation of space. This is proper time, registered by the observer. It is always positive ($d\tau > 0$) due to his recognition of the past and the future. Therefore, the coordinate time function of an object, the function $\frac{dt}{d\tau}$, manifests how this object travels along the time axis with respect to the observer.

When expressing the coordinate time function from the definition of $d\tau$ (3.1), we obtain

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{g_{00}}}\left(1 + \frac{1}{c^2}\, v_i\, \mathrm{v}^i\right), \qquad \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \tag{4.1}$$

where v^i is the chronometrically invariant (physically observable) velocity of the object we observe.

Substituting the observable velocity of photons $\mathrm{v}^i = c^i$ and the linear velocity of the light-speed rotation (3.15), specific to the isotropic space as we obtained above, we consider a case where the time basis vector is directed oppositely to the spatial basis vectors, so $\cos(x^0; x^k) = -1$ and, hence, $v_i = -c_i$. (The second case, $\cos(x^0; x^k) = +1$, leads to nonsense in the coordinate time function.) We obtain

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{g_{00}}}\left(1 - \sqrt{g_{00}}\right). \tag{4.2}$$

It is evident that the photon coordinate time stops, $\frac{dt}{d\tau} = 0$, when $\sqrt{g_{00}} = 1$ and, hence, the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ becomes $\mathrm{w} = 0$, implying the absence of gravitational fields. In the presence of gravitation we have $\sqrt{g_{00}} < 1$, so the photon coordinate time function increases with the value of the gravitational potential, and the isotropic space rotation is slowing down from the light speed.

The stopping of the photon coordinate time function reflects that they, the particles of the isotropic space, move at the velocity of light. Light signals are mediators in synchronization of clocks (Einstein's method of synchronization). In this process, a light signal transfers zero-point of the time coordinate from one clock to another. Thus, from the point of view of a regular observer, the isotropic space particles are "resting-in-time": their coordinate time is stopped with respect to his coordinate time, while physically observable time is not at rest due to their visible motion. In other words, photons rest in time, while we are moving along the time axis with respect to them. Therefore, the photon coordinate time function is always zero in the absence of gravitational fields. According to the formula (4.2), only gravitation is able to enforce the coordinate time of a photon to be flowing with respect to that of the observer.

Second consequence. It is interesting to ponder whether the light-speed rotation of the isotropic space has any influence on the space curvature. It is doubtful that this rotation can be attributed only to the curved space-time of General Relativity. To illustrate, consider the Riemann-Christoffel curvature tensor $R_{\alpha\beta\mu\nu}$. It is built on the second derivatives of the fundamental metric tensor $g_{\alpha\beta}$, and on its first derivatives, according to its definition

$$R_{\mu\nu\sigma\cdot}^{\cdots\alpha} = \frac{\partial\Gamma^{\alpha}_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\alpha}_{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\beta}_{\mu\sigma}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta}\,, \tag{4.3}$$

where $\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\sigma}\,\Gamma_{\mu\nu,\sigma} = \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)$.

In this formula, according to the definition of v_i (3.2), we should use $g_{0i} = -\frac{1}{c}\,v_i\sqrt{g_{00}}$. Hence, even if $\sqrt{g_{00}} = 1$ (no gravitational fields), v_i should have an influence on the Riemann-Christoffel tensor. But this is true, only if $g_{0i} \neq const$. In the isotropic space, in the absence of gravitation, as shown above, $v_i = -c_i$ and, hence, $g_{0i} = \frac{1}{c}\,c_i$. If rotation of the isotropic space is stationary and vorticeless, $v_i = -c_i$ is independent from the spatial coordinates and time, so its first and second derivatives are zero. In other words, there is not a goal of this rotation into the curvature tensor. Thus, with the diagonal spatial metric* where $g_{kk} = \{-1, -1, -1\}$ or $\{+1, +1, +1\}$, we arrive at the condition of Special Relativity, which is $R_{\mu\nu\sigma\cdot}^{\cdots\alpha} = 0$.

Therefore, even in the framework of the simplified conditions of Special Relativity, the isotropic space still rotates with the velocity of light.

*We know, according to the theorem introduced by Émile Cotton [9], that any three-dimensional square form can be reduced to the diagonal unit form. This means, in particular, that, if a four-dimensional space (space-time) is free of gravitation ($g_{00} = 1$) and its three-dimensional metric g_{ik} is stationary, the space-time metric is reducible to the diagonal unit form $g_{\alpha\alpha} = \{+1, -1, -1, -1\}$ or $\{-1, +1, +1, +1\}$ (see §46 of Petrov's *Einstein Spaces* [8]). This is the case considered by the Special Theory of Relativity, and is known as Minkowski's space (space-time).

§5. The origin of the fundamental rotation. Why does the isotropic space rotate at the velocity of light? In other words, wherefrom does the rotation originate? To answer this question, we should turn to the geometric structure of space-time.

The basic space-time of the General Theory of Relativity is the four-dimensional pseudo-Riemannian space, which metric is sign-alternating so that the time axis is emphasized as $x^0 = ct$. The space-time signature, $(+---)$ or $(-+++)$, was first pointed out by Hermann Minkowski in his famous *Raum und Zeit* [14] as the origin of the relativistic transformations of the spatial coordinates and time, which distinguishes relativistic physics from classical physics.

If a sign-definite signature, $(++++)$ or $(++++)$, the world would have four spatial coordinates where time is a spatial parameter as found in classical physics. In this case, no difference from the laws of classical physics would be observed, but simply four spatial coordinates instead of three ones. Accordingly, $ds^2 = 0$ that is the isotropic space condition which differs an isotropic region from a non-isotropic one, would mean that the space has been shrunk into a point. So, $ds^2 \neq 0$ is true everywhere in the space. No splitting into isotropic and non-isotropic regions is possible. All the space is a single non-isotropic spread.

In contrast, in a space of the sign-alternating signature as above, the isotropic space condition $ds^2 = 0$ is expanded as to contain non-zero time and spatial spreads, equal to each other in the length. As a result, the isotropic region $(ds^2 = 0)$ and the non-isotropic region $(ds^2 \neq 0)$ co-exist in the space.

Therefore the isotropic space (the home of photons), i.e. the region determined by the condition $ds^2 = 0$, is due only to the sign-alternating space metric which emphasizes time as a segregate axis of the space.

The next step in understanding the light-speed rotation of the isotropic space is visualized by consideration of the formula (3.12). This formula, $v_i = -c\, e_{(i)} \cos(x^0; x^i)$, shows how the linear velocity of the rotation of the observer's space depends on the lengths of the spatial basis vectors $\vec{e}_{(i)}$ (tangential to his real coordinate axes, inhomogeneous and curved), and on the angle between them and the time basis vector $\vec{e}_{(0)}$. The velocity of light appears in the formula, as well as in the other formulae of relativistic physics, due to the fact that time is presented here as the fourth coordinate axis, $x^0 = ct$, where the velocity plays a rôle of numerical coefficient.

If one assumes another numerical coefficient of the same dimension, say u cm/sec, so the time coordinate axis is presented as $x^0 = ut$, the formula has to be changed as*

$$v_i = -\,u\, e_{(i)} \cos\,(x^0;\, x^i)\,. \tag{5.1}$$

Once the isotropic space condition $ds^2 = 0$ applied to the space-time metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = e_{(\alpha)}\, e_{(\beta)} \cos\,(x^\alpha;\, x^\beta)$ as we did in §2 and §3, we obtain $\cos\,(x^0;\, x^i) = -1$ and, hence,

$$v_i = -\,u_i \tag{5.2}$$

in the space. In other word, when assuming $x^0 = ut$ instead of $x^0 = ct$, we immediately arrive at a result that the isotropic space rotates with a linear velocity equal to u.

As a result of what has been said above, we arrive at the conclusion that the isotropic space rotates with the velocity of light due to two purely geometric conditions:

a) The space-time metric is sign-alternating. The signature, $(+---)$ or $(-+++)$, emphasizes time as the fourth coordinate $x^0 = ct$ containing the velocity of light as a numerical coefficient.

b) The isotropic space condition $ds^2 = 0$. This is a sequel of the first condition. Namely, because the signature emphasizes the time axis $x^0 = ct$, there is in the space-time a region where the space-time spread is zero ($ds^2 = 0$), while the time spead and the spatial spread are non-zero, and are equal to each other.

The conditions are true in the framework of both General Relativity and Special Relativity, because the same signature condition exists, independent of the presence of the space curvature or the other factors which alter the basic geometries of the theories.

So, the light-speed rotation of the isotropic space has a purely geometrical origin due to the sign-alternate structure of the space-time metric, where time is presented as the fourth coordinate axis $x^0 = ct$.

§6. A topological interpretation of the result. How can we imagine that the isotropic space rotates with the velocity of light? In searching for a native illustration of this result, we turn our attention to the concepts of topology as the best way of understanding something in many-dimensional space geometry.

According to the concepts of topology [10], a finite symmetric system can be considered as a topological spread mapped into a spherical space. Can we apply these views to our Universe?

*In this case, the respective changes appear in Lorentz' transformations and in all other formulae of relativistic physics.

Observational astronomy manifest the presence of the event horizon in the cosmos, and also the homogeneity and symmetry of the Metagalaxy to within a first order approximation. Therefore the Universe is homogeneous and isotropic on the average.

Also, as was proved by Zelmanov in the 1950's, in the framework of the General Theory of Relativity, spatial infinitude of the homogeneous isotropic cosmological models depends on the frame of reference from which we observe the universe [11,12]. In other words, if a homogeneous isotropic universe, being observed in one reference frame, is infinite, it may be finite in another reference frame. Zelmanov enunciated this result as the *Infinite Relativity Principle*. Thus, being located in a universe of infinite spread, we can always move to a specific frame of reference wherein the universe seems finite.

So, we can consider the Universe as a finite spread, which is homogeneous and isotropic on the average. Therefore, we can apply the aforementioned topological views to the Universe as a whole.

In addition, we should take into account that only one geodesic line can be drawn through a given point in a given direction, and the unique geodesic line can be either non-isotropic or isotropic (see §6 of Petrov's *Einstein Spaces* [8] or §101 of Raschewski's *Riemannsche Geometrie und Tensoranalysis* [13] for detail). That is, the isotropic and non-isotropic regions of space-time have no common points.*

Therefore, we do consider the Universe as two segregate spreads (isotropic and non-isotropic), each mapped into a respective spherical space of the same radius of curvature. These two spherical spaces are equivalent to the surfaces of two concentric hyperspheres, which have the same radius, but are not coincident with each other. The surface of the isotropic hypersphere is the home of isotropic trajectories, while the non-isotropic hypershere's surface is the home of non-isotropic trajectories.

We are going to consider an observer who is located in the hypersphere's surface.

Any spherical formation of n dimensions (created by a spherical space of $n-1$ dimensions) is directed in its "parental" space of $n+1$ and higher dimensions. This can be easily understood, because in any

*This result can be illustrated in Minkowski's diagram, which is the plane paper (two-dimensional) representation of the four-dimensional pseudo-Riemannian space (space-time). Once a moving mass-bearing particle increases its velocity so much that it approaches the velocity of light, its non-isotropic trajectory in the diagram tries to reach the light cone (isotropic region) but never meets it as the particle never reaches the velocity of light. Even if the particle is moving infinitesimally close to the velocity of light, its trajectory is close to the light cone but never meets it. So, the isotropic and non-isotropic regions have no common points.

circle, the two-dimensional spherical formation created by a respective circumference (one-dimensional spherical space), is obviously directed in the three-dimensional space. If a hypothetical one-dimensional observer, located in a circumference, sees that every other (one-dimensional) object of the circumference moves with respect to him with a constant velocity, this is equivalent to the circumference rotating as a whole with respect to his position in it, with the same velocity along the direction in which he looks. Due to this rotation, an inertial force acts on all one-dimensional objects of the circumference, according to the angular velocity and the radius of it. This force has the same numerical value at all the objects, and is directed orthogonally to the circumference. As a result, all objects moving relative to the observer along the circumference are carried out, by the force, in the directions opposite to their motion. This is manifested as an additional acceleration braking the objects.

Analogously, a three-dimensional observer located in the isotropic hypersphere's surface (isotropic spherical space) sees that any other object of the surface moves with respect to him with the velocity of light along his direction of observation. This is equivalent to stating that the surface rotates as a whole with the velocity of light in the direction of his observation. Because the polar axis of the rotation is directed in the "parental" space of the hypersphere, the inertial force produced due to the rotation is directed orthogonally to the hypersphere's surface in each of its points, and is equally applied to all objects of the surface. Trying to carry the moving objects to the direction orthogonal to the surface, along which they travel, the force produces a braking acceleration on the objects. Because all objects of the isotropic space (massless particles, e.g. photons) move with the velocity of light, the additional braking acceleration cannot slow down their motion, but only change their energy (frequency). As was shown in my previous study [1], this "braking effect" is observed as Hubble redshift which is explained as energy loss of photons with distance, for work against the non-holonomity (rotation) field of the isotropic space wherein they travel.

§7. Conclusions. We have considered how the isotropic space (the home of photons) appears to a hypothetical "light-like" observer located in it. Such an observer cannot accompany his reference body to which he compares all his measurements (a real physical body, e.g. a cosmic rigid body, located in the non-isotropic space). Therefore the result of his observation differs from that obtained by a regular observer who always accompanies his reference body in the non-isotropic space. Meanwhile, this approach gives an advantage to see the real physical properties of the isotropic space. Here I would like to emphasize the most important of the obtained results:

1. In the four-dimensional pseudo-Riemannian space, which is the basic space-time of the General Theory of Relativity, the isotropic region (isotropic space, the home of photons) rotates, basically, with a linear velocity equal to the velocity of light. The fundamental rotation is slowed down in the presence of gravitation.
2. Even in a very simplified case of Minkowski's space — the basic space-time of the Special Theory of Relativity — the isotropic space of photons still rotates with the velocity of light. This means that a Galilean frame of reference (completely free of gravitation and rotation) is not possible in the isotropic space, since the latter always has an associated rotation.*
3. The fundamental rotation was found, in the frameworks of both General Relativity and Special Relativity, proceeding from only two obviously geometric conditions: a) the space-time metric is sign-alternating, $(+---)$ or $(-+++)$, where the time axis is emphasized as $x^0 = ct$, and b) $ds^2 = 0$ everywhere in the isotropic space. This means that the rotation has a purely geometrical origin due to the sign-alternating structure of the space-time metric.
4. In the framework of topology, the Universe can be presented as two segregate spreads (isotropic and non-isotropic) mapped onto two concentric hyperspheres, which have the same radius, but are not coinciding with each other. The fact that any object of the isotropic space moves relative to the observer with the velocity of light is eqivalent to an isotropic hypersphere which rotates with the velocity in its "parental" space of higher dimensions.

Thus, the isotropic space (the home of photons) rotates, at each of its points, with a linear velocity which is, basically, equal to the velocity of light. This fact was unfortunately overlooked during one hundred years commencing in Hermann Minkowski's 1908 famous presentation published posthumously†, in 1909, as *Raum und Zeit* [14]. Minkowski was the first person who pointed out that the Special Theory of Relativity,

*This is in contrast to the non-isotropic region (non-isotropic space) inhabited with mass-bearing particles. In Minkowski's space, as proven in the framework of the Special Theory of Relativity, we can reduce any motion to rectilinear and uniform form by transformations of the spatial coordinates and time. Therefore, a regular observer can find a Galilean frame of reference everywhere in the positions allowed for him in Minkowski's space.

†This presentation was delivered by Minkowski, a few months before the publication, at the *80th Assembly of German Natural Scientists and Physicians*, held

introduced by Albert Einstein three years before, in 1905, is explained in a four-dimensional space (space-time), where time is the fourth coordinate axis $x^0 = ct$, while the three-dimensional space of the observer moves with the velocity of light along it. Now, we clearly understand that this picture is not complete. It should be added to the light-speed rotation of the isotropic space (the home of photons). In other words, Minkowski's formula $x^0 = ct$ means not only the light-speed motion of the observer's space along the time axis, upstairs in Minkowski's diagram, but also the light-speed rotation of the surface of the isotropic cone which illustrates the isotropic space therein. It is significant that this understanding arrives on the anniversary of his *Raum und Zeit*, which was published exactly one hundred years ago. Therefore, I dedicate this paper to the memory of Hermann Minkowski (1864–1909), the pathfinder of space-time geometry.

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The Gravitational Field of a Condensed Matter Model of the Sun: The Space Breaking Meets the Asteroid Strip

Larissa Borissova

Abstract: This seminal study deals with the exact solution of Einstein's field equations for a sphere of incompressible liquid without the additional limitation initially introduced in 1916 by Karl Schwarzschild, according to which the space-time metric must have no singularities. The obtained exact solution is then applied to the Universe, the Sun, and the planets, by the assumption that these objects can be approximated as spheres of incompressible liquid. It is shown that gravitational collapse of such a sphere is permitted for an object whose characteristics (mass, density, and size) are close to the Universe. Meanwhile, there is a spatial break associated with any of the mentioned stellar objects: the break is determined as the approaching to infinity of one of the spatial components of the metric tensor. In particular, the break of the Sun's space meets the Asteroid strip, while Jupiter's space break meets the Asteroid strip from the outer side. Also, the space breaks of Mercury, Venus, Earth, and Mars are located inside the Asteroid strip (inside the Sun's space break).

Contents:

§1. Problem statement. The main task of this paper is to study the possibilities of applying condensed matter models in astrophysics and cosmology. A cosmic object consisting of condensed matter has a constant volume and a constant density. A sphere of incompressible liquid, being in the weightless state (as any cosmic object), is a kind of condensed matter. Thus, assuming that a star is a sphere of incompressible liquid, we can study the gravitational field of the star inside and outside it.

The Sun orbiting the center of the Galaxy meets the weightless condition (see Chapter 2 of [1] for detail)

$$\frac{GM}{r} = \mathrm{v}^2,$$

where $G = 6.67\times 10^{-8}$ cm^3/g × sec^2 is the Newtonian gravitational constant, M is the mass of the Galaxy, r is the distance of the Sun from the center of the Galaxy, and v is the Sun's velocity in its orbit. The planets of the Solar System also satisfy the weightless condition. Assuming that the planets have a similar internal constitution as the Sun, we can consider these objects as spheres of incompressible liquid being in a weightless state.

In addition to it, we assume that the Universe also is a sphere of incompressible liquid. Concerning the Universe, this problem is not solved in detail in this study: only several conditions specific to the liquid Universe model are considered in §4.

I will consider the problems by means of the General Theory of Relativity. First, it is necessary to obtain the exact solution of the Einstein field equations for the space-time metric induced by the gravitational field of a sphere of incompressible liquid.

The regular field equations of Einstein, with the λ-field neglected, have the form

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\,\varkappa T_{\alpha\beta}\,, \tag{1.1}$$

where $R_{\alpha\beta}$ is the Ricci tensor, R is the Riemann curvature scalar, $\varkappa = \frac{8\pi G}{c^2} = 18.6\times 10^{-28}$ cm/g is the Einstein gravitational constant, $T_{\alpha\beta}$ is the energy-momentum tensor, and $\alpha, \beta = 0, 1, 2, 3$ are the space-time indices. The gravitational field of spherical island of substance should possess spherical symmetry. Thus, it is described by the metric of spherical kind

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{1.2}$$

where e^{ν} and e^{λ} are functions of r and t.

In the case under consideration the energy-momentum tensor is that of an ideal liquid (incompressible, with zero viscosity), by the condition that its density is constant: $\rho = \rho_0 = const$. As known, the energy-momentum tensor in this case is

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p}{c^2}\right) b^{\alpha} b^{\beta} - \frac{p}{c^2}\, g^{\alpha\beta}, \tag{1.3}$$

where p is the pressure of the liquid, while

$$b^{\alpha} = \frac{dx^{\alpha}}{ds}\,, \qquad b_{\alpha}b^{\alpha} = 1 \tag{1.4}$$

is the four-dimensional velocity vector, which determines the reference frame of the given observer. The energy-momentum tensor should satisfy the conservation law

$$\nabla_{\sigma} T^{\alpha\sigma} = 0\,, \tag{1.5}$$

where ∇_{σ} is the four-dimensional symbol of covariant differentiation.

Formally, the problem we are considering is a generalization of the Schwarzschild solution produced for an analogous case (a sphere of incompressible liquid). Karl Schwarzschild [2] solved the Einstein field equations for this case, by the condition that the solution must be regular. He assumed that the components of the fundamental metric tensor $g_{\alpha\beta}$ must satisfy the signature conditions (the space-time metric must have no singularities). Thus, the Schwarzschild solution, according to his initial assumption, does not include space-time singularities.

This limitation of the space-time geometry, initially introduced in 1916 by Schwarzschild, will not be used by me in this study. Therefore, we will be able to study the singular properties of the space-time metric associated with a sphere of incompressible liquid. Then we will apply the obtained results to the cosmic objects such as the Sun, the planets, and, ultimately, the Universe as a whole.

It should be noted that the problem of space-time singularities plays a very important rôle in the General Theory of Relativity and astrophysics, because it is linked indirectly with the problem of black holes: such an object as a gravitational collapsar possesses a space-time singularity on its surface.

The term "black hole" was coined due to David Hilbert's study of 1917, which followed six months after Schwarzschild's original solution (and his tragic death in 1916). Hilbert analyzed the Schwarzschild solution for the gravitational field of a mass-point [3]. He wrote this solution in the form

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\left(d\theta^2 + \sin^2\!\theta\, d\varphi^2\right), \tag{1.6}$$

where $r_g = \frac{2GM}{c^2}$ is known as the Hilbert radius, while M is the mass of the field source (the mass-point). At $r = r_g$ the space-time region of the surface around the mass-point collapses: gravitational collapse, a state by which the component g_{00} is zero, occurs on the surface $r = r_g$.

It is easy to see that by $r = r_g$ two conditions are fulfilled

$$g_{00} = 0 \quad \text{and} \quad g_{11} \to \infty \,. \tag{1.7}$$

The first condition, $g_{00} = 0$, is known as the *collapse condition.* It is said that "time is stopped by gravitational collapse". This situation will be studied in detail in §4: it will be shown that the observed time is truly stopped, while the coordinate time continues its flow, uniformly. The second condition, $g_{11} \to \infty$, was completely ignored in the past, although it can be considered as the *condition of the breaking of the space.* From a formal viewpoint these conditions violate the requirements which determine the the space-time region of a real observer (a real observer has a real mass and can move only with a sublight velocity).

The conditions are linked with violation of the *space-time signature prescription.* This violation means that the given space-time has singularities in the regions (surfaces or volumes) wherein the aforementioned conditions are true. The signature conditions for a diagonal metric $(+---)$ have the form

$$\left.\begin{aligned} g_{00} &> 0 \\ g_{00}\, g_{11} &< 0 \\ g_{00}\, g_{11}\, g_{22} &> 0 \\ g = g_{00}\, g_{11}\, g_{22}\, g_{33} &< 0 \end{aligned}\right\} . \tag{1.8}$$

The first three are known as the *weak signature conditions.* The fourth is known as the *strong signature condition.* If one or all weak signature conditions are violated, while the strong signature condition is true, this is a *removable singularity.* If the strong signature condition is violated, the space-time has *unremovable singularity*: in this case the field solution is regularly failed from consideration, because it "has not physical meaning". Actually, someone did not see the physical meaning therein. However it is very meaningful mathematically. Therefore, I will direct my focus onto unremovable singularities in the Schwarzschild field of a sphere of incompressible liquid. The most important results obtained due to this approach will be discussed in §7.

The most known kind of space (space-time) containing a removable singularity is Schwarzschild space. It is an empty space (no continuous matter presented in the space, that means $T_{\alpha\beta} = 0$), filled with the spherically symmetric gravitational field of a mass-point (1.6). Given such a space, two weak signature conditions are violated ($g_{00} = 0$ and $g_{11} \to \infty$ by $r = r_g$) and the strong signature condition $g = -r^4 \sin^2 < 0$ is true by $r = r_g$ in it.

Because the Schwarzschild solution describes the gravitational field of a mass-point in a space free of distributed matter (empty space-time), and it includes the possibility of gravitational collapse, it is very popular amongst the theoretical physicists and astrophysicists working on the black hole problem. As a matter of fact, several cosmic objects like stars can collapse (the problem of gravitational collapse is often linked with stars at a later stage of their evolution). On the other hand, the aforementioned Schwarzschild solution means gravitational collapse of a mass-point's field, although stars are continuous bodies, not mass-points. Therefore, the problem of the space-time singularities (for instance, gravitational collapsars — black holes) should be solved by models of continuous bodies, not mass-points.

Thus, the main task of my study is to study singularities in the space filled with the gravitational field of a sphere of incompressible liquid, which can approximate the models an actual cosmic body like a star, a planet, or the Universe as a whole. In the framework of this approach, the Universe will be considered in §4, while the Sun and the planets will be considered in §7.

I will consider this problem employing the mathematical methods of physically observable quantities, known as chronometric invariants. This versatile mathematical apparatus was developed in 1944 by Abraham Zelmanov [4–6], then applied by him, very successfully, to relativistic cosmology. This mathematical technique gives the advantage that it is connected to a specific (chosen) observer and the physical standards of his laboratory, so we obtain the theoretical results expressed through the real quantities measurable in practice. All physically observable characteristics of the reference space of the space-time described by the metric (1.2) will be calculated in §2.

In §3, the exact solution of the field equations (1.1) will be obtained for the spherically symmetric metric (1.2) inside a sphere of incompressible liquid, which is described by the energy-momentum tensor (1.3). Because we do not limit the solution by that the metric must be regular, the obtained metric has two singularities: 1) collapse by $g_{00}=0$, and 2) break of the space by $g_{11}\to\infty$. It will be shown then that these singularities are unremovable, because the strong signature condition is also violated in both cases.

The singularities of the Schwarzschild field produced by a sphere of incompressible liquid, are studied in detail in §4. It will be shown that the conditions of collapse and breaking of the space depend on the density of the liquid sphere, and also on its total mass and radius. The spherical surface of the space breaking can either be inside the liquid sphere or outside it, depending on the numerical value of the mass density. Besides, the surface of the space breaking can meet the Schwarzschild sphere of collapse under particular conditions. The last situation realizes itself for the liquid model of the Universe. In the liquid model of the Sun, the surface of the space breaking is located outside the Sun itself. Also, we will arrive at the next conclusion: a liquid sphere of the Sun's radius, cannot be in the state of gravitational collapse. In contrast, the liquid Universe as a whole is represented as a collapsed cosmic object.

The properties of particles located on a collapsar's surface are the subject of study in §5. It will be shown that these particles have imaginary rest-mass and imaginary three-dimensional momentum. The term "relativistic mass" is inapplicable to the particles, because they do not move in the usual sense of this word.

Physically observable properties of the space inside a sphere of incompressible liquid will be calculated in §6. It will be shown a tricky situation therein: the three-dimensional space inside the sphere is a constant negative curvature space, while the four-dimensional Riemann-Christoffel curvature tensor does not satisfy the constant curvature condition (its three-dimensional components satisfy the constant positive curvature, while the component R_{0101} does not satisfy the constant curvature condition in general). The component R_{0101} is zero on the surface of collapse, and is positive inside the collapsar. Therefore, a collapsar's surface is a bridge connecting two spaces of the negative and the positive curvature.

In §7, these results will be applied to the Sun and the planets, which will be considered as spheres of incompressible liquid. It will be shown that the collapse condition is not satisfied for these objects. The surface of the space breaking is located outside such an object. According to the detailed calculations, the surface of the breaking of the Sun's space meets the Asteroid strip. Analogous calculations manifest the intersections of the planets' space breaking.

§2. Physically observable characteristics of the gravitational field inside a sphere of incompressible liquid. In the framework of the mathematical apparatus of physically observable quantities (chronometric invariants), two three-dimensional quantities are necessary for further derivation of the physically observable properties of a space [4–6]: a three-dimensional scalar — the gravitational potential of the field produced by the reference body of the observer

$$\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right), \tag{2.1}$$

and a three-dimensional vector — the linear velocity of the reference space's rotation in the point of observation

$$v_i = -\,c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \tag{2.2}$$

where Roman indices ($i = 1, 2, 3$ in the present case) are signed for the three-dimensional spatial coordinates.

We can easily see, therefore, that the collapse condition $g_{00} = 0$ is realized by $\mathrm{w} = c^2$. For the spherically symmetric metric (1.2), we have

$$\mathrm{w} = c^2\left(1 - e^{\frac{\nu}{2}}\right). \tag{2.3}$$

Because $g_{0i} = 0$ in the metric (1.2), i.e. the space does not rotate, we have $v_i = 0$. Hence the chr.inv.-tensor of the angular velocity of rotation of the reference space (the tensor of the space non-holonomity), determined in the theory of chronometric invariants, is zero

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right) = 0\,, \tag{2.4}$$

while the chr.inv.-vector of gravitational inertial force is

$$F_i = \frac{c^2}{c^2 - \mathrm{w}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) = -\,\frac{c^2}{2}\,\nu'\,, \tag{2.5}$$

where the prime denotes the differentiation along the r-coordinate.

With these, the chr.inv.-metric tensor [4–6]

$$h_{ik} = -\,g_{ik} + \frac{1}{c^2}\,v_i v_k \tag{2.6}$$

(it determines the physically observable metric of the observer's three-dimensional space), in the space of the spherically symmetric metric (1.2) has the following components

$$h_{11} = e^{\lambda}, \quad h_{22} = r^2, \quad h_{33} = r^2\sin^2\theta\,, \tag{2.7}$$

$$h^{11} = e^{-\lambda}, \quad h^{22} = \frac{1}{r^2}\,, \quad h_{33} = \frac{1}{r^2\sin^2\theta}\,, \tag{2.8}$$

$$h = \det\|\,h_{ik}\,\| = e^{\lambda} r^4 \sin^2\theta\,. \tag{2.9}$$

Thus, the chr.inv.-tensor of the rate of deformation of the space [4–6]

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\,\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \tag{2.10}$$

where

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t} \tag{2.11}$$

is the chr.inv.-operator of differentiation along the time coordinate, has the following non-zero components

$$D_{11} = \frac{\dot{\lambda}}{2}\, e^{\lambda - \frac{\nu}{2}}\,, \qquad D^{11} = -\,\frac{\dot{\lambda}}{2}\, e^{-\lambda - \frac{\nu}{2}}, \tag{2.12}$$

(the upper dot denotes differentiation along the time coordinate t).

We see, therefore, that a non-rotating spherical symmetric space contains gravitation, and can deform if the spatial metric h_{ik} does not depend on time. We will see later that the stationarity condition of the metric h_{ik} depends on the structure of the energy-momentum tensor of continuous matter filling the space.

Now, we calculate the characteristics of non-uniformity the space — the chr.inv.-Christoffel symbols of the first and second kinds

$$\Delta^k_{ij} = h^{km}\Delta_{ij,m} = \frac{1}{2}\, h^{km}\left(\frac{{}^*\partial h_{im}}{\partial x^j} + \frac{{}^*\partial h_{jm}}{\partial x^i} - \frac{{}^*\partial h_{ij}}{\partial x^m}\right), \tag{2.13}$$

where the chr.inv.-operator of differentiation along the spatial coordinates is

$$\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\, v_i\, \frac{{}^*\partial}{\partial t}\,. \tag{2.14}$$

We obtain, after algebra, non-zero components of $\Delta_{ij,m}$

$$\Delta_{11,1} = \frac{\lambda'}{2}\, e^{\lambda}\,, \quad \Delta_{22,1} = -\,r\,, \quad \Delta_{33,1} = -\,r\,\sin^2\theta\,, \tag{2.15}$$

$$\Delta_{12,2} = r\,, \qquad \Delta_{33,2} = -\,r^2 \sin\theta\cos\theta\,, \tag{2.16}$$

$$\Delta_{13,3} = r\,\sin^2\theta\,, \qquad \Delta_{23,3} = r^2 \sin\theta\cos\theta\,, \tag{2.17}$$

thus non-zero components of Δ^k_{ij} are

$$\Delta^1_{11} = \frac{\lambda'}{2}\,, \quad \Delta^1_{22} = -\,r\,e^{-\lambda}, \quad \Delta^1_{33} = -\,r\,\sin^2\theta\, e^{-\lambda}, \tag{2.18}$$

$$\Delta^2_{12} = \frac{1}{r}\,, \quad \Delta^2_{33} = -\,\sin\theta\cos\theta\,, \tag{2.19}$$

$$\Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^3_{23} = \cot\theta\,. \tag{2.20}$$

The three-dimensional observable curvature of the space is characterized by the chr.inv.-curvature tensor C_{lkij}, which possesses all the algebraic properties of the Riemann-Christoffel tensor [4–6]

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{kljij} - H_{iljk}\right), \tag{2.21}$$

where H_{lkij} is similar to the Schouten tensor of the theory of non-holonomic manifolds, and is derived from the non-commutativity of the second chr.inv.-derivatives of an arbitrary transferred three-dimensional vector Q_l along the spatial coordinates

$$^*\nabla_i{}^*\nabla_k Q_l - {}^*\nabla_k{}^*\nabla_i Q_l = \frac{2A_{ik}}{c^2}\frac{^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdots j}Q_j, \tag{2.22}$$

where the chr.inv.-covariant differential from the vector is

$$^*\nabla_k Q^i dx^k = dQ^i + \Delta^i_{kl}Q^k dx^l. \tag{2.23}$$

The tensor $H_{lki\cdot}^{\cdots j}$ has the form [4–6]

$$H_{lki\cdot}^{\cdots j} = \frac{^*\partial\Delta^j_{il}}{\partial x^k} - \frac{^*\partial\Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}\,, \tag{2.24}$$

it is connected with the curvature tensor C_{lkij} by

$$H_{lkij} = C_{lkij} + \frac{1}{c^2}\big(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + {} + A_{kl}D_{ij} + A_{li}D_{jk}\big), \tag{2.25}$$

while the contracted tensors $H_{lk} = H_{lki\cdot}^{\cdots i}$ and $C_{lk} = C_{lki\cdot}^{\cdots i}$ are connected, according to the theory of chronometric invariants, as

$$H_{lk} = C_{lk} + \frac{1}{c^2}\left(A_{kj}D^j_l + A_{lj}D^j_k + A_{kl}D\right). \tag{2.26}$$

We see that H_{lkij} and C_{lkij} are the same if the reference space is free of rotation and deformation. It is obvious that this condition is true for H_{lk} and C_{lk} as well. The tensor $C_{lk} = h^{ij}C_{ilkj}$ has the form

$$C_{lk} = \frac{^*\partial}{\partial x^k}\left(\frac{^*\partial\ln\sqrt{h}}{\partial x^l}\right) - \frac{^*\partial\Delta^i_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^i_{km} - \Delta^m_{kl}\frac{^*\partial\ln\sqrt{h}}{\partial x^m}\,. \tag{2.27}$$

Thus, we obtain non-zero components of C_{lk} for the spherically symmetric metric (1.2). They are

$$C_{11} = -\frac{\lambda'}{r}\,, \qquad C_{22} = \frac{C_{33}}{\sin^2\theta} = e^{-\lambda}\left(1 - \frac{r\lambda'}{2}\right) - 1. \tag{2.28}$$

So, we have calculated all the physically observable characteristics of the space, which are necessary for our further deduction of the exact solution of the Einstein field equations.

All that I have to add to these, are the physically observable components (chr.inv.-components) of the energy-momentum tensor of ideal liquid (1.3). Being calculated according to the theory of chronometric invariants, where $b^i = \frac{dx^i}{ds} = 0$ and $b^0 = \frac{1}{\sqrt{g_{00}}}$ [4–6], they are

$$\rho = \frac{T_{00}}{g_{00}} = \rho_0\,, \quad J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}} = 0\,, \quad U^{ik} = c^2 T^{ik} = p h^{ik}, \tag{2.29}$$

where ρ is the chr.inv.-density of the distributed matter, J^i is the chr.inv.-vector of the density of the momentum in the medium, U^{ik} is the chr.inv.-stress tensor. The condition $J^i = 0$ means that the observer's reference frame accompanies the mass, while $U^{ik} = ph^{ik}$ means that his reference frame accompanies the medium.

§3. The Einstein equations inside a sphere of incompressible liquid: the exact solution. In order to obtain the exact internal solution of the Einstein field equations with respect to a given distribution of matter, it is necessary to solve two systems of equations: the Einstein field equations (1.1), and the equations of the conservation law (1.5).

We will solve the equations in terms of physically observable quantities. The Einstein field equations expressed through the physically observable quantities (the chr.inv.-Einstein equations) are [4–6]

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{lj} + A_{jl}A^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}\,F_j\right)F^j = -\,\frac{\varkappa}{2}\left(\rho c^2 + U\right), \tag{3.1}$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}\,F_j A^{ij} = \varkappa J^i, \tag{3.2}$$

$$\begin{gathered} \frac{{}^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D_k^j + A_{k\cdot}^{\cdot j}\right) + DD_{ik} - D_{ij}D_k^j + \\ + 3A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}\,F_i F_k - c^2 C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right), \end{gathered} \tag{3.3}$$

where $U = h^{ik}U_{ik}$ is the trace of the stress-tensor U_{ik}. The chr.inv.-form of the conservation law is [4–6]

$$\frac{{}^*\partial\rho}{\partial t} + D\rho + \frac{1}{c^2}\,D_{ij}U^{ij} + {}^*\widetilde{\nabla}_i J^i - \frac{1}{c^2}\,F_i\,J^i = 0\,, \tag{3.4}$$

$$\frac{{}^*\partial J^k}{\partial t} + DJ^k + 2\left(D_i^k + A_{i\cdot}^{\cdot k}\right)J^i + {}^*\widetilde{\nabla}_i\,U^{ik} - \rho F^k = 0\,, \tag{3.5}$$

where the chr.inv.-differential operator ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}F_i$ is constructed on the basis of chr.inv.-divergence ${}^*\nabla_i$ which is according to (2.23).

To solve the system of the Einstein field equations we substitute, into (3.1–3.3), the chr.inv.-characteristics of the space obtained in §2. We substitute also the chr.inv.-components of the energy-momentum tensor (2.29), from which we conclude, additionally, that

$$U = 3p \tag{3.6}$$

in the spherically symmetric liquid model.

Then, after algebra, we obtain the chr.inv.-Einstein field equations in the spherically symmetric space (1.2) inside a sphere of incompressible liquid. The obtained equations, in component notation, are

$$e^{-\nu}\left(\ddot{\lambda} - \frac{\dot{\lambda}\dot{\nu}}{2} + \frac{\dot{\lambda}^2}{2}\right) - c^2 e^{-\lambda}\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{r} + \frac{(\nu')^2}{2}\right] = \\ = -\varkappa\left(\rho_0 c^2 + 3p\right), \tag{3.7}$$

$$\frac{\dot{\lambda}}{r}\, e^{-\lambda-\frac{\nu}{2}} = \varkappa J^1 = 0\,, \tag{3.8}$$

$$e^{\lambda-\nu}\left(\ddot{\lambda} - \frac{\dot{\lambda}\dot{\nu}}{2} + \frac{\dot{\lambda}^2}{2}\right) - c^2\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{(\nu')^2}{2}\right] + \frac{2c^2\lambda'}{r} = \\ = \varkappa\left(\rho_0 c^2 - p\right)e^{\lambda}, \tag{3.9}$$

$$\frac{c^2\left(\lambda' - \nu'\right)}{r}\, e^{-\lambda} + \frac{2c^2}{r^2}\left(1 - e^{-\lambda}\right) = \varkappa\left(\rho_0 c^2 - p\right). \tag{3.10}$$

The second equation manifests that $\dot{\lambda} = 0$ in this case. Hence, the space inside the sphere of incompressible liquid does not deform. Taking this circumstance into account, and also that the stationarity of λ, we reduce the field equations (3.7–3.10) to the final form

$$c^2 e^{-\lambda}\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{r} + \frac{(\nu')^2}{2}\right] = \varkappa\left(\rho_0 c^2 + 3p\right)e^{\lambda}, \tag{3.11}$$

$$-c^2\left[\nu'' - \frac{\lambda'\nu'}{2} + \frac{(\nu')^2}{2}\right] + \frac{2c^2\lambda'}{r} = \varkappa\left(\rho_0 c^2 - p\right)e^{\lambda}, \tag{3.12}$$

$$\frac{c^2\left(\lambda' - \nu'\right)}{r}\, e^{-\lambda} + \frac{2c^2}{r^2}\left(1 - e^{-\lambda}\right) = \varkappa\left(\rho_0 c^2 - p\right). \tag{3.13}$$

To solve the equations (3.11–3.13), a formula for the pressure p is necessary. To find the formula, we now deal with the conservation equations (3.4–3.5). Because, as was found, $J^i = 0$ and $D_{ik} = 0$ in the case under consideration, the chr.inv.-scalar conservation equation (3.4) leads to the trivial result $\frac{^*\partial\rho}{\partial t} = 0$. Thus $\rho = \rho_0 = const$ inside the sphere, and the chr.inv.-vectorial conservation equations (3.5) take the form

$$^*\nabla_i \left(ph^{ik}\right) - \left(\rho_0 + \frac{p}{c^2}\right) F^k = 0\,, \tag{3.14}$$

which, since $^*\nabla_i h^{ik} = 0$ in the case, reads

$$h^{ik}\frac{^*\partial p}{\partial x^i} - \left(\rho_0 + \frac{p}{c^2}\right) F^k = 0\,. \tag{3.15}$$

Taking into account that $\frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i}$ in the case, we obtain, this formula reduces to only a single nontrivial equation

$$p' e^{-\lambda} + \left(\rho_0 c^2 + p\right) \frac{\nu'}{2}\, e^{-\lambda} = 0\,, \tag{3.16}$$

where $p' = \frac{dp}{dr}$, $\nu' = \frac{d\nu}{dr}$, $e^{\lambda} \neq 0$. Dividing both parts of (3.16) by $e^{-\lambda}$, we arrive at

$$\frac{dp}{\rho_0 c^2 + p} = -\frac{d\nu}{2}\,, \tag{3.17}$$

which is a plain differential equation with separable variables. It can be easily integrated as

$$\rho_0 c^2 + p = Be^{-\frac{\nu}{2}}\,, \qquad B = const. \tag{3.18}$$

Thus we have to express the pressure p as the function of ν,

$$p = Be^{-\frac{\nu}{2}} - \rho_0 c^2. \tag{3.19}$$

In look for an r-dependent function $p(r)$, we integrate the field equations (3.11–3.13). Summarizing (3.11) and (3.12), we find

$$\frac{c^2\left(\lambda' + \nu'\right)}{r} = \varkappa Be^{\lambda - \frac{\nu}{2}}. \tag{3.20}$$

Then, expressing ν' from this equation, and substituting the result into (3.13), we obtain

$$\frac{2c^2}{r}\,\lambda' + \frac{2c^2}{r^2}\left(e^{\lambda} - 1\right) - \varkappa Be^{-\lambda - \frac{\nu}{2}} = \varkappa\left(\rho_0 c^2 - p\right) e^{\lambda}. \tag{3.21}$$

Substituting p from (3.19) into (3.21), we obtain the following differential equation with respect to λ

$$\lambda' + \frac{e^{\lambda} - 1}{r} - \varkappa\rho_0 r e^{\lambda} = 0\,. \tag{3.22}$$

We introduce a new variable $y = e^{\lambda}$. Thus $\lambda' = \frac{y'}{y}$. Substituting into this equation y and y', we obtain the Bernoulli equation (see Kamke [7], Part III, Chapter I, §1.34)

$$y' + f(r)\,y^2 + g(r)\,y = 0\,, \tag{3.23}$$

where

$$f(r) = \frac{1}{r} - \varkappa\rho_0\,r\,, \qquad g(r) = -\frac{1}{r}\,. \tag{3.24}$$

It has the following solution

$$\frac{1}{y} = E(r)\int\frac{f(r)\,dr}{E(r)}\,, \tag{3.25}$$

$$E(r) = e^{\int g(r)dr}. \tag{3.26}$$

Integrating (3.26), we obtain $E(r)$ which is

$$E(r) = e^{-\int\frac{dr}{r}} = e^{\ln\frac{L}{r}} = \frac{L}{r}\,, \qquad L = const > 0\,, \tag{3.27}$$

thus we obtain $\frac{1}{y} = e^{-\lambda}$ which is

$$e^{-\lambda} = \frac{L}{r}\int\frac{r}{L}\left(\frac{1}{r} - \varkappa\rho_0 r\right)dr = 1 - \frac{\varkappa\rho_0 r^2}{3} + \frac{Q}{r}\,, \quad Q = const. \tag{3.28}$$

To find Q, we rewrite equation (3.21) as

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \varkappa\rho_0\,. \tag{3.29}$$

This equation has a singularity at the point $r = 0$, therefore the numerical value of the right side term of the equation (the density of the liquid) grows to infinity by $r \to 0$, i.e. in the center of the sphere. This is in contradiction to the initially assumed condition $\rho_0 = const$, which is specific to incompressible liquids. As a matter of fact, this contradiction should not be in the theory. We remove this contradiction by re-writting (3.29) in the form

$$e^{-\lambda}\left(1 - r\lambda'\right) = \frac{d}{dr}\left(re^{-\lambda}\right) = 1 - \varkappa\rho_0 r^2. \tag{3.30}$$

After integration, we obtain

$$re^{-\lambda} = r - \frac{\varkappa\rho_0 r^3}{3} + A\,, \qquad A = const. \tag{3.31}$$

Because $A=0$ at the central point $r=0$, it should be zero at any other point as well. Dividing this equation by $r \neq 0$, we obtain

$$e^{-\lambda} = 1 - \frac{\varkappa \rho_0 r^2}{3} \,. \tag{3.32}$$

Comparing this solution with the value $e^{-\lambda}$ obtained earlier (3.28), we see that they meet each other if $Q=0$. Besides, we should suggest that $e^{\lambda_0}=1$ at the central point $r=0$, consequently $\lambda_0=0$.

Thus we have obtained the components $h^{11}=e^{-\lambda}$ and $h_{11}=e^{\lambda}$ of the chr.inv.-metric tensor h_{ik}, expressed through the coordinate r, i.e.

$$h^{11} = e^{-\lambda} = 1 - \frac{\varkappa \rho_0 r^2}{3}, \qquad h_{11} = e^{\lambda} = \frac{1}{1 - \frac{\varkappa \rho_0 r^2}{3}} \,. \tag{3.33}$$

So forth, we should introduce a boundary condition on the surface of the sphere. We have on the surface: $r=a$, where a is the radius of the sphere. Thus

$$e^{-\lambda_a} = 1 - \frac{\varkappa \rho_0 a^2}{3} \,. \tag{3.34}$$

On the other hand, the solution of this function is also the Schwarzschild solution in emptiness. Hence,

$$e^{-\lambda_a} = 1 - \frac{2GM}{c^2 a} \,, \tag{3.35}$$

where M is the mass of the sphere. Comparing both expressions, and taking into account that the Einstein gravitational constant is $\varkappa = \frac{8\pi G}{c^2}$, we find

$$M = \frac{4\pi a^3 \rho_0}{3} = \rho_0 V, \tag{3.36}$$

where $V = \frac{4\pi a^3}{3}$ is the volume of the sphere. Thus, we have obtained the regular relation between the mass and the volume of a homogeneous sphere.

Our next step is the looking for the solution $e^{-\lambda}$ outside the sphere, i.e. for $r > a$. Since outside the sphere the density of the substance (liquid) is $\rho_0 = 0$, we obtain, after integration of (3.30),

$$r\, e^{-\lambda} = \int_0^r dr - \int_0^a \varkappa \rho_0 r^2 \, dr = r - \frac{\varkappa \rho_0 a^3}{3} \,. \tag{3.37}$$

We obtain, from this formula, that

$$e^{-\lambda} = 1 - \frac{\varkappa \rho_0 a^3}{3r} \,. \tag{3.38}$$

Taking (3.38) into account, we obtain the Schwarzschild solution in emptiness

$$e^{-\lambda} = 1 - \frac{2GM}{c^2 r}\,. \tag{3.39}$$

To obtain ν we again use equation (3.20). Substituting, into this equation,

$$\lambda' = \frac{\frac{2\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} \tag{3.40}$$

and e^{λ}, we obtain, after transformations,

$$\nu' + \frac{\frac{2\varkappa\rho_0 r^2}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} - \frac{\varkappa B}{c^2}\frac{r e^{-\frac{\nu}{2}}}{1-\frac{\varkappa\rho_0 r^2}{3}} = 0\,. \tag{3.41}$$

We introduce a new variable $e^{-\frac{\nu}{2}} = y$. Thus, $\nu' = -\frac{2y'}{y}$. Substituting these into (3.41), we obtain the Bernoulli equation

$$y' + \frac{\varkappa B}{2c^2}\frac{r y^2}{1-\frac{\varkappa\rho_0 r^2}{3}} - \frac{\frac{\varkappa\rho_0 r}{3}\,y}{1-\frac{\varkappa\rho_0 r^2}{3}} = 0\,, \tag{3.42}$$

where

$$f(r) = \frac{\varkappa B}{2c^2}\frac{r}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \qquad g(r) = -\frac{\frac{\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}}\,. \tag{3.43}$$

Thus, we have the integral

$$\int g(r)\,dr = -\int \frac{\frac{\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} = \ln N\sqrt{\left|1-\frac{\varkappa\rho_0 r^2}{3}\right|}\,, \quad N = const, \tag{3.44}$$

then

$$E(r) = N\sqrt{\left|1-\frac{\varkappa\rho_0 r^2}{3}\right|}\,. \tag{3.45}$$

In the region where the signature condition $h_{11} = e^{\lambda} > 0$ is satisfied, we have

$$1 - \frac{\varkappa\rho_0 r^2}{3} > 0\,, \tag{3.46}$$

therefore we use the modulus of the function here.

Next, we look for $\frac{1}{y} = e^{\frac{\nu}{2}}$, which is

$$e^{\frac{\nu}{2}} = \frac{\varkappa B}{2c^2}\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\int\frac{r\,dr}{\sqrt{\left(1-\frac{\varkappa\rho_0 r^2}{3}\right)^3}}\,. \tag{3.47}$$

We obtain, after integration,

$$e^{\frac{\nu}{2}} = \frac{\varkappa B}{2c^2}\left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right), \qquad K = const. \tag{3.48}$$

Now, we look for the constants B and K. To find B, we rewrite the formula of p by the condition that $p=0$ on the surface of the sphere $(r=a)$. Thus, we obtain

$$B = \rho_0 c^2 e^{\frac{\nu_a}{2}}, \tag{3.49}$$

where $e^{\frac{\nu_a}{2}}$ is the value of the function $e^{\frac{\nu}{2}}$ on the surface. As a result, we have

$$e^{\frac{\nu}{2}} = \frac{\varkappa\rho_0}{2}\, e^{\frac{\nu_a}{2}}\left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right). \tag{3.50}$$

To find K, we take the value of $e^{\frac{\nu}{2}}$ on the surface $(r=a)$

$$e^{\frac{\nu_a}{2}} = \frac{\varkappa\rho_0\, e^{\frac{\nu_a}{2}}}{2}\left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}\right). \tag{3.51}$$

We obtain, from this formula, that

$$K = -\frac{1}{\varkappa\rho_0}\frac{1}{\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}\,. \tag{3.52}$$

The quantity $e^{\frac{\nu_a}{2}}$ means the numerical value of $e^{\frac{\nu}{2}}$ by $r=a$, therefore we can apply it to the Schwarzschild solution (a mass-point's field) in emptiness at $r=a$, i.e.

$$e^{\frac{\nu_a}{2}} = \sqrt{1-\frac{2GM}{c^2 a}}\,. \tag{3.53}$$

Taking the expressions for $e^{\frac{\nu_a}{2}}$, (3.34) and (3.35), into account, we obtain

$$e^{\frac{\nu}{2}} = \frac{1}{2}\, e^{\frac{\nu_a}{2}}\left(3 - \sqrt{\frac{1-\frac{\varkappa\rho_0 r^2}{3}}{1-\frac{\varkappa\rho_0 a^2}{3}}}\right) =$$

$$= \frac{1}{2}\left(3\sqrt{1-\frac{2GM}{c^2 a}} - \sqrt{1-\frac{2GMr^2}{c^2 a^3}}\right). \tag{3.54}$$

This formula on the surface $(r = a)$ meets the Schwarzschild solution in emptiness: $e^{\frac{\nu_a}{2}} = \sqrt{1-\frac{2GM}{c^2 a}} = \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}$.

Thus the space-time metric of the gravitational field inside a sphere of incompressible liquid is, since the formulae of ν and λ have already been obtained, as follows

$$ds^2 = \frac{1}{4}\left(3\,e^{\frac{\nu_a}{2}} - \sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1 - \frac{\varkappa\rho_0 r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \quad (3.55)$$

Taking (3.34) and (3.35) into account, we rewrite (3.55) as

$$ds^2 = \frac{1}{4}\left(3\,e^{\frac{\nu_a}{2}} - \sqrt{1 - \frac{2GMr^2}{c^2a^3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1 - \frac{2GMr^2}{c^2a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \quad (3.56)$$

Since $\frac{2GM}{c^2} = r_g$ is the Hilbert gravitational radius, we rewrite (3.56) in the form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r^2 r_g}{a^3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1 - \frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \quad (3.57)$$

It is therefore obvious that this "internal" metric completely coincides with the Schwarzschild metric in emptiness on the surface of the sphere of incompressible liquid ($r = a$).

Our next step is to obtain the space-time metric outside the sphere ($r > a$). We already obtained the "external" solution for $e^{-\lambda}$, which completely coincides with the "external" Schwarzschild solution for this function (3.39). Outside the sphere, (3.20) takes the form

$$\lambda' + \nu' = 0\,, \quad (3.58)$$

consequently where according to (3.39)

$$\lambda' = \frac{2GM}{c^2r^2}\frac{1}{1 - \frac{2GM}{c^2r}}\,. \quad (3.59)$$

Substituting (3.59) into (3.58) and integrating, we find

$$\nu = \ln\left(1 - \frac{2GM}{c^2r}\right) + P, \qquad P = const, \quad (3.60)$$

thus

$$e^{\nu} = P\left(1 - \frac{2GM}{c^2 r}\right). \tag{3.61}$$

Since this function is

$$e^{\nu} = 1 - \frac{2GM}{c^2 a}\,,$$

on the surface ($r = a$), we obtain $P = 1$. Thus we have established that the space-time outside a sphere of incompressible liquid is described by the Schwarzschild metric in emptiness, which is (1.6).

§4. Singular properties of the external and internal Schwarzschild solutions. The space-time of a sphere of incompressible liquid is described by the metric (3.55) or, in the equivalent form, by (3.57). The singular properties of the space-time will be studied here. This study is a generalization of the originally Schwarzschild solution for such a sphere [2], and means that Schwarzschild's requirement to the metric to be free of singularities will not be used here. Naturally, the metric (3.57) allows singularities; they will be studied here in detail. This problem will be solved by analogy with the singular properties of the Schwarzschild solution in emptiness (a mass-point's field), which already gave black holes. As will be shown, there is a big difference between the Schwarzschild solutions. The mass-point solution in emptiness [3] will be considered at first, because it plays a key rôle in physics of black holes (gravitational collapsars).

As is known, the Schwarzschild metric of a mass-point's field (1.6) has singularities by the condition that the radial coordinate r equals the Hilbert radius

$$r = \frac{2GM}{c^2} = r_g\,. \tag{4.1}$$

One considers (4.1) as the condition of collapse of real cosmic objects like stars. It is supposed that stars can collapse in the last stage of their evolution. Of course, no doubts that stars can collapse in this way. However they are not mass-points; they are continuous objects consisting of substance. Therefore, the Schwarzschild metric of a mass-point's field (1.6) does not characterize a collapsing continuous object, but states that the field of a continuous object collapses at the distance $r = r_g$ from its centre of gravity. This distance is known as the *radius of the Schwarzschild sphere*. It is easy to see that r_g depends on the object's mass M only, and not on its characteristics such as the density of substance or the radius of the object itself.

Two singular conditions are realized in the metric (1.6) by the condition (4.1)

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 = 1 - \frac{r_g}{r} = 0\,, \tag{4.2}$$

$$g_{11} = -\,h_{11} = \frac{1}{1 - \frac{r_g}{r}} \to \infty\,. \tag{4.3}$$

The first singularity (4.2) is known as *gravitational collapse*. In this case, the gravitational potential is $\mathrm{w} = c^2$. The state of collapse is connected indirectly with the physically observable time τ, which is determined by the theory of chronometric invariants [4–6] as

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i, \tag{4.4}$$

where t is the coordinate (ideal) time, which is according to $x^0 = ct$, and flows uniformly. As seen, τ depends on the gravitational potential and the rotation of the space. Since a space with the Schwarzschild metric does not rotate ($g_{0i} = 0$), we have

$$d\tau = \sqrt{g_{00}}\, dt = \sqrt{1 - \frac{r_g}{r}}\; dt\,, \tag{4.5}$$

consequently

$$\tau = 0 \tag{4.6}$$

on the surface of collapse in the field, which is located at the distance $r = r_g$ from the centre of gravity of the body. In other words, the observable time stops on the surface of collapse, being registered by a regular observer. (However the coordinate time t still be flowing uniformly.)

Consider the spatial part of the metric (1.6) by the condition (4.1). At first, we note that any four-dimensional metric ds^2 can be expressed through the interval of the physically observable time $d\tau$ and the physically observable space interval $d\sigma$ as [4–6]

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \qquad d\sigma^2 = h_{ik}\, dx^i dx^k. \tag{4.7}$$

Since the three-dimensional observed space (the observer's spatial section) is curved, only distances σ are observable, while r are coordinate (photometric) distances. A physically observable distance between two points with radial coordinates r_1 and r_2 along the radial direction, for the metric (1.6) has the form

$$\sigma_r = \sqrt{h_{11}}\, dr = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{r_g}{r}}}\,. \tag{4.8}$$

In the integration of this equation we should keep in mind that $r > r_g$ always for any regular observer, because the metric (1.6) does not describe the region inside the Schwarzschild sphere. The space-time inside the collapsar, created by a mass-point in emptiness, is described by another, non-stationary metric which is [8]

$$ds^2 = \frac{c^2 d\tilde{t}^2}{\frac{r_g}{c\tilde{t}} - 1} - \left(\frac{r_g}{c\tilde{t}} - 1\right) d\tilde{r}^2 - c^2 \tilde{t}^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right). \tag{4.9}$$

This metric is obtained from (1.6) by means of substitution among $r = c\tilde{t}$ and $ct = \tilde{r}$. We realize that, during a finite interval of time, $\tilde{t} = \frac{r_g}{c}$.

Let us find the observable distance between a point with the radial coordinate r_1 and a point on the Schwarzschild sphere $r = r_g$. Integrating (4.8) from r_g to r_1, we obtain

$$\sigma = \sqrt{r_1}\,\sqrt{r_1 - r_g} + r_g \ln \left| \frac{\sqrt{r_1} + \sqrt{r_1 - r_g}}{\sqrt{r_1}} \right|. \tag{4.10}$$

We have just integrated $d\sigma$ from $r_g = r_{sp}$ (a radial distance, where a breaking of the space takes place) to another radial coordinate which is $r = r_1 > r_g$, since we are presently considering only the space-time outside the collapsar. What is the collapsar according to the metric (1.6)? This is a region of the empty space-time inside the sphere of the radius $r = r_g$. We see, therefore, that the observed distance between the points with radial coordinates r_g and r_1 has a finite value, which becomes zero if $r_1 = r_g$. If $r_g \ll r_1$, we expand $\sqrt{1 - \frac{r_g}{r}}$ and $\ln \left|1 + \sqrt{1 - \frac{r_g}{r}}\right|$ into the series, then save only the terms of the first order with respect to $\frac{r_g}{r}$. We obtain, after algebra, for the metric (4.9), the approximate formula

$$\sigma_r = r_1 - \frac{r_g}{2} + r_g \ln \left| 2 - \frac{r_g}{2 r_1} \right| \simeq r_1 + 0.19\, r_g \,. \tag{4.11}$$

This formula is true for small r_g. Thus the observable distance σ_r between the points with the radial coordinates r_g and r_1 is larger than the coordinate (photometric) distance $r_1 - r_g$ between the points. It is important to remark that the elementary interval $d\sigma_r$ has a singularity by $r = r_g$, while the integral of it is continuous and has a finite value. The spatial interval of the metric (1.6) has the form

$$d\sigma^2 = \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right). \tag{4.12}$$

We see, therefore, that the three-dimensional metric form has a singularity at $r = r_g$. In this case, $h_{11} \to \infty$ (i.e. $d\sigma \to \infty$).

Thus the Schwarzschild metric of a mass-point's field (1.6) has two singularities: collapse and breaking of the space. Both singularities take place by the same condition $r = r_g$. We will refer to the state of the space-time by which the elementary observable spatial interval $d\sigma \to \infty$ as the *space breaking*, and denote the corresponding value of the radial coordinate as r_{br}. We see that $r_{br} = r_g$ in the space-time filled with the gravitational field of a mass-point. In other words, the space-time described by the Schwarzschild metric (1.6) has a singular surface, spherically covering the gravitating body at the distance $r = r_g$ from its centre of gravity (mass-point).

Now, we are going to study singularities in the space-time filled with the gravitational field of a sphere of incompressible (ideal) liquid. Inside such a sphere (its radius is $r = a$), space-time is described by the metric (3.55) or its equivalent form (3.57). As is seen, the metric has a spatial singularity (space breaking) by the condition

$$r_{br} = \sqrt{\frac{3}{\varkappa\rho_0}} = a\sqrt{\frac{a}{r_g}}\,, \tag{4.13}$$

thus we conclude something about the surface of the space breaking:

1) It meets the surface of the liquid sphere, if $a = r_g$;
2) It is located outside the liquid sphere, if $r_g < a$;
3) It is located inside the liquid sphere, if $r_g > a$.

Calculating the physically observable distance between the center of the liquid sphere and the spherical surface of the space breaking, in the r-direction, we obtain

$$\sigma_r = \int_0^{r_{br}} \sqrt{h_{11}}\, dr = \int_0^{r_{br}} \frac{dr}{\sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}} = \sqrt{\frac{3}{\varkappa\rho_0}} \arcsin\left(\sqrt{\frac{\varkappa\rho_0}{3}}\, r_{br}\right) = \frac{\pi}{2}\, r_{br}\,. \tag{4.14}$$

Thus, σ_r takes finite numerical values in the field of a liquid sphere. It is obvious that the physically observable distance $\frac{\pi}{2} r_{br}$ is larger than the coordinate (photometric) distance r_{br}.

Since r_g is determined only by the mass of the liquid sphere, r_{br} depends on ρ_0 as it does on the sphere's radius a. For example, considering the Sun as a sphere of incompressible liquid, whose density is $\rho_0 = 1.4$ g/cm^3, we obtain

$$r_{br} = 3.4\times 10^{13} \text{ cm}, \tag{4.15}$$

while the radius of the Sun is $a = 7 \times 10^{10}$ cm and its Hilbert radius is $r_g = 3 \times 10^5$ cm. Therefore, the surface of the Sun's space breaking is located outside the surface of the Sun, far distant from it in the near cosmos.

Consider another example. Assume our Universe to be a sphere of incompressible liquid, whose density is $\rho_0 = 10^{-31}$ g/cm^3. The radius of its space breaking, according to (4.13), is

$$r_{br} = 1.3 \times 10^{29} \text{ cm}. \tag{4.16}$$

Observational astronomy provides the following numerical value of the Hubble constant

$$H = \frac{c}{a} = (2.3 \pm 0.3) \times 10^{-18} \text{ sec}^{-1}, \tag{4.17}$$

where a is the observed radius of the Universe. It is easy obtain from here that

$$a = 1.3 \times 10^{28} \text{ cm}. \tag{4.18}$$

This value is comparable with (4.16), so the Universe's radius may meet the surface of its space breaking by some conditions. We calculate the mass of the Universe by (3.36) and (4.18). We have $M = 5 \times 10^{54}$ g. Thus, for the liquid model of the Universe, we obtain $r_g = 7.4 \times 10^{26}$ cm: the Hilbert radius (the radius of the surface of gravitational collapse) is located inside the liquid spherical body of the Universe.

Now, we are going to study the collapse condition of a sphere of incompressible liquid. On the first view, this problem statement makes nonsense, because the body of incompressible liquid cannot be compressed. Yes, it is true, if one would consider collapse as the process of compression of a liquid cosmic body. We do not do it. In contrast, we will consider a collapsar as a singular region of the space-time. In a particular case, a cosmic body consisting of incompressible liquid can be a collapsar, if the parameters of its field on its surface will correspond to the collapse condition $g_{00} = 0$ or the equivalent condition $\mathrm{w} = c^2$. But this rises to the occurrence of the physical conditions, not the evolutionary compression of a liquid cosmic body.

As is known, the collapse condition of a common case has the form

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 = 0\,, \tag{4.19}$$

thus a cosmic object is a collapsar, if the three-dimensional gravitational potential on its surface is

$$\mathrm{w} = c^2\,. \tag{4.20}$$

Consider the collapse condition for the space-time metric of the gravitational field inside a sphere of incompressible liquid (3.56). As is seen from the metric (3.56), the collapse condition (4.19) in this case is

$$3\,e^{\frac{\nu_a}{2}} = \sqrt{1-\frac{\varkappa\rho_0\,r^2}{3}}\,, \tag{4.21}$$

or, in terms of the Hilbert radius, when the metric takes the form (3.57), the collapse condition is

$$3\sqrt{1-\frac{r_g}{a}} = \sqrt{1-\frac{r_g\,r^2}{a^3}}\;. \tag{4.22}$$

We obtain that the numerical value of the radial coordinate r_c, by which the sphere's surface meets the surface of collapse, is

$$r_c = \sqrt{9a^2-8r_{br}^2} = a\sqrt{9-\frac{8a}{r_g}}\;. \tag{4.23}$$

Because we keep in mind really cosmic objects, the numerical value of r_c should be real. This requirement is obviously satisfied by

$$a < 1.125\,r_g\,. \tag{4.24}$$

If this condition holds not ($a \geqslant r_g$), the sphere, which is a spherical liquid body, has not the state of collapse.

It is obvious that the condition $a = r_g$ satisfies (4.24). Consider this interesting particular case in detail. We have, in this case, that

$$r_c = r_{br} = r_g = a\,. \tag{4.25}$$

This means that, in this case, given a sphere of incompressible liquid in the state of collapse, it has the radius of its surface a, the Hilbert radius r_g, and the radius of the space breaking r_{br} coinciding with the radius r_c characterizing its surface in the state of collapse.

Comparing (4.25) with (4.2–4.3), which characterize the Schwarzschild solution for a mass-point in emptiness, we see that a mass-point's field in emptiness satisfies the condition

$$r_g = r_{br}\,, \tag{4.26}$$

which is a particular case of (4.25): despite such characteristics as the proper radius a and the collapsed sirface's radius r_c are inapplicable to a mass-point, the common condition (4.25) still be working in the case, being represented in its particular form (4.26).

I repeat that the condition $a = r_g$ is only a partial case of (4.24). The common condition (4.24) includes three particular cases, concerning the location of the surface of a collapsed liquid sphere:

1) The radius of a collapsed liquid sphere is larger than the Schwarzschild sphere's radius ($a > r_g$);
2) The radius of a collapsed liquid sphere is lesser than the Schwarzschild sphere's radius ($a < r_g$);
3) The surface of a collapsed liquid sphere meets the Schwarzschild sphere ($a = r_g$).

It is obvious that r_c is imaginary for $r_g \ll a$, so collapse of such a sphere of incompressible liquid is impossible. For example, considering the Sun ($a = 7\times10^7$ cm, $M = 2\times10^{33}$ g, $r_g = 3\times10^5$ cm), we obtain from (4.24) that r_c has an imaginary value. This means that:

> A homogeneous sphere of incompressible liquid, whose parameters are the same as those of the Sun, cannot collapse.

One may ask: what does the condition $r_g \neq 0$ imply for the Sun? This means that the term r_g comes from another model of the Sun where it is approximated by a mass-point: the gravitational field of a mass-point includes a collapsed region inside the spherical surface of the radius r_g around the mass-point.

Another example. Consider the Universe as a sphere of incompressible liquid (the liquid model of the Universe). Assuming, according to the numerical value of the Hubble constant (4.17), that the Universe's radius is $a = 1.3\times10^{28}$ cm, we obtain the collapse condition, from (4.24),

$$r_g > 1.2\times10^{28} \text{ cm}, \tag{4.27}$$

and immediately arrive at the following conclusion:

> The observable Universe as a whole, being represented in the framework of the liquid model, is completely located inside its gravitational radius. In other words, the observable Universe is a collapsar — a huge black hole.

In another representation, this result means that a sphere of incompressible liquid can be in the state of collapse only if its radius approaches the radius of the observable Universe.

Compare the singularities of the liquid sphere's internal metric (3.57) and the mass-point's field metric (1.6). The weak signature conditions $g_{00} > 0$ and $g_{11} < 0$ are violated by $r = r_g$. The determinant of the fundamental metric tensor of the mass-point's field metric (1.6) equals

$$g = -\,r^4 \sin^2\theta < 0\,, \tag{4.28}$$

so the strong signature condition $g<0$ is fulfilled, hence the singularity of the mass-point's field metric is removable. This means that the space-time collapses and has the same breaking the condition $r_{br}=r_g$. Thus, the collapse surface coincides with the space breaking surface in a mass-point's field in emptiness: in this case, both collapse and the space breaking are realized by the same condition (4.26).

A few words more on the singularities of the liquid sphere's internal metric (3.57). In this case, the determinant of the fundamental metric tensor equals

$$g = -\frac{1}{4}\left(3\,e^{\frac{\nu_a}{2}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2 \frac{r^4\sin^2\theta}{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{4.29}$$

so the strong signature condition $g<0$ is always true for a sphere of incompressible liquid, except in two following cases: 1) in the state of collapse ($g_{00}=0$), and 2) by the breaking of space ($g_{11}\rightarrow\infty$). These particular cases violate the weak signature conditions $g_{00}>0$ and $g_{11}<0$ correspondingly. If both weak signature conditions are violated, g has a singularity of the kind $\frac{0}{0}$. If collapse occurs in the absence of the space breaking, we have $g=0$. If no collapse, while the space breaking is present, we have $g\rightarrow\infty$. In all the cases, the singularity is non-removable, because the strong singular condition $g<0$ is violated.

So, as was shown above, a spherical object consisting of incompressible liquid can be in the state of gravitational collapse only if it is as large and massive as the Universe. Meanwhile, the space breaking realizes itself in the fields of all cosmic objects, which can be approximated by spheres of incompressible liquid. Besides, since $r_{br}\sim\frac{1}{\sqrt{\rho_0}}$, the r_{br} is then greater while smaller is the ρ_0. Assuming all these, we arrive at the following conclusion:

> A regular sphere of incompressible liquid, which can be observed in the cosmos or an Earth-bound laboratory, cannot collapse but has the space breaking — a singular surface, distantly located around the liquid sphere.

This problem will be considered in detail in §7.

§5. Collapsar as a special state of substance. Let us now consider the properties of substance inside a collapsar and on its surface. As was already shown above in the study, this consideration is applicable, on the one hand, to the internal gravitational field of a homogeneous liquid sphere (3.57), and, on the other hand, to the Schwarzschild gravitational field of a mass-point in emptiness (1.6).

So, we need to understand what sorts of particles inhabit these singular regions (the regions inside a collapsar and on its surface). This problem will be solved here by analogy with the study in [1].

First, we study substance on a collapsar's surface. This spherical surface is characterized by the condition $r = r_g$ for the metric of a mass-point's field (1.6), and by the condition $r = r_c$ for the internal metric of a liquid sphere (3.57). Because the space-times metrics are free of rotation, and taking the collapse condition $g_{00} = 0$ into account, they can be written in the common chr.inv.-form

$$ds^2 = -\,d\sigma^2 = -\,h_{ik}\,dx^i dx^k, \tag{5.1}$$

based on the general formula $ds^2 = c^2 d\tau^2 - d\sigma^2$ (4.7), which comes from the theory of chronometric invariants [4–6] and can be applied to any space-time metric.

Space-time trajectories are characterized by the four-dimensional velocity vector, which on the surface of a collapsar takes the form

$$b^\alpha = \frac{dx^\alpha}{|ds|} = \frac{dx^\alpha}{d\sigma}\,, \qquad b_\alpha b^\alpha = -1, \tag{5.2}$$

so it is a space-like vector on a collapsar's surface. Multiplying it by rest-mass m_0, we obtain a momentum world-vector

$$P^\alpha = m_0\,\frac{dx^\alpha}{d\sigma}\,, \qquad P_\alpha P^\alpha = -\,m_0^2\,, \tag{5.3}$$

which is a space-like vector therein as well, while the rest-masses take imaginary numerical values in the case (on a collapsar's surface).

According to the theory of chronometric invariants, the physically observable components of the momentum vector P^α (5.3) should be

$$\frac{P_0}{\sqrt{g_{00}}} \longrightarrow \frac{0}{0}\,, \qquad P^i = i\,|m_0|\,\frac{dx^i}{d\sigma}\,, \tag{5.4}$$

which have analogous physical meaning as the respective components

$$\frac{P_0}{\sqrt{g_{00}}} = \pm m\,, \qquad P^i = \frac{m_0}{\sqrt{1-\mathrm{v}^2/c^2}}\,\frac{dx^i}{c\,d\tau} = \frac{1}{c}\,m\mathrm{v}^i \tag{5.5}$$

of the momentum world-vector of a regular, real rest-mass particle

$$P^\alpha = m_0\,\frac{dx^\alpha}{ds}\,, \qquad P_\alpha P^\alpha = m_0^2\,. \tag{5.6}$$

We conclude therefore that the observable quantity analogous to relativistic mass is not determined for the particles which inhabit the surface of a collapsar, while the observable quantity analogous to three-dimensional momentum is imaginary for them. Thus, this sort of particles has special physical properties: such a particle has imaginary rest-masses and three-dimensional momentum, while the characteristic known as relativistic mass is not applicable to them.

Thus the surface of a collapsar cannot be considered as that of a regular physical body: this is a space-time region where the signature conditions are violated, and is inhabited with a singular sort of substance. Therefore, a regular observer whose rest-mass takes real numerical values cannot be there.

Another question: what sort of substance exists inside a collapsar, under its collapsed surface?

A space-time filled with the Schwarzschild field of a mass-point in emptiness has the metric (1.6), which is stationary. This metric is written for a regular observer, who is located outside the surface of collapse, and is watching the collapsar from outside. It is known [8] that the same Schwarzschild metric written for an internal observer, located inside the collapsar, is obtained from (1.6) by means of substitution among $r = c\tilde{t}$ and $ct = \tilde{r}$. The resulting metric (4.9) is non-stationary [8]. Thus, despite the invariance of the space-time metric as a whole, its stationarity under the collapsed surface (inside the collapsar) depends on the observer's reference frame: from views of an external observer the space inside the collapsar is stationary, while it is non-stationary being observed by an internal observer inside the collapsar.

A sphere of incompressible liquid cannot expand or compress. Meanwhile, if its characteristics satisfy the collapse conditions, it can be in the state of collapse, i.e. be a gravitational collapsar (black hole). As was shown above in §4, this is possible if the liquid sphere is as large and massive as the Universe. In this case, the internal metric of a liquid sphere (3.57) will be the metric inside the collapsar. This metric is stationary. It is written for a regular observer who is watching the collapsar from outside. It is easy to re-write the metric for an internal observer by the same substitution of the coordinates as for the Schwarzschild metric of a mass-point field. The resulting metric will be non-stationary. Thus, the space inside a collapsar consisting of incompressible liquid can expand or compress, being observed from inside it.

This tricky situation is due to the fact that we consider a very specific case: two different space-time regions, which are located outside and inside a collapser respectively, and are separated by a singular surface. If considering a regular sphere of incompressible liquid, whose surface's radius differs from the radius of collapse for this mass, this situation would be impossible.

Since $g_{00} < 0$ under the surface of collapse, relativistic masses $\frac{P_0}{\sqrt{g_{00}}}$ take imaginary numerical values. This result can be easily obtained with the deduction analogous to as formula (5.4), but with replacement of $g_{00} = 0$ by $g_{00} < 0$. This is from the viewpoint of an external observer, located outside the collapsar. In the observer is located inside a collapsar, under the surface of collapse, in his reference frame all objects inside the collapsar will be observed as bearing real relativistic masses, while all objects outside the surface of collapse will be imaginary.

§6. Physical and geometric factors acting inside a sphere of incompressible liquid. Let us find the physical and geometric factors acting inside a sphere of incompressible liquid. The metric inside the sphere, expressed through the density of substance, is given by formula (3.55). According to the metric, the chr.inv.-vector of the gravitational inertial force (2.5) has only a single component which is non-zero

$$F^1 = -\frac{\varkappa\rho_0 c^2 r}{3}\,\frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{3e^{\frac{\nu_a}{2}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}} = -\frac{2GMr}{a^3}\,\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3e^{\frac{\nu_a}{2}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,. \qquad (6.1)$$

We see here that this is a force of attraction, which is proportional to distance r. Its numerical value is zero in the center of the sphere. In the state of collapse, $F^1 \to \infty$. Since the numerical value of the Einstein constant $\varkappa$ is very small ($\varkappa = 18.6\times10^{-28}$ cm/g), it is obvious that this force is significant only by large distances r, for instance, in the case of "cosmological" objects such as the Universe.

Now, we are going to consider the singularities of pressure p concerning a sphere of incompressible liquid. Substituting B (3.49) and $e^{\frac{\nu}{2}}$ (3.54) into p (3.19), we obtain the formula

$$p = \rho_0 c^2\,\frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}} - e^{\frac{\nu_a}{2}}}{3e^{\frac{\nu_a}{2}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,. \qquad (6.2)$$

From here we see that, for the liquid model, $p \to \infty$ for the state of collapse. Also, we see that the space breaking occurs by the pressure $p = -\frac{\rho_0 c^2}{3}$. This is a *negative pressure of radiation*, because $p = \frac{\rho_0 c^2}{3}$ is the equation of state of radiation. It is obvious that this situation is possible only if the spherical surface case of the space breaking is located inside the liquid sphere (by $r < a$). Therefore, this particular case is important for our further understanding of the internal constitution of the cosmic objects which could be approximated by spheres of incompressible liquid.

Consider the space-time regions outside the singularities. Because $r \leqslant a$ means the space inside the liquid sphere, the numerator is positive outside the region of collapse always, except on the surface of the liquid sphere ($r = a$) where it equals zero.

So forth we consider the sign of this function in the region outside the collapse. The denominator is always positive in this region. Since

$$e^{\frac{\nu_a}{2}} = \sqrt{1 - \frac{\varkappa\rho_0 a^2}{3}} = \sqrt{1 - \frac{2GM}{c^2 a}}\,,$$

the numerator is positive by $r \geqslant 0$, that is inside the sphere except the region inside the sphere of breaking ($r = r_{br}$, the numerator is strongly negative in this case). It follows from (4.13), if the sphere of incompressible liquid is not a collapsar, the sphere of the space breaking is located outside it ($r_{br} > a$). Consequently, $\rho = p = 0$ in the layer.

Consider the pressure near the surface of the liquid sphere. The constant $\varkappa = 18.6\times 10^{-28}$ cm/g is a very small value. Therefore, if ρ_0 is not very large, $\varkappa\rho_0$ is also very small. Supposing that

$$\sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}} \approx 1 - \frac{\varkappa\rho_0 r^2}{6}\,,$$

we obtain, after algebra, the approximate formula for p, which is

$$p \approx \frac{\varkappa\rho_0^2 c^2\left(a^2 - r^2\right)}{12} = \frac{\rho_0 GM}{2a^2}\left(\frac{a^2 - r^2}{a}\right), \tag{6.3}$$

where $\frac{GM}{a^2} = g$ is the free-fall acceleration.

Now we calculate the pressure of the liquid, with taking into account that the liquid has the density ρ_0, while the parameter $h = a - r$ is the distance from the surface of the sphere to the point of the measurement. Assuming that $h \ll r$, i.e. the measurement is processed in the upper layer of the sphere, near its surface, we obtain

$$a^2 - r^2 = (a - r)\,(a + r) = h\,(2a + h) \approx 2ah\,.$$

Thus, we arrive at the regular formula for the pressure

$$p = \rho_0 g h\,. \tag{6.4}$$

Let us study the geometric properties of the three-dimensional space of a sphere of incompressible liquid. Calculating the components of the tensor H_{lkij} by the formula (2.25) for the metric (3.55), we obtain its non-zero components

$$H_{1212} = C_{1212} = -\frac{\varkappa\rho_0}{3}\,\frac{r^2}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{6.5}$$

$$H_{1313} = C_{1313} = -\frac{\varkappa\rho_0}{3}\,\frac{r^2\sin^2\theta}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{6.6}$$

$$H_{2323} = C_{2323} = -\frac{\varkappa\rho_0}{3}\,r^4\sin^2\theta\,. \tag{6.7}$$

We see, therefore, that the non-zero components of the observable space curvature tensor C_{iklj} satisfy the condition

$$C_{iklj} = -\frac{\varkappa\rho_0}{3}\left(h_{kl}h_{ij} - h_{il}h_{kj}\right), \tag{6.8}$$

where the constant $-\frac{\varkappa\rho_0}{3}$ is the observable three-dimensional curvature in the two-dimensional direction. This means that this is a constant negative curvature three-dimensional space. Calculating the observable scalar curvature $C = h^{ik}C_{ik}$, where non-zero components of C_{ik} are

$$C_{11} = -\frac{2\varkappa\rho_0}{3}\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{6.9}$$

$$C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2\varkappa\rho_0 r^2}{3}\,, \tag{6.10}$$

we obtain

$$C = -\,2\varkappa\rho_0 = const < 0\,. \tag{6.11}$$

Consequently, the components of the three-dimensional observable tensor of curvature C_{iklj} have the form

$$C_{iklj} = \frac{C}{6}\left(h_{kl}h_{ij} - h_{il}h_{kj}\right). \tag{6.12}$$

Thus the physically observable three-dimensional space has a constant negative curvature. The radius of the curvature $\Re$ in this case is imaginary. It is linked with C by the relation

$$C = -\,2\varkappa\rho_0 = \frac{1}{\Re^2}\,, \tag{6.13}$$

thus

$$\Re = \frac{i}{2\varkappa\rho_0}\,. \tag{6.14}$$

Let us estimate the numerical value of $|\Re|$ for this liquid model of the Universe. Assuming the density of the Universe to be $\rho_0 = 10^{-31}$ g/cm^3, we obtain $|\Re| = 2\times 10^{27}$ cm. Thus, the numerical value of $\Re$ is comparable with the Hubble radius of the Universe $a = 1.3\times 10^{28}$ cm (4.18).

We see from (6.5) and (6.6) that the three-dimensional space has breakings in the direction of two surfaces (x^1, x^2) and (x^1, x^3). Both breakings are realized by the condition $r = r_{br} = \sqrt{\frac{3}{\varkappa\rho_0}} = a\sqrt{\frac{a}{r_g}}$. It was shown above that $g_{11} = -h_{11} \to \infty$ by this condition.

Now let us study the geometrical properties of the space-time described by the metric (3.55). First, we calculate the components of the Riemann-Christoffel tensor

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_{\beta\gamma}\, g_{\alpha\delta} + \partial_{\alpha\delta}\, g_{\beta\gamma} - \partial_{\alpha\gamma}\, g_{\beta\delta} - \partial_{\beta\delta}\, g_{\alpha\gamma}\right) + \\ + g^{\sigma\tau}\left(\Gamma_{\alpha\delta,\sigma}\Gamma_{\beta\gamma,\tau} - \Gamma_{\beta\delta,\sigma}\Gamma_{\alpha\gamma,\tau}\right), \tag{6.15}$$

where $\Gamma_{\alpha\beta,\delta}$ are the four-dimensional Christoffel symbols of the 1st kind. We have, for the metric (3.55), $g_{ik} = -h_{ik}$ and $\Gamma_{ik,j} = -\Delta_{ik,j}$. Thus, calculating the other components of $\Gamma_{\alpha\beta,\delta}$, which are non-zero,

$$\Gamma_{01,0} = -\Gamma_{00,1} = \frac{\varkappa\rho_0 r}{12}\,\frac{3e^{\frac{\nu_a}{2}} - \sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}}{\sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{6.16}$$

$$\Gamma_{11,1} = -\frac{\varkappa\rho_0 r}{3}\,\frac{1}{\left(1 - \frac{\varkappa\rho_0 r^2}{3}\right)^2}\,, \tag{6.17}$$

and substituting these into (6.5), we obtain

$$R_{0101} = -\frac{\varkappa\rho_0}{12}\,\frac{3e^{\frac{\nu_a}{2}} - \sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}}{\sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{6.18}$$

$$R_{1212} = -C_{1212}\,, \quad R_{1313} = -C_{1313}\,, \quad R_{2323} = -C_{2323}\,. \tag{6.19}$$

We see from here that the four-dimensional space inside a sphere of incompressible liquid, described by the metric (3.55), is not a constant curvature space. This is because the component R_{0101}, determining the four-curvature in the radial-time (x^0, x^1)-direction, does not satisfy the condition

$$R_{\alpha\beta\gamma\delta} = Q\left(g_{\beta\gamma}g_{\alpha\delta} - g_{\beta\delta}g_{\alpha\gamma}\right), \qquad Q = const, \tag{6.20}$$

which determines a four-dimensional constant curvature space.

Also, we see that $R_{0101} \to \infty$ by the space breaking, while $R_{0101} = 0$ in the state of collapse. It is seen from (6.19) that all the spatial (three-dimensional) components of the Riemann-Christoffel tensor are positive. The mixed (space-time) component R_{0101} (6.18) is negative, except in the case of collapse where it equals zero. Because the numerator of (6.18) is proportional to $\sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$, the component R_{0101} will be positive inside the collapsar (since $\sqrt{g_{00}} < 0$ therein). Thus the four-curvature in the space-time direction (x^0, x^1) changes its sign by the state of collapse. Therefore we arrive at the conclusion that the surface of collapse is a bridge connecting two spaces of the negative and the positive curvature.

§7. The internal constitution of the Solar System: the Sun and the planets as spheres of incompressible liquid. First, we are going to consider the Sun as a sphere of incompressible liquid. Schwarzschild [2] was the first person who considered the gravitational field of a sphere of incompressible liquid. He however limited this consideration by an additional condition that the space-time metric should not have singularities. In this study the metric (3.55) will be used. It allows singularities, in contrast to the limited case of Schwarzschild: 1) collapse of the space, and 2) the space breaking.

We calculate the radius of the space breaking by formula (4.13), where we substitute the Sun's density $\rho_0 = 1.41$ g/cm^3. We obtain

$$r_{br} = 3.4 \times 10^{13} \text{ cm} = 2.3 \text{ AU}, \tag{7.1}$$

where 1 AU $= 1.49 \times 10^{13}$ cm (Astronomical Unit) is the average distance between the Sun and the Earth. So, we have obtained that the spherical surface of the Sun's space breaking is located inside the Asteroid strip, very close to the orbit of the maximal concentration of substance in it (as is known, the Asteroid strip is hold from 2.1 to 4.3 AU from the Sun). Thus we conclude that:

> The space of the Sun (actually — its gravitational field), as that of a sphere of incompressible liquid, has a breaking. The space breaking is distantly located from the Sun's body, in the space of the Solar System, and meets the Asteroid strip near the maximal concentration of the asteroids.

In addition to it, we conclude:

> The Sun, approximated by a mass-point according to the Schwarzschild solution for a mass-point's field in emptiness, has a space breaking located inside the Sun's body. This space breaking coincides with the Schwarzschild sphere — the sphere of collapse.

Object	Mass, gram	Proper radius, cm	Hilbert radius, cm
Sun	1.98×10^{33}	6.95×10^{10}	2.9×10^{5}
Mercury	2.21×10^{26}	2.36×10^{8}	0.03
Venus	4.93×10^{27}	6.19×10^{8}	0.73
Earth	5.97×10^{27}	6.38×10^{8}	0.88
Mars	6.45×10^{26}	3.44×10^{8}	0.10
Jupiter	1.90×10^{30}	7.11×10^{9}	2.8×10^{2}
Saturn	5.68×10^{29}	6.00×10^{9}	84
Uranus	8.72×10^{28}	2.55×10^{9}	13
Neptune	1.03×10^{29}	2.74×10^{9}	15
Pluto	1.31×10^{25}	1.20×10^{8}	0.002

Table 1: The proper radius and the Hilbert radius of the Sun and the planets, calculated in the framework of the model where they are approximated by spheres of incompressible liquid.

What is the Schwarzschild sphere? It is an imaginary spherical surface of the Hilbert radius $r_g = \frac{2GM}{c^2}$, which is not a radius of a physical body in a general case (despite it can be such one in the case of a black hole — a physical body whose radius meets the Hilbert radius calculated for its mass). The numerical value of r_g is determined only by the mass of the body, and does not depend on its other properties. The physical meaning of the Hilbert radius in a general case is as follows: this is the boundary of the region in the gravitational field of a mass-point M, where real particles exist; particles in the boundary (the Hilbert radius) bear the singular properties as shown in §5. In the region wherein $r \leqslant r_g$, real particles cannot exist. The Hilbert radius r_g calculated for the Sun and the planets is given in Table 1.

Let us turn back to the Sun approximated by a sphere of incompressible liquid. The space-time metric is (3.55) in this case. Substituting into (4.23) the Sun's mass $M = 2\times10^{33}$ g, radius $a = 7\times10^{7}$ cm, and the Hilbert radius $r_g = 3\times10^{5}$ cm calculated for its mass, we obtain that the numerical value of the radial coordinate r_c by which the Sun's surface meets the surface of collapse of its mass is imaginary. Thus, we arrive at the conclusion that a sphere of incompressible liquid, whose parameters are the same as those of the Sun, cannot collapse. This conclusion is as that before, see Page 247.

One can ask: then what does the Hilbert radius r_g mean for the Sun, in this context? Here is the answer: r_g is the photometric distance in the radial direction, separating the "external" region inhabited with real particles and the "internal" region under the radius wherein all particles bear imaginary masses. Particles which inhabit the boundary surface (its radius is r_g) bear singular physical properties. Note that no one real (external) observer can register events inside the singularity.

Now we apply this research method to the planets of the Solar System. Thus, we approximate the planets by spheres of incompressible liquid. All results of the calculation are given in Table 2.

The numerical values of r_c, calculated for the planets according to the same formula (4.23) as that for the liquid model of the Sun, are imaginary. Therefore, the planets being approximated by spheres of incompressible liquid cannot collapse as well as the Sun.

According to Table 1, the Hilbert radius r_g calculated for the planets is much smaller than the sizes of their physical bodies, and is in the order of 1 cm. This means that, given any of the planets of the Solar System, the singulary surface separating our world and the imaginary mass particles world in its gravitational field draws the sphere of the radius about one centimetre around its centre of gravity.

Table 2 gives the numerical values of the radius of the space breaking, calculated for each of the planets through the average density of substance inside the planet according to the formula (4.13).

The results of the summarizing and substraction associated with the planets, according to Table 2, lead to the next conclusions:

1. The spheres of the singularity breaking of the spaces of Mercury, Venus, and the Earth are completely located inside the sphere of the singularity breaking of the Sun's space;
2. The spheres of the singularity breaking of the internal spaces of all planets intersect among themselves, when being in the state of a "parade of planets";
3. The spheres of the singularity breaking of the Earth's space and Mars' space reach the Asteroid strip;
4. The sphere of the singularity breaking of Mars' space intersects with the Asteroid strip near the orbit of Phaeton (the hypothetical planet which was orbiting the Sun, according to the Titius–Bode law, at $r = 2.8$ AU, and whose distraction in the ancient time gave birth to the Asteroid strip).
5. Jupiter's singularity breaking surface intersects the Asteroid strip near Phaeton's orbit, $r = 2.8$ AU, and meets Saturn's singularity breaking surface from the outer side;
6. The singularity breaking surface of Saturn's space is located between those of Jupiter and Uranus;

Object	Density, gram/cm^3	Orbit, AU	Radius of the space breaking*, AU	Location of the space breaking sphere
Sun	1.41	—	2.3	Asteroid strip
Mercury	4.10	0.39	1.3	Completely inside the Sun's space breaking
Venus	5.10	0.72	1.2	Completely inside the Sun's space breaking
Earth	5.52	1.00	1.1	Completely inside the Sun's space breaking
Mars	3.80	1.52	1.4	Meets the Sun's space breaking at the outer side
Asteroid strip	—	2.5†	—	—
Jupiter	1.38	5.20	2.3	Meets the Sun's space breaking at one side and Saturn's space breaking at the opposite side
Saturn	0.720	9.54	3.2	Between Jupiter's space breaking and Uranus' space breaking
Uranus	1.30	19.2	2.4	Between Saturn's space breaking and Neptune's space breaking
Neptune	1.20	30.1	2.4	On the lower boundary of the Kuiper belt
Pluto	2.0	39.5	1.9	Completely on the lower strip of the Kuiper belt
Kuiper belt	—	30–100	—	—

*The distance (radius) of the singulary breaking of the respective cosmic body's space, measured from the body (in Astronomical Units).
†The density of the Asteroid strip's substance has a maximum registered at 2.5 AU, while the strip itself continues from 2.1 to 4.3 AU.

Table 2: Singularity breakings of the local spaces of the Sun and the planets.

7. The singularity breaking surface of Uranus's space is located between those of Saturn and Neptune;
8. The singularity breaking surface of Neptune's space meets, from the outer side, the lower boundary of the Kuiper belt (the strip of the aphelia of the Solar System's comets);
9. The singularity breaking surface of Pluto is completely located inside the lower strip of the Kuiper belt.

Just two small notes in addition to these. The intersections of the space breakings of the planets, discussed here, take place for only that case where the planets thenselves are in the state of a "parade of planets". However the conclusions concerning the location of the space breaking spheres, for instance — that the space breaking spheres of the internal planets are located inside the sphere of the Sun's space breaking, while the space breaking spheres of the external planets are located outside it, — are true for any position of the planets.

What does the "space breaking" mean from the physical viewpoint? Has this breaking a real action on a physical body appeared in it, or is it only a mathematical fiction? As was obtained in §6, the space (space-time) of a sphere of incompressible liquid has a breaking of its four-curvature $R_{\alpha\beta\gamma\delta}$ by the condition $r = r_{br}$: the quantity R_{0101} (6.18), which is the four-curvature of the space in the radial-time direction 0101, has a breaking $R_{0101} \to \infty$ (the curvature becomes infinite) at the distance $r = r_{br}$ from the centre of gravity of the liquid sphere. (See top of Page 255.) Because the curvature determines the gravitational field filling the space, the aforementioned breaking means the breaking in the gravitational field of the liquid sphere at $r = r_{br}$. This is the physical meaning of the space breaking we studied here.

The fact that the space breaking of the Sun meets the Asteroid strip, near Phaeton's orbit, allows us to say: yes, the space breaking considered in this study has a really physical meaning. As probable the Sun's space breaking did not permit the Asteroids to be joined into a common physical body, Phaeton. Alternatively, if Phaeton was an already existing planet of the Solar System, the common action of the space breaking of the Sun and that of another massive cosmic body, appeared near the Solar System in the ancient ages (for example, another star passing near it), has led to the distraction of Phaeton's body.

Thus the internal constitution of the Solar System was formed by the structure of the Sun's space (space-time) filled with its gravitational field, and according to the laws of the General Theory of Relativity.

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The Generalized Warp Drive Concept in the EGR Theory

Patrick Marquet*

Abstract: In this paper, we briefly review the basic theory of the Alcubierre drive, known as the Warp Drive Concept, and its subsequent improvements. By using the Arnowitt-Deser-Misner formalism we then re-formulate an extended extrinsic curvature which corresponds to the extra curvature of the Extended General Relativity (EGR). With this preparation, we are able to generalize the Alcubierre metric wherein the space-like hypersurfaces are Riemannian, and the characteristic Alcubierre function is associated with the EGR geometry. This results in a reduced energy density tensor, whose form displays a potential ability to avoid the weak energy condition.

Contents:

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Notations:

To completely appreciate this article, it is imperative to define some notations employed.

INDICES. Throughout this paper, we adopt the Einstein summation convention whereby a repeated index implies summation over all values of this index:

4-tensor or 4-vector: small Latin indices $a, b, \ldots = 1, 2, 3, 4;$

3-tensor or 3-vector: small Greek indices $\alpha, \beta, \ldots = 1, 2, 3;$

4-volume element: d^4x;

3-volume element: d^3x.

SIGNATURE OF SPACE-TIME METRIC:

$(-+++)$ unless otherwise specified.

OPERATIONS:

Scalar function: $U(x^a)$;

Ordinary derivative: $\partial_a U$;

Covariant derivative in GR: ∇_a;

Covariant derivative in EGR: D_a or $'$, (alternatively).

NEWTON'S CONSTANT:

$\mathfrak{G} = c = 1$.

Introduction

The physical restriction related to the finite nature of the light velocity has so far been a stumbling block to exploring the superluminal speed possibility of long-term space journeys.

However, recent theoretical works have lent support to plausible interstellar hyperfast travels, without physiological human constraints.

How is this possible? The principle of space travel while locally "at rest", is analogous to galaxies receding away from each other at extreme velocities due to the expansion (and contraction) of the Universe.

Instead of moving a spaceship from a planet A to a planet B, we modify the space between them. The spaceship can be carried along by a local spacetime "singular region" and is thus "surfing" through space with a given velocity with respect to the rest of the Universe.

In 1994, a Mexican physicist Miguel Alcubierre [1], working at the Physics and Astronomy Department of Cardiff University in Wales, Great Britain, published a short paper describing such a propulsion mode, known today under the name *Warp Drive*.

Based on this theory, a faster than light travel could be for the first time considered without violating the laws of relativity.

Many problems (open questions) remain to be investigated, among which two major problems are reflected in the following statements:

a) Produce a sufficiently large negative energy to create a local space distortion without violating the energy conditions resulting from the laws of General Relativity [2];

b) Maintain contact (control) between the spaceship and the outside of the distorsion (causality connection).

The problem a) can be avoided if one considers a non-Riemannian geometry that governs the laws of our Universe [3] which could eliminate the negative energy density required by the Alcubierre metric to sustain a realistic Warp Drive.

The difficulty b) may be theoretically circumvented by introducing certain types of transformations which may allow us to use the warped regions for the removal of the singularities or "event horizons". Some of these transformations are briefly reviewed in the course of this study.

Pre-requisite: time-like unit four-vector

As is well known [4], the covariant derivative of a time-like vector field u^a (whose square is $u^a u_a = -1$), may be expressed in an invariant manner in terms of tensor fields which describe the kinematics of the congruence of curves generated by the vector field u^a.

One may write

$$u_{a;b} = \varsigma_{ab} + \omega_{ab} + \frac{1}{3}\left(\theta h_{ab}\right) + \dot{u}_a u_b\,,$$

where $\dot{u}_a = u_{a;b}u^b$ is the acceleration of the flow lines, τ is the proper time, $\omega_{ab} = h^c_a h^d_b u_{[c;d]}$ is the vorticity tensor, $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor, $\theta_{ab} = h^c_a h^d_b u_{(c;d)}$ is the expansion tensor, $\theta = h^{ab}\theta_{ab} = u^a_{;a}$ is the expansion scalar, $\varsigma_{ab} = \theta_{ab} - \frac{1}{3}\left(h_{ab}\theta\right)$ is the shear tensor.

The kinematic quantities are completely orthogonal to u^a, i.e.,

$$h_{ab}u^b = \omega_{ab}u^b = \varsigma_{ab}u^b = 0\,, \quad \dot{u}_a u^b = \omega_a u^b = 0\,.$$

Physically, the time-like vector field u^a is often taken to be the four-velocity of a fluid. The volume element expansion θ extracted from this decomposition can be thus seen as a hydrodynamic picture: it is of major importance in the foregoing.

Chapter 1. Basics of Warp Drive Physics

§1.1. Description of the Alcubierre concept

§1.1.1 Space-time bubble

The Universe is approximated as a Minkowskian space: we choose an arbitrary curve and deform the space-time in the immediate vicinity in such a way that the curve becomes a time-like geodesic somewhat like a "ripple", in order to generate a perturbed or singular local region in which one may fit a spaceship and its occupants.

Let x_s be the center of the region where the spaceship stays, and x any coordinate within this region so that $x = x_s$ for the spaceship.

Within an orthonormal coordinate frame, such a region, which is referred to as a *bubble*, is transported forward with respect to distant observers, along a given direction (x in this text).

With respect to the same distant observers, the apparent velocity of the bubble center is given by

$$v_s(t) = \frac{dx_s(t)}{dt}\,, \tag{1.1}$$

where $x_s(t)$ is the trajectory of the region along the x-direction, and

$$r_s(t) = \sqrt{\left(x - x_s(t)\right)^2 + y^2 + z^2} \tag{1.2}$$

is the variable distance outward from the center of the spaceship until $\Re$ which may be called the *radius of the singular region.*

The spaceship is at rest inside the bubble and has no local velocity.

§1.1.2. Characteristics

From these first elements, we must now select the exact form of a metric that will "push" the spaceship along a trajectory described by an arbitrary function of time (x_s, t).

Furthermore, this trajectory should be a time-like geodesic, whatever $v_s(t)$. By substituting $x = x_s(t)$ in the new metric to be defined, we should expect to find

$$d\tau = dt\,. \tag{1.3}$$

The proper time of the spaceship is equal to coordinate time which is also the proper time of distant observers.

Since these observers are situated in the flat region, we conclude that the spaceship suffers no time dilation as it moves. It will be easy to prove that this spaceship moves along a time-like geodesic and its proper acceleration is zero.

§1.2. The physics that leads to Warp Drive

§1.2.1. The (3+1) Formalism: the Arnowitt-Deser-Misner (ADM) technique

In 1960, Arnowitt, Deser, and Misner [5] suggested a technique based on decomposing the space-time into a family of space-like hypersurfaces and parametrized by the value of an arbitrarily chosen time coordinate x^4.

This "foilation" displays a proper-time element $d\tau$ between two nearby hypersurfaces labelled $x^4 = const$ and $x^4 + dx^4 = const$. The proper-time element $d\tau$ must be proportional to dx^4. Thus we write

$$d\tau = N\big(x^\alpha, x^4\big)\, dx^4. \tag{1.4}$$

In the ADM terminology, N is called the *lapse function* and more specifically the *time lapse*.

Consider now the three-vector whose spatial coordinates x^α are lying in the hypersurface ($x^4 = const$) and which is normal to it.

We want to evaluate this vector on the second hypersurface, which is $x^4 + dx^4 = const$, where these coordinates now become $N^\alpha dx^4$. This N^α vector is known as the *shift vector*.

The ADM four-metric tensor is decomposed into covariant components

$$(g_{ab})_{\text{ADM}} = \left\{ \begin{array}{cc} -N^2 - N_\alpha N_\beta\, g^{\alpha\beta}, & N_\beta\,, \\ N_\alpha\,, & g_{\alpha\beta}\,. \end{array} \right. \tag{1.5}$$

The line element corresponding to the hypersurfaces' separation is therefore written

$$(ds^2)_{\text{ADM}} = (g_{ab})_{\text{ADM}}\, dx^a dx^b$$

or

$$(ds^2)_{\text{ADM}} = -N^2 \left(dx^4\right)^2 + g_{ab} \left(N^\alpha dx^4 + dx^\alpha\right)\left(N^\beta dx^4 + dx^\beta\right) = \\ = \left(-N^2 - N_\alpha N^\alpha\right)\left(dx^4\right)^2 + 2N_\beta\, dx^4 dx^\beta + g_{\alpha\beta}\, dx^\alpha dx^\beta, \quad (1.6)$$

where $g_{\alpha\beta}$ is the 3-metric tensor of the hypersurfaces.

The ADM metric tensor has the contravariant components

$$(g^{ab})_{\text{ADM}} = \begin{cases} -N^{-2}, & \dfrac{N^\beta}{N^2}\,, \\ \dfrac{N^\alpha}{N^2}\,, & g^{\alpha\beta} - N^\alpha \dfrac{N^\beta}{N^2}\,. \end{cases} \quad (1.7)$$

As a result, the hypersurfaces have a unit time-like normal vector with components

$$n^a = N^{-1}\left(1,\, -N^\alpha\right), \qquad n_a = (-N,\, 0)\,. \quad (1.8)$$

When the fundamental three-tensor satisfies $g^{\alpha\beta} = \delta^{\alpha\beta}$ the metric (1.6) becomes

$$ds^2 = -\left(N^2 - N_\alpha N^\alpha\right) dt^2 - 2N^\alpha dx\, dt + dx^\alpha dx^\beta$$

or

$$ds^2 = -N^2 dt^2 - \left(dx + N^\alpha dt\right)^2 + dy^2 + dz^2. \quad (1.9)$$

§1.2.2. Curvatures in the ADM formalism

The Einstein action can be written in terms of the metric tensor $(g_{ab})_{\text{ADM}}$ (1.5) and (1.7), as [6]

$$S_{\text{ADM}} = \int R\sqrt{-g}\; d^4x = \\ = \int dt \int N\left(K_{\alpha\beta}K^{\alpha\beta} - K^2 + {}^{(3)}R\right)\sqrt{-g}\; d^3x + \\ + \text{boundary terms}\left(K^\alpha_\alpha K^\beta_\beta = K^2\right), \quad (1.10)$$

where $g = \det \| g_{\alpha\beta} \|$, while ${}^{(3)}R$ stands for the *intrinsic curvature* tensor of the hypersurface $x^4 = const$

$$K_{\alpha\beta} = (2N)^{-1}\left(-N_{\alpha;\beta} - N_{\beta;\alpha} + \partial_t\, g_{\alpha\beta}\right). \quad (1.11)$$

The tensor (1.11) (in which ; refers to covariant differentiation with respect to the three-metric), represents the *extrinsic curvature*, and as such, describes the manner in which that surface is embedded in the surrounding four-dimensional space-time.

The determinant ${}^{(4)}g$ of the four-metric is shown to be related to the determinant ${}^{(3)}g$ by

$$\sqrt{-\,{}^{(4)}g} = N\,\sqrt{{}^{(3)}g}\ .$$

The rate of change of the three-metric tensor $g_{\alpha\beta}$ with respect to the time label can be decomposed into "normal" and "tangential" contributions:

- The normal change is proportional to the extrinsic curvature $\frac{-2N}{K_{\alpha\beta}}$ of the hypersurface;
- The tangential change is given by the Lie derivative of $g_{\alpha\beta}$ along the shift vector N^{α}, namely

$$\underset{N}{\mathrm{L}}\, g_{\alpha\beta} = 2N_{(\alpha;\beta)}\,. \tag{1.12}$$

The main advantage of the ADM formalism is that the time derivative is isolated and it can be used in further specific computations. Furthermore we verify that

$$K_{\alpha\beta} = -\,n_{\alpha;\beta}\,, \tag{1.13}$$

which is sometimes called the *second fundamental form* of the three-space [7]. Six of the ten Einstein equations imply for K^{α}_{β} to evolve according to [8]

$$\frac{\partial K^{\alpha}_{\beta}}{\partial t} + \underset{N}{\mathrm{L}}\, K^{\alpha}_{\beta} = \nabla^{\alpha}\nabla_{\beta} N + \\ + N\left[R^{\alpha}_{\beta} + K^{\alpha}_{\alpha}K^{\alpha}_{\beta} + 4\pi\,(T-C)\,\delta^{\alpha}_{\beta} - 8\pi T^{\alpha}_{\beta}\right], \tag{1.14}$$

where R^{α}_{β} is the three-Ricci tensor, and $C = T_{ab}\,n^a n^b$ is the material energy density in the rest frame of normal congruence (time-like vector field) with $T = T^{\alpha}_{\alpha}$.

It is convenient to introduce the three-momentum current density $I_{\alpha} = -\,n_c\,T^{c}_{\alpha}$. So the remaining four equations finally form the so-called *constraint equations*

$$H = \frac{1}{2}\left(R - K^{\alpha}_{\beta}K^{\beta}_{\alpha} + K^2\right) - 8\pi C = 0\,, \tag{1.15}$$

$$H_{\beta} = \nabla_{\alpha}\bigl(K^{\alpha}_{\beta} - K\delta^{\alpha}_{\beta}\bigr) - 8\pi I_{\beta} = 0\,. \tag{1.16}$$

Equation (1.15) will be of central importance in the present theory.

Chapter 2. The Alcubierre Warp Drive

§2.1. The Alcubierre metric

In view of building a space warp progressing along the x-direction, one may choose with Alcubierre

$$\left.\begin{aligned} &N = 1 \\ &N^1 = -\,v_s(t)\, f(r_s,t) \\ &N^2 = N^3 = 0 \end{aligned}\right\}, \tag{2.1}$$

we then have

$$(ds^2)_{\text{AL}} = -\,dt^2 + \big[dx - v_s\, f(r_s,t)\, dt\big]^2 + dy^2 + dz^2; \tag{2.2}$$

this interval is known as the *Alcubierre metric*.

The function $f(r_s,t)$ is so defined as to cause space-time to contract on the forward edge and equally expanding on the trailing edge of the singular region. It is often referred to as a "*top hat*" *function*.

Let us now write down the Alcubierre metric under the following equivalent form

$$(ds^2)_{\text{AL}} = -\,\big[1 - v_s^2 f^2(r_s,t)\big]\, dt^2 - 2\,v_s f\, dt\, dx + dx^2 + dy^2 + dz^2, \tag{2.3}$$

which puts in evidence the covariant components of the Alcubierre metric tensor

$$\left.\begin{aligned} &(g_{44})_{\text{AL}} = -\,\big[1 - v_s^2 f^2(r_s,t)\big] \\ &(g_{41})_{\text{AL}} = (g_{14})_{\text{AL}} = -\,v_s\, f(r_s,t) \\ &(g_{22})_{\text{AL}} = (g_{33})_{\text{AL}} = 1 \end{aligned}\right\}. \tag{2.4}$$

§2.2 Analyzing the "top hat" function

We now turn our attention to the "top hat" function $f(r_s,t)$ itself, which allows for the bubble to develop. Alcubierre originally chosen the following form

$$f(r_s,t) = \frac{\tanh\big[\sigma\,(r_s + \Re)\big] - \tanh\big[\sigma\,(r_s - \Re)\big]}{2\tanh\,(\sigma R)}\,, \tag{2.5}$$

where $\Re > 0$ is the "radius" of the "region", while σ is a "bump" parameter which can be used to "tune" the "wall" thickness of the singular region.

The larger this parameter, the greater the contained energy density, so its shell thickness decreases. Moreover, the absolute increase of σ means a faster approach of the condition

$$\lim_{\sigma\to\infty} f(r_s,t) = \begin{cases} 1 \text{ for } r_s \in [-\Re,\, \Re\,], \\ 0 \text{ otherwise.} \end{cases} \tag{2.6}$$

Note that $r_s = 0$ at the center of the singular region (spaceship location). For $r_s > \Re$, the function $f(r_s,t)$ should rapidly verify $f(r_s,t)=0$ and we recover the Minkowski space-time.

As outlined earlier, any function will suffice so long as the above conditions are fullfilled. For simplified calculations, it is convenient to introduce the equivalent *piecewise continuous function* as established by Pfenning and Ford [9]

$$f_{\text{p.c.}}(r_s,t) = \begin{cases} 1 \text{ for } r_s < \Re - \frac{\Delta}{2}\,, \\ \left(\frac{-1}{\Delta}\right)\left(r_s - \Re - \frac{\Delta}{2}\right) \text{ for } \Re - \frac{\Delta}{2} < r_s < \Re + \frac{\Delta}{2}\,, \\ 0 \text{ for } r_s > \Re + \frac{\Delta}{2}\,, \end{cases} \tag{2.7}$$

where the variable Δ is the region shell "thickness".

Setting the slopes of the functions $f(r_s,t)$ and $f_{\text{p.c.}}(r_s,t)$ to be equal at $r_s = \Re$, leads to the following result

$$\Delta = \frac{1+\tanh^2(\sigma\Re)^2}{2\sigma\,[\,\tanh(\sigma\Re)]}\,. \tag{2.8}$$

For large $\sigma\Re$, one may admit the approximation

$$\Delta \approx \frac{2}{\sigma}\,. \tag{2.9}$$

§2.3. Eulerian observer

§2.3.1 Definition

With the choice of the three-vector $N^\alpha = 0$, we have a particular coordinate frame called *normal coordinates*, according to (1.8). Such a choice of coordinates constitutes an "Eulerian" gauge.

In the Alcubierre formalism, $N^1 \neq 0$ characterizes a special type of observer who "measures" the warped shell and the associated region when they cross through.

His four-velocity is normal to the hypersurfaces. This observer, who also is referred to as *Eulerian observer*, is initially at rest. Just the front wall of the disturbance reaches the observer, he begins to accelerate, in the progressing direction of the singular region, relative to observers located at large distance from him.

Once during his "stay" inside the region, the Eulerian observer travels with a nearly constant velocity given by

$$\frac{dx(t)}{dt} = v_s(t_\rho, \rho)\, f(\rho)\,, \tag{2.10}$$

where t_ρ is the time measured at the coordinate

$$\rho = \sqrt{y^2 + z^2}\ . \tag{2.11}$$

This velocity will always be less than the region's velocity unless $\rho = 0$, i.e. when the observer is at the center of the spaceship.

After reaching the region's equator, the Eulerian observer decelerates, and is left at rest while going out of the rear edge of the "wall".

If using the piecewise continuous function of Pfenning for $r_s < \Re - \frac{\Delta}{2}$, any observer moves along the singular region with the same speed. Inside the warped regions ("shells"), i.e. for

$$\Re - \frac{\Delta}{2} \,<\, \rho \,<\, \Re + \frac{\Delta}{2}\,,$$

we recover the conditions deduced from the "top hat" function (2.5), as viewed by the Eulerian observer. The singular regions have toroidal geometry concentrated on either part of the longitudinal direction of travel x, and are thus perpendicular to the plane defined by ρ.

§2.3.2. Specific characteristics

Following Alcubierre, such an observer has a four-velocity normal to the hypersurfaces $t = const$.

With the condition $d\tau = dt = ds$, it is straightforward to show that this four-velocity has the following components

$$\left.\begin{aligned} (u^a)_{\text{AL}} &= \big[\,1,\, v_s\, f(r_s, t),\, 0,\, 0\,\big] \\ (u_a)_{\text{AL}} &= \big[\,-1,\, 0,\, 0,\, 0\,\big] \end{aligned}\right\}. \tag{2.12}$$

The Eulerian observer follows time-like geodesics orthogonal to the Euclidean hypersurfaces.

From the metric (2.2), inspection shows that the Eulerian observer is in free fall, i.e. his four-acceleration is zero

$$(a^b)_{\text{AL}} = (u^a)_{\text{AL}}\,(u^b_{;a})_{\text{AL}} = 0\,,$$

which confirms the postulate of §1.1.2.

In this case $\delta^{\alpha\beta} = g^{\alpha\beta}$, $N = 1$, and (1.11) reduces to

$$K_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha N_\beta + \partial_\beta N_\alpha\right).$$

The contracted tensor, which is defined by

$$\theta = -\,\mathrm{trace}\, K^{\alpha\beta}, \tag{2.13}$$

is the *expansion scalar* defined above; it means the expansion of the three-volume element which, taking account of (2.1), is

$$\theta = v_s \,\frac{df}{(dx)_{\mathrm{AL}}}\,, \tag{2.14}$$

where $(x)_{\mathrm{AL}} = x - x_s(t)$ is the single derivative variable.

Hence, we find

$$\theta = v_s \left(\frac{df}{dr_s}\right)\left[\frac{dr_s}{d(x - x_s)}\right] \tag{2.15}$$

and by using the classical derivative formula of functions of functions, it is not difficult to show that this last formula becomes

$$\theta = v_s \left(\frac{df}{dr_s}\right)\left(\frac{x_s}{r_s}\right). \tag{2.16}$$

Obviously, the shape of the function f, (2.5) induces both a volume contraction and expansion ahead of, as well as behind, the singular region.

§2.4. Negative energy requirement

§2.4.1. The Alcubierre-Einstein tensor

Before determining the form of the Alcubierre-Einstein tensor, we recall briefly the so-called *energy conditions.*

Let us consider at a point p on the manifold (M, g_{ab}), an energy-momentum tensor T^{ab}.

For any time-like vector $u^a \in \mathrm{T}_p$ (tangent space at p), one must have the inequality

$$C = T_{ab}\, u^b u^b \geqslant 0\,, \tag{2.17}$$

known as the *weak energy condition.*

In addition, the "*dominant*" *energy condition* stipulates that for any time-like four-vector $u^a > 0$, the four-vector $Q^a = T^a_b u^b$ is a non-space-like vector.

By continuity, the weak energy condition implies the *null energy condition* which asserts that for any null vector k^a

$$T_{ab}\, k^a k^b \geqslant 0\,.$$

Lastly, we consider the *strong energy condition* for any time-like four-vector u^a

$$\left(T_{ab} - \frac{1}{2}\, g_{ab}\, T\right) u^a u^b \geqslant 0\,.$$

NOTE: The dominant energy condition implies the weak energy condition and therefore the null energy condition, but not necessarily the strong energy condition, which itself implies the null energy condition but not necessarily the weak energy condition.

From the components of the metric tensor (2.4), it is possible to form the contravariant components of the Ricci tensor $(R^{ab})_{\rm AL}$ of the Alcubierre metric.

The resulting Einstein tensor

$$(G^{ab})_{\rm AL} = (R^{ab})_{\rm AL} - \frac{1}{2}\,(g^{ab})_{\rm AL}\, R$$

contains the time component $(R^{44})_{\rm AL}$ and

$$(G^{44})_{\rm AL} = -\left(\frac{v_s^2}{4r_s^2}\right)\rho^2\left(\frac{df}{dr_s}\right)^2.$$

Using $(G^{44})_{\rm AL}$ to define the energy density $(T^{44})_{\rm AL}$, one finds

$$C = \frac{1}{8\pi}\,(G^{44})_{\rm AL}\,(u_4\, u_4)_{\rm AL} = -\frac{1}{32\,\pi}\left(\frac{v_s^2\rho^2}{r_s^2}\right)\left(\frac{df}{dr_s}\right)^2. \tag{2.18}$$

This formula is always negative as seen by the Eulerian observers, and therefore it is not compatible with the energy condition (2.17).

Another way of writing this equation is obtained by using the Gauss-Codazzi relations to form the Einstein tensor as a function of both the intrinsic and extrinsic curvatures, which eventually leads to [10]

$$C = T_{ab}\, n^a n^b = \frac{1}{16\pi}\left({}^{(3)}R + K^2 - K_{\alpha\beta}K^{\alpha\beta}\right). \tag{2.19}$$

By choosing $N^1 = -v_s\, f(r_s)$, $N^2 = N^3 = 0$, and ${}^{(3)}R = 0$ the Alcubierre formulation is obtained again.

The energy density as measured by the Eulerian observer is given by

$$(C)_{\rm AL} = \frac{1}{16\pi}\left(K^2 - K_{\alpha\beta}K^{\alpha\beta}\right), \tag{2.20}$$

thus referring to (2.13), we find back

$$\theta = -\,\partial_1 N^1 = v_s\, f'(r_s)\, \frac{x - x_s}{r_s} \tag{2.21}$$

and

$$(C)_{\text{AL}} = \frac{1}{16\pi}\left[\left(\partial_1 N^1\right)^2 - \left(\partial_1 N^1\right)^2 - 2\left(\frac{\partial_2 N^1}{2}\right)^2 - 2\left(\frac{\partial_3 N^1}{2}\right)^2\right], \tag{2.22}$$

$$(C)_{\text{AL}} = -\frac{1}{32\pi}\, v_s^2\, f'^2(r_s)\, \frac{y^2 + z^2}{r_s^2}\,. \tag{2.23}$$

§2.4.2. Negative energy

We now write down the form of the total negative energy required to sustain the Alcubierre metric.

Without loss of generality, we may simplify the case by assuming a constant velocity for the singular region, i.e.

$$x(t) = v_s\,(t) \tag{2.24}$$

at $t = 0$, we have

$$r_s(t = 0)\,, \qquad \sqrt{(x^\alpha)^2} = r\,. \tag{2.25}$$

Under these conditions, we must calculate the integral of the local energy density over the proper volume $d^3x = dV$ (hypersurface)

$$E = \int \sqrt{y}\; T^{44}\, dV, \tag{2.26}$$

where y is the determinant of the spatial metric on the hypersurface $t = const$, which, in our case, is $y = 1$.

One finds

$$E = -\,\frac{1}{32\pi}\, v_s^2 \int \frac{\rho^2}{r^2}\left[\frac{df(r_s,t)}{dr}\right]^2 dV. \tag{2.27}$$

With the piecewise function of Pfenning (2.7), the energy is, in the spherical coordinates

$$E = -\,\frac{1}{12}\, v_s^2 \int_{\Re-\Delta/2}^{\Re+\Delta/2} r^2 \left(-\frac{1}{\Delta}\right)^2 dr. \tag{2.28}$$

The contributions to the energy come only from the singular region's "shell" areas.

We then see that one needs a special type of negative energy (matter) to travel faster than the speed of light by means of a Warp Drive. Such an exotic matter has never been detected so far.

Chapter 3. Causality

§3.1. Horizon formation

We regard the speed of the spaceship v as constant, and r_s is then

$$r_s = \sqrt{(x - vt)^2 + y^2 + z^2} \tag{3.1}$$

reducing the metric (2.3) to two dimensions, $y = z = 0$, we obtain

$$ds^2 = -\left(1 - v^2 f^2\right) dt^2 - 2vf\,dx\,dt + dx^2 \tag{3.2}$$

for which now

$$x > v\,t\,, \tag{3.3}$$

$$r = x - v\,t = x' \tag{3.4}$$

this new variable defines, in the original Alcubierre metric, the proper spatial coordinate

$$dx = dx' + v\,dt$$

of the spaceship frame from which are observed the events in order to ensure a control communication.

Adopting the new coordinate

$$dx' = dx - v\,dt \tag{3.5}$$

and setting

$$S(r,t) = 1 - f\,(r,t)\,, \tag{3.6}$$

we may keep the metric (3.2) under the same form

$$(ds^2)_{\rm HS} = -\left[1 - v^2 S(r,t)^2\right] dt^2 - 2vS(r,t)\,dx'dt + dx^2. \tag{3.7}$$

We will refer to it as the *Hiscock metric* after William A. Hiscock [11]. It can be written as

$$(ds^2)_{\rm HS} = (g_{44})_{\rm HS}\,dt^2 + 2\,(g_{41})_{\rm HS}\,dx'dt + dx'^2 \tag{3.8}$$

with the covariant components of the fundamental tensor

$$\left.\begin{aligned} (g_{44})_{\rm HS} &= -\left(1 - v^2 S^2\right) \\ (g_{41})_{\rm HS} &= (g_{14})_{\rm HS} = -\,v\,S \\ (g_{11})_{\rm HS} &= (g_{22})_{\rm HS} = (g_{33})_{\rm HS} = 1 \end{aligned}\right\}. \tag{3.9}$$

The spaceship frame metric (3.7) is also expressed by

$$(ds^2)_{\text{HS}} = -\,H(r)\left(\frac{dt - vS}{H(r)}\,dx'\right)^2 + \frac{dx'^2}{H(r)}\,, \tag{3.10}$$

where

$$(g_{44})_{\text{HS}} = -\,H(r)\,,$$

we then introduce a new time coordinate

$$dt' = \frac{vS}{H(r)}\,dx', \tag{3.11}$$

which is manifestly the spaceship's proper time since $H(r) = 1$ (thus $f = 1$) as $r = 0$.

At the same time, the coordinates are not asymptotically normalized. Indeed, for large r distant from the spaceship, $H(r)$ approaches $1 - v^2$ rather than 1. One may solve the problem by defining yet one more set of coordinates

$$T' = \sqrt{1 - v^2}\;t'\,, \qquad X = x'\sqrt{1 - v^2}\;. \tag{3.12}$$

By examining the form of the metric (3.10), the coordinate system seems to be valid only for $r > 0$, i.e. if $v < 1$ as per (3.3).

However, when $v > 1$ (superluminal velocity), there exists a coordinate singularity, that is, an event horizon at the location r_0 for the metric (3.10), such that

$$H(r_0) = 0$$

or

$$f(r_0) = 1 - \frac{1}{v}\,. \tag{3.13}$$

This horizon first appears for the occupants of the spaceship, who are unable to "see" beyond the distortion, and therefore cannot communicate with the outer universe.

§3.2. Reducing the energy

Based on the works produced by W. Hiscock, F. Loup, D. Waite and also E. Halerewicz et al. [12, 13], it has been proposed a particular metric which allows for the use of the warped region in order to "causally connect" the inside of the spaceship and the outside of the singular bubble region.

This generalized Hiscock metric (3.7) can also dramatically lower the negative energy requirements.

§3.2.1. The ESAA metric

By lowering the energy requirement, the proposed model intends to show that the Warp Drive metric is much more realistic than that originally shown by Pfenning and Ford.

We refer to this new space-time metric as *Ex Somnium Ad Astra* (ESAA), which literally translates as *From a Dream to the Stars* (Simon Jenks).

We are going to introduce the change $\rho = r_s$ of the variables. Independently of this change, the ESAA metric differs from (3.7) by the fundamental tensor whose covariant components are

$$\left.\begin{aligned} (g_{44})_{\text{ESAA}} &= -\left[\,N^2(\rho) - v_s\,S(\rho)^2\right] \\ (g_{41})_{\text{ESAA}} &= (g_{14})_{\text{ESAA}} = -\,v_s\,S(\rho) \\ (g_{11})_{\text{ESAA}} &= (g_{22})_{\text{ESAA}} = (g_{33})_{\text{ESAA}} = 1 \end{aligned}\right\}, \tag{3.14}$$

thus from these we readily note that the "time lapse" function is no longer equal to 1.

In cylindrical coordinates (following x), the ESAA metric is

$$\begin{aligned} (ds^2)_{\text{ESAA}} = -\big[N(\rho) - v_s(r)\,S(\rho)\big]^2 dt^2 - {} \\ - 2v_s\,S(\rho) + dx'^2 + dr^2 + r^2 d\phi^2 . \end{aligned} \tag{3.15}$$

Let us set

$$r = \rho\sin\theta\,, \qquad x' = r\cos\theta\,,$$

it is then easy to see that (3.15) becomes

$$\begin{aligned} (ds^2)_{\text{ESAA}} = \left[N^2(t,\rho) - v_s(t)\,S^2(\rho)\right] dt^2 + 2v_s(t)\,S(\rho)\cos\theta\,dt\,dr - {} \\ - 2v_s(t)\,S(\rho)\,\rho\sin\theta\,d\theta\,dt + dr^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta\,d\phi^2 . \end{aligned} \tag{3.16}$$

§3.2.2 Required energy

The energy density of the spaceship frame is given by

$$(T^{44})_{\text{ESAA}} = -\,\frac{v_s}{32\,\pi}\left(\frac{dS}{d\rho}\right)^2 \frac{(\sin\theta)^2}{N^4(\rho,t)}\,. \tag{3.17}$$

Clearly, an arbitrarily large N reduces the (negative) energy density requirement of the spaceship frame.

In our given coordinate system, the volume element is given by

$$dV = \rho^2\sin\theta\,d\rho\,d\theta\,d\phi\,,$$

r_s	f	S	N
0	1	0	1.023
20	0.997	0.0023	3.428
50	0.5	0.5	2×10^{75}
100	4.5×10^{-5}	0.9999	1.0950

Table 1: Numerical estimates for the lapse function.

thus reinstating Newton's constant $\mathfrak{G}$ and c, the total energy required to sustain the distortion is finally given here by

$$E = -\int_0^\infty \left[\frac{v_s c^4}{12\mathfrak{G}} \left(\frac{dS}{d\rho}\right)^2 \frac{1}{N^4} \right] \rho^2 d\rho \,. \tag{3.18}$$

Another modification of the Alcubierre geometry has been suggested by Van den Broeck [14], in order to reduce the amount of needed negative energy.

The Van den Broeck metric is

$$ds^2 = -\,dt^2 + B^2(r_s)\big[\,dx - v_s(t)\,f(r_s)\,dt\big]^2 + dy^2 + dz^2,$$

where $B(r_s)$ is a twice differentiable polynomial such that its numerical value is $-1 < B(r_s) \leqslant 1+\alpha$ for $\Re' \leqslant r_s \leqslant \Re' + \Delta'$, and $B(r_s) = 1$ for $\Re' + \Delta' \leqslant r_s$ (here $\Re'$ is the radius of an internal "blown pocket" within the Alcubierre region with thickness Δ).

This modification keeps the surface area of the bubble itself microscopically small, while at the same time expanding the spatial volume inside the region caused by the factor α.

One can show that the energy density given by the tensor T_{44} is much lower than the one calculated by Alcubierre.

As an example, reinstating again the factor $c^2/\mathfrak{G}$, to get the kilogram units, for a bubble of $\Re = 100\,\text{m}$, the standard Alcubierre value for the total negative energy would be $E \approx -6.2\times10^{62}\,v_s$ Kg, which is theoretically enormous, but with the Van den Broeck solution ($v_s \approx 1$), this energy is reduced to 4.9×10^{30} Kg, that is a few solar masses: this shows that reasonable energy levels can be reached by investigating new models.

It is however difficult to establish energy level comparisons. This is because each model is characterized by different and newly introduced parameters.

In the case of the ESAA metric, we can, as an indication, compute some values for the functions f and S with the resulting lapse function

N, setting the initial values for the bump parameter as $\sigma = 0.1$ and bubble radius $\Re = 50$ m.

We first notice that in the warped regions ($r_s = \Re$), the lapse function N takes on very large values, which appears as a severe drawback, but interestingly, for $r_s > \Re$, the ESAA model yields back a lapse function $N \to 1$, which is in full accordance with the free fall condition (1.3).

§3.3. Causally connected spaceship

§3.3.1. The spaceship frame of reference

As we defined the Pfenning piecewise function (2.7) corresponding to the Alcubierre "top hat" function, we may establish the similar type of function with the time lapse N inserted

$$f(r_s)_{\text{p.c.}} = \begin{cases} 1 \ \text{ for } \ r_s < \Re - \frac{\Delta}{2}\,, \\ 1 - \left(\frac{1}{N}\right) r_s - \Re \ \text{ for } \ \Re - \frac{\Delta}{2} < r_s < \Re + \frac{\Delta}{2}\,, \\ 0 \ \text{ for } \ r_s > \Re + \frac{\Delta}{2}\,. \end{cases}$$

The "free fall" condition demands

$$N(r_s) = \begin{cases} 1 \ \text{ for } \ r_s < \Re - \frac{\Delta}{2}\,, \\ 1 \ \text{ for } \ r_s > \Re + \frac{\Delta}{2}\,. \end{cases}$$

The spaceship frame Hiscock-ESAA horizon is thus defined as

$$H(r_s) = \begin{cases} 1 \ \text{ for } \ r_s < \Re - \frac{\Delta}{2}\,, \quad H(r_s) > 0\,, \\ N^2 - \left(\frac{v_s}{N}\right)^2 \ \text{ for } \ r_s = \Re - \frac{\Delta}{2}\,, \quad H(r_s) > 0\,, \\ N^2 \ \text{ for } \ r_s = \Re\,, \quad H(r_s) > 0\,, \\ N^2 - \left(\frac{v_s}{N}\right)^2 \ \text{ for } \ r_s = \Re + \frac{\Delta}{2}\,, \quad H(r_s) > 0\,, \end{cases}$$

where we emphasize that N does not depend on the speed v_s.

Three cases are to be considered:

Subliminal velocities: For large values of N, the spaceship will always be connected to the domain from $r_s = 0$ (center of the spaceship) to the exterior part of the bubble $r_s = \Re + \frac{\Delta}{2}$, and since $H(r_s) > 0$, there is no horizon;

Luminal velocity: For the same domain, $H(r_s) = 0$, since $N = 1$ and $S(r) = 1$, a horizon will appear in front of the spaceship, which becomes causally disconnected from the part beyond the bubble.

Provided that N *is not a function of the speed and has been engineered at subluminal speeds*, it is always connected to the spaceship and the warped region $\int_{\Re-\Delta/2}^{\Re+\Delta/2}$ can be "controlled" by the "astronauts";

Superluminal velocities: The same argument applies here.

§3.3.2. A remote frame of reference

With the function $N(t, r_s)$, the Alcubierre metric is written

$$(d\tau^2)_{\text{ALN}} = (ds^2)_{\text{ALN}} = -N^2 dt^2 - \left[dx - v_s f(r_s, t)\, dt\right]^2$$

or

$$\begin{aligned}(ds^2)_{\text{ALN}} &= -\left(N^2 - v_s f^2(r_s, t)\right) dt^2 - 2 v_s f(r_s, t)\, dt\, dx + dx^2 = \\ &= -M(r_s)\, dt^2 - 2 v_s f(r_s)\, dx\, dt + dx^2, \end{aligned} \qquad (3.19)$$

where $M(r_s) = N^2 - v_s^2 f(r_s)^2$.

We will refer to (3.19) as the *ESAA-Alcubierre metric*, as observed from a remote frame of reference.

The remote metric of Hiscock, analogous to (3.10), is thus given by

$$(ds^2)_{\text{ALN}} = -M(r_s)\, dt'^2 + \frac{N^2}{M(r_s)}\, dx^2,$$

lending

$$dt'^2 = -dt^2 - \frac{2 v_s f(r_s)\, dx\, dt}{M(r_s)} + \frac{N^2 - M(r_s)}{M(r_s)^2}\, dx^2. \qquad (3.20)$$

If $v_s < 1$ (subliminal), $M(r_s) > 0$ then the domain is causally connected to the spaceship's remote frame.

If $v_s = 1$ (luminal), $M(r_s) = 0$, a horizon appears for the remote frame.

If $v_s > 1$ (superluminal), $M(r_s) < 0$, a horizon appears somewhere between $\Re - \frac{\Delta}{2}$ and $r_s < \Re - \frac{\Delta}{2}$.

Using the continuous "top hat" function in (3.20) for the warped region of Pfenning $\left[\Re - \frac{\Delta}{2},\, \Re + \frac{\Delta}{2}\right]$, one obtains

$$M(r_s) = N^2 - h$$

with

$$h = \sqrt{1 - \left[v_s^2\,(r_s - \Re)^2\right] N^2(t, r_s)}\ .$$

Given that $N^2 \gg v_s f^2(r_s)$, then $M(r_s) > 0$ and the warped region will be always connected to the remote frame.

In other words, for large N, a signal can be sent by the spaceship to $r_s = \Re + \frac{\Delta}{2}$, and a signal sent by a remote observer can reach $r_s = \Re - \frac{\Delta}{2}$.

Therefore the region between

$$\Re - \frac{\Delta}{2} \leqslant r_s \leqslant \Re + \frac{\Delta}{2}$$

is observed from both frames, and may allow us to engineer the spaceship (speed control). Reverting now to the Alcubierre function

$$f(r_s, t) = \frac{\tanh\left[\sigma(r_s + \Re)\right] - \tanh\left[\sigma(r_s - \Re)\right]}{2\tanh(\sigma\Re)},$$

we know that it is 1 in the spaceship and 0 far from it. There exists an open interval where $f(r_s, t)$ starts to decrease from 1 to 0, precisely where the negative energy is located.

In order to maintain the "free fall" condition (1.3), N should reduce to 1 in the spaceship and far from it outside the singular region.

In order to fulfill this condition, we suggest here the following form for N which differs from the formula (33) of [13]

$$N = \exp\left(\tanh\left[\sigma(r_s - \Re)\right]^2\right). \tag{3.21}$$

This has the advantage of taking higher "peak" value near the spaceship where the excessive proper time $Nd\tau$ is thus rapidly shortened as $r_s \to \Re$.

Chapter 4. The EGR-Like Picture

§4.1. A particular extended Lie derivative

Instead of considering the Alcubierre function f associated with a local Riemannian structure emerging from a background Euclidean space-time, we choose here to express f in the EGR-like formulation.

Unlike the classical theory, this singular region will now be distinguished from a non-flat background space-time i.e. a "weak" Riemannian background manifold, which is physically more appropriate.

Our aim is to find an additional energy decrease with a way to possibly avoid violating the weak energy condition.

We begin by defining an extended Lie derivative of g_{ab} that leads to a new extrinsic curvature.

Let us consider the infinitesimal coordinates shift

$$x'^a = x^a + N^a, \tag{4.1}$$

the relevant metric variation is classically given by

$$\delta g_{ab} = -\,g_{ac}\,\frac{\partial N^c}{\partial x^b} - g_{cb}\,\frac{\partial N^c}{\partial x_a} - \frac{\partial g_{ab}}{\partial x^c}\,N^c. \tag{4.2}$$

Furthermore it can be shown that [15]

$$\delta g_{ab} = (N_{a;b} + N_{b;a}) = \underset{N}{\mathrm{L}}\, g_{ab}\,. \tag{4.3}$$

When $\underset{N}{\mathrm{L}}\, g_{ab} = 0$, we have the Killing equations which preserve the metric (a condition referred to as infinitesimal isometry) under (4.1).

In the EGR theory, the metric undergoes an additional variation ζ upon (4.1) due to the covariant derivative of the metric, and we expect to find for the Killing equations the following expression

$$\underset{N}{\mathrm{L}}\, g_{ab} = \zeta\, g_{ab}\,. \tag{4.4}$$

We need now to define the explicit form of the infinitesimal variation ζ. To this effect we will first consider a vector l with components A^i such that

$$l^2 = g_{ik}\, A^i A^k$$

upon (4.1) this vector is varied by

$$l'^2 = (1+\zeta)\, l^2,$$

i.e.

$$dl^2 = \zeta\, l^2.$$

Obviously we have

$$dl^2 = (\mathrm{D}_c\, g_{ik})\, A^i A^k\, dx^c,$$

where, as stipulated in the EGR theory,

$$\mathrm{D}_c\, g_{ik} = \frac{1}{3}\left(J_k\, g_{ci} + J_i\, g_{ck} - J_c\, g_{ik}\right),$$

thus

$$dl^2 = l^2\, g^{ik}\,(\mathrm{D}_c\, g_{ik})\, dx^c$$

and so

$$\zeta = g^{ik}\,(\mathrm{D}_c\, g_{ik})\, dx^c$$

setting

$$g^{ik}\,(\mathrm{D}_c\, g_{ik}) = B_c$$

we write

$$\zeta\, g_{ab} = g_{ab}\, B_c\, dx^c.$$

Within a sufficient approximation, we may set

$$dx^c = N^c,$$

hence we define the "extended Lie derivative" of g_{ab} as

$$\underset{N'}{\mathrm{L}}\, g_{ab} \equiv \underset{N}{\mathrm{L}}\, g_{ab}\, B_c N^c, \tag{4.5}$$

where N' is the rescaled shift vector.

At this stage, we want to stress that the assumed extension is here always considered in a Riemannian scheme.

The definition (4.5) formally holds for a Lie derivative of g_{ab}, provided the last term is "likened" to a Riemannian correction.

Indeed a "non-Riemannian" Lie derivative (i.e. defined in the framework of the EGR theory) is not applicable, due to the algebraic nature of this operation.

The EGR theory however provides a justification as to the origin of the extra term in (4.5).

§4.2. Extended extrinsic curvature and associated energy density

We are now able to define the "extended" extrinsic curvature as

$$K'_{\alpha\beta} = (2N')^{-1}\left(\nabla_\alpha N'_\beta + \nabla_\beta N'_\alpha + \frac{\partial g_{\alpha\beta}}{\partial t}\right). \tag{4.6}$$

Accordingly, we still consider the classical field equations as inferred from the Hilbert-Einstein action

$$S = \int R\sqrt{-g}\; d^4x\,.$$

By doing so, we set forth a close one-to-one correspondence between the EGR scalar curvature $\mathsf{R} = R - \frac{1}{3}\left(\nabla_e J^e + \frac{1}{2} J^2\right)$ and the modified Riemannian scalar curvature R depicted in Riemannian geometry.

In this perspective, the equation (2.19) becomes here

$$C' = \frac{1}{6\pi}\left({}^{(3)}R' - K'_{\alpha\beta}K'^{\alpha\beta} + K'^2\right). \tag{4.7}$$

Now we are going to generalize the Alcubierre metric by following the same pattern which has led to (2.20).

However, based on the extended formulation, we now choose

$$N'^1 = -\, v_s\, f(r_s)\,, \tag{4.8}$$

$$N'^2 = N^2, \qquad N'^3 = N^3 \tag{4.9}$$

and

$$^{(3)}R' = {}^{(3)}R\,.$$

An immediate and important consequence appears when one observes the form of the expression

$$(C')_{\text{AL}} = \frac{1}{16\pi}\left[{}^{(3)}R - K_{\alpha\beta}(N'^1, N)\,K^{\alpha\beta}(N'^1, N) + K^2(N'^1, N)\right]. \tag{4.10}$$

In contrast to the classical Alcubierre scheme, the non-vanishing initial Riemannian scalar curvature of the hypersurfaces may have now a significant impact on the negative energy density reduction.

In addition, the term K^2, which should not cancel off here, contributes even further to lowering this energy.

Discussion and Concluding Remarks

First observation: The expansion of the volume element $\theta = -K^\alpha_\alpha$ is attached to the bubble which it generates and is thus a local property;

Second observation: The free fall condition (1.3) requires obviously a flat space (flat Universe), instead of a Riemannian one.

However, in the EGR context, the non-vanishing scalar curvature $^{(3)}R$ may be also regarded here as sufficiently "local" with respect to the (quasi) Euclidean space as a whole, wherein the Eulerian observers are situated.

Indeed, if the three-volume of each hypersurface $t = const$ is extremalized, the condition $K = const$ results (see André Lichnérowicz [16] and also subsequently maximum slicing conditions by Yvonne Choquet Bruhat [17]).

It is then possible to impose this condition, with respect to using equation (1.16), to eliminate $^{(3)}R$ from the trace of equation (1.15): in this case it is shown that the lapse function can be taken to be $N \to 1$ as an asymptotic boundary condition, which leads to an asymptotically flat space-time.

This condition is physically satisfied when one considers the scale of distances in our observable Universe as compared to the bubble warping dimensions, so that (4.10) holds with an asymptotically flat universe wherefrom the distant observers are located.

Hence, we can always imagine a situation where stellar massive objects arranged in such a required configuration are coming into play, and where the influence of their curvature given by $^{(3)}R$ may then be used to balance the negative energy, which renders the Warp Drive compatible with the weak energy condition.

With these two observations our theory tends to run counter to the zero expansion Warp Drive suggested by José Natario [18].

Needless to say, all arguments regarding the piecewise function causality constraints detailed above are equally valid in our extended formulation.

Within the standard Alcubierre metric, it is however possible to avoid the problem of causally disconnecting the spaceship from the outer edge of the bubble.

A somewhat recent two dimensional metric concept has been proposed by Serge V. Krasnikov [19] in which the time for a round trip to a distant planet as measured by clocks located on the Earth can be made arbitrarily short.

To connect the Earth to the planet, a space-time extension of this metric leads to the creation of a "tube" wherein the space-time is flat, but the light cones are opened out so as to allow superluminal travel [20].

In some cases, these metrics are shown not to lead to the fatally closed time-like curves.

Appendix. Detailing a stellar round trip example according to Alcubierre

A1. Stellar journey

Consider two quasi-static planets A and B, which are apart from each other at a distance D in the Euclidean space-time.

A spaceship starts off on its own (self-propulsion) from A at an initial moment of time $t = t_A$, with a subluminal velocity $v < c$.

At a distance d away from A, $d \ll D$, the spaceship stops at a point where the bubble is being created, which then drags the spaceship towards the planet B, thus inducing a coordinate three-acceleration $\mathbf{a}$ that varies rapidly from $\mathbf{a} = 0$ to $\mathbf{a} = const \neq 0$.

Halfway, between A and B, the bubble is controlled so as to invert this acceleration from $\mathbf{a}$ to $-\mathbf{a}$.

As the absolute values of acceleration and deceleration are assumed equal, the spaceship will eventually be at rest at a distance d away from the planet B at the time the disturbance will disappear ($v_s = 0$) and the journey is further completed at a "physical" speed $v < c$.

The total coordinate time elapsed in the one-way trip from the planet A to the planet B is: $T = t_{\text{self-propulsion}} + t_{\text{bubble}}$. Had the acceleration been constant along the distance $(D-2d)$, we would have

$$(D-2d) = \frac{\mathbf{a}\, t^2_{\text{bubble}}}{2}\,,$$

where

$$t^2_{\text{bubble}} = \frac{2\,(D-2d)}{\mathbf{a}}\,.$$

In fact, during the accelerating stage of the bubble, we will have

$$^{(+)}t^2_{\text{bubble}} = \frac{(D-2d)}{\mathbf{a}}$$

and during the decelerating stage

$$^{(-)}t^2_{\text{bubble}} = \frac{(D-2d)}{\mathbf{a}}\,,$$

which in total yields

$$T = t_{\text{self-propulsion}} + \sqrt{\frac{2\,(D-2d)}{\mathbf{a}}}$$

that is

$$T = 2\left(\frac{d}{v} + \sqrt{\frac{(D-2d)}{2\mathbf{a}}}\right).$$

A2. Deceleration stage

Remember that we considered planets A and B as static in a quasi-flat space. In this case $dx = dy = dz = 0$. This means that their proper time is equal to their coordinate time (reinstating c): $t = \tau = \frac{x^4}{c}$.

The proper time τ measured in the spaceship, on the other hand, must take into account the Lorentz transformations

$$\tau_{\text{ship}} = 2\left(\frac{d}{\gamma v} + \sqrt{\frac{(D-2d)}{2\mathbf{a}}}\right),$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,.$$

If the radius $\Re$ of the bubble satisfies, as it should, $\Re \ll d \ll D$, one may admit the approximation

$$\tau \approx T \approx \sqrt{\frac{2D}{\mathbf{a}}}\,.$$

This clearly shows that T can be chosen as small as we like, by increasing the value of $\mathbf{a}$.

As outlined by Alcubierre, since a round trip will only take twice as long, we can be back on the planet A after an arbitrarily short proper time, both from the point of view of an observer on board of the spaceship and from the point of view of an observer located on the planet.

The spaceship will then be able to travel much faster than the speed of light while remaining on a time-like trajectory (which is inside its local light-cone).

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