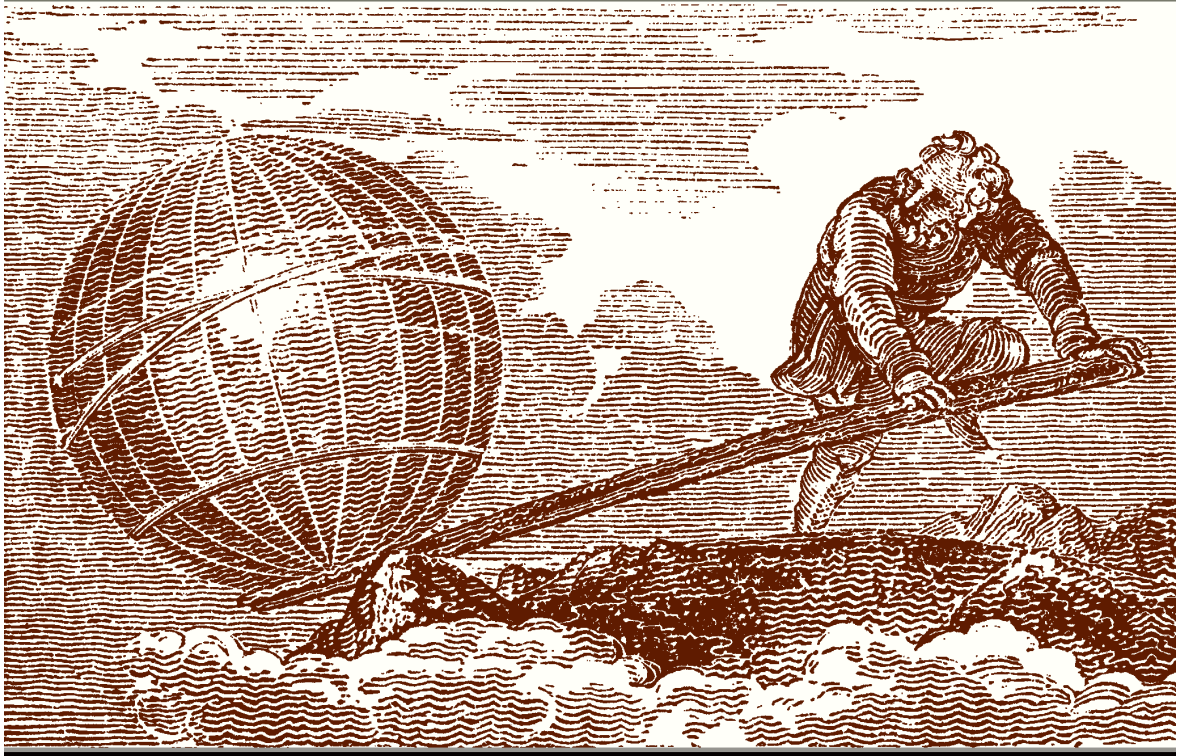


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A Theory of Frozen Light According to General Relativity

Dmitri Rabounski and Larissa Borissova

Abstract: We suggest a theory of frozen light, which was first registered in 2000 by Lene Hau, who pioneered this experimental research, which was then approved by two other groups of experimentalists. Frozen light is explained here as a new state of matter, which differs from the others (solid, gas, liquid, plasma). The explanation is given through space-time terms of the General Theory of Relativity, employing the mathematical apparatus of chronometric invariants (physically observable quantities) which are the respective projections of space-time quantities onto the line of time and the three-dimensional spatial section of an observer. We suggest to consider a region of space (space-time), where the metric is fully degenerate. It is shown that this is the ultimate case of the isotropic region (home of light-like massless particles, e.g. photons), where the metric is particularly degenerate so that the space-time interval is zero, while the observable time and three-dimensional intervals are nonzero and equal to each other. Both the space-time interval, the observable time interval, and the observable three-dimensional interval are zero in a fully degenerate region. This means that, from the point of view of a regular observer, any particle of a fully degenerate region travels instantly. Therefore, we refer to such a region and the particles inhabiting it as zero-space and zero-particles. Moving to coordinate quantities inside zero-space shows that the real speed therein is that of light, depending on the gravitational potential and the rotation of space. It is shown that the eikonal equation for zero-particles, expressed through physically observable quantities, is a standing wave equation: zero-particles appear to a regular “external” observer as standing light waves (stopped, or frozen light), while zero-space is filled with a system of standing light waves (light-like holograms). In the internal reference frame of zero-space, momentum does not conserve. This is solely a property of virtual photons of Quantum Electrodynamics. Therefore zero-particles (we can observe them as standing light waves) should play a rôle of virtual photons. Thus the frozen light experiments are an experimental “foreword” to discovery of zero-particles, which are virtual photons.

A thesis of this presentation has been submitted to the *APS March Meeting 2011*, planned on March 21–25, 2011, in Dallas, Texas.

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§1. Frozen light. An introduction. In the summer of 2000, Lene V. Hau, who pioneered light-slowing experiments over many years in the 1990's at Harvard University, first obtained light slowed down to rest state. In her experiment, light was stored, for milliseconds, in ultracold atoms of sodium (with a gaseous cloud of the atoms cooled down to within a millionth of a degree of absolute zero). This state was then referred to as *frozen light* or *stopped light*. An anthology of the primary experiments is given in her publications [1–5]. After the first success of 2000, Lene Hau still continues the study: in 2009, light was stopped for 1.5 second at her laboratory [6].

Then frozen light was approved, during one year, by two other groups of experimentalists. A group headed by Ronald L. Walsworth and Mikhail D. Lukin of the Harvard-Smithsonian Center for Astrophysics stopped light in a room-temperature gas [7]. In experiments conducted by Philip R. Hemmer at the Air Force Research Laboratory in Hanscom (Massachusetts), light was stopped in a cooled-down solid [8].

The best-of-all survey of all experiments on this subject was given in Lene Hau's *Frozen Light*, which was first published in 2001, in *Scientific American* [4]. Then an extended version of this paper was reprinted in 2003, in a special issue of the journal [5].

On the other hand, the frozen light problem was met by our theoretical study of the 1990's, which was produced independently of the experimentalists (we knew nothing about the experiments until January 2001, when the first success in stopping light was widely advertised in the scientific press). Our task was to reveal what kinds of particles

could theoretically inhabit the space (space-time) of the General Theory of Relativity. We have obtained that, aside for mass-bearing and massless (light-like) particles, those of the third kind may also exist. Such particles inhabit a space with a fully degenerate metric, which is the ultimate case of the light-like (particularly degenerate) space. This means that the particles are the ultimate case of photons. It was shown that, from the viewpoint of a regular observer, they should be perceived as standing light waves (or frozen light, in other words).

These theoretical results were presented, among the others, in our book [9], which was first published in 2001 and then reprinted in 2008. However they were very fragmented along the book, where many problems (such as geodesic motion, gravitational collapse, and others) were discussed commonly for all particles. Therefore we have decided to join the results in this single paper, thus giving a complete presentation of our theory of frozen light.

§2. Introducing fully degenerate space (zero-space) as the ultimate case of (particularly degenerate) light-like space.

Once we want to reveal a descriptive picture of any physical theory, we need to express the results through real physical quantities (physical observables), which can be measured in experiments. In the General Theory of Relativity, a complete mathematical apparatus for calculating physically observable quantities was introduced in 1944 by Abraham Zelmanov [10, 11], and is known as the *theory of chronometric invariants*. Its essence consists of projecting four-dimensional quantities onto the line of time and the three-dimensional spatial section of an observer. As a result, we obtain quantities observable in practice.

Expressing the four-dimensional (space-time) interval through physically observable quantities, we can reveal what principal kinds of space (space-time) are conceivable in the General Theory of Relativity. We show here how to do it, and the result we have obtained.

The operator of projection onto the time line of an observer is the world-vector of his four-dimensional velocity

$$b^\alpha = \frac{dx^\alpha}{ds}, \quad \alpha = 0, 1, 2, 3, \quad (2.1)$$

with respect to his reference body (the vector is tangential to the world-trajectory of the observer). The theory assumes the observer to be resting with respect to his references. Thus $b^i = 0$ ($i=1, 2, 3$), while the rest components of b^α are: $b^0 = \frac{1}{\sqrt{g_{00}}}$, $b_0 = g_{0\alpha}b^\alpha = \sqrt{g_{00}}$, $b_i = g_{i\alpha}b^\alpha = \frac{g_{0i}}{\sqrt{g_{00}}}$. The operator of projection onto the three-dimensional spatial section of

the observer is the four-dimensional symmetric tensor

$$h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta, \quad (2.2)$$

while the properties of the operators are: $b_\alpha b^\alpha = 1$, $h_\alpha^i b^\alpha = 0$, $h_i^\alpha h_\alpha^k = \delta_i^k$.

Thus, any world-vector Q^α has two (observable) chr.inv.-projections, while any 2nd-rank world-tensor $Q^{\alpha\beta}$ has three ones, respectively,

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}, \quad h_\alpha^i Q^\alpha = Q^i, \quad (2.3)$$

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}, \quad h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q_0^i}{\sqrt{g_{00}}}, \quad h_\alpha^i h_\beta^k Q^{\alpha\beta} = Q^{ik}. \quad (2.4)$$

For instance, projecting a world-coordinate interval dx^α we obtain the interval of the physically observable time

$$d\tau = \sqrt{g_{00}} dt + \frac{g_{0i}}{c\sqrt{g_{00}}} dx^i, \quad (2.5)$$

and the three-dimensional coordinate interval dx^i . The physically observable velocity is the three-dimensional chr.inv.-vector

$$v^i = \frac{dx^i}{d\tau}, \quad v_i v^i = h_{ik} v^i v^k = v^2, \quad (2.6)$$

which along isotropic (light-like) trajectories becomes the physically observable velocity of light c^i , whose square is $c_i c^i = h_{ik} c^i c^k = c^2$.

The chr.inv.-metric tensor h_{ik} with the components

$$h_{ik} = -g_{ik} + b_i b_k, \quad h^{ik} = -g^{ik}, \quad h_k^i = -g_k^i = \delta_k^i \quad (2.7)$$

is obtained after projecting the fundamental metric tensor $g_{\alpha\beta}$ onto the observer's three-dimensional spatial section. The chr.inv.-operators of differentiation along the line of time and the spatial section

$$\frac{* \partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}, \quad \frac{* \partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0}, \quad (2.8)$$

are non-commutative

$$\frac{* \partial^2}{\partial x^i \partial t} - \frac{* \partial^2}{\partial t \partial x^i} = \frac{1}{c^2} F_i \frac{* \partial}{\partial t}, \quad \frac{* \partial^2}{\partial x^i \partial x^k} - \frac{* \partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{* \partial}{\partial t} \quad (2.9)$$

thus determine the gravitational inertial force F_i acting in the space, and the angular velocity A_{ik} of the space rotation

$$F_i = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \quad (2.10)$$

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} (F_i v_k - F_k v_i), \quad (2.11)$$

where $w = c^2(1 - \sqrt{g_{00}})$ is the gravitational potential, while $v_i = -\frac{cg_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation (its contravariant component $v^i = -cg^{0i}\sqrt{g_{00}}$ is determined through $v_i = h_{ik}v^k$ and $v^2 = h_{ik}v^i v^k$).

We now express the four-dimensional interval ds through physically observable quantities. We express $g_{\alpha\beta}$ from $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$. Thus,

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = b_\alpha b_\beta dx^\alpha dx^\beta - h_{\alpha\beta} dx^\alpha dx^\beta, \quad (2.12)$$

where $b_\alpha dx^\alpha = c d\tau$, so the first term is $b_\alpha b_\beta dx^\alpha dx^\beta = c^2 d\tau^2$. The term $h_{\alpha\beta} dx^\alpha dx^\beta$ is the same as the square of the physically observable three-dimensional interval

$$d\sigma^2 = h_{ik} dx^i dx^k, \quad (2.13)$$

because the theory of chronometric invariants assumes the observer to be resting with respect to his references ($b^i = 0$). Thus the four-dimensional interval being expressed through physical observables has the form

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \quad (2.14)$$

According to this formula, three principal kinds of subspace are possible in the space (space-time) of the General Theory of Relativity.

First. The subspace, where

$$ds^2 = c^2 d\tau^2 - d\sigma^2 \neq 0, \quad c^2 d\tau^2 \neq d\sigma^2 \neq 0, \quad (2.15)$$

is known as the *non-isotropic space*. This is the home of non-isotropic (i.e. nonzero four-dimensional) trajectories and mass-bearing particles, which are both regular subluminal particles and hypothetical superluminal tachyons. Such trajectories lie “within” the light hypercone (the home of subluminal particles), and also “outside” the light hypercone (the home of tachyons).

Second. The subspace, where

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = 0, \quad c^2 d\tau^2 = d\sigma^2 \neq 0, \quad (2.16)$$

is known as the *isotropic space*. This is the home of isotropic (i.e. zero four-dimensional) trajectories. Along such trajectories, the space-time interval is zero, while the interval of the physically observable time and the three-dimensional physically observable interval are nonzero. Isotropic trajectories lie on the surface of the light hypercone, which is the surface of the light speed. Thus the isotropic space hosts particles travelling at the velocity of light. Such particles have zero rest-mass.

They are massless particles, in other words. These are, in particular, photons. For this reason, particles of the isotropic space are also known as massless light-like particles.

These two kinds of space (space-time) are originally well-known commencing in the beginning of the 20th century, once the theory of space-time-matter had been introduced.

We however suggest to consider a third kind of subspace (and particles), which are also theoretically possible in the space (space-time) of the General Theory of Relativity. Consider isotropic (light-like) trajectories in the ultimate case, where, apart from $ds^2=0$, they meet even more stricter conditions $c^2d\tau^2=0$ and $d\sigma^2=0$, i.e.

$$ds^2 = c^2d\tau^2 - d\sigma^2 = 0, \quad c^2d\tau^2 = 0, \quad d\sigma^2 = 0. \quad (2.17)$$

This means that not only the space-time interval is zero along such trajectories ($ds^2=0$ in any isotropic space). In addition to it, the observable interval of time between any events and all observable three-dimensional lengths are zero therein (being registered by a regular subluminal observer). Therefore, the space wherein such trajectories lie is the ultimate case of the isotropic (light-like) space.

So forth, we go insightfully into the details of the conditions, which characterize a space of this exotic kind. Taking into account the formulae of $d\tau$ (2.5) and $d\sigma$ (2.13), and also the fact that $h_{00}=h_{0i}=0$, we express the conditions $c^2d\tau^2=0$ and $d\sigma^2=0$ in the extended form

$$cd\tau = \left[1 - \frac{1}{c^2} (\mathbf{w} + v_i u^i) \right] cdt = 0, \quad dt \neq 0, \quad (2.18)$$

$$d\sigma^2 = h_{ik} dx^i dx^k = 0, \quad (2.19)$$

where $u^i = \frac{dx^i}{dt}$ is the three-dimensional coordinate velocity, which is not a physically observable chr.inv.-quantity.

As is known, the necessary and sufficient condition of full degeneration of a space means zero value of the determinant of the metric tensor, which characterizes the space. For the degenerate three-dimensional physically observable metric $d\sigma^2 = h_{ik} dx^i dx^k = 0$ this condition is

$$h = \det \|h_{ik}\| = 0. \quad (2.20)$$

On the other hand, as was shown by Zelmanov [10], the determinant $g = \det \|g_{\alpha\beta}\|$ of the fundamental (four-dimensional) metric tensor $g_{\alpha\beta}$ is connected to the determinant of the chr.inv.-metric tensor h_{ik} through the relation

$$g = -hg_{00}. \quad (2.21)$$

Hence degeneration of the three-dimensional metric form $d\sigma^2$, which is characterized by the condition $h = 0$, means degeneration of the four-dimensional metric form ds^2 , i.e. the condition $g = 0$, as well. Therefore a four-dimensional space of the third kind we have herein suggested to consider is a *fully degenerate space*. Respectively, the conditions (2.18) and (2.19) which characterize such a space are the *physical conditions of full degeneration*.

Also, we suggest to refer further to any regular isotropic space as a *particularly degenerate space*. This is because the space-time interval is zero therein, $ds^2 = 0$, but $c^2 d\tau^2 \neq 0$ and $d\sigma^2 \neq 0$ thus the fundamental metric tensor is not degenerate: $g = \det \|g_{\alpha\beta}\| \neq 0$. In other words, a regular isotropic space is “particularly degenerate”.

As has been said above, full degeneration requires not only $ds^2 = 0$ but also $c^2 d\tau^2 = 0$ and $d\sigma^2 = 0$. Therefore, we suggest to refer further to any fully degenerate space (space-time) as *zero-space*.

Substituting $h_{ik} = -g_{ik} + b_i b_k = -g_{ik} + \frac{1}{c^2} v_i v_k$ into the second condition (2.19) of those two characterizing a fully degenerate space, then dividing it by dt^2 , we obtain the physical conditions of full degeneration, (2.18) and (2.19), in the final form

$$w + v_i u^i = c^2, \quad g_{ik} u^i u^k = c^2 \left(1 - \frac{w}{c^2}\right)^2, \quad (2.22)$$

where $v_i u^i$ is the scalar product of the linear velocity of the space rotation v_i and the coordinate velocity u^i in the space.

On the basis of the conditions of full degeneration, three subkinds of fully degenerate space (zero-space) are conceivable:

- 1) If such a space is free of gravitational fields ($w = 0$), the first condition of the conditions of full degeneration (2.22) means $v_i u^i = c^2$, while the second condition of (2.22) becomes $g_{ik} u^i u^k = c^2$. In this particular case, the fully degenerate space rotates with the velocity of light, and all speeds of motion therein are that of light;
- 2) Once a gravitational field appears in such a space, the space rotation and speeds of motion become slower than light therein according to the conditions of full degeneration (2.22). This is a general case of fully degenerate space;
- 3) If a fully degenerate space does not rotate ($v_i = 0$), the gravitational potential is $w = c^2$ therein. This means, according to the definition $w = c^2(1 - \sqrt{g_{00}})$ of the potential, that $g_{00} = 0$ which is the condition of gravitational collapse. Also, according to the second condition of full degeneration (2.22), the equality $w = c^2$ means $g_{ik} dx^i dx^k = 0$. This state, $g_{ik} dx^i dx^k = 0$, may realize itself

in three cases: a) the three-dimensional coordinate metric g_{ik} degenerates ($\det \|g_{ik}\| = 0$); b) all trajectories within the space are shrunk into a point ($dx^i = 0$); c) when both these conditions are commonly present in the space. A fully degenerate space of this subkind is collapsed: this is a fully degenerate black hole, in other words. This particular case will be detailed in §4.

About the zero-space metric. As has been said above, all intervals (space-time, time, and spatial ones) are zero in a fully degenerate space from the point of view of an “external” observer located in a regular (non-degenerate) space. The space-time (four-dimensional) interval is invariant, thus its equality to zero remains unchanged in any reference frame. However this is not true about non-invariant quantities, which are the interval of the coordinate time dt and the three-dimensional coordinate interval $g_{ik} dx^i dx^k$. As follows from the conditions of full degeneration (2.22), the coordinate quantities can be nonzero in such a space (except in the case of gravitational collapse, where $g_{ik} dx^i dx^k = 0$). So, we can move from the quantities registered by a regular observer to the coordinate quantities within a fully degenerate space, thus satisfying our curiosity to see what happens therein.

The interval $d\mu^2$ inside a fully degenerate space (i.e. the *zero-space metric*) can be obtained from the second condition of full degeneration (2.22), due to the fact that the three-dimensional coordinate metric g_{ik} does not degenerate. Thus, the zero-space metric has the form

$$d\mu^2 = g_{ik} dx^i dx^k = \left(1 - \frac{w}{c^2}\right)^2 c^2 dt^2 \neq 0, \quad (2.23)$$

which, due to the first condition of full degeneration is $w + v_i u^i = c^2$, can be equally expressed as

$$d\mu^2 = g_{ik} dx^i dx^k = \frac{v_i v_k u^i u^k}{c^2} dt^2 \neq 0. \quad (2.24)$$

The zero-space metric manifests that, everywhere in such a space, the following condition

$$g_{ik} \overset{*}{u}^i \overset{*}{u}^k = c^2, \quad (2.25)$$

is true. Here $\overset{*}{u}^i = \frac{1}{\sqrt{g_{00}}} \frac{dx^i}{dt} = \frac{dx^i}{dt}$ is the physical coordinate velocity we introduce through the “starry” derivative with respect to time in analogy to the respective “starry” chr.inv.-derivative (2.8).

According to (2.25), the physical velocities inside a fully degenerate space are always equal to the velocity of light.

The zero-space metric $d\mu^2$ (2.23) is not invariant: $d\mu^2 \neq inv$. This means that the geometry inside a fully degenerate space region is non-Riemannian*. As a result, from the viewpoint of a hypothetical observer located in such a space, the length of the four-dimensional velocity vector does not conserve along its trajectory therein

$$u_\alpha u^\alpha = u_k u^k = g_{ik} u^i u^k = \left(1 - \frac{w}{c^2}\right)^2 c^2 \neq const \quad (2.26)$$

but depends on the distribution of the gravitational potential. This fact, in common with the circumstance that the physical velocities therein are equal to the velocity of light, will lead us in §7 to the conclusion that particles, whose home is zero-space, can be associated with virtual photons known due to Quantum Electrodynamics.

§3. The geometric structure of zero-space. So, a regular observer perceives the entire fully degenerate space (zero-space) as a point-like region determined by the observable conditions of full degeneration, which are $d\tau = 0$ and $d\sigma^2 = h_{ik} dx^i dx^k = 0$. These conditions mean that he perceives any two events in the zero-space as simultaneous, and also all three-dimensional lengths therein are perceived as zero. Such an observation can be processed at any point of our regular non-degenerate (four-dimensional pseudo-Riemannian) space. This is only possible, if we assume that our space meets the entire zero-space at each point, as it is “stuffed” with the zero-space.

Let us now turn to the geometric interpretation of the conditions of full degeneration. To obtain an illustrated view of the zero-space geometry, we are going to use a *locally geodesic frame of reference*. The fundamental metric tensor within the infinitesimal vicinity of a point in such a frame is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2} \left(\frac{\partial^2 \tilde{g}_{\alpha\beta}}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} \right) (\tilde{x}^\mu - x^\mu) (\tilde{x}^\nu - x^\nu) + \dots, \quad (3.1)$$

i.e. the numerical values of its components in the vicinity of a point differ from those at this point itself only in the 2nd-order terms or the higher other terms, which can be neglected. Therefore, at any point in a local geodesic frame of reference, the fundamental metric tensor $g_{\alpha\beta}$ is constant (within the 2nd order terms withheld), while the first derivatives of the metric are zero.

*As is known, Riemannian spaces are, by definition, those where: a) the space metric has the square Riemannian form $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$, and b) the metric is invariant $ds^2 = inv$.

It is obvious that a local geodesic frame of reference can be set up within the infinitesimal vicinity of any point in a Riemannian space. As a result, at any point belonging to the local geodesic frame of reference, a flat space can be set up tangential to the Riemannian space so that the local geodesic frame in the Riemannian space is a global geodesic frame in the tangential flat space. Because the fundamental metric tensor is constant in a flat space, the quantities $\tilde{g}_{\alpha\beta}$ converge to those of the tensor $g_{\alpha\beta}$ in the tangential flat space, in the vicinity of any point in the Riemannian space. This means that, in the tangential flat space, we can set up a system of basis vectors $\vec{e}_{(\alpha)}$ tangential to the curved coordinate lines of the Riemannian space. Because the coordinate lines of a Riemannian space are curved (in a general case), and, in the case where the space is non-holonomic*, are not even orthogonal to each other, the lengths of the basis vectors are sometimes substantially different from the unit length.

Consider the world-vector $d\vec{r}$ of an infinitesimal displacement, i.e. $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$. Then $d\vec{r} = \vec{e}_{(\alpha)} dx^\alpha$, where the components are

$$\left. \begin{aligned} \vec{e}_{(0)} &= (e_{(0)}^0, 0, 0, 0), & \vec{e}_{(1)} &= (0, e_{(1)}^1, 0, 0) \\ \vec{e}_{(2)} &= (0, 0, e_{(2)}^2, 0), & \vec{e}_{(3)} &= (0, 0, 0, e_{(3)}^3) \end{aligned} \right\}. \quad (3.2)$$

The scalar product of the vector $d\vec{r}$ with itself gives $d\vec{r}d\vec{r} = ds^2$. On the other hand, it is $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. So, we obtain a formula

$$g_{\alpha\beta} = \vec{e}_{(\alpha)} \vec{e}_{(\beta)} = e_{(\alpha)} e_{(\beta)} \cos(x^\alpha; x^\beta), \quad (3.3)$$

which facilitates our better understanding of the geometric structure of different regions within the space. According to (3.3), therefore,

$$g_{00} = e_{(0)}^2, \quad (3.4)$$

where, as is known, g_{00} is included into the formula of the gravitational potential $w = c^2(1 - \sqrt{g_{00}})$. Hence the time basis vector $\vec{e}_{(0)}$ (tangential to the line of time $x^0 = ct$) has the length $e_{(0)} = 1 - \frac{w}{c^2}$. Thus the lesser the length of $\vec{e}_{(0)}$ is (than 1), the greater the gravitational potential w . In the case of gravitational collapse ($w = c^2$), the length of the time basis vector $\vec{e}_{(0)}$ becomes zero.

Next, according to (3.3), the quantity g_{0i} is

$$g_{0i} = e_{(0)} e_{(i)} \cos(x^0; x^i), \quad (3.5)$$

*The non-holonomy of a space (space-time) means that the lines of time are non-orthogonal to the three-dimensional spatial section therein. It manifests as the three-dimensional rotation of the space.

while, on the other hand, $g_{0i} = -\frac{1}{c} v_i \left(1 - \frac{w}{c^2}\right) = -\frac{1}{c} v_i e_{(0)}$. Hence, the linear velocity of the space rotation, determined as $v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}$, is

$$v_i = -c e_{(i)} \cos(x^0; x^i), \quad (3.6)$$

and manifests the angle of inclination of the lines of time towards the spatial section. Then, according to the general formula (3.3), we have

$$g_{ik} = e_{(i)} e_{(k)} \cos(x^i; x^k), \quad (3.7)$$

hence the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ has the form

$$h_{ik} = e_{(i)} e_{(k)} \left[\cos(x^0; x^i) \cos(x^0; x^k) - \cos(x^i; x^k) \right]. \quad (3.8)$$

From formula (3.6), we see that, from the geometrical viewpoint, v_i is the projection (scalar product) of the spatial basis vector $\vec{e}_{(i)}$ onto the time basis vector $\vec{e}_{(0)}$, multiplied by the velocity of light. If the spatial sections are everywhere orthogonal to the lines of time (giving holonomic space), $\cos(x^0; x^i) = 0$ and $v_i = 0$. In a non-holonomic space, the spatial sections are not orthogonal to the lines of time, so $\cos(x^0; x^i) \neq 0$. Generally $|\cos(x^0; x^i)| \leq 1$, hence the linear velocity of the space rotation v_i can not exceed the velocity of light.

First, consider the geometric structure of the isotropic (light-like) space. It is characterized by the condition $c^2 d\tau^2 = d\sigma^2 \neq 0$. According to this condition, time and regular three-dimensional space meet each other. Geometrically, this means that the time basis vector $\vec{e}_{(0)}$ meets all three spatial basis vectors $\vec{e}_{(i)}$, i.e. time “falls” into space (this fact does not mean that the spatial basis vectors coincide, because the time basis vector is the same for the entire spatial frame). In other words, $\cos(x^0; x^k) = \pm 1$ everywhere in the isotropic space. At $\cos(x^0; x^i) = +1$ the time basis vector is co-directed with the spatial ones: $\vec{e}_{(0)} \uparrow \vec{e}_{(i)}$. If $\cos(x^0; x^i) = -1$, the time and spatial basis vectors are oppositely directed: $\vec{e}_{(0)} \updownarrow \vec{e}_{(i)}$. The condition $\cos(x^0; x^k) = \pm 1$ can be expressed through the gravitational potential $w = c^2 (1 - \sqrt{g_{00}})$, because, in a general case, $e_{(0)} = \sqrt{g_{00}}$ (3.4). Finally, we obtain the geometric conditions which characterize the isotropic space. They are

$$\cos(x^0; x^k) = \pm 1, \quad e_{(i)} = e_{(0)} = \sqrt{g_{00}} = 1 - \frac{w}{c^2}, \quad (3.9)$$

and, hence,

$$v_i = \mp c e_{(i)} = \mp \sqrt{g_{00}} c_i = \mp \left(1 - \frac{w}{c^2}\right) c_i, \quad (3.10)$$

$$h_{ik} = \left(1 - \frac{w}{c^2}\right)^2 [1 - \cos(x^i; x^k)], \quad (3.11)$$

where c^i is the chronometrically invariant three-dimensional vector of the physically observable velocity of light, $c_i c^i = h_{ik} c^i c^k = c^2$.

According to the obtained formula (3.10), we conclude, as well as it was primarily concluded by one of us in a previous study [12]:

The isotropic space rotates at each point with a linear velocity, which is basically equal, to the velocity of light, and is slowing down in the presence of the gravitational potential.

The isotropic space exists at any point in the four-dimensional regular space as a light hypercone — a hypersurface whose metric is

$$g_{\alpha\beta} dx^\alpha dx^\beta = 0, \quad (3.12)$$

or, in the extended form,

$$\left(1 - \frac{w}{c^2}\right)^2 c^2 dt^2 - 2\left(1 - \frac{w}{c^2}\right) v_i dx^i dt + g_{ik} dx^i dx^k = 0, \quad (3.13)$$

according to the formulae of the gravitational potential $w = c^2(1 - \sqrt{g_{00}})$ and the linear velocity of the space rotation $v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}$.

This is a subspace of the four-dimensional space which hosts massless (light-like) particles travelling at the velocity of light. Because the space-time interval in such a region is zero, all four-dimensional directions inside it are equal (in other words, they are isotropic). Therefore this subspace is commonly referred to as the *isotropic hypercone*.

Let us now turn to the geometric structure of the zero-space. Because w and v_i , being written in the basis form, are $w = c^2(1 - e_{(0)})$ and $v_i = -c e_{(i)} \cos(x^0; x^i)$, the condition of full degeneration $w + v_i u^i = c^2$ can be written in the basis form as well

$$c e_{(0)} = -e_{(i)} u^i \cos(x^0; x^i). \quad (3.14)$$

This formula can be regarded as the *geometric condition of full degeneration*.

Because the four-dimensional metric is also equal to zero in the zero-space, such a space exists at any point of the isotropic (light) hypercone as a fully degenerate subspace of it. Such a *fully degenerate isotropic hypercone* is described by a somewhat different equation

$$\left(1 - \frac{w}{c^2}\right)^2 c^2 dt^2 - g_{ik} dx^i dx^k = 0, \quad (3.15)$$

or, due to the zero-space metric, which can equally be presented as (2.23) and (2.24), by the equation

$$\frac{v_i v_k u^i u^k}{c^2} dt^2 - g_{ik} dx^i dx^k = 0. \quad (3.16)$$

The difference between such a fully degenerate isotropic hypercone and the regular isotropic (light) hypercone is that the first satisfies the condition of full degeneration $w + v_i u^i = c^2$. Because v_i is expressed in both cases in the same form (3.10), we arrive at the conclusion:

The fully degenerate isotropic hypercone is a cone of light-speed rotation as well as the regular isotropic hypercone. In other words, the zero-space rotates at its each point with a linear velocity equal to the velocity of light. Its rotation becomes slower than light in the presence of the gravitational potential.

Finally, we conclude that the regular isotropic (light) hypercone contains the degenerate isotropic hypercone, which is the entire zero-space, as a subspace embedded into it at its each point. This is a clear illustration of the fractal structure of the world presented here as a system of the isotropic cones found inside each other.

§4. Gravitational collapse in a zero-space region. Fully degenerate black holes. As is known, *gravitational collapsar* or *black hole* is a local region of space (space-time), wherein the condition $g_{00} = 0$ is true. Because the gravitational potential is defined as $w = c^2 (1 - \sqrt{g_{00}})$, the gravitational collapse condition $g_{00} = 0$ means that the gravitational potential is $w = c^2$ in the region. We are going to consider how this condition can be realized in zero-space.

The first condition of full degeneration (2.22) is $w + v_i u^i = c^2$. According to the condition, if $v_i u^i = 0$ in a local zero-space region, the gravitational potential is $w = c^2$ therein. This means that, in the case of $v_i u^i = 0$, the gravitational potential is strong enough to bring the local region of zero-space to gravitational collapse. We suggest to refer to such a region as a *fully degenerate gravitational collapsar* or, equivalently, as a *fully degenerate black hole*.

The second condition of full degeneration becomes $g_{ik} dx^i dx^k = 0$ in this case. Together with the previous, this means that three physical and geometric conditions are realized in fully degenerate black holes

$$w = c^2, \quad v_i u^i = 0, \quad g_{ik} dx^i dx^k = 0, \quad (4.1)$$

whose physical meaning is as follows:

- 1) The gravitational potential inside fully degenerate black holes is strong enough to stop the regular light-speed rotation of the local region of zero-space, i.e.

$$v_i = \mp c e_{(i)} = \mp \sqrt{g_{00}} c_i = \mp \left(1 - \frac{w}{c^2}\right) c_i = 0; \quad (4.2)$$

- 2) In this case, the time basis vector $\vec{e}_{(0)}$ has zero length (intervals of time are zero inside fully degenerate black holes)

$$e_{(0)} = \sqrt{g_{00}} = 1 - \frac{w}{c^2} = 0; \quad (4.3)$$

- 3) In any case of zero-space, the condition $\cos(x^0; x^k) = \pm 1$ is true: the time basis vector $\vec{e}_{(0)}$ meets all three spatial basis vectors $\vec{e}_{(i)}$ (time “falls” into space). Therefore, the previous condition $e_{(0)} = 0$ means that all three three-dimensional (spatial) basis vectors $\vec{e}_{(i)}$ have zero length inside fully degenerate black holes as well, i.e.

$$e_{(i)} = e_{(0)} = \sqrt{g_{00}} = 1 - \frac{w}{c^2} = 0; \quad (4.4)$$

- 4) The condition $e_{(i)} = 0$ means that the entire three-dimensional space inside fully degenerate black holes is shrunk into a point (all three-dimensional coordinate intervals are $dx^i = 0$). Hence, the third condition $g_{ik} dx^i dx^k = 0$ of the conditions inside fully degenerate black holes (4.1) is due to $dx^i = 0$, while the three-dimensional coordinate metric is not degenerate therein

$$\det \|g_{ik}\| \neq 0. \quad (4.5)$$

Hence fully degenerate black holes are point-like objects, which keep light stored inside themselves due to their own ultimately strong gravitation. In other words, they are “absolute black holes” of all gravitational collapsars theoretically conceivable due to the General Theory of Relativity.

§5. Zero-space: the gate for teleporting photons. As we mentioned above, a regular observer may connect to the entire fully degenerate space (zero-space) at any point or local region of the regular space once the observable conditions of full degeneration, which are $d\tau = 0$ and $d\sigma^2 = h_{ik} dx^i dx^k = 0$, are realized therein. The physical meaning of the first condition $d\tau = 0$ is that the regular observer perceives any two events in the zero-space region as simultaneous, at whatever distance from them they are located. We will further refer to such a way of instantaneous transfer of information as the *long-range action*. A process in which a particle (a mediator of the interaction) may realize the long-range action will be referred to as *teleportation*.

Therefore, the first condition of full degeneration $d\tau = 0$, which can also be extended due to the definition of $d\tau$ (2.5) as

$$d\tau = \left(1 - \frac{w}{c^2}\right) dt - \frac{1}{c^2} v_i dx^i = 0, \quad (5.1)$$

thus expressed in the form $w + v_i u^i = c^2$ (2.22), has also the physical meaning of the *teleportation condition*.

Mediators of the long-range action are particles, which are a sort of photons. This is because, as was detailed in page 8, the physical conditions inside a zero-space region are the ultimate case of the conditions of the regular isotropic (light-like) space, which is the home of photons. In other words, the long-range action is transferred by special “fully degenerate photons”, which exist under the physical conditions of full degeneration. Such particles, what they are and how they seem from the point of view of a regular observer, will be discussed in §6–§8.

Once a photon has entered into a local zero-space region at one location of our regular space, it can be instantly connected to another photon which has simultaneously entered into another zero-space “gate” at another distant location. From the point of view of a regular “external” observer, such a connexion is realized instantly. However, inside the zero-space itself, fully degenerate photons transfer interaction between these two locations with the velocity of light (see comments to formula 2.25 in page 10, for details).

Thus, we conclude that instant transfer of information is naturally permitted in the framework of the General Theory of Relativity, despite the real speeds of particles not exceeding the velocity of light. This is merely a “space-time trick”, which may only be due to the space-time geometry and topology: we only see that the information is transferred instantaneously, while it is transferred by not-faster-than-light particles travelling in another space which seems to us, the “external” observers, as that wherein all intervals of time and all three-dimensional spatial intervals are zero.

Until this day, teleportation has had an explanation given only by Quantum Mechanics [13]. It was previously achieved only in the strict quantum fashion — quantum teleportation of photons in 1998 [14] and of atoms in 2004 [15, 16]. Now the situation changes: with our theory we can find physical conditions for teleportation of photons in a non-quantum way, which is not due to the probabilistic laws of Quantum Mechanics but according to the exact (non-quantum) laws of the General Theory of Relativity following the space-time geometry. We therefore suggest to refer to this fashion as *non-quantum teleportation*.

§6. Zero-particles: particles which inhabit zero-space. As is obvious, the fully degenerate space can only host such particles for which the physical conditions of full degeneration are true. The properties of such particles will now be under focus. We will start this consideration

from the regular (non-degenerate) particles, then apply the physical conditions of full degeneration, thus determining the characteristics of the particles hosted by the fully degenerate space (zero-space).

According to the General Theory of Relativity [17], any mass-bearing particle is characterized by the four-dimensional vector of momentum

$$P^\alpha = m_0 \frac{dx^\alpha}{ds}, \quad (6.1)$$

where m_0 is the rest-mass characterizing the particle. In the framework of de Broglie's wave-particle duality, we can represent the same mass-bearing particle as a wave characterized by the four-dimensional wave vector

$$K^\alpha = \frac{\omega_0}{c} \frac{dx^\alpha}{ds}, \quad (6.2)$$

while ω_0 is the rest-frequency of the de Broglie wave. The square of the momentum vector P^α and the wave vector K^α along the trajectory of each single mass-bearing particle is constant, which is nonzero

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\alpha P^\beta = m_0^2 = \text{const} \neq 0, \quad (6.3)$$

$$K_\alpha K^\alpha = g_{\alpha\beta} K^\alpha K^\beta = \frac{\omega_0^2}{c^2} = \text{const} \neq 0, \quad (6.4)$$

i.e. P^α and K^α are non-isotropic vectors in this case.

As is seen, the space-time interval ds is applied as the derivation parameter for mass-bearing particles. It works, because such particles travel along non-isotropic trajectories, where, as is known, $ds \neq 0$. Massless (light-like) particles inhabit the isotropic space. They travel along isotropic trajectories, where $ds^2 = c^2 d\tau^2 - d\sigma^2 = 0$ and $c^2 d\tau^2 = d\sigma^2 \neq 0$. The space-time interval is $ds = 0$ therein, and thus cannot be applied as the derivation parameter. Zelmanov [10] had removed this problem by suggesting the observable three-dimensional interval, which is $d\sigma \neq 0$ along isotropic trajectories. Moreover, $d\sigma$ and $d\tau$ are chronometric invariants: they are invariant along the three-dimensional spatial section of the observer. Therefore they can be used as derivation parameters along both isotropic and non-isotropic trajectories, in the framework of the chronometrically invariant formalism.

Since ds^2 in the chr.inv.-form (2.14) can be expressed through the physically observable chr.inv.-velocity v^i (2.6) as

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = c^2 d\tau^2 \left(1 - \frac{v^2}{c^2} \right), \quad (6.5)$$

we can write down the regular formulae of P^α (6.1) and K^α (6.2) as

$$P^\alpha = m \frac{dx^\alpha}{d\sigma} = \frac{m}{c} \frac{dx^\alpha}{d\tau}, \quad (6.6)$$

$$K^\alpha = \frac{\omega}{c} \frac{dx^\alpha}{d\sigma} = \frac{k}{c} \frac{dx^\alpha}{d\tau}, \quad (6.7)$$

where m is the relativistic mass (derived for massless particles through their relativistic energy $E = mc^2$), ω is the relativistic frequency, while $k = \frac{\omega}{c}$ is the wave number.

In the case of massless particles (isotropic trajectories), the square of the momentum vector P^α and the wave vector K^α is zero

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\alpha P^\beta = \frac{m^2}{c^2} \frac{g_{\alpha\beta} dx^\alpha dx^\beta}{d\sigma^2} = \frac{m^2}{c^2} \frac{ds^2}{d\sigma^2} = 0, \quad (6.8)$$

$$K_\alpha K^\alpha = g_{\alpha\beta} K^\alpha K^\beta = \frac{\omega^2}{c^2} \frac{g_{\alpha\beta} dx^\alpha dx^\beta}{d\sigma^2} = \frac{\omega^2}{c^2} \frac{ds^2}{d\sigma^2} = 0, \quad (6.9)$$

i.e. P^α and K^α are isotropic vectors in this case.

Calculation of the contravariant components of P^α and K^α gives

$$P^0 = m \frac{dt}{d\tau}, \quad P^i = \frac{m}{c} \frac{dx^i}{d\tau} = \frac{1}{c} m v^i, \quad (6.10)$$

$$K^0 = k \frac{dt}{d\tau}, \quad K^i = \frac{k}{c} \frac{dx^i}{d\tau} = \frac{1}{c} k v^i, \quad (6.11)$$

where $m v^i$ is the three-dimensional chr.inv.-momentum vector, while $k v^i$ is the three-dimensional chr.inv.-wave vector.

The function $\frac{dt}{d\tau}$ can be obtained from the equation of the square of the four-dimensional velocity, which is $g_{\alpha\beta} u^\alpha u^\beta = +1$ for subluminal velocities, $g_{\alpha\beta} u^\alpha u^\beta = 0$ for the velocity of light, and $g_{\alpha\beta} u^\alpha u^\beta = -1$ for superluminal velocities. Extending $g_{\alpha\beta} u^\alpha u^\beta$ to component notation, then substituting the definitions of h_{ik} , v_i , v^i into each of these three formulae, we arrive at the same quadratic equation

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{2v_i v^i}{c^2 \left(1 - \frac{w}{c^2}\right)} \frac{dt}{d\tau} + \frac{1}{\left(1 - \frac{w}{c^2}\right)^2} \left(\frac{1}{c^4} v_i v_k v^i v^k - 1\right) = 0, \quad (6.12)$$

which solves (to within positive roots) as

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{w}{c^2}} \left(\frac{1}{c^2} v_i v^i + 1\right). \quad (6.13)$$

With this solution, we obtain the covariant components P_i and K_i , then — the chr.inv.-projections of P^α and K^α onto the line of time

$$P_i = -\frac{m}{c}(v_i + v_i), \quad \frac{P_0}{\sqrt{g_{00}}} = m, \quad (6.14)$$

$$K_i = -\frac{k}{c}(v_i + v_i), \quad \frac{K_0}{\sqrt{g_{00}}} = k. \quad (6.15)$$

According to the chronometrically invariant formalism (see formula (2.3) for detail), any world-vector Q^α has two physically observable projections: $\frac{Q_0}{\sqrt{g_{00}}}$ and Q^i . Hence, the physical observables are

- 1) the relativistic mass m ,
- 2) the three-dimensional momentum mv^i ,

which are represented, in the framework of de Broglie's wave-particle duality, respectively by

- 1) the wave number $k = \frac{\omega}{c}$,
- 2) the three-dimensional wave vector kv^i .

In the case of massless particles (isotropic trajectories), v^i is equal to the physically observable chr.inv.-velocity of light c^i .

Now, we apply the physical conditions of full degeneration to the obtained formulae, thus considering the particles hosted by the fully degenerate space.

Using the definition of $d\tau$ (2.5), we obtain the relation between the coordinate velocity u^i and the physical observable velocity v^i

$$v^i = \frac{u^i}{1 - \frac{1}{c^2}(w + v_k u^k)}, \quad (6.16)$$

which takes the first condition of full degeneration $w + v_i u^i = c^2$ (2.22) into account. Thus, we express ds^2 in the form

$$ds^2 = c^2 d\tau^2 \left(1 - \frac{v^2}{c^2}\right) = c^2 dt^2 \left\{ \left[1 - \frac{1}{c^2}(w + v_k u^k)\right]^2 - \frac{u^2}{c^2} \right\}, \quad (6.17)$$

containing the first condition of full degeneration as well. Hence, the four-dimensional vector of momentum can be expressed in the form

$$P^\alpha = m_0 \frac{dx^\alpha}{ds} = \frac{M}{c} \frac{dx^\alpha}{dt}, \quad (6.18)$$

$$M = \frac{m_0}{\sqrt{\left[1 - \frac{1}{c^2}(w + v_k u^k)\right]^2 - \frac{u^2}{c^2}}}. \quad (6.19)$$

Such a mass M depends not only on the three-dimensional velocity of the particle with respect to the observer, but also on the gravitational potential w , and on the linear velocity of the rotation v_i of space at the point of observation.

Substituting, into the formula of M , the quantity $v^2 = h_{ik} v^i v^k$ derived from (6.16), and m_0 expressed through m , we arrive at the relation between the relativistic mass m and the mass M

$$M = \frac{m}{1 - \frac{1}{c^2} (w + v_i u^i)}. \quad (6.20)$$

From the obtained formula we see that M , under the first condition of full degeneration $w + v_i u^i = c^2$, becomes a ratio between two quantities, each one is equal to zero, but the ratio itself is not zero: $M \neq 0$. This fact is not a surprise. The same is true for the relativistic mass m in the case of $v = c$, which is the case of massless (light-like) particles. Once there $m_0 = 0$ in the numerator, and the relativistic square-root term is zero in the denominator (due to $v = c$), the ratio of these quantities is still $m \neq 0$.

In analogy to the momentum vector P^α , we can represent the wave vector K^α is the form

$$K^\alpha = \frac{\omega_0}{c} \frac{dx^\alpha}{ds} = \frac{\Omega}{c^2} \frac{dx^\alpha}{dt}, \quad (6.21)$$

$$\Omega = \frac{\omega_0}{\sqrt{\left[1 - \frac{1}{c^2} (w + v_k u^k)\right]^2 - \frac{u^2}{c^2}}} = \frac{\omega}{1 - \frac{1}{c^2} (w + v_i u^i)}, \quad (6.22)$$

which also takes the first condition of full degeneration into account.

It is easy to obtain that the components of the momentum vector in the fully degenerate space (zero-space) are

$$P^0 = M \neq 0, \quad P^i = \frac{1}{c} M u^i \neq 0, \quad P_i = -\frac{1}{c} M u_i \neq 0, \quad (6.23)$$

$$\frac{P_0}{\sqrt{g_{00}}} = M \left[1 - \frac{1}{c^2} (w + v_i u^i) \right] = m = 0, \quad (6.24)$$

while the components of the wave vector are

$$K^0 = \frac{\Omega}{c} \neq 0, \quad K^i = \frac{1}{c^2} \Omega u^i \neq 0, \quad K_i = -\frac{1}{c^2} \Omega u_i \neq 0, \quad (6.25)$$

$$\frac{K_0}{\sqrt{g_{00}}} = \frac{\Omega}{c} \left[1 - \frac{1}{c^2} (w + v_i u^i) \right] = \frac{\omega}{c} = 0. \quad (6.26)$$

As is seen, the physically observable quantities $\frac{P_0}{\sqrt{g_{00}}}$ (6.24) and $\frac{K_0}{\sqrt{g_{00}}}$ (6.26), which are the projections of the world-vectors P^α and K^α onto the line of time, become zero under the first condition of full degeneration $w + v_i u^i = c^2$. This is because, despite the quantities M and Ω being nonzero, their multiplier in the brackets becomes zero under the condition. This means, according to the obtained formulae (6.24) and (6.26), that the relativistic mass m and the relativistic frequency ω (which corresponds to the relativistic mass within de Broglie's wave-particle duality) are zero in the fully degenerate space.

As a result, we can conclude something about the physically observable characteristics of the particles hosted by the fully degenerate space (zero-space):

- 1) Such fully degenerate particles bear zero relativistic mass ($m = 0$) and zero relativistic de Broglie frequency ($\omega = 0$);
- 2) They also bear zero rest-mass ($m_0 = 0$). This follows from the fact that the physical conditions inside a zero-space region are the ultimate case of the conditions of the regular isotropic (light-like) space, which is the home of photons (see page 8 for detail).

Therefore, the particles hosted by the fully degenerate space (zero-space) are the ultimate case of photons, which exist under the conditions of full degeneration. They are “fully degenerate photons”, in other words. Since not only their rest-mass m_0 , but also the relativistic mass m and frequency ω are zero, we suggest to refer further to such fully degenerate particles as *zero-particles*.

§7. Insight into zero-space: zero-particles as virtual photons.

As is well-known, the Feynman diagrams are a graphical description of interactions between elementary particles. The diagrams show that the actual carriers of the interactions are virtual particles. In other words, almost all physical processes rely on the emission and the absorption of virtual particles (e.g. virtual photons) by real particles of our world.

Hence, to give a geometric interpretation of the Feynman diagrams in the space-time of the General Theory of Relativity, we only need a formal definition for virtual particles. Here is how to do it.

According to Quantum Electrodynamics, virtual particles are those for which, contrary to regular ones, the regular relation between energy and momentum

$$E^2 - c^2 p^2 = E_0^2, \quad (7.1)$$

where $E = mc^2$, $p^2 = m^2 v^2$, $E_0 = m_0 c^2$, is not true. In other words, for

virtual particles,

$$E^2 - c^2 p^2 \neq E_0^2. \quad (7.2)$$

In a pseudo-Riemannian space, the regular relation (7.1) is true. It follows from the condition $P_\alpha P^\alpha = m_0^2 = \text{const} \neq 0$ for mass-bearing particles (non-isotropic trajectories), and from the condition $P_\alpha P^\alpha = 0$ for massless particles (isotropic trajectories). Substituting the respective components of the momentum vector P^α , we obtain the regular relation, in the chr.inv.-form, for mass-bearing particles,

$$E^2 - c^2 m^2 v_i v^i = E_0^2, \quad (7.3)$$

and that for massless ones, $E^2 - c^2 m^2 v_i v^i = 0$, that is the same as

$$h_{ik} v^i v^k = c^2. \quad (7.4)$$

But this is not true in the fully degenerate space (zero-space). This is because the zero-space metric $d\mu^2$ (2.23) is not invariant: $d\mu^2 \neq \text{inv}$. As a result, from the viewpoint of a hypothetical observer who is located therein, a degenerate four-velocity vector being transferred in parallel to itself does not conserve its length: $u_\alpha u^\alpha \neq \text{const}$ (2.26). Therefore, the regular relation between energy and momentum $E^2 - c^2 p^2 = \text{const}$ (7.1) is not applicable to zero-particles, but another relation, which is a sort of $E^2 - c^2 p^2 \neq \text{const}$ (7.2), is true. Because the latter is the main property of virtual particles, we arrive at the conclusion:

Zero-particles may play a rôle of virtual particles, which, according to Quantum Electrodynamics, are material carriers of interaction between regular particles of our world. If so, the entire zero-space is an “exchange buffer” in whose capacity zero-particles transfer interactions between regular mass-bearing and massless particles of our world.

As has been shown on page 22, zero-particles are fully degenerate photons. They can also exist in a collapsed region of zero-space, wherein the condition of gravitational collapse is true (see §4). Hence, virtual particles of two kinds can be presupposed:

- 1) Virtual photons — regular fully degenerate photons;
- 2) Virtual collapsars — fully degenerate photons, which are hosted by the collapsed regions of zero-space.

All that we have suggested here is for yet the sole explanation of virtual particles and virtual interactions given by the geometric methods of the General Theory of Relativity, and according to the geometric structure of the four-dimensional space (space-time).

§8. Zero-particles from the point of view of a regular observer: standing light waves. The following important question rises: if zero-particles bear zero rest-mass and zero relativistic mass, how can they be perceived by a regular observer like us who are located in the regular (non-degenerate) space? To answer this question, we now consider zero-particles in framework of the geometric optics approximation.

As is known [17], the four-dimensional wave vector of massless particles in the geometric optics approximation is

$$K_\alpha = \frac{\partial\psi}{\partial x^\alpha}, \quad (8.1)$$

where ψ is the wave phase (eikonal). In analogy to K_α , we suggest to introduce the four-dimensional vector of momentum

$$P_\alpha = \frac{\hbar}{c} \frac{\partial\psi}{\partial x^\alpha}, \quad (8.2)$$

where \hbar is Planck's constant, while the coefficient $\frac{\hbar}{c}$ equates the dimensions of both parts of the equation. We obtain the physically observable projections of these world-vectors onto the line of time

$$\frac{K_0}{\sqrt{g_{00}}} = \frac{1}{c} \frac{{}^*\partial\psi}{\partial t}, \quad \frac{P_0}{\sqrt{g_{00}}} = \frac{\hbar}{c^2} \frac{{}^*\partial\psi}{\partial t}. \quad (8.3)$$

Equating these to the respective formulae obtained in §6, we obtain that the relativistic frequency and mass are formulated, in the framework of the geometric optics approximation, as

$$\omega = \frac{{}^*\partial\psi}{\partial t}, \quad m = \frac{\hbar}{c^2} \frac{{}^*\partial\psi}{\partial t}, \quad (8.4)$$

and, respectively, the generalized frequency and mass are

$$\Omega = \frac{1}{1 - \frac{1}{c^2} (\mathbf{w} + v_i u^i)} \frac{{}^*\partial\psi}{\partial t}, \quad M = \frac{\hbar}{c^2 [1 - \frac{1}{c^2} (\mathbf{w} + v_i u^i)]} \frac{{}^*\partial\psi}{\partial t}. \quad (8.5)$$

Thus, we have a possibility of obtaining the respective formulae for the energy and momentum of a particle, expressed through its wave phase in the framework of the geometric optics approximation. In the fully degenerate space (zero-space), the relativistic mass, momentum, frequency, and energy are zero. However, the generalized mass M , momentum Mu^i , frequency Ω , and energy E are nonzero therein (see §6 for detail). As a result of (8.5), we obtain the formulae

$$E = \hbar \Omega = Mc^2 = \frac{\hbar}{1 - \frac{1}{c^2} (\mathbf{w} + v_i u^i)} \frac{{}^*\partial\psi}{\partial t}, \quad (8.6)$$

$$Mu^i = -\hbar h^{ik} \frac{*\partial\psi}{\partial x^k}, \quad (8.7)$$

which, in the regular (non-degenerate) space, transform into

$$E = \hbar\omega = mc^2 = \hbar \frac{*\partial\psi}{\partial t}, \quad mv^i = -\hbar h^{ik} \frac{*\partial\psi}{\partial x^k}. \quad (8.8)$$

As is known [17], the condition $K_\alpha K^\alpha = 0$, which is specific to massless particles (isotropic trajectories), has the form

$$g^{\alpha\beta} \frac{\partial\psi}{\partial x^\alpha} \frac{\partial\psi}{\partial x^\beta} = 0, \quad (8.9)$$

which is the basic equation of geometric optics (the eikonal equation). After formulating the regular differential operators through the chr.inv.-differential operators (2.8), and taking into account the main property $g_{\alpha\sigma} g^{\beta\sigma} = \delta_\alpha^\beta$ of the tensor $g_{\alpha\beta}$, which gives $g^{00} = \frac{1}{g_{00}} (1 - \frac{1}{c^2} v_i v^i)$, we arrive at the chr.inv.-eikonal equation for massless particles

$$\frac{1}{c^2} \left(\frac{*\partial\psi}{\partial t} \right)^2 - h^{ik} \frac{*\partial\psi}{\partial x^i} \frac{*\partial\psi}{\partial x^k} = 0. \quad (8.10)$$

In the same way, proceeding from the condition $P_\alpha P^\alpha = m_0^2$ characterizing mass-bearing particles (non-isotropic trajectories), we obtain the chr.inv.-eikonal equation for mass-bearing particles

$$\frac{1}{c^2} \left(\frac{*\partial\psi}{\partial t} \right)^2 - h^{ik} \frac{*\partial\psi}{\partial x^i} \frac{*\partial\psi}{\partial x^k} = \frac{m_0^2 c^2}{\hbar^2}, \quad (8.11)$$

which when $m_0 = 0$ becomes the same as the former one.

To obtain the chr.inv.-eikonal equation for zero-particles, we apply the conditions $m_0 = 0$, $m = 0$, $\omega = 0$, and $P_\alpha P^\alpha = 0$, which characterize the fully degenerate space (zero-space). After some algebra we obtain the chr.inv.-eikonal equation for zero-particles

$$h^{ik} \frac{*\partial\psi}{\partial x^i} \frac{*\partial\psi}{\partial x^k} = 0. \quad (8.12)$$

As is seen, this is a standing wave equation. This fact, and also that zero-particles are the ultimate case of light-like particles (see page 22 for details), allows us to conclude how zero-particles could be registered experimentally:

Zero-particles should seem from the point of view of a regular observer as *standing light waves* — the waves of stopped light, in other words. So, the entire zero-space should appear filled with a system of standing light waves (light-like holograms).

§9. Conclusion. What is frozen light? So, the geometric structure of the four-dimensional space (space-time) of the General Theory of Relativity manifests the possibility of the ultimate case of photons, for which not only the rest-mass is zero (as for regular photons), but also the relativistic mass is zero. We therefore refer to them as *zero-particles*. Such particles are hosted by a space with fully degenerate metric, which is the ultimate case of the light-like (particularly degenerate) space. They are fully degenerate photons, in other words.

Zero-particles can be hosted by both regular regions and collapsed regions of the fully degenerate space. In the latter case, they exist under the condition of gravitational collapse (see §4).

The fully degenerate space looks like a local volume, wherein all observable intervals of time and all three-dimensional observable intervals are zero. Once a photon has entered into such a zero-space “gate” at one location of our regular space, it can be instantly connected to another photon which has entered into a similar “gate” at another location. This is a way for *non-quantum teleportation* of photons (see §5).

Also, the regular relation between energy and momentum is not true for zero-particles. This means that zero-particles may play a rôle of virtual particles, which are material carriers of interaction between regular particles of our world (see §7).

From the point of view of a regular observer, zero-particles should appear as standing light waves — the waves of stopped light (see §8). The latter meets that which has been registered in the frozen light experiment: there, a light beam being stopped is “stored” in atomic vapor, and remains invisible to the observer until that moment of time when it is set free again in its regularly “travelling state”. (See Introduction and the original reports about the experiments referred therein.)

This means that the frozen light experiment pioneered at Harvard by Lene Hau is an experimental “foreword” to the discovery of zero-particles and, hence, a way for non-quantum teleportation.

With these we can mean frozen light as a new state of matter, which differs from the others (solid, gas, liquid, plasma).

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A Three-Dimensional Charged Black Hole Inspired by Non-Commutative Geometry

Juan Manuel Tejeiro and Alexis Larrañaga*

Abstract: We find a new charged black hole solution in three-dimensional anti-de Sitter (AdS) space using an anisotropic perfect fluid inspired by a non-commutative black hole as the source of matter and a Gaussian distribution of electric charge. We deduce the thermodynamical quantities of this black hole and compare them with those of a charged Banados-Teitelboim-Zanelli (BTZ) solution.

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§1. Introduction. The theoretical discovery of radiating black holes [1] disclosed the first physically relevant window regarding the mysteries of quantum gravity. The string-black hole correspondence principle [2] suggests that in this extreme regime stringy effects cannot be neglected. One of the most interesting outcomes of string theory is that target space-time coordinates become non-commuting operators on a D-brane [3]. Thus, string-brane coupling has put in evidence the necessity of space-time quantization. Recently, an improved version of field theory of a non-commutative space-time manifold has been proposed as a cheaper way to reproduce the string phenomenology, at least in the low-energy limit. In this proposal, non-commutativity is encoded in the commutator

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (1)$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix which determines the fundamental cellular discretization of space-time much in the same way as the Planck constant \hbar discretizes the phase space. This proposal provides a black hole with a minimum scale $\sqrt{\theta}$ known as the non-commutative black hole [4–8], whose commutative limit is the Schwarzschild metric. The thermodynamics and evaporation process of the non-commutative black hole has been studied in [9], while the entropy issue is discussed in

*National Astronomical Observatory, National University of Colombia, Ciudad Universitaria, Bogota D.C. 111321, Colombia. E-mail: ealarranaga@unal.edu.co.

[10, 11] and the Hawking radiation in [12]. Charged non-commutative black holes have been studied in [13, 14] and recently, a non-commutative three-dimensional black hole whose commutative limit is revealed by the non-rotating Banados-Teitelboim-Zanelli (BTZ) solution was studied in [15], while the three-dimensional rotating counterpart of it was deduced in [16].

In this paper, we construct a new charged black hole in AdS₃ space-time using an anisotropic perfect fluid inspired by the four-dimensional non-commutative black hole as the source of matter while considering a Gaussian distribution of electric charge. The resulting solution exhibits two horizons that degenerate into one in the extremal case. We compare the thermodynamics of this non-commutative black hole with that of the charged BTZ solution [17, 18].

§2. Derivation of the charged solution in three dimensions.

In the analysis of black holes in the framework of non-commutative spaces, one has to solve the corresponding field equations. As argued in [6, 19] it is not necessary to change the Einstein tensor part of the field equations because the non-commutative effects act only on the matter source. The underlying philosophy of this approach is to modify the distribution of point-like sources in favour of smeared objects. This is in agreement with the conventional procedure for the regularization of ultra-violet divergences by introducing a cut-off. As a gravitational analogue of the non-commutative modification of quantum field theory [4], we conclude that in General Relativity, the effect of non-commutativity can be taken into account by keeping the standard form of the Einstein tensor on the left-hand side of the field equations as well as by introducing a modified energy-momentum tensor as a source on the right-hand side.

Therefore, one way of implementing the effect of smearing is the following substitution rule: in three dimensions, the Dirac delta function $\delta^{3D}(r)$ is replaced by a Gaussian distribution with minimal width $\sqrt{\theta}$,

$$\rho(r) = \frac{M}{4\pi\theta} e^{-r^2/4\theta} \tag{2}$$

giving a mass distribution in the form

$$m(r) = 2\pi \int_0^r r' \rho(r') dr' = M \left(1 - e^{-r^2/4\theta}\right). \tag{3}$$

As coordinate non-commutativity is a property of the space-time fabric itself, and not of its material content, the same smearing effect is

expected to operate on electric charge [13,14]. Thus, a point-charge Q is spread throughout a minimal-width Gaussian charge cloud according to

$$\rho_e(r) = \frac{Q}{4\pi\theta} e^{-r^2/4\theta}. \quad (4)$$

For a static, circularly symmetric charge distribution, the current density J_μ is non-vanishing only along the time direction, i.e.

$$J^\mu = (\rho_e, 0, 0). \quad (5)$$

In order to find a black hole solution in AdS₃ space-time, we recall the Einstein-Maxwell equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi (T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{\text{electr}}) + \frac{1}{\ell^2} g_{\mu\nu}, \quad (6)$$

$$\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} F^{\mu\nu}) = J^\nu, \quad (7)$$

where ℓ is related with the cosmological constant by

$$\Lambda = -\frac{1}{\ell^2}. \quad (8)$$

The energy-momentum tensor for matter will take the anisotropic form

$$(T_\nu^\mu)^{\text{matter}} = \text{diag}(-\rho, p_r, p_\perp). \quad (9)$$

In order to completely define this tensor, we rely on the covariant conservation condition $T^{\mu\nu}{}_{;\nu} = 0$. This gives the source as an anisotropic fluid of density ρ , radial pressure

$$p_r = -\rho \quad (10)$$

and tangential pressure

$$p_\perp = -\rho - r \partial_r \rho. \quad (11)$$

The electromagnetic energy-momentum tensor $T_{\mu\nu}^{\text{electr}}$ is defined in terms of $F_{\mu\nu}$ as

$$T_{\mu\nu}^{\text{electr}} = -\frac{2}{\pi} \left(F_{\mu\sigma} F_{\nu}^{\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right). \quad (12)$$

By solving the Maxwell equations (7) with source (5), we obtain the electric field

$$E(r) = \frac{1}{r} \int_0^r r' \rho_e(r') dr' = \frac{Q}{2\pi r} \left(1 - e^{-r^2/4\theta} \right). \quad (13)$$

Using the static, circularly symmetric line-element

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\varphi^2, \quad (14)$$

the Einstein field equations (6) are written accordingly as

$$\frac{1}{r} \frac{df}{dr} = -16\pi\rho - \frac{1}{2} E^2 + \frac{2}{\ell^2}, \quad (15)$$

$$\frac{d^2 f}{dr^2} = 16\pi\rho_{\perp} + \frac{1}{2} E^2 + \frac{2}{\ell^2}. \quad (16)$$

Solving the above equations, we find

$$f(r) = -8M \left(1 - e^{-r^2/4\theta}\right) + \frac{r^2}{\ell^2} - \frac{Q^2}{8\pi^2} \left[\ln|r| + \frac{1}{2} \text{Ei}\left(-\frac{r^2}{2\theta}\right) - \text{Ei}\left(-\frac{r^2}{4\theta}\right) \right], \quad (17)$$

where $\text{Ei}(z)$ represents the exponential integral function,

$$\text{Ei}(z) = - \int_{-z}^{\infty} \frac{e^{-t}}{t} dt. \quad (18)$$

Note that when $\frac{r^2}{4\theta} \rightarrow \infty$, either when considering a large black hole ($r \rightarrow \infty$) or the commutative limit ($\theta \rightarrow 0$), we obtain the charged BTZ solution,

$$f^{\text{BTZ}}(r) = -8M + \frac{r^2}{\ell^2} - \frac{Q^2}{8\pi^2} \ln|r|. \quad (19)$$

The line-element (14, 17) describes the geometry of a non-commutative black hole with the corresponding event horizons given by the following condition imposed on $f(r)$

$$f(r) = -8M \left(1 - e^{-r_{\pm}^2/4\theta}\right) + \frac{r_{\pm}^2}{\ell^2} - \frac{Q^2}{8\pi^2} \left[\ln|r_{\pm}| + \frac{1}{2} \text{Ei}\left(-\frac{r_{\pm}^2}{2\theta}\right) - \text{Ei}\left(-\frac{r_{\pm}^2}{4\theta}\right) \right]. \quad (20)$$

This equation cannot be solved in closed form. However, by plotting $f(r)$ one can see obvious intersections with the r -axis and determine numerically the existence of horizons and their radii. Fig. 1 shows that, instead of a single event horizon, there are different possibilities for this black hole:

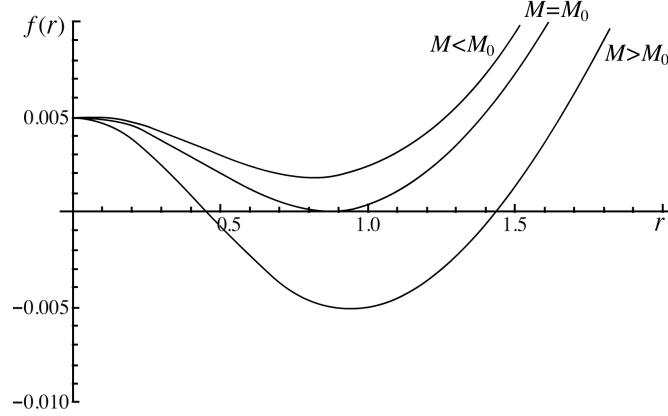


Fig. 1: Metric function f as a function of r . We have taken the values $\theta = 0.1$, $\ell = 10$ and $Q = 1$. The minimum mass is $M_0 \approx 0.00127$.

- 1) Two distinct horizons for $M > M_0$;
- 2) One degenerate horizon (extremal black hole) for $M = M_0$;
- 3) No horizon for $M < M_0$.

In view of this, there can be no black hole if the original mass is less than the lower-limit mass M_0 . The horizon of the extremal black hole is determined by the conditions $f = 0$ and $\partial_r f = 0$, giving

$$\left[4 \left(1 - e^{r_0^2/4\theta} \right) \theta + r^2 \right] \left[\frac{1}{2} \text{Ei} \left(-\frac{r_0^2}{2\theta} \right) - \text{Ei} \left(-\frac{r_0^2}{4\theta} \right) + \ln |r_0| + \frac{2\theta}{r_0^2} \left(3 + e^{-r_0^2/2\theta} - 3e^{-r_0^2/4\theta} - e^{r_0^2/4\theta} \right) \right]^{-1} = \frac{Q^2 \ell^2}{8\pi^2} \quad (21)$$

and subsequently the mass of the extremal black hole can be written as

$$M_0 = \frac{\frac{2r_0^2}{\ell^2 \theta} - \frac{Q^2}{8\pi^2 \theta} \left(1 + e^{-r^2/2\theta} - 2e^{-r^2/4\theta} \right)}{4e^{-r_0^2/4\theta}}. \quad (22)$$

§3. Thermodynamics. The Hawking temperature of the non-commutative black hole is

$$T_H = \frac{1}{4\pi} \partial_r f|_{r_+} = \frac{r_+}{2\pi \ell^2} \left[1 - \frac{2M_H \ell^2}{\theta} e^{-r_+^2/4\theta} - \frac{Q^2 \ell^2}{16\pi^2 r_+^2} \left(1 + e^{-r_+^2/2\theta} - 2e^{-r_+^2/4\theta} \right) \right], \quad (23)$$

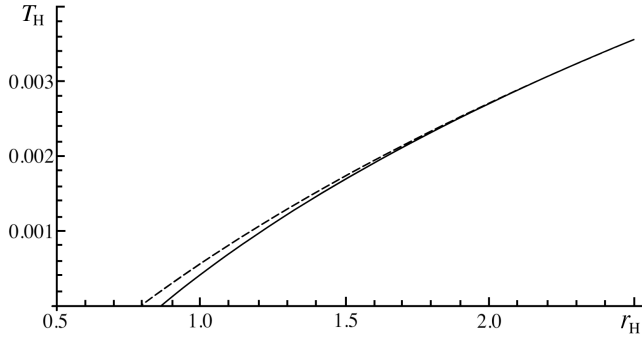


Fig. 2: The Hawking temperature versus r_H . The solid line represents the temperature for the non-commutative black hole with $\theta=0.1$. There is no difference with respect to the charged BTZ solution (dashed line) for large r_H . In both cases, we use the values $\ell=10$ and $Q=1$.

where

$$M_H = \frac{r_+^2}{8\ell^2 (1 - e^{-r_+^2/4\theta})} - \frac{Q^2}{64\pi^2 (1 - e^{-r_+^2/4\theta})} \times \left[\ln|r_+| + \frac{1}{2} \text{Ei}\left(-\frac{r_+^2}{2\theta}\right) - \text{Ei}\left(-\frac{r_+^2}{4\theta}\right) \right]. \quad (24)$$

For large black holes, i.e. $\frac{r_+^2}{4\theta} \gg 0$, one recovers the temperature of the rotating BTZ black hole,

$$T_H^{\text{BTZ}} = \frac{r_+}{2\pi\ell^2} \left(1 - \frac{Q^2\ell^2}{64\pi^2 r_+^2} \right). \quad (25)$$

As shown in Fig. 2, the Hawking temperature is a monotonically increasing function of the horizon radius for large black holes. For large black holes, there is indeed no difference with respect to the charged BTZ solution.

The first law of thermodynamics for a charged black hole reads

$$dM = T_H dS + \Phi dQ, \quad (26)$$

where the electrostatic potential is given by

$$\Phi = \left(\frac{\partial M}{\partial Q} \right)_{r_+} = -\frac{Q}{32\pi^2 (1 - e^{-r_+^2/4\theta})} \times \left[\ln|r_+| + \frac{1}{2} \text{Ei}\left(-\frac{r_+^2}{2\theta}\right) - \text{Ei}\left(-\frac{r_+^2}{4\theta}\right) \right]. \quad (27)$$

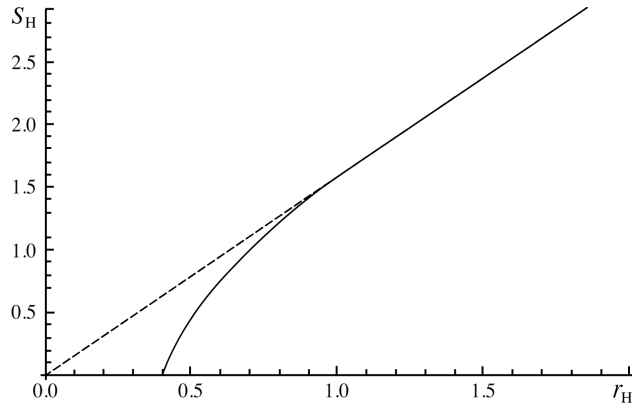


Fig. 3: Entropy versus r_H . The solid line represents the entropy of the non-commutative black hole with $\theta=0.1$. The dashed line represents the entropy of the charged BTZ black hole.

We calculate the entropy as

$$S = \int_{r_0}^{r_+} \frac{1}{T_H} dM, \quad (28)$$

which gives

$$S = \frac{\pi}{2} \int_{r_0}^{r_+} \left(\frac{1}{1 - e^{-\xi^2/4\theta}} \right) d\xi. \quad (29)$$

The entropy as a function of r_+ is depicted in Fig.3. Note that, in the large black hole limit, the entropy function corresponds to the Bekenstein-Hawking entropy (area law), $S_{\text{BH}} = \frac{\pi r_+}{2}$.

§4. Conclusion. We have constructed a non-commutative electrically charged black hole in AdS_3 space-time using an anisotropic perfect fluid inspired by the four-dimensional non-commutative black hole and a Gaussian distribution of electric charge. The result yields two horizons that degenerate into one in the extreme case. We have compared the thermodynamics of this black hole with that of a charged Banados-Teitelboim-Zanelli (BTZ) black hole. The Hawking temperature and entropy for a large non-commutative charged black hole approach those of the charged BTZ solution.

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Instanton Representation of Plebanski Gravity. Gravitational Instantons from the Classical Formalism

Eyo Eyo Ita III*

Abstract: We present a reformulation of General Relativity as a “generalized” Yang-Mills theory of gravity, using a $SO(3,C)$ gauge connection and the self-dual Weyl tensor as dynamical variables. This formulation uses Plebanski’s theory as the starting point, and obtains a new action called the instanton representation of Plebanski gravity (IRPG). The IRPG has yielded a collection of various new results, which show that it is a new approach to General Relativity intrinsically different from existing approaches. Additionally, the IRPG appears to provide a realization of the relation amongst General Relativity, Yang-Mills theory and instantons.

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*Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom. E-mails: eei20@cam.ac.uk; ita@usna.edu.

§1. Introduction. In the 1980's there was a major development in General Relativity due to Abhay Ashtekar, which provided a new set of Yang-Mills like variables known as the Ashtekar variables (see e.g. [1,2] and [3]). These variables have re-invigorated the efforts at achieving a quantum theory of gravity using techniques from Yang-Mills theory. Additionally, the relation of General Relativity to Yang-Mills theory by its own right is an interesting and active area of research [4,5]. The purpose of the present paper is two-fold. First, we will provide a new formulation of General Relativity which shows that its relation to Yang-Mills theory can be taken more literally in a certain well-defined context. The degrees of freedom of General Relativity will be explicitly embedded in a Yang-Mills like action resembling an instanton, and this formulation will be referred to as the instanton representation of Plebanski gravity. Secondly, in this paper we will focus just on some of the classical aspects of the theory, and make contact with existing results of General Relativity as well as provide various new results.

The organization of this paper is as follows. In §2 we will first provide a review of Plebanski's theory of gravity I_{Pleb} and the mechanism by which the Einstein equations follow from it. The Plebanski action contains a self-dual connection one-form A^a , where $a = 1, 2, 3$ denotes an $\text{SO}(3, \mathbb{C})$ index with respect to which the (internal) self-duality is defined, a matrix $\psi_{ae} \in \text{SO}(3, \mathbb{C}) \otimes \text{SO}(3, \mathbb{C})$, and a triple of self-dual two-forms Σ^a , also self-dual in the $\text{SO}(3, \mathbb{C})$ sense. The Ashtekar action I_{Ash} arises upon elimination of ψ_{ae} via a new mechanism, which basically restricts one to a functional submanifold of the space of actions defined by I_{Pleb} . Using this same mechanism, in §3 we show that elimination of certain components of the two forms Σ^a in favor of ψ_{ae} yields a new action I_{Inst} , the instanton representation of Plebanski gravity. This shows that I_{Ash} and I_{Inst} are in a sense complementary within I_{Pleb} , which suggests that the latter is also a theory of General Relativity. We prove this rigorously in §4 by demonstrating that I_{Inst} does indeed reproduce the Einstein equations, combined with a prescription for writing a solution subject to the initial value constraints.

In §5 we provide an analysis of the I_{Inst} equations of motion beyond the Einstein equations. A Hodge duality condition emerges on-shell, which as shown in §7 explicitly provides the spacetime metric.* In §6 we clarify the similarities and differences between I_{Inst} and the pure spin

*The implication is that the metrics from §4 and §7 must be equal to each other as a consistency condition. This should provide a practical method for constructing General Relativity solutions via what we will refer to as the instanton representation method.

connection formulation of Capovilla, Dell and Jacobson (CDJ) in [6]. There are common notions in the community that a certain antecedent of the CDJ action is essentially the same action as I_{Inst} . The present paper shows that I_{Inst} is in fact a new action for General Relativity. This will as well be independently corroborated by various follow-on papers which apply the instanton representation method to the construction of solutions. §7 delineates the reality conditions on I_{Inst} , which appear to be intertwined with the signature of spacetime. §8 and §9 clarify a hidden relation of General Relativity to Yang-Mills theory, which brings into play the concept of gravitational instantons.

The author has not been able to find, amongst the various sources in the literature, a uniform definition of what a gravitational instanton is. Some references, for example as in [7] and [8], define gravitational instantons as General Relativity solutions having a vanishing Weyl tensor with nonvanishing cosmological constant. This would seem to imply, in the language of the present paper, that gravitational instantons can exist only for spacetimes of Petrov Type O.* On the other hand, other references (for example [9]) allow for Type D gravitational instantons. In spite of this a common element, barring topological considerations, appears to be that of a solution to the vacuum Einstein equations having self-dual curvature. We hope in the present paper to shed some light on the concept of gravity as a “generalized” Yang-Mills instanton, which can exist as a minimum for Petrov Type I in addition to Types D and O. §10 contains a summary of the main results of this paper and some future directions of research, touching briefly on the quantum theory.

On a final note prior to proceeding, we will establish the following index conventions for this paper. Lowercase symbols from the beginning part of Latin alphabet a, b, c, \dots will denote internal $\text{SO}(3, \mathbb{C})$ indices and those from the middle i, j, k, \dots will denote spatial indices, each taking values 1, 2 and 3. $\text{SL}(2, \mathbb{C})$ indices will be labelled by capital letters A and A' taking values 0 and 1, and four-dimensional spacetime indices by Greek symbols μ, ν, \dots . For the internal $\text{SO}(3, \mathbb{C})$ indices (a, b, c, \dots, h) the raised and lowered index positions are equivalent since the $\text{SO}(3)$ group metric is taken to be the unit matrix (e.g. $\delta_{ab} \equiv \delta_b^a \equiv \delta_a^b \equiv \delta^{ab}$). For spatial indices (i, j, k, \dots) and spacetime indices (μ, ν, \dots), the raised and the lowered index positions are not equivalent, since the corresponding covariant metrics h_{ij} and $g_{\mu\nu}$ are in general different from the unit matrix. For multi-indexed quantities we will

*The definitions of the various Petrov Types can be found in [10] and in [11]. The purpose of the instanton representation of Plebanski gravity is to be able to classify General Relativity solutions according to their Petrov Type.

normally separate $\text{SO}(3, \mathbb{C})$ from the other types of indices by placing them in opposing positions. So for example, the objects A_i^a and B_a^i respectively will be used to denote a $\text{SO}(3, \mathbb{C})$ gauge connection and its associated magnetic field.

§2. Plebanski's theory of gravity. The starting Plebanski action [12] writes General Relativity using self-dual two forms in lieu of the spacetime metric $g_{\mu\nu}$ as the basic variables. We adapt the starting action to the language of the $\text{SO}(3, \mathbb{C})$ gauge algebra as

$$I = \int_{\mathbb{M}} \delta_{ae} \Sigma^a \wedge F^e - \frac{1}{2} (\delta_{ae} \varphi + \psi_{ae}) \Sigma^a \wedge \Sigma^e, \quad (1)$$

where $\Sigma^a = \frac{1}{2} \Sigma_{\mu\nu}^a dx^\mu \wedge dx^\nu$ are a triplet of $\text{SO}(3, \mathbb{C})$ two forms and $F^a = \frac{1}{2} F_{\mu\nu}^a dx^\mu \wedge dx^\nu$ is the field-strength two form for gauge connection one form $A^a = A_\mu^a dx^\mu$. Also, ψ_{ae} is symmetric and traceless and φ is a numerical constant. The field strength is written in component form as $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$, with $\text{SO}(3, \mathbb{C})$ structure constants $f^{abc} = \epsilon^{abc}$. The equations of motion resulting from (1) are (see e.g. [13] and [14])

$$\left. \begin{aligned} \frac{\delta I}{\delta A^g} &= D\Sigma^g = d\Sigma^g + \epsilon_{fh}^g A^f \wedge \Sigma^h = 0 \\ \frac{\delta I}{\delta \psi_{ae}} &= \Sigma^a \wedge \Sigma^e - \frac{1}{3} \delta^{ae} \Sigma^g \wedge \Sigma_g = 0 \\ \frac{\delta I}{\delta \Sigma^a} &= F^a - \Psi_{ae}^{-1} \Sigma^e = 0 \longrightarrow F_{\mu\nu}^a = \Psi_{ae}^{-1} \Sigma_{\mu\nu}^e \end{aligned} \right\}. \quad (2)$$

The first equation of (2) states that A^g is the self-dual part of the spin connection compatible with the two forms Σ^a , where $D = dx^\mu D_\mu = dx^\mu (\partial_\mu + A_\mu)$ is the exterior covariant derivative with respect to A^a . The second equation implies that the two forms Σ^a can be constructed from tetrad one-forms $e^I = e_\mu^I dx^\mu$ in the form*

$$\Sigma^a = i e^0 \wedge e^a - \frac{1}{2} \epsilon_{afg} e^f \wedge e^g. \quad (3)$$

Equation (3) is a self-dual combination of tetrad wedge products, which enforces the equivalence of (1) to General Relativity. Note that equation (3) implies [14]

$$\frac{i}{2} \Sigma^a \wedge \Sigma^e = \delta^{ae} \sqrt{-g} d^4x, \quad (4)$$

*In the tetrad formulation of gravity, this corresponds to spacetimes of Lorentzian signature when e^0 is real, and Euclidean signature when e^0 is pure imaginary.

with the spacetime volume element as the proportionality factor. The third equation of motion in (2) states that the curvature of A^a is self-dual as a two form, which implies that the metric $g_{\mu\nu} = \eta_{IJ} e_\mu^I e_\nu^J$ derived from the tetrad one-forms e^I satisfies the vacuum Einstein equations.

If one were to eliminate the two forms Σ^a and the matrix ψ_{ae} from the action (1) while leaving the connection A_μ^a intact, then one would obtain the CDJ action [6], corresponding to the pure spin connection formulation of General Relativity. But we would like to obtain a formulation of General Relativity which preserves these fields to some extent, since they contain fundamental gravitational degrees of freedom and also provide a mechanism for implementing the initial value constraints.

The most direct way to preserve the ability to implement the constraints in a totally constrained system is to first perform a 3+1 decomposition of the action. The starting action (1) in component form is given by

$$I[\Sigma, A, \Psi] = \frac{1}{4} \int_M d^4x \left(\Sigma_{\mu\nu}^a F_{\rho\sigma}^a - \frac{1}{2} \Psi_{ae}^{-1} \Sigma_{\mu\nu}^a \Sigma_{\rho\sigma}^e \right) \epsilon^{\mu\nu\rho\sigma}, \quad (5)$$

where $\epsilon^{0123} = 1$ and we have defined $\Psi_{ae}^{-1} = \delta_{ae} \varphi + \psi_{ae}$. For $\varphi = -\frac{\Lambda}{3}$, where Λ is the cosmological constant, then we have that

$$\Psi_{ae}^{-1} = -\frac{\Lambda}{3} \delta_{ae} + \psi_{ae}. \quad (6)$$

The matrix ψ_{ae} , presented in [5], is the self-dual part of the Weyl curvature tensor in $SO(3, C)$ language. The eigenvalues of ψ_{ae} determine the algebraic classification of spacetime which is independent of coordinates and of tetrad frames [10, 11].* Ψ_{ae}^{-1} is the matrix inverse of Ψ_{ae} which we will refer to as the CDJ matrix, and is the result of appending to ψ_{ae} a trace part. In the CDJ formulation this field becomes eliminated in addition to the two forms Σ^a .

§2.1. The Ashtekar variables. Assuming a spacetime manifold of topology $M = \Sigma \times \mathbb{R}$, where Σ refers to 3-space, let us perform a 3+1 decomposition of (5). Defining $\tilde{\sigma}_a^i \equiv \frac{1}{2} \epsilon^{ijk} \Sigma_{jk}^a$ and $B_a^i \equiv \frac{1}{2} \epsilon^{ijk} F_{jk}^a$ for the spatial parts of the self-dual and curvature two forms, this is given by

$$I = \int dt \int_\Sigma d^3x \tilde{\sigma}_a^i \dot{A}_i^a + A_0^a D_i \tilde{\sigma}_a^i + \Sigma_{0i}^a (B_a^i - \Psi_{ae}^{-1} \tilde{\sigma}_e^i), \quad (7)$$

where we have integrated by parts, using $F_{0i}^a = \dot{A}_i^a - D_i A_0^a$ from the tem-

*This includes principal null directions and properties of gravitational radiation.

poral component of the curvature. The operator D_i is the spatial part of the $SO(3, \mathbb{C})$ covariant derivative, which in (1) acts as a covariant divergence. The following action ensues on any $SO(3, \mathbb{C})$ -valued vector v_a , given by $D_i v_a = \partial_i v_a + f_{abc} A_i^b v_c$. We will use (2) and (3) to redefine the two form components in (7).

Define e_i^a as the spatial part of the tetrads e_μ^I and make the identification

$$e_i^a = \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_b^j \tilde{\sigma}_c^k (\det \|\tilde{\sigma}\|)^{-1/2} = \sqrt{\det \|\tilde{\sigma}\|} (\tilde{\sigma}^{-1})_i^a. \quad (8)$$

For a special case $e_i^0 = 0$, known as the time gauge, then the temporal components of the two forms (3) are given by (see e.g. [13, 15])

$$\Sigma_{0i}^a = \frac{i}{2} \underline{N} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_b^j \tilde{\sigma}_c^k + \epsilon_{ijk} N^j \tilde{\sigma}_a^k, \quad (9)$$

where $\underline{N} = N (\det \|\tilde{\sigma}\|)^{-1/2}$ with N and N^i being a set of four nondynamical fields. In the steps leading to the CDJ action of [6], the fields $N^\mu = (N, N^i)$ become eliminated along with the process of eliminating the 2-forms $\Sigma_{\mu\nu}^a$.

Substituting (9) into (7), we obtain the action

$$I = \int dt \int_\Sigma d^3x \tilde{\sigma}_a^i \dot{A}_i^a + A_0^a G_a - N^i H_i - iNH. \quad (10)$$

The fields (A_0^a, N, N^i) are auxiliary fields whose variations yield respectively the following constraints

$$\left. \begin{aligned} G_a &= D_i \tilde{\sigma}_a^i \\ H_i &= \epsilon_{ijk} \tilde{\sigma}_a^j B_a^k + \epsilon_{ijk} \tilde{\sigma}_a^j \tilde{\sigma}_e^k \Psi_{ae}^{-1} \\ H &= (\det \|\tilde{\sigma}\|)^{-1/2} \times \\ &\quad \times \left[\frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j B_c^k - \frac{1}{6} (\text{tr } \Psi^{-1}) \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \tilde{\sigma}_c^k \right] \end{aligned} \right\}. \quad (11)$$

Rather than attempt to perform a canonical analysis, we will proceed from (10) as follows. Think of $I = I_{\tilde{\sigma}, \Psi}[A]$ as an infinite dimensional functional manifold of theories parametrized by the fields $\tilde{\sigma}_a^i$ and Ψ_{ae} , and then restrict attention to a submanifold corresponding to the theory of General Relativity.

Following suit, say that we impose the following conditions on Ψ_{ae}^{-1}

$$\epsilon^{bae} \Psi_{ae}^{-1} = 0, \quad \text{tr } \Psi^{-1} = -\Lambda \quad (12)$$

with no restrictions on $\tilde{\sigma}_a^i$, where Λ is the cosmological constant. Then Ψ_{ae}^{-1} becomes eliminated and equation (10) reduces to the action for General Relativity in the Ashtekar variables (see e.g. [1–3])

$$I_{\text{Ash}} = \frac{i}{G} \int dt \int_{\Sigma} d^3x \tilde{\sigma}_a^i \dot{A}_i^a + A_0^a D_i \tilde{\sigma}_a^i - \epsilon_{ijk} N^i \tilde{\sigma}_a^j B_c^k + \frac{i}{2} \underline{N} \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \left(B_c^k + \frac{\Lambda}{3} \tilde{\sigma}_c^k \right), \quad (13)$$

where $\underline{N} = N (\det \|\tilde{\sigma}\|)^{-1/2}$ is the densitized lapse function. The action (13) is written on the phase space $\Omega_{\text{Ash}} = (\tilde{\sigma}_a^i, A_i^a)$ and the variable Ψ_{ae}^{-1} has been eliminated. The auxiliary fields A_0^a , N and N^i respectively are the $\text{SO}(3, \mathbb{C})$ rotation angle, the lapse function and the shift vector. These auxiliary fields play the role of Lagrange multipliers smearing their associated initial value constraints G_a , H , and H_i , respectively the Gauss' law, Hamiltonian and vector (sometimes known as diffeomorphism) constraints. Note that $\tilde{\sigma}_a^i$ in the original Plebanski action was part of an auxiliary field $\Sigma_{\mu\nu}^a$, but now in (13) it has become promoted to the status of a momentum space dynamical variable. At the level of (13), one could further eliminate the 2-forms Σ^a to obtain the CDJ pure spin connection action appearing in [6]. However, (13) is already in a form suitable for quantization and for implementation of the initial value constraints via the temporal parts of these 2-forms.

§3. The instanton representation. Having shown that Plebanski's action (1) contains (13), an action known to describe General Relativity, as a direct consequence of (12), we will now show that (1) also contains an alternate formulation of General Relativity based on the field Ψ_{ae} , which can also be derived directly from (5).

Instead of equation (12), let us impose the following conditions in the constraints (11)

$$\epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j B_c^k = -\frac{\Lambda}{3} \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \tilde{\sigma}_c^k, \quad \epsilon_{ijk} \tilde{\sigma}_a^j B_c^k = 0 \quad (14)$$

with no restriction on Ψ_{ae} . Substitution of (14) into (11) yields

$$\left. \begin{aligned} H_i &= \epsilon_{ijk} \tilde{\sigma}_a^j \tilde{\sigma}_e^k \Psi_{ae}^{-1} \\ H &= (\det \|\tilde{\sigma}\|)^{-1/2} \left[-\frac{\Lambda}{6} \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \tilde{\sigma}_c^k - \right. \\ &\quad \left. - \frac{1}{6} (\text{tr } \Psi^{-1}) \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \tilde{\sigma}_c^k \right] = -\sqrt{\det \|\tilde{\sigma}\|} (\Lambda + \text{tr } \Psi^{-1}) \end{aligned} \right\}. \quad (15)$$

Hence substituting (15) into (10), we obtain an action given by

$$I = \int dt \int_{\Sigma} d^3x \tilde{\sigma}_a^i \dot{A}_i^a + A_0^a D_i \tilde{\sigma}_a^i + \epsilon_{ijk} N^i \tilde{\sigma}_a^j \tilde{\sigma}_e^k \Psi_{ae}^{-1} - iN \sqrt{\det \|\tilde{\sigma}\|} (\Lambda + \text{tr} \Psi^{-1}). \quad (16)$$

But (16) still contains $\tilde{\sigma}_a^i$, therefore we will completely eliminate $\tilde{\sigma}_a^i$ by substituting the spatial restriction of the third equation of motion of (2), given by

$$\tilde{\sigma}_a^i = \Psi_{ae} B_e^i, \quad (17)$$

into (16). This substitution, which also appears in [6] in the form of the so-called CDJ ansatz, yields the action*

$$I_{\text{Inst}} = \int dt \int_{\Sigma} d^3x \Psi_{ae} B_e^i \dot{A}_i^a + A_0^a B_e^i D_i \Psi_{ae} + \epsilon_{ijk} N^i B_a^j B_e^k \Psi_{ae} - iN \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}), \quad (18)$$

which depends on the CDJ matrix Ψ_{ae} and the Ashtekar connection A_i^a , with no appearance of $\tilde{\sigma}_a^i$. In the original Plebanski theory Ψ_{ae} was an auxiliary field which could be eliminated. But now Ψ_{ae} has become promoted to the status of a full dynamical variable, analogously to the case for $\tilde{\sigma}_a^i$ in I_{Ash} .

There are a few items of note regarding (18). Note that it contains the same auxiliary fields (A_0^a, N, N^i) as in the Ashtekar theory (13). Since we have imposed the constraints $H_\mu = (H, H_i)$ on the Ashtekar phase space within the starting Plebanski theory in order to obtain I_{Inst} , then this suggests that the initial value constraints (G_a, H, H_i) should play an analogous role in (18) as their counterparts in (13). This relation holds only when Ψ_{ae} is nondegenerate, which limits one to spacetimes of Petrov Types I, D and O where Ψ_{ae} has three linearly independent eigenvectors.[†] Lastly, note that by further elimination of Ψ_{ae} and N^i from (18) one can obtain the CDJ action in [6]. However, we would like to preserve Ψ_{ae} since it contains gravitational degrees of freedom relevant to the instanton representation, and the shift vector N^i as we will see also assumes an important role.

*Equation (17) is valid when B_a^i and Ψ_{ae} are nondegenerate as 3×3 matrices. Hence all results of this paper will be confined to configurations where this is the case.

[†]We refer to (18) as the instanton representation of Plebanski gravity because it follows directly from Plebanski's action (1). We will in this sense use (18) as the starting point for the reformulation of gravity thus presented. The association of (18) with gravitational instantons will be made more precise later in this paper.

§4. Equations of motion of the instanton representation. We will now show that Einstein equations follow from the instanton representation action I_{Inst} in the same sense that they follow from the original Plebanski action (1). More precisely, we will demonstrate consistency of the equations of motion of (18) with equations (2) and (3). After integrating by parts and discarding boundary terms, the starting action (18) is given by

$$I_{\text{Inst}} = \int dt \int_{\Sigma} d^3x \Psi_{ae} B_e^k (F_{0k}^a + \epsilon_{kjm} B_a^j N^m) - iN \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}). \quad (19)$$

The equation of motion for the shift vector N^i is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N^i} = \epsilon_{ijk} B_a^j B_e^k \Psi_{ae} = 0, \quad (20)$$

which implies on the solution to the equations of motion that $\Psi_{ae} = \Psi_{(ae)}$ is symmetric.

The equation of motion for the lapse function N is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N} = \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) = 0. \quad (21)$$

Nondegeneracy of Ψ_{ae} and of the magnetic field B_e^i implies that on-shell, the following relation must be satisfied

$$\Lambda + \text{tr} \Psi^{-1} = 0, \quad (22)$$

which implies that λ_3 can be written explicitly in terms of λ_1 and λ_2 , regarded as physical degrees of freedom. The equation of motion for Ψ_{ae} is

$$\begin{aligned} \frac{\delta I_{\text{Inst}}}{\delta \Psi_{ae}} &= B_e^k F_{0k}^a + \epsilon_{kjm} B_e^k B_a^j N^m + \\ &+ iN \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Psi^{-1} \Psi^{-1})^{ea} = 0, \end{aligned} \quad (23)$$

where we have used (22). The symmetric and the antisymmetric parts of (23) must separately vanish. The antisymmetric part is given by

$$B_{[e}^k F_{0k}^{a]} + \epsilon_{mkj} N^m B_e^k B_a^j = 0, \quad (24)$$

which can be used to solve for the shift vector N^i . Using the relation $\epsilon_{ijk} B_a^j B_e^k = \epsilon_{aed} (B^{-1})_i^d (\det\|B\|)$ for nondegenerate 3×3 matrices, we have

$$N^i = -\frac{1}{2} \epsilon^{ijk} F_{0j}^g (B^{-1})_k^g. \quad (25)$$

The symmetric part of (23) is given by

$$B_{(e}^k F_{0k}^a) + iN \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Psi^{-1}\Psi^{-1})^{(ea)} = 0, \quad (26)$$

where we have used that Ψ_{ae} on-shell is symmetric from (20).

§4.1. Verification of the Einstein equations. To make a direct connection from the instanton representation to Einstein's General Relativity, we will show that the equations of motion for I_{Inst} imply the Einstein equations. Let us use the relation

$$\sqrt{-g} = N\sqrt{h} = N\sqrt{\det\|\tilde{\sigma}\|} = \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|}, \quad (27)$$

which writes the determinant of the spacetime metric $g_{\mu\nu}$ in terms of dynamical variables (A, Ψ) using the 3+1 decomposition, and uses the determinant of (17). Defining $\epsilon^{0ijk} \equiv \epsilon^{ijk}$ and using the symmetries of the four-dimensional epsilon tensor $\epsilon^{\mu\nu\rho\sigma}$, then the following identities hold

$$B_{(e}^k F_{0k}^a) = \frac{1}{2} \epsilon^{klm} F_{lm}^{(e} F_{0k}^a) = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^e. \quad (28)$$

Using (28) and (27), then equation (26) can be re-written as

$$\frac{1}{8} F_{\mu\nu}^b F_{\rho\sigma}^f \epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g} (\Psi^{-1}\Psi^{-1})^{(bf)} = 0. \quad (29)$$

Left and right multiplying (29) by Ψ , which is symmetric after implementation of (20), we obtain

$$\frac{1}{4} (\Psi^{bb'} F_{\mu\nu}^{b'}) (\Psi^{ff'} F_{\rho\sigma}^{f'}) \epsilon^{\mu\nu\rho\sigma} = -2i\sqrt{-g} \delta^{bf}. \quad (30)$$

Note that this step and the steps that follow require that Ψ_{ae} be nondegenerate as a 3×3 matrix. Let us make the definition

$$\Sigma_{\mu\nu}^a = \Psi_{ae} F_{\mu\nu}^e = \Sigma_{\mu\nu}^a [\Psi, A], \quad (31)$$

which retains Ψ_{ae} and A_μ^a as fundamental, with the two forms $\Sigma_{\mu\nu}^a$ being derived quantities. Upon using the third line of (2) as a re-definition of variables, which amounts to using the curvature and the CDJ matrix to construct a two form, (30) reduces to

$$\frac{1}{4} \Sigma_{\mu\nu}^b \Sigma_{\rho\sigma}^f dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \Sigma^b \wedge \Sigma^f = -2i\sqrt{-g} \delta^{bf} d^4x. \quad (32)$$

One recognizes (32) as the condition that the two forms thus constructed, which are now derived quantities, be derivable from tetrads,

which is the analogue of (4). Indeed, one can conclude as a consequence of (32) that there exist one forms $e^I = e^I_\mu dx^\mu$ where $I = 0, 1, \dots, 3$, such that

$$\Psi_{ae} F^e = ie^0 \wedge e^a - \frac{1}{2} \epsilon_{afg} e^f \wedge e^g \equiv P_{fg}^a e^f \wedge e^g. \quad (33)$$

We have defined P_{fg}^a as a projection operator onto the self-dual combination of one-form wedge products, self-dual in the $SO(3, \mathbb{C})$ sense. To complete the demonstration that the instanton representation yields the Einstein equations, it remains to show that the connection A^a is compatible with the two forms Σ^a as constructed in (31).

Using the fact that Ψ_{ae} is symmetric on solutions to (20), the starting action (19) can be written as*

$$I_{\text{Inst}} = \int_{\text{M}} d^4x \frac{1}{8} \Psi_{ae} F_{\mu\nu}^a F_{\rho\sigma}^e \epsilon^{\mu\nu\rho\sigma} - i\sqrt{-g} (\Lambda + \text{tr} \Psi^{-1}). \quad (34)$$

The equation of motion for the connection A_μ^a from (34) is given by

$$\frac{\delta I_{\text{Inst}}}{\delta A_\mu^a} \sim \epsilon^{\mu\sigma\nu\rho} D_\sigma (\Psi_{ae} F_{\nu\rho}^e) - \frac{i}{2} \delta_i^\mu D_{da}^{ij} \left[N (B^{-1})_j^d \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) \right], \quad (35)$$

where we have used that Ψ_{ae} is symmetric and we have defined

$$\left. \begin{aligned} \overline{D}_{ea}^{ji}(x, y) &\equiv \frac{\delta B_e^j(y)}{\delta A_i^a(x)} = \epsilon^{jki} (-\delta_{ae} \partial_k + f_{eda} A_k^d) \delta^{(3)}(x, y) \\ \overline{D}_{ea}^{0i} &\equiv 0 \end{aligned} \right\}. \quad (36)$$

The term in square brackets in (35) vanishes on-shell, since it is proportional to the equation of motion (21) and its spatial derivatives, which leaves us with

$$\epsilon^{\mu\sigma\nu\rho} D_\sigma (\Psi_{ae} F_{\nu\rho}^e) = 0. \quad (37)$$

Equation (37) states that when (20) and (22) are satisfied, then the two forms $\Sigma_{\mu\nu}^a$ constructed from Ψ_{ae} and $F_{\mu\nu}^e$ as in (31) are compatible with the connection A_μ^a . This is the direct analogue of the first equation from (2).

Using (19) as the starting point, which uses Ψ_{ae} and A_μ^a as the dynamical variables, we have obtained the Einstein equations in the same

*The same action was written down in [6], which arises from elimination of the self-dual 2-forms directly from Plebanski's action. In the approach of the present paper, we have eliminated only the spatial part of the 2-forms, and have used the antisymmetric part of Ψ_{ae} to solve for the shift vector N^i .

sense that the starting Plebanski theory (1) implies the Einstein equations. The first equation of (2) has been reproduced via (37), which holds provided that (22) and (20) are satisfied. The second equation of (2) has been reproduced via (32), which follows from (29) when (20) is satisfied. The third equation of (2) may be regarded as a defining relation for the instanton representation. Since the Einstein equations have arisen from the instanton representation, then it follows that I_{Inst} is another representation for General Relativity for nondegenerate Ψ_{ae} and B_e^i .

On the solution to (20) and (22) and using (33), the action for the instanton representation can be written in the language of two forms as

$$I_{\text{Inst}} = \frac{1}{2} \int_{\text{M}} \Psi_{bf} F^b \wedge F^f = \frac{1}{2} \int_{\text{M}} P_{fg}^a e^f \wedge e^g \wedge F^a, \quad (38)$$

which upon the identification of one forms e^I with tetrads, is nothing other than the self-dual Palatini action [17].

Note that the Palatini action implies the Einstein equations with respect to the metric defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{IJ} e^I \otimes e^J, \quad (39)$$

where η_{IJ} is the Minkowski metric, which provides additional confirmation that the instanton representation I_{Inst} describes Einstein's General Relativity when Ψ_{ae} is nondegenerate.

§4.2. Discussion: constructing a solution. We have shown how the Einstein equations follow from the instanton representation (18), which uses Ψ_{ae} and A_μ^a as the dynamical variables. Equation (30) implies the existence of a tetrad, which imposes equivalence of I_{Inst} with General Relativity, but it does not explain how to construct the tetrad. Since the spacetime metric $g_{\mu\nu}$ is the fundamental variable in Einstein's theory, we will bypass the tetrad and construct $g_{\mu\nu}$ directly as follows.

Perform a 3+1 decomposition of spacetime $\text{M} = \Sigma \times \text{R}$, where Σ is a three-dimensional spatial hypersurface. The line element is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij} \omega^i \otimes \omega^j, \quad (40)$$

where h_{ij} is the induced 3-metric on Σ , and we have defined the one form

$$\omega^i = dx^i - N^i dt. \quad (41)$$

The shift vector is given by (25), rewritten here for completeness

$$N^i = -\frac{1}{2} \epsilon^{ijk} F_{0j}^g (B^{-1})_k^g, \quad (42)$$

and the lapse function N can apparently be chosen freely.

To complete the construction of $g_{\mu\nu}$ using I_{Inst} as the starting point we must write the 3-metric h_{ij} using Ψ_{ae} and A_μ^a . The desired expression is given by

$$h_{ij} = (\det\|\Psi\|)(\Psi^{-1}\Psi^{-1})^{ae}(B^{-1})_i^a(B^{-1})_j^e(\det\|B\|) = h_{ij}[\Psi, A], \quad (43)$$

where the following conditions must be satisfied

$$B_e^i D_i \Psi_{ae} = 0, \quad \epsilon_{dae} \Psi_{ae} = 0, \quad \Lambda + \text{tr} \Psi^{-1} = 0. \quad (44)$$

Equations (44) will be referred to as the Gauss' law, diffeomorphism and Hamiltonian constraints, which follow from variation of Lagrange multipliers A_0^a , N^i and N in the action (18). Note that equations (44) involve only Ψ_{ae} and the spatial part of the connection A_μ^a , objects which determine a spatial metric in (43).

The spacetime metric $g_{\mu\nu}$ solving the Einstein equations is given by

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & -N_j \\ -N_i & h_{ij} \end{pmatrix},$$

where $N_i = h_{ik} N^k$. There are a few things to note regarding this:

- 1) From (42), the shift vector N^i depends only on A_μ^a , which contains gauge degrees of freedom in the temporal component A_0^a ;
- 2) Secondly, the lapse function N is freely specifiable;
- 3) Third, each A_i^a and Ψ_{ae} satisfying the initial value constraints (44) determines a 3-metric h_{ij} , which when combined with a choice of A_0^a and lapse function N should provide a solution $g_{\mu\nu}$ for spacetimes of Petrov Type I, D and O.

Note, when one uses the CDJ ansatz $\tilde{\sigma}_a^i = \Psi_{ae} B_e^i$ that (43) implies $h h^{ij} = \tilde{\sigma}_a^i \tilde{\sigma}_a^j$, which is the relation of the Ashtekar densitized triad to the contravariant 3-metric h^{ij} [1]. Upon implementation of (44) on the phase space Ω_{Inst} , then one is left with the two degrees of freedom per point of General Relativity, and h_{ij} becomes expressed explicitly in terms of these degrees of freedom.

§5. Analysis of the equations of motion. We will now provide a rudimentary analysis of the physical content of the equations of motion of I_{Inst} beyond the Einstein equations. The first equation, re-written here for completeness, is (23)

$$B_f^i F_{0i}^b + i\sqrt{-g}(\Psi^{-1}\Psi^{-1})^{fb} + \epsilon_{ijk} B_f^i B_b^j N^k = 0. \quad (45)$$

Also, when (20) and (22) are satisfied, then (35) implies (37), also written here

$$\epsilon^{\mu\sigma\nu\rho} D_\sigma (\Psi_{ae} F_{\nu\rho}^e) = 0. \quad (46)$$

We have shown that when Ψ_{bf} is symmetric after determination of N^i as in (25), that the symmetric part of (45) in conjunction with (46) imply the Einstein equations. We will now show under this condition that (45) and (46) form a self-consistent system. Act on (46) with D_μ and use the definition of curvature as the commutator of covariant derivatives, yielding

$$\epsilon^{\mu\nu\rho\sigma} D_\mu D_\nu (\Psi_{ae} F_{\rho\sigma}^e) = f_{abc} \Psi_{ce} (F_{\mu\nu}^b F_{\rho\sigma}^e \epsilon^{\mu\nu\rho\sigma}) = 0. \quad (47)$$

Then substituting the symmetric part of (45) into (47), up to an insignificant numerical factor we get

$$f_{abc} \Psi_{ce} \left[i\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{(eb)} \right] = i\sqrt{-g} f_{abc} (\Psi^{-1})^{cb} = 0, \quad (48)$$

which is simply the statement that Ψ_{ce} is symmetric in c and e which is consistent with (20) for $\det\|B\| \neq 0$. This can also be seen at the level of 2-forms by elimination of the curvature from (47) to obtain

$$f_{abc} F_{\mu\nu}^b \Sigma_{\rho\sigma}^c \epsilon^{\mu\nu\rho\sigma} \longrightarrow f_{abc} (\Psi^{-1})^{bf} \Sigma_{\mu\nu}^f \Sigma_{\rho\sigma}^c \epsilon^{\mu\nu\rho\sigma} \sim 0 \quad (49)$$

due to (32), on account of antisymmetry of the structure constants.

We will now multiply (45) by $(B^{-1})_k^f$, in conjunction with using the identity $(B^{-1})_j^d B_b^j = \delta_b^d$ since B_f^i is nondegenerate. Then equation (45) can be written as

$$F_{0k}^b + iN (\det\|B\|)^{-1/2} (\det\|\Psi\|)^{-1/2} (\det\|B\| \det\|\Psi\|) \times \\ \times \left[(\Psi^{-1} \Psi^{-1})^{df} (B^{-1})_j^d (B^{-1})_k^f \right] B_b^j + \epsilon_{kjm} B_b^j N^m = 0. \quad (50)$$

We can now use (43) in the second term of (50), which defines the spatial 3-metric in terms of Ψ_{ae} and the spatial connection A_i^a solving the constraints (44). Using this in conjunction with the relation $N (\det\|B\|)^{-1/2} (\det\|\Psi\|)^{-1/2} = Nh^{-1/2} = \underline{N}$, then equation (50) becomes

$$F_{0i}^b + i\underline{N} h_{ij} B_b^j + \epsilon_{ijk} B_b^j N^k = 0. \quad (51)$$

We will show in the next subparagraph that (51) is simply the statement that the curvature $F_{\mu\nu}^a$ is Hodge self-dual with respect to a metric $g_{\mu\nu}$ whose spatial part is h_{ij} , whose lapse function is N and whose shift vector is N^i .

It may appear via (30) that only the symmetric part of (45) is needed in order for I_{Inst} to imply the Einstein equations for Petrov Types I, D and O. But we have utilized the equation of motion (45) to arrive at (51), which includes information derived using the antisymmetric part of Ψ_{ae} . The reconciliation is in the observation that part of the process of solving the Einstein equations involves computing the shift vector via (25), which simultaneously eliminates the antisymmetric part of (45). Since (51) then is consistent with the Einstein equations, then the implication is that each such solution is included within the class of configurations under which the curvature $F_{\mu\nu}^a$ is Hodge self-dual with respect to the corresponding metric $g_{\mu\nu}$. The spatial part h_{ij} of this metric is defined on the configurations (Ψ_{ae}, A_i^a) satisfying (44).

§5.1. Dynamical Hodge self-duality operator. We will now prove that equation (51) is indeed the statement that the curvature $F_{\mu\nu}^a$ is Hodge self-dual with respect to $g_{\mu\nu} = g_{\mu\nu}[\Psi, A]$. To show this, we will derive the Hodge self-duality condition for Yang-Mills theory in curved spacetime, using the 3+1 decomposition of the associated metric.

The following relations will be useful

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = -\frac{N^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}, \quad (52)$$

where N is real for Lorentzian signature spacetimes and pure imaginary for Euclidean signature.

The Hodge self-duality condition for the curvature $F_{\mu\nu}^a$ can be written in the following form

$$\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}^a = \frac{\beta}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a, \quad (53)$$

where β is a numerical constant which we will determine. Expanding (53) and using $F_{00}^a = 0$, we have

$$\begin{aligned} N\sqrt{h} \left[(g^{\mu 0} g^{\nu j} - g^{\nu 0} g^{\mu j}) F_{0j}^a + g^{\mu i} g^{\nu j} \epsilon_{ijk} B_a^k \right] = \\ = \frac{\beta}{2} (2\epsilon^{\mu\nu 0i} F_{0i}^a + \epsilon^{\mu\nu ij} \epsilon_{ijm} B_a^m). \end{aligned} \quad (54)$$

We will now examine the individual components of (54). The $\mu = 0, \nu = 0$ component yields $0 = 0$, which is trivially satisfied. Moving on to the $\mu = 0, \nu = k$ component, we have

$$N\sqrt{h} \left[(g^{00} g^{kj} - g^{k0} g^{0j}) F_{0j}^a + g^{0i} g^{kj} \epsilon_{ijm} B_a^m \right] = \beta B_a^k. \quad (55)$$

Making use of (52) as well as the antisymmetry of the epsilon symbol, after some algebra* we obtain

$$F_{0j}^a + \epsilon_{jmk} B_a^m N^k + \beta \underline{N} h_{jk} B_a^k = 0, \quad (56)$$

where we have defined $\underline{N} = Nh^{-1/2}$. Note that (56) is the same as (51) for $\beta = i$, which establishes Hodge self-duality with respect to the spatio-temporal components.

We must next verify Hodge self-duality with respect to the purely spatial components of the curvature. For the $\mu = m, \nu = n$ component of (53), we have

$$N\sqrt{h} \left[(g^{m0} g^{nj} - g^{n0} g^{mj}) F_{0j}^a + g^{mi} g^{nj} \epsilon_{ijk} B_a^k \right] = \beta \epsilon^{mn0i} F_{0i}^a. \quad (57)$$

Substitution of (52) into (57) after some algebra yields[†]

$$\begin{aligned} \frac{\sqrt{h}}{N} (N^n h^{mj} - N^m h^{nj}) (F_{0j}^a + \epsilon_{jkl} B_a^k N^l) &= \\ &= \epsilon^{mnl} (\beta F_{0l}^a - \underline{N} h_{lk} B_a^k). \end{aligned} \quad (58)$$

Using $h^{ij} h_{jk} = \delta_k^i$ and simplifying, then (58) reduces to

$$F_{0k}^a + \epsilon_{kmn} B_a^m N^n = \frac{1}{\beta} \underline{N} h_{kl} B_a^l. \quad (59)$$

Consistency of (59) with (56) implies that $\frac{1}{\beta} = -\beta$, or $\beta = \pm i$. Comparison of (56) and (59) with (51) shows that the Hodge self-duality condition arises dynamically from the equations of motion (18) of I_{Inst} . Moreover, the curvature $F_{\mu\nu}^a$ is Hodge self-dual with respect to this operator, which can be written as[‡]

$$H_{\pm}^{\mu\nu\rho\sigma} = \frac{1}{2} \left[\sqrt{-g} (g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma}) \pm i \epsilon^{\mu\nu\rho\sigma} \right], \quad (60)$$

where $g_{\mu\nu} = g_{\mu\nu}[\Psi, A]$ is defined in terms of instanton representation variables.

The results can then be summarized as follows. The instanton representation I_{Inst} on-shell implies that the $\text{SO}(3, \mathbb{C})$ gauge curvature $F_{\mu\nu}^a$

*See Appendix A leading to equation (130).

†See Appendix A leading to equation (138).

‡It appears that $\beta = \pm i$ follows from our choice of a Lorentzian signature metric corresponding to real N , and that one can make a Wick rotation $N \rightarrow iN$, and analogously require $\beta = \pm 1$ for Euclidean signature. However, we will show in this paper that the reality conditions play a role in the signature of spacetime, more so than does the choice of lapse function N .

is Hodge self-dual with respect to a metric $g_{\mu\nu}$. But I_{Inst} also implies on-shell that $g_{\mu\nu}$ solves the Einstein equations, which in turn identifies $F_{\mu\nu}^a$ with the Riemann curvature $\mathfrak{Riem} \equiv R_{\mu\nu\rho\sigma}$. Hence \mathfrak{Riem} is also Hodge self-dual on any solution, which implies that the solutions of I_{Inst} correspond to gravitational instantons.*

§6. Relation to the CDJ pure spin connection formulation.

There is an action for General Relativity derived by Capovilla, Dell and Jacobson (CDJ), which can be written almost entirely in terms of the spin connection [6]. The authors used Plebanski's action (1) as the starting point, from which they proceed to eliminate the 2-forms $\Sigma_{\mu\nu}^a$ and the matrix ψ_{ae} , leading for $\Lambda = 0$ to the action

$$I_{\text{CDJ}} = \int_{\mathcal{M}} d^4x \operatorname{tr} \left[M \left(M - \frac{1}{2} \operatorname{tr} M \right) \right], \quad (61)$$

where we have defined

$$M^{bf} = -\frac{i}{8\sqrt{-g}} F_{\mu\nu}^b F_{\rho\sigma}^f \epsilon^{\mu\nu\rho\sigma}. \quad (62)$$

Note that equation (62) for [6] is the same as (29), which is the symmetric part of (51). The action (34) serves in [6] as an intermediate step in obtaining the action (61) from (1).[†] But in our context, equation (34) follows from (18) after elimination of $\Psi_{[ae]}$ and N^i through their equations of motion.

Given that the CDJ action essentially follows from (18) after elimination of Ψ_{ae} , then this implies that Ψ_{ae} should satisfy equation (2.20b) of [6] on any solution for $\Lambda = 0$. We will show this by following the same steps in [6]. To obtain Ψ_{ae} in terms of A_μ^a , one would need to take the square root of M^{bf} in (62). This introduces various complications, which are circumvented in [6] by using the characteristic equation for (a symmetric) Ψ_{ae}

$$\Psi^{-3} - (\operatorname{tr} \Psi^{-1}) \Psi^{-2} + \frac{1}{2} \left[(\operatorname{tr} \Psi^{-1})^2 - \operatorname{tr} \Psi^{-2} \right] \Psi^{-1} - \det \|\Psi\|^{-1} = 0. \quad (63)$$

*The gauge curvature $F_{\mu\nu}^a$ takes its values in the $\text{SO}(3, \mathbb{C})$ Lie algebra corresponding to the self-dual half of the Lorentz group $\text{SO}(3, 1)$. The equivalence of internal self-duality with Hodge self-duality makes sense when one has a tetrad e_μ^I , which intertwines between internal and spacetime indices. But since tetrads are now derived quantities in I_{Inst} , this feature appears to be more fundamentally related to the Yang-Mills aspects of the theory. We will show in a few paragraphs that this is indeed the case.

[†]Note that Ψ in the present paper, after the elimination of the shift vector N^i is actually defined as Ψ^{-1} in [6].

One must then use $\Psi^{-1}\Psi^{-1} = M$ from (29) as well as $\text{tr}\Psi^{-1} = -\Lambda$ from (22), which when substituted into (63) yields the equation

$$\left[M + \frac{1}{2}(-\text{tr}M + \Lambda^2) \right] \Psi^{-1} = -\Lambda M + I\sqrt{\det\|M\|}, \quad (64)$$

where I is the unit 3×3 matrix. Then assuming that the left hand side of (64) is invertible, one can solve for $\Psi_{(ae)}$ as

$$\Psi_{(ae)} = \left(-\Lambda_{af} + \delta_{af}\sqrt{\det\|M\|} \right)^{-1} \left[M_{fe} + \frac{1}{2}\delta_{fe}(\Lambda^2 - \text{tr}M) \right]. \quad (65)$$

Then upon substitution of (65) into (34) one obtains the CDJ action (62) for $\Lambda = 0$. For $\Lambda \neq 0$ one can expand (65) in powers of Λ using a geometric series, yielding

$$\Psi_{ae} = -\frac{1}{\Lambda} \left\{ \left[\delta_{ae} - \frac{\Lambda(\Lambda^2 - \text{tr}M)}{2\sqrt{\det\|M\|}} + 1 \right] \left[\delta_{ae} - \frac{\Lambda M_{ae}}{\sqrt{\det\|M\|}} \right]^{-1} \right\}. \quad (66)$$

Then one obtains the analogue of equation (3.9) of [6], which we will not display here.

Let us now comment on the differences between (18) and (34), namely equation (2.8) in [6]. Equation (34) can be obtained by elimination of the 2-forms $\Sigma_{\mu\nu}^a$ directly from (1). Then the CDJ action (61) follows by further elimination of the field Ψ_{ae} . But (18) is the result of eliminating only Σ_{ij}^a , the spatial part of the 2-forms, and preserving the temporal components Σ_{0i}^a as well as Ψ_{ae} .^{*} By complete elimination of the 2-form Σ^a as in [6], one also eliminates the flexibility of implementing the Hamiltonian and diffeomorphism constraints in (44). These are necessary for the construction of the metric $g_{\mu\nu}$, which plays the dual roles of solving the Einstein equations and enforcing Hodge duality. Additionally, in equation (2.8) in [6] the matrix Ψ does not have an antisymmetric part, whereas $\Psi_{[ae]}$ was necessary in order to obtain (45) as well as the shift vector N^i . These two features constitute a vital part of the Hodge duality condition (51).

^{*}The exception to this is the time gauge $e_i^0 = 0$, from which (18) follows. This has the effect of fixing the boost parameters corresponding to the local Lorentz frame. Since the $\text{SO}(3, \mathbb{C})$ and $\text{SU}(2)$ Lie algebras are isomorphic, (1) can be regarded as being based on the self-dual $\text{SU}(2)_-$ part of the Lorentz algebra, which leaves open the interpretation of the antiself dual part $\text{SU}(2)_+$. Since only $\text{SU}(2)_-$ is needed in order to obtain General Relativity, it could be that e_i^0 is somehow associated with $\text{SU}(2)_+$. On a separate note, we have preserved the temporal 2-form components Σ_{0i}^a in I_{inst} , in order to preserve the freedom to implement the initial value constraints.

The Einstein equations for $\Lambda = 0$ can be derived from (61), which is shown as equations (2.19a), (2.19b) and (2.20a) in [6]. But the statement that the metric (equations (2.2) and (2.4) in [6]) arises as a solution to these Einstein equations appears to the best of the present author's knowledge to be a separate postulate not derivable directly from (61). We will show explicitly in the present paper that this metric is the same one arising from the Hodge duality condition (45), and complete the missing link in this loop regarding the Einstein equations.

§7. The spacetime metric: revisited. We have shown that the instanton representation I_{Inst} , on-shell, implies a Hodge self-duality condition for the $\text{SO}(3, \mathbb{C})$ curvature $F_{\mu\nu}^a$ with respect to a spacetime metric $g_{\mu\nu}$ solving the Einstein equations which also follow from I_{Inst} . All that is needed to construct the 3-metric h_{ij} for this spacetime metric are the spatial connection A_i^a and the CDJ matrix Ψ_{ae} solving the initial value constraints (44). The specification of the shift vector N^i via $A_0^a \subset A_\mu^a = (A_0^a, A_i^a)$, combined with a lapse function N , then completes the construction of $g_{\mu\nu}$ via (40). We will see that I_{Inst} provides an additional simple formula for constructing $g_{\mu\nu}$ via the concept of Hodge duality. The Hodge self-duality condition (59) is given by

$$\epsilon_{ijk} B_a^j N^k + i \underline{N} h_{ij} B_a^j = -F_{0i}^a. \quad (67)$$

Multiplying (67) by $(B^{-1})_m^a$, we obtain the relation

$$\epsilon_{ijk} N^k + i \underline{N} h_{ij} = -F_{0i}^a (B^{-1})_j^a. \quad (68)$$

Equation (68) provides a prescription for writing the spacetime metric explicitly in terms of the connection as follows.* The antisymmetric part of (68) yields the shift vector

$$N^k = -\frac{1}{2} \epsilon^{kij} F_{0i}^a (B^{-1})_j^a, \quad (69)$$

and the symmetric part yields the 3-metric up to a conformal factor

$$i \underline{N} h_{ij} = -F_{0(i}^a (B^{-1})_{j)}^a \equiv -c_{(ij)}, \quad (70)$$

where we have defined $c_{ij} = F_{0i}^a (B^{-1})_j^a$. The determinant of (70) yields

$$-i \frac{N^3}{\sqrt{h}} = -\det \|c_{(ij)}\| \equiv -c \longrightarrow i \underline{N} = \frac{c}{N^2}. \quad (71)$$

*In other words, the physical degrees of freedom from the initial value constraint contained in (44) become absorbed into the definition of the 3-metric h_{ij} .

Substituting this relation back into (70) enables us to solve for h_{ij}

$$h_{ij} = -\frac{N^2}{c} c_{(ij)}. \quad (72)$$

Let us define the following densitized object $\underline{c}_{(ij)} = c^{-1} c_{(ij)}$. Then the line element (40) can also be written as*

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2(dt^2 + \underline{c}_{(ij)} \omega^i \otimes \omega^j), \quad (73)$$

where we have defined the one forms $\omega^i = dx^i - N^i dt$, with N^i given by equation (69). Starting from a spacetime of Lorentzian (Euclidean) signature for the lapse function N real (imaginary), we have obtained a line element (73). This implies the following consistency conditions

$$\left. \begin{array}{l} \underline{c}_{ij} > 0 \longrightarrow N \text{ imaginary} \longrightarrow \text{Euclidean signature} \\ \underline{c}_{ij} < 0 \longrightarrow N \text{ real} \longrightarrow \text{Lorentzian signature} \end{array} \right\}. \quad (74)$$

The result is that every connection A_μ^a with nondegenerate magnetic field B_a^i , combined with a lapse function N , determines a spacetime metric $g_{\mu\nu}$ of signature given by (74) solving the Einstein equations.

An elegant formula was constructed by Urbantke, which determines the metric with respect to which a given $SU(2)$ Yang-Mills curvature, is self-dual in the spacetime sense. The formula is given by [16]

$$\sqrt{-g} g_{\mu\nu} = \frac{4}{3} \eta f_{abc} F_{\mu\rho}^a F_{\alpha\beta}^b F_{\sigma\nu}^c \epsilon^{\rho\alpha\beta\sigma}. \quad (75)$$

Since we are treating General Relativity in analogy with Yang-Mills theory, it is relevant to perform a 3+1 decomposition of (75). The result of this decomposition is given by[†]

$$g_{00} \propto \det \|F_{0i}^a\|, \quad g_{0k} \propto \epsilon_{klm} (F^{-1})_c^{0l} B_c^m, \quad g_{ij} \propto F_{0(i}^a (B^{-1})_{j)}^a. \quad (76)$$

Comparison of (76) with (69) and (70) reveals that on-shell, the instanton representation of Plebanski gravity reproduces the Urbantke metric purely from an action principle. When the spatial part of the Urbantke metric is built from variables solving the constraints (44), then the Urbantke metric also solves the Einstein equations by construction.

*More precisely, since (40) as defined by (44) forms a subset of the line element defined by (73), then the equality of (40) with (73) must be regarded as a consistency condition. Since (67) contains a velocity \dot{A}_i^a and (44) does not, then the interpretation is that the equality between the line elements (40) and (73) must enforce the time evolution of initial data satisfying the initial value constraints (44).

[†]See Appendix B for the details of the derivation.

§7.1. Reality conditions. Since the connection A_μ^a is allowed to be complex, then the line element (73) in general allows for complex metrics $g_{\mu\nu}$. General Relativity should correspond to the restriction of this to real-valued metrics, which implies certain conditions on A_μ^a such that \underline{c}_{ij} be real-valued in (74). So the imposition of reality conditions requires that the undensitized matrix $c_{ij} = F_{0i}^a (B^{-1})_j^a$ be either real or pure imaginary, which leads to two cases

$$\left. \begin{array}{l} c_{(ij)} \text{ real} \longrightarrow \text{Euclidean signature} \\ c_{(ij)} \text{ imaginary} \longrightarrow \text{Lorentzian signature} \end{array} \right\}. \quad (77)$$

We will see that (77) places restrictions on the connection A_μ^a for a spacetime of fixed signature. For a general A_μ^a satisfying the reality conditions, there is apparently no constraint fixing the signature of the spatial part of the metric h_{ij} .*

The metric is clearly real if one is restricted to connections having a real curvature $F_{\mu\nu}^a$. When $F_{\mu\nu}^a$ is complex then we must impose reality conditions requiring \underline{c}_{ij} to be real as in (74). The symmetric part of this enforces reality of the 3-metric h_{ij} and the antisymmetric part enforces reality of the shift vector N^i . The lapse function N must always be chosen to be either real or pure imaginary. The signature of spacetime, which in either case apparently may change, might be more directly related to the reality of the metric. This is unlike the case in the Ashtekar variables, where for Euclidean signature spacetimes one is restricted to real variables.

We will now delineate the reality conditions on the spacetime metric for the case where the curvature $F_{\mu\nu}^a$ is complex. First let us perform the following split of the connection A_μ^a into the real and imaginary parts of its spatial and temporal components

$$A_i^a = (\Gamma - iK)_i^a, \quad A_0^a = (\eta - i\zeta)^a. \quad (78)$$

Corresponding to this 3+1 split, there is an analogous 3+1 split induced upon $F_{\mu\nu}^a$ into spatial and temporal components. The spatial part of this defines the magnetic field B_a^i given by

$$B_a^i = (R - iT)_a^i, \quad (79)$$

*Hence there is a caveat associated with the labels “Euclidean” and “Lorentzian” used in (77). The lapse function N is freely specifiable, since it is not constrained by A_μ^a . But it is still conceivable in (77) that different components of $\underline{c}_{(ij)}$ could have different signs based on the initial data of A_μ^a . If this were to be the case, then this could bring in the possibility of topology changes for spacetimes described by I_{Inst} if the signature were not preserved under time-evolution.

where we have defined

$$\left. \begin{aligned} R_a^i &= \epsilon^{ijk} \partial_j \Gamma_k^a + \frac{1}{2} \epsilon^{ijk} f_{abc} \Gamma_j^b \Gamma_k^c - \frac{1}{2} \epsilon^{ijk} f^{abc} K_j^b K_k^c \\ T_a^i &= \epsilon^{ijk} D_j K_k^a = \epsilon^{ijk} (\partial_j K_k^a + f^{abc} \Gamma_j^b K_k^c) \end{aligned} \right\}. \quad (80)$$

The quantity T_a^i is the covariant curl of K_i^a using Γ_i^a as a connection. The temporal part of the curvature $F_{\mu\nu}^a$ is given by

$$F_{0i}^a = (f - ig)_i^a, \quad (81)$$

where we have defined

$$\left. \begin{aligned} f_i^a &= \dot{\Gamma}_i^a - D_i \eta^a + f^{abc} K_i^b \zeta^c \\ g_i^a &= D_0 K_i^a - D_i \zeta^a \end{aligned} \right\}. \quad (82)$$

The operator D_i is the covariant derivative with respect to Γ_i^a as in the second line of (80), and D_0 is given by

$$D_0 K_i^a = \dot{K}_i^a + f^{abc} \eta^b K_i^c. \quad (83)$$

For the general complex case, reality conditions require that $c_{ij} = (B^{-1})_i^a F_{0j}^a$ be either real or pure imaginary as in (77). It will be convenient to use the following matrix identity, suppressing the indices

$$B^{-1} = (R - iT)^{-1} = (1 + iRT)[1 - (R^{-1}T)^2]^{-1}R^{-1}, \quad (84)$$

which splits the inverse of a complex matrix into its real and imaginary parts. Then upon contraction of the internal indices, c_{ij} is given by

$$\begin{aligned} (f - ig)(R - iT)^{-1} &= [f + gR^{-1}T + i(-g + fR^{-1}T)] \times \\ &\quad \times [1 - (R^{-1}T)^2] R^{-1}. \end{aligned} \quad (85)$$

The last two matrices in (85) are real and the first matrix is in general complex. For Lorentzian signature spacetimes we must require the real part of the first matrix to be zero, and for Euclidean signature we must require the imaginary part to be zero. This leads to the matrix equations

$$\left. \begin{aligned} \text{Euclidean signature:} & \quad g^{-1}f = -R^{-1}T \\ \text{Lorentzian signature:} & \quad f^{-1}g = R^{-1}T \end{aligned} \right\}. \quad (86)$$

The aforementioned caveats still apply with respect to the stability of the signature. But in either case the reality conditions constitute

9 equations in 24 unknowns, namely the 12 complex components of the four-dimensional $SO(3, \mathbb{C})$ connection A_μ^a . After implementation of these reality conditions, then this leaves $24 - 9 = 15$ real degrees of freedom in A_μ^a .*

§8. Gravity as a “generalized” Yang-Mills theory. We will now show how the concept of Hodge self-duality stems at a more fundamental level from internal duality with respect to gravitational degrees of freedom. Let us start off by considering the following action which resembles $SO(3, \mathbb{C})$ Yang-Mills theory in curved spacetime

$$I = \int_M d^4x \left(-\frac{1}{4} \sqrt{-g} g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^f \Psi_{bf} + \frac{1}{G} \sqrt{-g} R \right), \quad (87)$$

where $R = R[g]$ is meant to signify that R is the curvature of the same metric which appears in the Yang-Mills term.

The quantity $g^{\mu\nu}$ is the covariant metric corresponding to the background spacetime upon which a Yang-Mills field A_μ^a propagates, and $F_{\mu\nu}^a$ is the curvature of A_μ^a , given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c, \quad (88)$$

where $f^{abc} = \epsilon^{abc}$ are the structure constants of $SO(3, \mathbb{C})$.

Equation (87) is different from the usual Yang-Mills theory in that the two curvatures $F_{\mu\nu}^a$ additionally couple to a field Ψ_{bf} taking its values in two copies of $SO(3, \mathbb{C})$. In the special case $\Psi_{ae} = k \delta_{ae}$ for some numerical constant k , Ψ_{ae} plays the role of the Cartan-Killing metric for the $SO(3, \mathbb{C})$ Lie algebra. There is a wide array of literature concerning gravity and Yang-Mills theory, where one attempts to solve (87) for the Yang-Mills field A_μ^a as well as for the metric $g_{\mu\nu}$. But in the gravitational context, $\Psi_{ae} = -\frac{3}{\Lambda} \delta_{ae}$ implies that the metric $g_{\mu\nu}$ must be restricted to spacetimes of Petrov Type O, since Ψ_{ae} then has three equal eigenvalues [10].

The implication is that when one solves (87) in the case $\Psi_{ae} = -\frac{3}{\Lambda} \delta_{ae}$, then one is solving the coupled Yang-Mills theory only for conformally flat spacetimes. But we would like to incorporate more general geometries. On the one hand in vacuum Yang-Mills theory one already has a Yang-Mills solution for known metrics by virtue of Hodge duality and the Bianchi identity. On the other hand, the generalization of Ψ_{ae} to

*For example in (78), then one possibility is to regard the nine components of $\Re\{A_i^a\}$ as freely specifiable, and then use (86) to determine the nine components of $\Im\{A_i^a\}$ in terms of them.

include gravitational degrees of freedom, as we will see, enables one to identify the Yangs-Mills theory with the gravity theory that it is coupling to. To see this, let us split the Yang-Mills part of the Lagrangian of (87) into its spatial and temporal parts

$$L_{\text{YM}} = \frac{\sqrt{-g}}{2} \left(g^{00} g^{ij} F_{0i}^b F_{0j}^f - g^{0i} g^{0j} F_{0i}^b F_{0j}^f + 2g^{0i} g^{jk} F_{ij}^b F_{0k}^f + \frac{1}{2} g^{ik} g^{jl} F_{ij}^b F_{kl}^f \right) \Psi_{bf}, \quad (89)$$

where $F_{0i}^a = \dot{A}_i^a - D_i A_0^a$ is the temporal component of the curvature.

The electric field is the momentum canonically conjugate to the Yang-Mills spatial connection

$$\Pi_b^i = \frac{\delta I_{\text{YM}}}{\delta \dot{A}_i^b} = \sqrt{-g} \left(g^{00} g^{ij} F_{0j}^f - g^{0i} g^{0j} F_{0j}^f + g^{0m} g^{ni} F_{mn}^f \right) \Psi_{bf}. \quad (90)$$

Next, we will make use of the 3+1 decomposition of the spacetime metric

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{0j} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix},$$

where $N^\mu = (N, N^i)$ are the lapse function and shift vector, and $\sqrt{-g} = N\sqrt{h}$ is the determinant of $g_{\mu\nu}$. Substitution into (90) yields

$$\Pi_b^i = \frac{\sqrt{h}}{N} \left(-h^{ij} F_{0j}^f + N^m h^{ni} F_{mn}^f \right) \Psi_{bf}, \quad (91)$$

and substitution into (89) yields

$$L_{\text{YM}} = -\frac{1}{2} N \sqrt{h} \left[-\frac{1}{N^2} h^{ij} F_{0i}^b F_{0j}^f + 2 \frac{N^i}{N^2} \left(h^{jk} - \frac{N^j N^k}{N^2} \right) F_{ij}^b F_{0k}^f + \frac{1}{2} h^{ik} \left(h^{jl} - \frac{2N^j N^l}{N^2} \right) F_{ij}^b F_{kl}^f \right] \Psi_{bf}. \quad (92)$$

We will now eliminate the velocities \dot{A}_i^a from (92) by inverting (91)

$$F_{0j}^f = h_{jk} \left[-\frac{N}{\sqrt{h}} \Pi_b^k (\Psi^{-1})^{bf} + N^m h^{nk} F_{mn}^f \right]. \quad (93)$$

Upon substitution of (93) into (92) after several long but straightforward algebraic steps, we obtain

$$L_{\text{YM}} = \frac{1}{2} \frac{N}{\sqrt{h}} h_{ij} \Pi_b^i \Pi_f^j (\Psi^{-1})^{bf} + \frac{1}{4} N \sqrt{h} h^{ik} h^{jl} F_{ij}^b F_{kl}^f \Psi_{bf}. \quad (94)$$

Defining the $\text{SO}(3, \mathbb{C})$ magnetic field by $F_{ij}^a = \epsilon_{ijk} B_a^k$, and using the relation

$$\frac{1}{2} \epsilon_{ijm} \epsilon_{klm} h^{ik} h^{jl} = \frac{1}{h} h_{mn}, \quad (95)$$

and presupposing the 3-metric h_{ij} to be nondegenerate, then (94) yields

$$L_{\text{YM}} = \frac{1}{2} \underline{N} h_{ij} \left[(\Psi^{-1})^{bf} \Pi_b^i \Pi_f^j - \Psi_{bf} B_b^i B_f^j \right]. \quad (96)$$

This is the electromagnetic decomposition of the generalized Yang–Mills action, with Ψ_{bf} replacing the invariant Cartan–Killing form for the $\text{SO}(3)$ gauge group. But for geometries not of Petrov Type O, then Ψ_{bf} is in general no longer $\text{SO}(3, \mathbb{C})$ invariant.

To see how General Relativity follows from this “generalized” Yang–Mills theory, let us impose the following relation between the electric and the magnetic fields of the latter

$$\Pi_a^i = \beta \Psi_{ae} B_e^i \quad (97)$$

for some numerical constant β . Then for nondegenerate Ψ_{bf} , substitution of (97) into (90) implies that

$$\beta B_f^i = N \sqrt{h} \left(g^{00} g^{ij} F_{0j}^f - g^{0i} g^{0j} F_{0j}^f + g^{0m} g^{ni} F_{mn}^f \right). \quad (98)$$

The right hand side of (98) is given by

$$N \sqrt{h} \left[-\frac{1}{N^2} \left(h^{ij} - \frac{N^i N^j}{N^2} \right) F_{0j}^f - \frac{N^i N^j}{N^4} F_{0j}^f - \frac{N^m}{N^2} \left(h^{ni} - \frac{N^n N^i}{N^2} \right) F_{mn}^f \right], \quad (99)$$

which simplifies to

$$\frac{\sqrt{h}}{N} \left(h^{ij} F_{0j}^f + N^k h^{ij} F_{kj}^f \right) = -\beta B_f^i. \quad (100)$$

Equation (100) can be rewritten as

$$F_{0j}^f + \epsilon_{jmk} B_f^m N^k + \beta \underline{N} h_{ji} B_f^i = 0. \quad (101)$$

The choice $\beta = \pm i$ would imply that equation (97) automatically imposes Hodge self-duality of the Yang–Mills curvature $F_{\mu\nu}^f$ with respect to the metric $g_{\mu\nu}$ which it couples to, namely

$$H^{\mu\nu\rho\sigma} F_{\rho\sigma}^b = 0, \quad (102)$$

where we have defined the Hodge self-duality operator

$$H^{\mu\nu\rho\sigma} = \frac{1}{2} [\sqrt{-g} (g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma}) + \beta \epsilon^{\mu\nu\rho\sigma}]. \quad (103)$$

Comparison of (97) with the spatial restriction of equation the third equation of (2), and comparison of (101) with (51), implies that (97) is the internal analogue of Hodge self-duality. Indeed, the fact that the metric defining (102) solves the Einstein equations transforms (34) on-shell into (87). Since the solutions to ordinary vacuum Yang-Mills theory include Yang-Mills instantons, then this suggests that I_{Inst} is a theory which should include gravitational instantons.

§9. Gravitational instantons: revisited. We will now put into context the points raised in the introduction paragraph regarding the apparent ambiguity in the definition of gravitational instantons. It has been noted by Ashtekar and Renteln [1] that the ansatz

$$B_a^i = -\frac{\Lambda}{3} \tilde{\sigma}_a^i, \quad (104)$$

solves the initial value constraints of the Ashtekar variables arising from (13). It was noted that this corresponds to the conformally flat space-times.* There is a covariant form of the action (13) provided by Samuel [18, 19] in which the basic variables are two forms $\Sigma^b = \frac{1}{2} \Sigma_{\mu\nu}^b dx^\mu \wedge dx^\nu$, given by

$$I = \int_{\mathcal{M}} d^4x \left(\Sigma_{\mu\nu}^b F_{\rho\sigma}^b + \frac{\Lambda}{6} \Sigma_{\mu\nu}^b \Sigma_{\rho\sigma}^b \right) \epsilon^{\mu\nu\rho\sigma}. \quad (105)$$

Equation (105) leads to General Relativity with cosmological constant through the equations of motion

$$\epsilon^{\mu\nu\rho\sigma} D_\nu \Sigma_{\rho\sigma}^b = 0, \quad F_{\mu\nu}^b = -\frac{\Lambda}{3} \Sigma_{\mu\nu}^b, \quad (106)$$

where the two form is constructed from $\text{SL}(2, \mathbb{C})$ one forms

$$\Sigma_{\mu\nu}^{AB} = i \left(e_\mu^{AA'} e_{\nu A'}^B - e_\nu^{AA'} e_{\mu A'}^B \right) \quad (107)$$

in self-dual combination. The class of solutions described by the second equation of (106) are the evolution of (104), which is the data set on the initial spatial hypersurface. The observation that the first equation

*We will see that (97) is the generalization of (104) which incorporates more general geometries including Types D and O, when Ψ_{ae} becomes identified with the CDS matrix Ψ_{ae} of I_{Inst} .

of (106) follows identically from the second due to the Bianchi identity, combined with the self duality in (107) allows an association of gravitation with Yang-Mills instantons to be inferred [18].

It was postulated that there might be other Yang-Mills field strengths which satisfy (106), but one is limited to conformally flat metrics since not all two forms Σ^a are constructible from tetrad one forms $e_\mu^{AA'}$ as in (107). The problem of relating (106) to the Yang-Mills self-duality condition $*F = F$ resides in the observation that the metric $g_{\mu\nu}$ must first be known. In [7], Jacobson eliminates the tetrad from the self-duality condition to address the sector with vanishing self-dual Weyl curvature, by proposing the following condition on the curvature

$$F^b \wedge F^f - \frac{1}{3} \delta^{bf} \text{tr} F \wedge F = 0. \quad (108)$$

Given a connection A_μ^a which solves (108), the tetrads in (107) associated with the 2-forms Σ^b determine a metric which is a self-dual Einstein solution with cosmological constant Λ . Moreover, the curvature satisfying (108) is self-dual with respect to this metric. Since (108) is the same as the second equation of (2) when $\Psi_{ae} \propto \delta_{ae}$, then the problem of “finding the metric” as pointed out by Samuel in [18] translates into the problem of finding the connection in (108).

Hence the aforementioned developments have been shown only for the conformally self-dual case where the self-dual Weyl tensor ψ_{ae} vanishes, whence the metric is explicitly constructible. This limits one to spacetimes of Petrov Type O.* The proposition of the present paper has been to extend the library of solutions to include the Petrov Types I and D cases using I_{Inst} .

§9.1. Generalization beyond Petrov Type O instantons. We have seen that the CDJ ansatz, the spatial restriction of the third equation of (2), imposes the condition of Hodge self-duality on the “generalized” $\text{SO}(3, \mathbb{C})$ Yang-Mills fields in (97). When Ψ_{ae} is chosen to satisfy the constraints (44), then the implication is that this Yang-Mills theory becomes a theory of General Relativity. Since vacuum Yang-Mills theory in conformally flat spacetimes describes instantons, then this suggests that the gravitational analogue of pure Yang-Mills theory must describe gravitational instantons, specifically incorporating the physical

*In [8] gravitational instantons are defined as spacetimes with vanishing self-dual Weyl curvature, and nonvanishing cosmological constant. This falls within the Petrov Type O case with $\Psi_{ae} = -\frac{3}{\Lambda} \delta_{ae}$, with no restrictions on the connection A_i^a . We would like to generalize this to incorporate Type D and Type I spacetimes.

degrees of freedom from (44). To examine the implications for gravity let us recount the action (34), repeated here for completeness

$$I_{\text{Inst}} = \int_{\text{M}} d^4x \frac{1}{8} \Psi_{ae} F_{\mu\nu}^a F_{\rho\sigma}^e \epsilon^{\mu\nu\rho\sigma} - i\sqrt{-g} (\Lambda + \text{tr} \Psi^{-1}), \quad (109)$$

which corresponds to (19) at the level after elimination of $\Psi_{[ae]}$ and the shift vector N^i .

Recall also that the equation of motion for Ψ_{ae} prior to elimination of N^i and in (45) implies the Hodge self-duality condition

$$\beta \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a = \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}^a \quad (110)$$

once one has made the identification of $h_{ij} = h_{ij}[\Psi, A]$. Substitution of (110) into the first term of (109) yields

$$I_{\text{Inst}} = \int_{\text{M}} d^4x \frac{\beta}{4} \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}^a F_{\rho\sigma}^e \Psi_{ae} - i\sqrt{-g} (\Lambda + \text{tr} \Psi^{-1}), \quad (111)$$

which is nothing other than the action for gravity coupled to a ‘‘generalized’’ $\text{SO}(3, \mathbb{C})$ Yang-Mills theory of gravity (87). On the other hand, the equation of motion for Ψ_{ae} derived from (109) is

$$\frac{1}{8} F_{\mu\nu}^b F_{\rho\sigma}^f \epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{fb} = 0. \quad (112)$$

Comparison of (112) with (43) indicates that dynamically on the solution to the equations of motion,

$$\begin{aligned} \frac{1}{8} F_{\mu\nu}^b F_{\rho\sigma}^f \epsilon^{\mu\nu\rho\sigma} &= -i\beta^{-1/2} N (\det \|B\|)^{-1/2} (\det \|\Psi\|)^{-1/2} \times \\ &\times h_{ij} B_b^i B_f^j = -i\beta \underline{N} h_{ij} B_b^i B_f^j, \end{aligned} \quad (113)$$

where $\underline{N} = N h^{-1/2}$. Since the initial value constraints must be consistent with the equations of motion we can insert (113) into (109), which yields

$$I_{\text{Inst}} = \frac{\beta}{2} \int_{\text{M}} \Psi_{ae} F^a \wedge F^e = -i\beta \int_{\text{M}} \underline{N} h_{ij} \Psi_{ae} B_a^i B_e^j d^4x. \quad (114)$$

But equation (114) is only the magnetic part of a Yang-Mills theory in curved spacetime. To obtain the respective electric part we use the relation $B_e^i = \frac{1}{\beta} \Psi_{ae}^{-1} \tilde{\sigma}_a^i$, which shows on-shell that the following objects are equivalent

$$\begin{aligned} -i\beta \underline{N} h_{ij} B_b^i B_f^j \Psi_{bf} &= -i \underline{N} h_{ij} \tilde{\sigma}_b^i B_f^j = \\ &= -i\beta \underline{N} h_{ij} (\Psi^{-1})^{bf} \tilde{\sigma}_b^i \tilde{\sigma}_f^j. \end{aligned} \quad (115)$$

So we can use (115) to eliminate B_a^i from (114), yielding

$$I_{\text{Inst}} = \frac{\beta}{2} \int_{\text{M}} \Psi_{ae} F^a \wedge F^e = -i\beta \int_{\text{M}} \frac{1}{\beta^2} \underline{N} h_{ij} (\Psi^{-1})^{ea} \tilde{\sigma}_a^i \tilde{\sigma}_e^j d^4x. \quad (116)$$

The action for the instanton representation I_{Inst} evaluated on a classical solution can be written as the average of the actions (114) and (116), which yields

$$\begin{aligned} I_{\text{Inst}} &= \frac{i\beta}{2} \int dt \int_{\Sigma} d^3x \underline{N} h_{ij} \left[-\frac{1}{\beta^2} (\Psi^{-1})^{bf} \tilde{\sigma}_b^i \tilde{\sigma}_f^j - \Psi_{bf} B_b^i B_f^j \right] = \\ &= i\beta \int dt \int_{\Sigma} d^3x \left[\underline{N} h_{ij} T^{ij} - \frac{i}{2} \beta \left(1 + \frac{1}{\beta^2} \right) \underline{N} h_{ij} \tilde{\sigma}_b^i \tilde{\sigma}_f^j (\Psi^{-1})^{bf} \right] \end{aligned} \quad (117)$$

with T^{ij} given by

$$T^{ij} = \frac{1}{2} \left[(\Psi^{-1})^{ae} \tilde{\sigma}_a^i \tilde{\sigma}_e^j - \Psi_{ae} B_a^i B_e^j \right]. \quad (118)$$

With the exception of the term proportional to β , (117) would be the action for a “generalized” Yang-Mills theory. Note that it is a genuine Yang-Mills theory only for $\Psi_{ae} = k\delta_{ae}$, which covers only the Type O sector of gravity.

Upon making the identification $\tilde{\sigma}_a^i \equiv \Pi_a^i$ from (96), then we have on the solution to the equations of motion that

$$\frac{1}{8} \int_{\text{M}} d^4x \Psi_{bf} F_{\mu\nu}^b F_{\rho\sigma}^f \epsilon^{\mu\nu\rho\sigma} = i\beta \int_{\text{M}} d^4x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}^b F_{\rho\sigma}^f + Q, \quad (119)$$

where Q is the second term in the bottom line of (117). The identification between the Yang-Mills and the instanton representation actions can be made only for $\beta^2 = -1$. In this case $Q = 0$ and equation (119) implies on the solution to the equations of motion that

$$\frac{1}{8} \int_{\text{M}} d^4x \left(\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma} \pm i \epsilon^{\mu\nu\rho\sigma} \right) F_{\mu\nu}^b F_{\rho\sigma}^f \Psi_{bf} = 0. \quad (120)$$

In order for this to be true for all curvatures, we must have

$$\pm \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^f = \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}^f, \quad (121)$$

namely that the curvature of the starting theory must be self-dual in the Hodge sense in any solution to the equations of motion. In this case, it can be said that General Relativity is literally a Yang-Mills theory coupled gravitationally to itself.

§10. Summary. In this paper we have presented the instanton representation of Plebanski gravity, a new formulation of General Relativity. The basic dynamical variables are an $SO(3, \mathbb{C})$ gauge connection A_μ^a and a matrix Ψ_{ae} taking its values in two copies of $SO(3, \mathbb{C})$. The consequences of the associated action I_{Inst} were determined via its equations of motion with the following results:

- 1) The two equations of motion for I_{Inst} imply the Einstein equations when the initial value constraints are satisfied;
- 2) When these constraints are satisfied, then one can define a spatial 3-metric $h_{ij}[\Psi, A]$ using Ψ_{ae} and A_i^a , the spatial part of the connection A_μ^a ;
- 3) The first equation of motion for I_{Inst} is consistent with the second equation when the initial value constraints are satisfied;
- 4) The first equation of motion of I_{Inst} implies that the curvature $F_{\mu\nu}^a$ is Hodge self-dual with respect to the metric $g_{\mu\nu}$ which solves the Einstein equations as a consequence of the initial value constraints.

Each of these results hinges crucially on the existence of solutions to the initial value constraints. So it remains to be verified that once the initial value constraints are satisfied on an initial spatial hypersurface, then the equations of motion should preserve these constraints for all time. We will relegate demonstration of this for a future publication.

Additionally, we have clarified the relation between I_{Inst} and I_{CDJ} in [5]. The two formulations are not the same as it may naively appear for the following reasons:

- 1) The action I_{CDJ} at the level prior to elimination of Ψ_{ae} from I_{Pleb} is missing the 2-forms $\Sigma_{\mu\nu}^a$ as well as the antisymmetric part of Ψ_{ae} . However, I_{Inst} contains Σ_{0i}^a , the temporal part of $\Sigma_{\mu\nu}^a$ as well as $\Psi_{[ae]}$;
- 2) The Hodge duality condition follows directly as an equation of motion for I_{Inst} , a crucial part of which involves $N^\mu = (N, N^i)$ from Σ_{0i}^a which are needed both for constructing General Relativity solutions as well as for implementing the initial value constraints*;
- 3) The reality conditions in I_{Inst} appear to be intimately connected with the signature of spacetime as well as initial data, which is unlike the usual formulations of General Relativity. The implications of this should be borne out when one attempts to construct solutions.

*The advantages of these features should become more apparent when one proceeds to construct General Relativity solutions and in the quantum theory.

The instanton representation I_{Inst} has exposed an interesting relation between General Relativity and Yang-Mills theory, which suggests that this is indeed a theory of “generalized” Yang-Mills instantons. In the conformally flat case, the CDJ matrix Ψ_{ae} has three equal eigenvalues and thus plays the role of a Cartan-Killing $\text{SO}(3, \mathbb{C})$ invariant metric. The generalization of this to more general geometries presents an interesting physical interpretation, since Ψ_{ae} contains gravitational degrees of freedom. In the Petrov Type D case for example, where Ψ_{ae} has two equal eigenvalues, then the Yang-Mills $\text{SO}(3)$ symmetry becomes broken down to $\text{SO}(3, \mathbb{C})$. In the algebraically general Type I case, where $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the $\text{SO}(3, \mathbb{C})$ symmetry becomes completely broken. A possible future direction is to investigate possible mechanisms which could induce such a breaking of this symmetry.

Nevertheless, the first order of business in future research will be to check for consistency of the initial value constraints of I_{Inst} under time evolution. Then next will be to use I_{Inst} reconstruct as many of the known General Relativity solutions as possible and to construct new solutions. Additionally, we will examine the quantum theory with a view to addressing many of the unresolved questions in quantum gravity.

§10.1. Preview into the quantum theory. Instantons in Yang-Mills theory can be associated with transitions between topologically inequivalent vacua, induced by tunnelling classical solutions upon Wick rotation between Lorentzian and Euclidean signature spacetimes. A future direction of research will be to investigate the analogue of this feature for I_{Inst} , in addition to the quantum aspects of the theory. For instance, upon substitution of contraction of (112) with Ψ_{bf} one obtains the relation

$$\frac{1}{8} \Psi_{bf} F_{\mu\nu}^b F_{\rho\sigma}^f \epsilon^{\mu\nu\rho\sigma} = -i \sqrt{-g} \text{tr} \Psi^{-1} = i \sqrt{-g} \Lambda, \quad (122)$$

where we have used the Hamiltonian constraint from variation of N in (109). Substitution of (122) back into (109) yields

$$I_{\text{Inst}} = i \Lambda \int_{\text{M}} d^4x \sqrt{-g} = i \Lambda \text{Vol}(\text{M}), \quad (123)$$

where $\text{Vol}(\text{M})$ is the spacetime volume. The exponentiation of this in units of $\hbar G$ yields

$$\psi = e^{i\Lambda(\hbar G)^{-1} \text{Vol}(\text{M})}, \quad (124)$$

which forms the dominant contribution to the path integral for gravity due to gravitational instantons [20]. On the other hand, substitution of

$\Psi_{ae} = -\frac{3}{\Lambda} \delta_{ae}$ into the starting action (19) produces a total derivative leading via Stokes' theorem to a Chern-Simons boundary term I_{CS} . The exponentiation of this boundary term in units of $\hbar G$ yields

$$\psi_{\text{Kod}} = e^{\pm 3(2\hbar G\Lambda)^{-1} \int_M \text{tr} F \wedge F} = e^{\pm 3(\hbar G\Lambda)^{-1} I_{CS}[A]}, \quad (125)$$

which is known as the Kodama state which describes de Sitter spacetime [21, 22]. One of the results of the quantum theory of I_{Inst} should be to clarify the role of (125) in quantum gravity, and to attempt to find its counterparts for Ψ_{ae} corresponding to more general spacetime geometries. The generalization of the left hand side of (125) is

$$\psi_{\text{Inst}} = e^{(2\hbar G)^{-1} \int_M \Psi_{ae} F^a \wedge F^e}. \quad (126)$$

As part of the investigation of the quantum theory one would like to find the analogue of the right hand side of (125) for (126).

Appendix A. Components of the Hodge self-duality operator.

From the equation

$$N\sqrt{\hbar} \left[(g^{00} g^{kj} - g^{k0} g^{0j}) F_{0j}^a + g^{0i} g^{kj} \epsilon_{ijm} B_a^m \right] = \beta B_a^k \quad (127)$$

from (55), we have

$$N\sqrt{\hbar} \left\{ \left[-\frac{1}{N^2} \left(h^{kj} - \frac{N^k N^j}{N^2} \right) - \left(\frac{N^k N^j}{N^2} \right) \right] F_{0j}^a - \frac{N^i}{N^2} \left(h^{kj} - \frac{N^k N^j}{N^2} \right) \epsilon_{ijm} B_a^m \right\} = \beta B_a^k. \quad (128)$$

Cancelling off the terms multiplying F_{0j}^a which are quadratic in N^i , we have

$$-\frac{\sqrt{\hbar}}{N} h^{kj} (F_{0j}^a + \epsilon_{jmi} B_a^m N^i) = \beta B_a^k. \quad (129)$$

Multiplying (129) by $\underline{N} = Nh^{-1/2}$ and by h_{lk} , this yields

$$F_{0l}^a + \epsilon_{lmi} B_a^m N^i + \beta \underline{N} h_{lk} B_a^k = 0. \quad (130)$$

From the equation

$$N\sqrt{\hbar} \left[(g^{m0} g^{nj} - g^{n0} g^{mj}) F_{0j}^a + g^{mi} g^{nj} \epsilon_{ijk} B_a^k \right] = \beta \epsilon^{mn0j} F_{0j}^a, \quad (131)$$

from (57), we have

$$N\sqrt{h} \left\{ \left[-\frac{N^m}{N^2} \left(h^{nj} - \frac{N^n N^j}{N^2} \right) + \frac{N^n}{N^2} \left(h^{mj} - \frac{N^m N^j}{N^2} \right) \right] F_{0j}^a + \left(h^{mi} - \frac{N^m N^i}{N^2} \right) \left(h^{nj} - \frac{N^n N^j}{N^2} \right) \epsilon_{ijk} B_a^k \right\} = \beta \epsilon^{0mnj} F_{0j}^a. \quad (132)$$

Expanding and using the vanishing of the term quadratic in the shift vector N^i , we have

$$\frac{\sqrt{h}}{N} \left\{ (h^{mj} N^n - h^{nj} N^m) F_{0j}^a + N\sqrt{h} h^{mi} h^{nj} \epsilon_{ijk} B_a^k - (h^{mi} N^n N^j + h^{nj} N^m N^i) \epsilon_{ijk} B_a^k \right\} = \beta \epsilon^{0mnj} F_{0j}^a. \quad (133)$$

From the third term on the left hand side of (133), we have the following relation upon relabelling indices $i \leftrightarrow j$ on the first term in brackets

$$-h^{mi} N^n N^j \epsilon_{ijk} B_a^k - h^{nj} N^m N^i \epsilon_{ijk} B_a^k = -h^{mj} N^n N^i \epsilon_{jik} B_a^k - h^{nj} N^m N^i \epsilon_{ijk} B_a^k = \epsilon_{ijk} (h^{mj} N^n - h^{nj} N^m) N^i B_a^k. \quad (134)$$

Note that the combination $h^{mj} N^n - h^{nj} N^m$ on the right hand side of (134) is the same term multiplying F_{0j}^a in the left hand side of (133). Using this fact, then (133) can be written as

$$\frac{\sqrt{h}}{N} \left[(h^{mj} N^n - h^{nj} N^m) (F_{0j}^a + \epsilon_{jki} B_a^k N^i) \right] + \underline{N} \epsilon^{mnl} h_{lk} B_a^k = \beta \epsilon^{mnj} F_{0j}^a, \quad (135)$$

where $\epsilon^{0mnj} = \epsilon^{mnj}$. Using $F_{0j}^a + \epsilon_{jki} B_a^k N^i = -\beta \underline{N} h_{jk} B_a^k$ from (130) in (135), then we have

$$-\frac{\sqrt{h}}{N} (h^{mj} N^n - h^{nj} N^m) \beta \underline{N} h_{jk} B_a^k + \underline{N} \epsilon^{mnl} h_{lk} B_a^k = \beta \epsilon^{mnj} F_{0j}^a. \quad (136)$$

This simplifies to

$$-\beta (\delta_k^m N^n - \delta_k^n N^m) B_a^k + \underline{N} \epsilon^{mnl} h_{lk} B_a^k = \beta \epsilon^{mnj} F_{0j}^a \longrightarrow \beta (\epsilon^{mnj} F_{0j}^a + B_a^m N^n - B_a^n N^m) = \underline{N} \epsilon^{mnj} h_{jk} B_a^k. \quad (137)$$

Contracting (137) with ϵ_{mnl} and dividing by 2, we obtain the relation

$$F_{0l}^a + \epsilon_{lmn} B_a^m N^n - \frac{1}{\beta} \underline{N} h_{lk} B_a^k = 0. \quad (138)$$

Consistency of (138) with (130) implies that $\beta^2 = -1$, or that $\beta = \pm i$.

Appendix B. Urbantke metric components. We now perform a 3+1 decomposition of the Urbantke metric*

$$g_{\mu\nu} = f_{abc} F_{\mu\rho}^a F_{\alpha\beta}^b F_{\sigma\nu}^c \epsilon^{\rho\alpha\beta\sigma}. \quad (139)$$

In what follows we define $\epsilon^{0123} = 1$, and make use of the fact that the structure constants $f_{abc} = \epsilon_{abc}$ for $\text{SO}(3, \mathbb{C})$ are numerically the same as the three-dimensional epsilon symbol. Also, we will use the definition $B_a^i = \frac{1}{2} \epsilon^{ijk} F_{jk}^a$ of the Ashtekar magnetic field. The main result of this appendix is that due to the symmetries of the four-dimensional epsilon tensor, each term in the expansion is the same to within a numerical constant. We will show this by explicit calculation.

1. Starting from the time-time component we have

$$g_{00} = f_{abc} F_{0\rho}^a F_{\alpha\beta}^b F_{\sigma 0}^c \epsilon^{\rho\alpha\beta\sigma}. \quad (140)$$

The time-time component of $g_{\mu\nu}$ reduces from two terms to one term

$$\begin{aligned} f_{abc} F_{0i}^a F_{0j}^b F_{k0}^c \epsilon^{i0jk} + f_{abc} F_{0i}^a F_{j0}^b F_{k0}^c \epsilon^{ijk0} &= \\ = 2 f_{abc} \epsilon^{ijk} F_{0i}^a F_{0j}^b F_{0k}^c &= 12 \det \| F_{0i}^a \|. \end{aligned} \quad (141)$$

2. Moving on to the space-time components, we have

$$\begin{aligned} g_{0k} &= f_{abc} F_{0\rho}^a F_{\alpha\beta}^b F_{\sigma k}^c \epsilon^{\rho\alpha\beta\sigma} = f_{abc} F_{0i}^a F_{\alpha\beta}^b F_{\sigma k}^c \epsilon^{i\alpha\beta\sigma} = \\ &= f_{abc} F_{0i}^a F_{0j}^b F_{lk}^c \epsilon^{i0jl} + f_{abc} F_{0i}^a F_{j0}^b F_{lk}^c \epsilon^{ij0l} + f_{abc} F_{0i}^a F_{jl}^b F_{0k}^c \epsilon^{ijl0} = \\ &= -2 f_{abc} \epsilon^{ijl} \epsilon_{lkm} B_c^m F_{0i}^a F_{0j}^b - 2 f_{abc} F_{0i}^a F_{0k}^c B_b^i = \\ &= -2 f_{abc} (\delta_k^i \delta_m^j - \delta_m^i \delta_k^j) F_{0i}^a F_{0j}^b B_c^m - 2 f_{abc} F_{0i}^a F_{0k}^c B_b^i = \\ &= 2 f_{abc} F_{0m}^a F_{0k}^b B_c^m = 2 (\det \| F_{0i}^a \|) \epsilon_{mkl} (F^{-1})_{0l}^c B_c^m. \end{aligned} \quad (142)$$

3. The spatial components are given by

$$g_{ij} = f_{abc} F_{i\rho}^a F_{\alpha\beta}^b F_{\sigma j}^c \epsilon^{\rho\alpha\beta\sigma} \quad (143)$$

which decomposes into a sum of four terms

$$\begin{aligned} g_{ij} &= f_{abc} F_{i0}^a F_{mk}^b F_{lj}^c \epsilon^{0mkl} + f_{abc} F_{im}^a F_{0n}^b F_{ij}^c \epsilon^{m0nl} + \\ &+ f_{abc} F_{im}^a F_{n0}^b F_{lj}^c \epsilon^{mn0l} + f_{abc} F_{im}^a F_{nl}^b F_{0j}^c \epsilon^{mnl0}. \end{aligned} \quad (144)$$

Using the fact that the middle two terms are equal, we have

$$\begin{aligned} g_{ij} &= -f_{abc} F_{0i}^a B_b^l \epsilon_{ljm} B_c^m - 2 f_{abc} \epsilon^{mnl} \epsilon_{imk} B_a^k F_{0n}^b \epsilon_{ljp} B_c^p - \\ &- 2 f_{abc} \epsilon_{imk} B_a^k B_b^m F_{0j}^c. \end{aligned} \quad (145)$$

*We have omitted the conformal factor for simplicity, which can always be re-inserted at the end of the derivations.

Applying epsilon symbol identities to the second term of (145) and simplifying, we get

$$\begin{aligned}
& -f_{abc}(\det\|B\|)F_{0i}^a\epsilon_{cbd}(B^{-1})_j^d - 2f_{abc}(\delta_j^m\delta_p^n - \delta_p^m\delta_j^n) \times \\
& \times \epsilon_{imk}B_a^kF_{0n}^bB_c^p - 2f_{abc}(\det\|B\|)\epsilon_{bad}(B^{-1})_i^dF_{0j}^c = \\
& = 2(\det\|B\|)\left[F_{0i}^d(B^{-1})_j^d + F_{0j}^d(B^{-1})_i^d\right] - \\
& - 2(\det\|B\|)\epsilon^{nkm}(B^{-1})_m^bF_{0n}^b\epsilon_{ijk} + 4(\det\|B\|)(B^{-1})_i^dF_{0j}^b. \quad (146)
\end{aligned}$$

Note that the third term on the right hand side of (146), upon application of epsilon identities, is given by

$$\begin{aligned}
& 2(\det\|B\|)(\delta_i^n\delta_j^m - \delta_j^n\delta_i^m)(B^{-1})_m^bF_{0n}^b = \\
& = 2(\det\|B\|)\left[F_{0i}^b(B^{-1})_j^b - F_{0j}^b(B^{-1})_i^b\right]. \quad (147)
\end{aligned}$$

Substituting (147) back into the right had side of (146) and after some cancellations, we get that the spatial part of $g_{\mu\nu}$ is given by

$$g_{ij} = 4(\det\|B\|)\left[F_{0i}^b(B^{-1})_j^b + F_{0j}^b(B^{-1})_i^b\right], \quad (148)$$

which is symmetric as expected.

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Instanton Representation of Plebanski Gravity. Application to the Schwarzschild Metric

Eyo Eyo Ita III*

Abstract: In this paper we apply the instanton representation method to the construction of spherically symmetric blackhole solutions. The instanton representation implies the existence of additional Type D solutions which are axially symmetric. We explicitly construct these solutions, and show that they are fully consistent with Birkhoff's theorem.

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§1. Introduction. In [1] a new formulation of General Relativity has been introduced, known as the instanton representation of Plebanski gravity. The basic variables are a $SO(3, \mathbb{C})$ gauge connection A_μ^a and a 3×3 matrix Ψ_{ae} which takes its values in two copies of $SO(3, \mathbb{C})$. The equations of motion of the instanton representation imply the Einstein

*Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom. E-mails: eei20@cam.ac.uk; ita@usna.edu.

equations when the initial value constraints of General Relativity are satisfied, and imply that the gauge curvature of A_μ^a is Hodge self-dual with respect to the same metric $g_{\mu\nu}$ solving these equations.* As a consistency condition on this formulation, one should require that the 3-metric h_{ij} determined using the constraint solutions and the 3-metric defined by the Hodge duality condition be equal to one another. In this way, which we will refer to as the instanton representation method, one has a new recipe for constructing General Relativity solutions.

The initial value constraint solutions of General Relativity can be classified according to the Petrov classification of spacetime, which depends on the multiplicity of eigenvalues and eigenvectors of Ψ_{ae} (see e.g. [2, 3]). The instanton representation is concerned with the cases where Ψ_{ae} has three linearly independent eigenvectors, such as for Petrov Types I, D and O where its equivalence with General Relativity is manifest. In the Petrov Type D case there are two distinct eigenvalues of Ψ_{ae} , which can be permuted in three different ways. In the Type O case there is only one distinct eigenvalue and permutation, whereas in the Petrov Type I case there are three distinct eigenvalues with six possible permutations. The instanton representation method implies that there should be a separate General Relativity solution associated with each permutation of eigenvalues of Ψ_{ae} .

In this paper we apply the instanton representation method to the construction of spherically symmetric General Relativity solutions. According to Birkhoff's theorem [4], any spherically symmetric vacuum solution of the Einstein field equations must be static and must agree with the Schwarzschild solution. The Schwarzschild metric is a Type D vacuum solution, which as we will show in the instanton representation corresponds to a particular permutation $\vec{\lambda}_{(1)}$ of eigenvalues solving the initial value constraints. There are two additional permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ of this same set of eigenvalues. The instanton representation implies that these latter permutations should also correspond to solutions, which leads to the following obvious question. Are the $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ solutions consistent with the Birkhoff theorem or do they lead to a contradiction? In other words, is the Hodge duality condition of the instanton representation subject to the initial value constraints consistent with the ansatz of spherical symmetry and time-independence for any metrics other than the Schwarzschild metric? In this paper we find that the $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ metrics are different from the Schwarzschild metric, yet in a sense which this paper will make precise are not in contradiction

*The latter actually follows from the equations of motion, and does not have to be added in as a separate postulate.

with Birkhoff's theorem.

The organization of this paper is as follows. In §2 we present some basic background on the initial value constraints problem in terms of the instanton representation phase space variables. In §3 we specialize the constraints to Type D spacetimes for a diagonal Ψ_{ae} for simplicity. §4 puts in place the ingredients necessary to produce spherically symmetric solutions. This uses a particular ansatz for the spatial connection A_i^a of a certain form, which includes time-independence of its components. §5, §6 and §7 apply the aforementioned instanton representation method to the construction of the metrics for the eigenvalue permutations $\vec{\lambda}_{(1)}$, $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$. The $\vec{\lambda}_{(1)}$ permutation leads to the Schwarzschild metric, and the remaining permutations lead to metrics which do not meet the conditions under which Birkhoff's theorem holds. §8 provides a summary and a brief discussion of these results.

§2. The initial value constraints. The dynamical variables in the instanton representation of Plebanski gravity are a $\text{SO}(3, \mathbb{C})$ gauge connection A_μ^a and a 3×3 complex matrix $\Psi_{ae} \in \text{SO}(3, \mathbb{C}) \otimes \text{SO}(3, \mathbb{C})^*$. The variables are subject to the following constraints on each three-dimensional spatial hypersurface Σ

$$\mathbf{w}_e\{\Psi_{ae}\} = 0, \quad \epsilon_{dae}\Psi_{ae} = 0, \quad \Lambda + \text{tr}\Psi^{-1} = 0, \quad (1)$$

where Λ is the cosmological constant.[†] We require that $\det\|\Psi\| \neq 0$, which means that the eigenvalues λ_1 , λ_2 and λ_3 of Ψ_{ae} must be nonvanishing. The first equation of (1) is defined as

$$\mathbf{w}_e\{\Psi_{ae}\} = \mathbf{v}_e\{\Psi_{ae}\} + C_{be}(f_{abf}\delta_{ge} + f_{ebg}\delta_{af})\Psi_{fg} = 0, \quad (2)$$

where f_{abc} are the $\text{SO}(3)$ structure constants, and we have defined the vector fields \mathbf{v}_a and a magnetic helicity density matrix C_{ae} given by

$$\mathbf{v}_a = B_a^i \partial_i, \quad C_{ae} = A_i^a B_e^i. \quad (3)$$

In (3) we have defined the magnetic field B_a^i , which we assume to have nonvanishing determinant $\det\|B\| \neq 0$, as

$$B_a^i = \epsilon^{ijk} \partial_j A_k^a + \frac{1}{2} \epsilon^{ijk} f_{abc} A_j^b A_k^c. \quad (4)$$

*For index conventions we use lower case symbols from the beginning of the Latin alphabet a, b, c, \dots to denote internal $\text{SO}(3, \mathbb{C})$ indices, and from the middle i, j, k, \dots for spatial indices. Spacetime indices are denoted by μ, ν, \dots

[†]The constraints in (1) are respectively the Gauss' law, diffeomorphism and Hamiltonian constraints. These constraints were also written down by Capovilla, Dell and Jacobson in the context of the initial value problem of General Relativity [5].

These variables define a spacetime metric $g_{\mu\nu}$, written in 3+1 form, as follows

$$ds^2 = -N^2 dt^2 + h_{ij} \omega^i \otimes \omega^j, \quad (5)$$

where h_{ij} is the spatial 3-metric with one forms $\omega^i = dx^i - N^i dt$, where $N^\mu = (N, N^i)$ are the lapse function and shift vector. The 3-metric h_{ij} can be constructed from the constraint solutions, and is given by

$$(h_{ij})_{\text{Constraints}} = (\det\|\Psi\|)(\Psi^{-1}\Psi^{-1})^{ae}(B^{-1})_i^a(B^{-1})_j^e(\det\|B\|), \quad (6)$$

where Ψ_{ae} and A_i^a are solutions to (1). The constraints (1) do not fix N^μ , and make use only of the spatial part of the connection A_μ^a .

From the four-dimensional curvature $F_{\mu\nu}^a$ and using $F_{0i}^a = \dot{A}_i^a - D_i A_0^a$ for the temporal component one can construct a matrix c_{ij} , given by

$$c_{ij} = F_{0i}^a (B^{-1})_j^a, \quad c \equiv \det\|c_{(ij)}\|. \quad (7)$$

The separation of c_{ij} into symmetric and antisymmetric parts defines a 3-metric $(h_{ij})_{\text{Hodge}}$ and a shift vector N^i , given by

$$(h_{ij})_{\text{Hodge}} = -\frac{N^2}{c} c_{(ij)}, \quad N^i = -\frac{1}{2} \epsilon^{ijk} c_{jk}. \quad (8)$$

Equation (8) arises from the Hodge duality condition implied by the instanton representation [1]. Equations (8) and (6) are 3-metrics constructed using two separate criteria, and as a consistency condition must be set equal to each other. This is the basic feature of the instanton representation method in constructing General Relativity solutions in practice, which enables one to also write (5) as

$$ds^2 = -N^2 \left[dt^2 + \frac{1}{c} c_{(ij)} \left(dx^i + \frac{1}{2} \epsilon^{imn} c_{mn} dt \right) \left(dx^j + \frac{1}{2} \epsilon^{jrs} c_{rs} dt \right) \right]. \quad (9)$$

Since Ψ_{ae} is a nondegenerate complex matrix by supposition, then it is diagonalizable when there are three linearly independent eigenvectors [2]. This enables one to classify solutions according to the Petrov type of the self-dual Weyl tensor ψ_{ae} . The matrix ψ_{ae} is symmetric and traceless, and related to Ψ_{ae} in the following way

$$\Psi_{ae}^{-1} = -\frac{\Lambda}{3} \delta_{ae} + \psi_{ae}. \quad (10)$$

So for this paper we assume that Ψ_{ae} is invertible, which requires the existence of three linearly independent eigenvectors. Hence, the results of this paper are limited to Petrov Types I, D and O. For each such Ψ_{ae} , combined with a connection A_i^a solving the constraints (1), the Hodge

duality condition (8) should yield a metric solving the vacuum Einstein equations.

§3. Application to Petrov Type D spacetimes. For the purposes of this paper we will restrict attention to the case where $\Psi_{ae} = \text{diag}(\Psi_{11}, \Psi_{22}, \Psi_{33})$ is diagonal. Then from equation (1) the diffeomorphism constraint is automatically satisfied since a diagonal matrix is already symmetric. We can then associate the elements of Ψ_{ae} with its eigenvalues, and the Hamiltonian constraint is given by

$$\Lambda + \frac{1}{\Psi_{11}} + \frac{1}{\Psi_{22}} + \frac{1}{\Psi_{33}} = 0. \quad (11)$$

The Gauss' law constraint can be written as

$$\mathbf{v}_e\{\Psi_{ae}\} + C_{be}(f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}) = 0. \quad (12)$$

Since restricting to diagonal Ψ_{ae} , we need only consider the terms of (12) with $e = a$ on the first term, $e = f$ on the second and $a = g$ on the third. This is due to the fact that a is a free index while the remaining are dummy indices. Then we get the following equations

$$\left. \begin{aligned} \mathbf{v}_1\{\Psi_{11}\} + C_{23}(\Psi_{33} - \Psi_{11}) + C_{32}(\Psi_{11} - \Psi_{22}) &= 0 \\ \mathbf{v}_2\{\Psi_{22}\} + C_{31}(\Psi_{11} - \Psi_{22}) + C_{13}(\Psi_{22} - \Psi_{33}) &= 0 \\ \mathbf{v}_3\{\Psi_{33}\} + C_{12}(\Psi_{22} - \Psi_{33}) + C_{21}(\Psi_{33} - \Psi_{11}) &= 0 \end{aligned} \right\}. \quad (13)$$

Equation (13) is a set of three differential equations which can be put into the operator-valued matrix form

$$\begin{pmatrix} \mathbf{v}_1 - C_{[23]} & -C_{32} & C_{23} \\ C_{31} & \mathbf{v}_2 - C_{[31]} & -C_{13} \\ -C_{21} & C_{12} & \mathbf{v}_3 - C_{[12]} \end{pmatrix} \begin{pmatrix} \Psi_{11} \\ \Psi_{22} \\ \Psi_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where we have defined $C_{[ae]} = C_{ae} - C_{ea}$. Since we have already removed three degrees of freedom by choosing Ψ_{ae} to be diagonal, and Gauss' law is a set of three conditions, we would rather not overconstrain Ψ_{ae} any further. In other words, we will regard the Gauss' law constraint as a set of conditions fixing three elements of the connection A_i^a , with Ψ_{ae} constrained only by the Hamiltonian constraint (11). We will from now on make the identifications

$$\Psi_{11} = \varphi_1, \quad \Psi_{22} = \varphi_2, \quad \Psi_{33} = \varphi_3, \quad (14)$$

defined as the eigenvalues of Ψ_{ae} . We will now specialize to the Petrov Type D case, where two of the eigenvalues are equal with no vanishing eigenvalues.

§3.1. The Hamiltonian constraint. Denote the eigenvalues of Ψ_{ae} by $\lambda_f = (\varphi_1, \varphi, \varphi)$ and all permutations thereof. Then the Hamiltonian constraint (11) reduces to

$$\frac{1}{\varphi_1} + \frac{2}{\varphi} + \Lambda = 0. \quad (15)$$

Equation (15) yields the following relations which we will use later

$$\varphi_1 = -\left(\frac{\varphi}{\Lambda\varphi + 2}\right), \quad \varphi_1 - \varphi = -\varphi\left(\frac{\Lambda\varphi + 3}{\Lambda\varphi + 2}\right). \quad (16)$$

The diagonalized self-dual Weyl curvature for a spacetime of Type D is of the form $\psi_{ae} = \text{diag}(-2\Psi, \Psi, \Psi)$ for some function Ψ . The corresponding CDJ matrix is given by adding to this a cosmological contribution as in (10), which in matrix form is given by

$$\Psi_{ae}^{-1} = \begin{pmatrix} -\frac{\Lambda}{3} - 2\Psi & 0 & 0 \\ 0 & -\frac{\Lambda}{3} + \Psi & 0 \\ 0 & 0 & -\frac{\Lambda}{3} + \Psi \end{pmatrix}.$$

One can then read off the value of φ in (15) as

$$\varphi = \frac{1}{-\frac{\Lambda}{3} + \Psi}, \quad \Lambda\varphi + 2 = \left(\frac{\frac{\Lambda}{3} + 2\Psi}{-\frac{\Lambda}{3} + \Psi}\right), \quad \Lambda\varphi + 3 = \frac{3\Psi}{-\frac{\Lambda}{3} + \Psi}. \quad (17)$$

From (17) the following quantities Φ and ψ can be constructed

$$\left. \begin{aligned} \Phi &= \frac{\varphi(\Lambda\varphi + 3)^2}{(\Lambda\varphi + 2)^3} = 9 \left(\frac{1}{2\Psi^{1/3} + \frac{\Lambda}{3}\Psi^{-2/3}} \right)^3 \\ \psi &= \varphi^2(\Lambda\varphi + 3) = 3 \left(\frac{1}{-\frac{\Lambda}{3}\Psi^{-1/3} + \Psi^{2/3}} \right)^3 \end{aligned} \right\}, \quad (18)$$

which will become useful later in this paper.

§3.2. The Gauss' law constraint. Next, we must set up the Gauss' law constraint (13) for the Type D case. There are three distinct permutations of eigenvalues to consider

$$\vec{\lambda}_{(1)} = (\varphi_1, \varphi, \varphi), \quad \vec{\lambda}_{(2)} = (\varphi, \varphi_1, \varphi), \quad \vec{\lambda}_{(3)} = (\varphi, \varphi, \varphi_1), \quad (19)$$

which we will treat individually. The steps which follow will refer to $\vec{\lambda}_{(1)}$, with the remaining cases obtainable by cyclic permutation. The Gauss'

law constraint for permutation $\vec{\lambda}_{(1)}$ reduces to

$$\begin{pmatrix} \mathbf{v}_1 - C_{[23]} & -C_{32} & C_{23} \\ C_{31} & \mathbf{v}_2 - C_{[31]} & -C_{13} \\ -C_{21} & C_{12} & \mathbf{v}_3 - C_{[12]} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which leads to the following equations

$$\left. \begin{aligned} \mathbf{v}_1\{\varphi_1\} &= C_{[23]}(\varphi_1 - \varphi) \\ \mathbf{v}_2\{\varphi\} &= C_{31}(\varphi - \varphi_1) \\ \mathbf{v}_3\{\varphi\} &= C_{21}(\varphi_1 - \varphi) \end{aligned} \right\}. \quad (20)$$

Using the results from (16), the first equation of (20) implies that

$$-\mathbf{v}_1 \left\{ \left(\frac{\varphi}{\Lambda\varphi + 2} \right) \right\} = -C_{[23]} \varphi \left(\frac{\Lambda\varphi + 3}{\Lambda\varphi + 2} \right). \quad (21)$$

Since the vector fields \mathbf{v}_a are first-derivative operators, equation (21) can be written as

$$\begin{aligned} \frac{1}{\varphi} \left(\frac{\Lambda\varphi + 2}{\Lambda\varphi + 3} \right) \mathbf{v}_1 \left\{ \left(\frac{\varphi}{\Lambda\varphi + 2} \right) \right\} &= C_{[23]} = \\ &= \frac{1}{\varphi} \left(\frac{\Lambda\varphi + 2}{\Lambda\varphi + 3} \right) \left[\frac{(\Lambda\varphi + 2) \mathbf{v}_1\{\varphi\} - \varphi \mathbf{v}_1\{\Lambda\varphi + 2\}}{(\Lambda\varphi + 2)^2} \right], \end{aligned} \quad (22)$$

where we have used the Leibniz rule. Equation (22) then simplifies to

$$\frac{2 \mathbf{v}_1\{\varphi\}}{\varphi(\Lambda\varphi + 2)(\Lambda\varphi + 3)} = \frac{1}{3} \mathbf{v}_1\{\ln \Phi\} = C_{[23]}, \quad (23)$$

which gives

$$\mathbf{v}_1\{\ln \Phi\} = 3C_{[23]} \quad (24)$$

with Φ given by (18).

The second equation of (20) implies that

$$\mathbf{v}_2\{\varphi\} = C_{31} \varphi \left(\frac{\Lambda\varphi + 3}{\Lambda\varphi + 2} \right). \quad (25)$$

Using (16), equation (25) simplifies to

$$\frac{1}{\varphi} \left(\frac{\Lambda\varphi + 2}{\Lambda\varphi + 3} \right) \mathbf{v}_2\{\varphi\} = \frac{1}{3} \mathbf{v}_2\{\varphi^2(\Lambda\varphi + 3)\} = C_{31}, \quad (26)$$

which gives

$$\mathbf{v}_2\{\ln \psi\} = 3C_{31}. \quad (27)$$

The manipulations of the third equation of (20) are directly analogous to (26) and (27), which implies that

$$\mathbf{v}_3\{\varphi\} = -C_{21}\varphi \left(\frac{\Lambda\varphi + 3}{\Lambda\varphi + 2} \right) \longrightarrow \mathbf{v}_3\{\ln\psi\} = -3C_{21}. \quad (28)$$

Hence the three equations for $\vec{\lambda}_{(1)}$ can be written as

$$\mathbf{v}_1\{\ln\Phi\} = 3C_{[23]}, \quad \mathbf{v}_2\{\ln\psi\} = 3C_{31}, \quad \mathbf{v}_3\{\ln\psi\} = -3C_{21}, \quad (29)$$

where Φ and ψ are given by (18).

For the second permutation of eigenvalues $\vec{\lambda}_{(2)}$ we have $\vec{\varphi} = (\varphi, \varphi_1, \varphi)$, which leads to the Gauss' law equations

$$\mathbf{v}_2\{\ln\Phi\} = 3C_{[31]}, \quad \mathbf{v}_3\{\ln\psi\} = 3C_{12}, \quad \mathbf{v}_1\{\ln\psi\} = -3C_{32}. \quad (30)$$

For the third permutation of eigenvalues $\vec{\lambda}_{(3)}$ we have $\vec{\varphi} = (\varphi, \varphi, \varphi_1)$, which leads to the Gauss' law equations

$$\mathbf{v}_3\{\ln\Phi\} = 3C_{[12]}, \quad \mathbf{v}_1\{\ln\psi\} = 3C_{23}, \quad \mathbf{v}_2\{\ln\psi\} = -3C_{13}. \quad (31)$$

The implication of this is the following. If there exists a General Relativity solution for a particular eigenvalue permutation, say $\vec{\lambda}_{(1)}$, then there must exist solutions corresponding to the remaining permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$.

§4. The spherically symmetric case. We are now ready to proceed with the instanton representation method. We must first choose a connection A_μ^a which will play dual roles. On the one hand A_μ^a will define a metric based on the Hodge duality condition, and on the other hand its spatial part A_i^a will in conjunction with Ψ_{ae} form a metric based on the solution to the Gauss' law and the Hamiltonian constraints. For the purposes of this paper we will choose a connection A_μ^a which is known to produce spherically symmetric blackhole solutions. This paragraph will show that the requirements on $(h_{ij})_{\text{Hodge}}$ and on $(h_{ij})_{\text{Constraints}}$ are in a sense complementary. Then in the subsequent paragraphs of this paper we will equate these two metrics, which, as we will see, imposes stringent conditions on the form of the final solution.

§4.1. Ingredients for the Hodge duality condition. Let the connection A_μ^a be defined by the following one-forms

$$A^1 = i \frac{f'}{g} dt + (\cos\theta) d\phi, \quad A^2 = -\left(\frac{\sin\theta}{g}\right) d\phi, \quad A^3 = \frac{d\theta}{g}, \quad (32)$$

where $f = f(r)$ and $g = g(r)$ are at this stage arbitrary functions of radial distance r and a prime denotes differentiation with respect to r . Equation (32) yields the curvature 2-forms $F^a = dA^a + \frac{1}{2} f^{abc} A^b \wedge A^c$, given by

$$\left. \begin{aligned} F^1 &= -\left(\frac{if'}{g}\right)' dt \wedge dr - \sin \theta \left(1 - \frac{1}{g^2}\right) d\theta \wedge d\phi \\ F^2 &= -\frac{g'}{g^2} \sin \theta d\phi \wedge dr - \frac{if'}{g^2} dt \wedge d\theta \\ F^3 &= -\frac{g'}{g^2} dr \wedge d\theta - \frac{if'}{g^2} \sin \theta dt \wedge d\phi \end{aligned} \right\}. \quad (33)$$

From this we can read off the nonvanishing components of the magnetic field B_a^i and the temporal component of the curvature F_{0i}^a , given by

$$\left. \begin{aligned} B_1^1 &= \sin \theta \left(1 - \frac{1}{g^2}\right), & B_2^2 &= -\frac{g'}{g^2} \sin \theta, & B_3^3 &= -\frac{g'}{g^2} \\ F_{01}^1 &= -\left(\frac{if'}{g}\right)', & F_{02}^2 &= -\left(\frac{if'}{g^2}\right)', & F_{03}^3 &= -\left(\frac{if'}{g^2}\right) \sin \theta \end{aligned} \right\}. \quad (34)$$

Since (34) form diagonal matrices, then the antisymmetric part of $(B^{-1})_i^a F_{0j}^a$ is zero which according to (8) makes the shift vector N^i equal to zero. Then following suit with (7) we have

$$c_{ij} = F_{0i}^a (B^{-1})_j^a = -i \begin{pmatrix} \frac{(f'/g)'}{\sin \theta \left(1 - \frac{1}{g^2}\right)} & 0 & 0 \\ 0 & (f'/g') \frac{1}{\sin \theta} & 0 \\ 0 & 0 & (f'/g') \sin \theta \end{pmatrix}.$$

The determinant of $c_{(ij)}$ is given by

$$c = \det \|(B^{-1})_0^a F_{0j}^a\| = i \frac{(f'/g)' (f'/g')^2}{\left(1 - \frac{1}{g^2}\right) \sin \theta}. \quad (35)$$

So Hodge duality for the chosen connection A_μ^a implies, using (8), that the 3-metric $(h_{ij})_{\text{Hodge}}$ is given by

$$(h_{ij})_{\text{Hodge}} = -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\ 0 & \frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)' (f'/g')} & 0 \\ 0 & 0 & \frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)' (f'/g')} \sin^2 \theta \end{pmatrix}.$$

According to Birkhoff's theorem, any spherically symmetric solution for vacuum General Relativity must be given by the Schwarzschild solution. Hodge duality alone is insufficient to select this solution, since it presently allows for three free functions f , g and N . Let us determine the minimal set of additional conditions necessary to obtain the Schwarzschild solution. Spherical symmetry ($g_{\theta\theta}=r^2$ and $g_{\phi\phi}=r^2 \sin^2\theta$) in conjunction with the choice $N=f$ leads to the condition

$$\frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)'(f'/g')} = r^2, \quad (36)$$

which still contains one degree of freedom in the choice of g . For example let us further choose $g = \frac{1}{f}$. Then $g' = -\frac{f'}{f^2}$, which yields

$$\frac{1}{2} \frac{d^2 f^2}{dr^2} = \frac{1}{r^2} (f^2 - 1). \quad (37)$$

Defining $u = \ln r$, then this leads to the equation

$$\left(\frac{d^2}{du^2} - \frac{d}{du} - 2\right) f^2 = -2 \quad (38)$$

with solution $f^2 = 1 + k_1 e^{-u} + k_2 e^{2u}$ for arbitrary constants k_1 and k_2 . This yields the solution

$$f^2 = 1 + k_1 r^{-1} + k_2 r^2. \quad (39)$$

Upon making the identification $k_1 \equiv -2GM$ and $k_2 \equiv -\frac{\Lambda}{3}$ one recognizes (39) as the solution for a Schwarzschild-de Sitter black hole.*

§4.2. Ingredients for the Gauss' law constraint. The conditions determining $(h_{ij})_{\text{Constraints}}$ are fixed by the spatial connection A_i^a and Ψ_{ae} solving the constraints (1). Note in (32) that A_i^a depends only on g and not on f . This means that only g can be fixed by the Gauss' law constraint, and that f must be fixed by equality of (8) with (6). We will now proceed to solve the Gauss' law constraint for our connection (32), with spatial part given in the matrix form

$$A_i^a = \begin{pmatrix} A_r^1 & A_\theta^1 & A_\phi^1 \\ A_r^2 & A_\theta^2 & A_\phi^2 \\ A_r^3 & A_\theta^3 & A_\phi^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & -\frac{\sin\theta}{g} \\ 0 & \frac{1}{g} & 0 \end{pmatrix},$$

*We will show that the set of conditions leading to (39) arise precisely from the equality of (8) with (6), namely that the Hodge-duality metric solve the Einstein equations. Without this, the solution is not unique.

where $g=g(r)$ is an arbitrary function only of radial distance r from the origin. By this choice we have also made the choice of a coordinate system (r, θ, ϕ) to whose axes various quantities will be referred. From (4), one can construct the magnetic field B_a^i

$$B_a^i = \begin{pmatrix} -\left(1 - \frac{1}{g^2}\right) \sin \theta & 0 & 0 \\ 0 & \sin \theta \frac{d}{dr} g^{-1} & 0 \\ 0 & 0 & \frac{d}{dr} g^{-1} \end{pmatrix},$$

and the magnetic helicity density matrix C_{ae} , given by

$$C_{ae} = A_i^a B_e^i = \frac{\partial}{\partial r} \begin{pmatrix} 0 & 0 & \frac{\cos \theta}{g} \\ 0 & 0 & \frac{\sin \theta}{2} \left(1 - \frac{1}{g^2}\right) \\ 0 & -\frac{\sin \theta}{2} \left(1 - \frac{1}{g^2}\right) & 0 \end{pmatrix}.$$

The vector field $\mathbf{v}_a = B_a^i \partial_i$ can be read off from the magnetic field matrix

$$\left. \begin{aligned} \mathbf{v}_1 &= -\sin \theta \left(1 - \frac{1}{g^2}\right) \frac{\partial}{\partial r} \\ \mathbf{v}_2 &= \frac{d}{dr} \left(\frac{1}{g}\right) \sin \theta \frac{\partial}{\partial \theta}, \quad \mathbf{v}_3 = \frac{d}{dr} \left(\frac{1}{g}\right) \frac{\partial}{\partial \phi} \end{aligned} \right\}. \quad (40)$$

These will constitute the differential operators in the Gauss' law constraint. The ingredients for (6) for the configuration chosen are

$$(\det \|\Psi\|)(\Psi^{-1} \Psi^{-1})^{ae} = - \begin{pmatrix} \frac{\frac{\Lambda}{3} + 2\Psi}{(-\frac{\Lambda}{3} + \Psi)^2} & 0 & 0 \\ 0 & \frac{1}{\frac{\Lambda}{3} + 2\Psi} & 0 \\ 0 & 0 & \frac{1}{\frac{\Lambda}{3} + 2\Psi} \end{pmatrix}$$

for the part involving Ψ_{ae} , and

$$\begin{aligned} \eta_{ij}^{ae} &\sim (B^{-1})_i^a (B^{-1})_j^e (\det \|B\|) \longrightarrow \\ &\longrightarrow - \begin{pmatrix} \frac{\left(\frac{d}{dr} g^{-1}\right)^2}{1 - \frac{1}{g^2}} & 0 & 0 \\ 0 & 1 - \frac{1}{g^2} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2}\right) \sin^2 \theta \end{pmatrix} \end{aligned}$$

for the part involving the magnetic field B_a^i . We have, in an abuse of notation, anticipated the result of multiplying the matrices needed for (6) for this special case where the matrices are diagonal. We will be particularly interested in the $\Lambda=0$ case, as it is the simplest case to test for the Hodge duality condition.* For $\Lambda=0$ the 3-metric based on the initial value constraints (6) is given by

$$(h_{ij})_{\Lambda=0} = \frac{1}{2\Psi} \begin{pmatrix} \frac{4\left(\frac{d}{dr}g^{-1}\right)^2}{1-\frac{1}{g^2}} & 0 & 0 \\ 0 & 1-\frac{1}{g^2} & 0 \\ 0 & 0 & \left(1-\frac{1}{g^2}\right)\sin^2\theta \end{pmatrix}.$$

We are now ready to apply the instanton representation method to the construction of solutions.

§5. First permutation of eigenvalues $\vec{\lambda}_{(1)}$. We will now produce some of the known blackhole solutions corresponding to the eigenvalue permutation $\vec{\lambda}_{(1)}$. The first equation of (29) for the chosen connection reduces to

$$\mathbf{v}_1\{\ln\Phi\} = 3C_{[23]} \longrightarrow -\sin\theta\left(1-\frac{1}{g^2}\right)\frac{\partial\ln\Phi}{\partial r} = 3\sin\theta\frac{\partial}{\partial r}\left(1-\frac{1}{g^2}\right) \quad (41)$$

where we have used (40), which integrates to

$$\Phi = c(\theta, \phi)\left(1-\frac{1}{g^2}\right)^{-3}, \quad (42)$$

where c at this stage is an arbitrary function of two variables not to be confused with the c in (7). The second equation of (29) is given by

$$\mathbf{v}_2\{\ln\psi\} = 3C_{31} \longrightarrow \left(\frac{d}{dr}g^{-1}\right)\sin\theta\frac{\partial\ln\psi}{\partial\theta} = 0, \quad (43)$$

which implies that $\psi = \psi(r, \phi)$. The third equation of (29) is given by

$$\mathbf{v}_3\{\ln\psi\} = -3C_{21} \longrightarrow \left(\frac{d}{dr}g^{-1}\right)\frac{\partial\ln\psi}{\partial\phi} = 0. \quad (44)$$

In conjunction with the results from (43), one has that $\psi = \psi(r)$ must be a function only of r . Note that this is consistent with Φ being solely a function of r as in (42), which requires that $c(\theta, \phi) = c$ be a num-

*The $\Lambda \neq 0$ case will be relegated for future research.

erical constant. Continuing from (42) we have

$$\left(\frac{1}{2\Psi^{1/3} + \frac{\Lambda}{3}\Psi^{-2/3}} \right)^3 = c \left(1 - \frac{1}{g^2} \right)^{-3}, \quad (45)$$

which upon redefining the parameter c yields the solution

$$g^2 = \left(1 - \frac{2}{c}\Psi^{1/3} - \frac{\Lambda}{3c}\Psi^{-2/3} \right)^{-1}. \quad (46)$$

So knowing Ψ , which comes directly from the CDJ matrix for Petrov Type D, enables us to determine the connection A_i^a explicitly in this case.

We can now proceed to compute the 3-metric h_{ij} for the chosen configuration. We would rather like to express the metric directly in terms of Ψ , which is the fundamental degree of freedom for the given Petrov Type. Hence from (45) we have

$$1 - \frac{1}{g^2} = \frac{1}{c}\Psi^{-2/3} \left(2\Psi + \frac{\Lambda}{3} \right), \quad (47)$$

which yields

$$\frac{d}{dr} g^{-1} = -\frac{1}{3c}\Psi^{-5/3} \left(1 - \frac{2}{c}\Psi^{1/3} - \frac{\Lambda}{3c}\Psi^{-2/3} \right)^{-1/2} \left(-\frac{\Lambda}{3} + \Psi \right) \Psi', \quad (48)$$

where $\Psi' = \frac{d\Psi}{dr}$. Then the magnetic field part η_{ij}^{ae} of the metric can be written explicitly in terms of Ψ

$$\eta_{ij}^{ae} = -\frac{1}{c} \begin{pmatrix} \frac{1}{9} \frac{\Psi^{-8/3}(\Psi')^2}{1 - \frac{2}{c}\Psi^{1/3} - \frac{\Lambda}{3c}\Psi^{-2/3}} \frac{(-\frac{\Lambda}{3} + \Psi)^2}{2\Psi + \frac{\Lambda}{3}} & 0 & 0 \\ 0 & \Psi^{-2/3} \left(2\Psi + \frac{\Lambda}{3} \right) & 0 \\ 0 & 0 & \Psi^{-2/3} \left(2\Psi + \frac{\Lambda}{3} \right) \sin^2\theta \end{pmatrix}.$$

Multiplying this matrix by $(\det\|\Psi\|)(\Psi^{-1}\Psi^{-1})^{ae}$, we obtain the 3-metric

$$(h_{ij})_{\vec{\lambda}_{(1)}} = \frac{1}{c} \begin{pmatrix} \frac{1}{9} \frac{\Psi^{-8/3}(\Psi')^2}{1 - \frac{2}{c}\Psi^{1/3} - \frac{\Lambda}{3c}\Psi^{-2/3}} & 0 & 0 \\ 0 & \Psi^{-2/3} & 0 \\ 0 & 0 & \Psi^{-2/3} \sin^2\theta \end{pmatrix}.$$

This is a general solution for the permutation sequence $\vec{\lambda}_{(1)}$ for the chosen connection. As a doublecheck, let us eliminate the constant of

integration c via the rescaling $\Psi \rightarrow \Psi c^{-3/2}$. But the shift vector N^i and the lapse N have remained undetermined based purely on the initial value constraints. For $N^i=0$, which is a result of the Hodge duality condition, this yields a spacetime metric of

$$ds^2 = -N^2 dt^2 + \frac{1}{9} \left(\frac{\Psi^{-8/3} (\Psi')^2}{1 - 2\Psi^{1/3} c^{-3/2} - \frac{\Lambda}{3} \Psi^{-2/3}} \right) dr^2 + \Psi^{-2/3} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (49)$$

Already, it can be seen that (49) can lead to some known General Relativity solutions. Taking $\Psi = \frac{1}{r^3}$, $c = (GM)^{-2/3}$, $N^i=0$ and $N^2 = 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2$ where N is the lapse function, we obtain

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix},$$

which is the solution for a Euclidean Schwarzschild-de Sitter blackhole. For $\Lambda=0$, $g_{\mu\nu}$ reduces to the Schwarzschild metric and for $G=0$, it reduces to the de Sitter metric.* There clearly exist solutions corresponding to $\vec{\lambda}_{(1)}$, since it is known that the Einstein equations admit blackhole solutions. On the other hand, the instanton representation implies that there must be additional solutions corresponding to the remaining permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$. We must now check for consistency of these additional solutions, if they exist, with Birkhoff's theorem. Let us examine the different eigenvalue permutations in turn.

§5.1. Hodge duality condition for $\lambda_{(1)}$ for $\Lambda = 0$. Note that the lapse function N at the level of (5) is freely specifiable and not fixed by (6). To make progress we will need to impose the Hodge duality condition, namely the equality of (6) with (8). From the Gauss' law constraint we can read off from (47) in the $\Lambda=0$ case that

$$\Psi = \frac{1}{8} \left(1 - \frac{1}{g^2} \right)^3. \quad (50)$$

So upon implementation of the Hodge duality condition, then the

*Setting $M=0$ corresponds to a transition from Type D to Type O spacetime, where $\Psi=0$.

3-metric must satisfy the condition $(h_{ij})_{\text{Constraints}} = (h_{ij})_{\text{Hodge}}$, or

$$\begin{aligned} (h_{ij})_{\Lambda=0} &= - \begin{pmatrix} 16 \left(\frac{d}{dr} g^{-1} \right)^2 \left(1 - \frac{1}{g^2} \right)^{-4} & 0 & 0 \\ 0 & 4 \left(1 - \frac{1}{g^2} \right)^{-2} & 0 \\ 0 & 0 & 4 \left(1 - \frac{1}{g^2} \right)^{-2} \sin^2 \theta \end{pmatrix} = \\ &= -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\ 0 & \frac{\left(1 - \frac{1}{g^2} \right)}{(f'/g)'} (f'/g')} & 0 \\ 0 & 0 & \frac{\left(1 - \frac{1}{g^2} \right)}{(f'/g)'} (f'/g')} \sin^2 \theta \end{pmatrix}. \end{aligned}$$

As a consistency condition on the radial component g_{rr} we must require that

$$N^2 \left(\frac{g'}{f'} \right)^2 = 16 \frac{g'}{g^2} \left(1 - \frac{1}{g^2} \right)^{-4}, \quad (51)$$

and as a consistency condition on $g_{\theta\theta}$ we must require that

$$N^2 \frac{\left(1 - \frac{1}{g^2} \right)}{(f'/g)'} (f'/g')} = 4 \left(1 - \frac{1}{g^2} \right)^{-2}. \quad (52)$$

Equations (51) and (52) are a set of two equations in three unknowns N , g and f . Upon dividing equation (52) into (51), then N^2 drops out and we have the following relation between f and g

$$\frac{1}{f'} \left(\frac{f'}{g} \right)' = \frac{4}{g^2 - 1} \rightarrow \frac{f''}{f'} = \frac{g'}{g} + \frac{4g}{g^2 - 1}. \quad (53)$$

Integration of (53) determines $f = f[g]$ explicitly in terms of g , and substitution of the result into (51) determines $N = N[g]$ via

$$\left. \begin{aligned} f &= k_2 + k_1 \int dr g \exp \left[4 \int \frac{g dr}{g^2 - 1} \right] \\ N^2 &= \frac{16}{g' g^2} \left(1 - \frac{1}{g^2} \right)^{-2} \left(k_2 + k_1 \int dr g \exp \left[4 \int \frac{g dr}{g^2 - 1} \right] \right)^2 \end{aligned} \right\}. \quad (54)$$

Recall that g is fixed by the Gauss' law constraint on the spatial hypersurface Σ , and that f and N have to do with the temporal part of the metric. The function g is apparently freely specifiable, and each g

determines f and N . So the Hodge duality condition determines the temporal parts of $g_{\mu\nu}$ from the spatial part.

There are an infinite number of solutions parametrized by the function g . But according to Birkhoff's theorem there should be only one static spherically symmetric vacuum solution, namely the Schwarzschild solution. The Hodge duality condition by itself is insufficient to select this solution. First, we must impose the spherically symmetric form $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2\theta$, which implies

$$4\left(1 - \frac{1}{g^2}\right)^{-2} = r^2 \longrightarrow g = \left(1 - \frac{2}{r}\right)^{-1/2}, \quad g' = -r^{-2} \left(1 - \frac{2}{r}\right)^{-3/2} \quad (55)$$

in units where $GM = 1$. Substitution of (55) into (51) and (52) yields

$$N^2 = r^4 \left(1 - \frac{2}{r}\right) \phi^2, \quad N^2 = -\frac{1}{2} r^5 \left(1 - \frac{2}{r}\right) \phi' \phi, \quad \phi = f' \left(1 - \frac{2}{r}\right)^{1/2}. \quad (56)$$

Equating the first and second equations of (56) leads to the condition that $\phi = r^{-2}$. Putting this into the third equation allows us to find f

$$f = \int dr r^{-2} \left(1 - \frac{2}{r}\right)^{-1/2} = -\left(1 - \frac{2}{r}\right) \longrightarrow N^2 = 1 - \frac{2}{r}, \quad (57)$$

as well as the lapse function N . Putting (57) back into (51) then determines g_{rr} , given by

$$g_{rr} = -\frac{1}{1 - \frac{2}{r}}. \quad (58)$$

The final result is that the condition of spherical symmetry $g_{\theta\theta} = r^2$ in addition to Hodge duality of the curvature of the chosen A_i^a fixes the lapse function N , which yields the spacetime line element

$$-ds^2 = \left(1 - \frac{2}{r}\right) dt^2 + \left(1 - \frac{2}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (59)$$

The result is the Euclidean Schwarzschild metric, as predicted by Birkhoff's theorem.

§6. Second permutation of eigenvalues $\vec{\lambda}_{(2)}$. We have found spherically symmetric blackhole solutions using the first permutation $\vec{\lambda}_{(1)}$. According to the Birkhoff theorem there should be no additional spherically symmetric time-independent solutions. But we will nevertheless proceed with the construction of any solutions implied by the

instanton representation for the second permutation $\vec{\lambda}_{(2)}$. The Gauss' law constraint equations associated with $\vec{\lambda}_{(2)}$ are given by (30)

$$\mathbf{v}_2\{\ln \Phi\} = 3C_{[31]}, \quad \mathbf{v}_3\{\ln \psi\} = 3C_{12}, \quad \mathbf{v}_1\{\ln \psi\} = -3C_{32} \quad (60)$$

with Φ and ψ given by (18). The first equation of (60) yields

$$\mathbf{v}_2\{\ln \Phi\} = 3C_{[31]} \longrightarrow \left(\frac{d}{dr} g^{-1}\right) \sin \theta \frac{\partial \ln \Phi}{\partial \theta} = 3 \frac{\partial}{\partial r} \left(-\frac{\cos \theta}{g}\right) \quad (61)$$

which integrates to

$$\Phi = c(r, \phi) \sin^{-3} \theta, \quad (62)$$

where c is at this stage an arbitrary function of two variables. The second equation of (60) yields

$$\mathbf{v}_3\{\ln \psi\} = 3C_{12} = 0 \longrightarrow \left(\frac{d}{dr} g^{-1}\right) \frac{\partial \ln \psi}{\partial \phi} = 0, \quad (63)$$

which implies that $\psi = \psi(r, \theta)$. The third equation of (60) yields

$$\begin{aligned} \mathbf{v}_1\{\ln \psi\} = -3C_{32} &\longrightarrow \\ &\longrightarrow -\sin \theta \left(1 - \frac{1}{g^2}\right) \frac{\partial \ln \psi}{\partial r} = \frac{3}{2} \sin \theta \frac{\partial}{\partial r} \left(1 - \frac{1}{g^2}\right), \end{aligned} \quad (64)$$

which integrates to

$$\psi = k(\theta, \phi) \left(1 - \frac{1}{g^2}\right)^{-3/2}. \quad (65)$$

For consistency of (65) with the results of (62) and (63), we must have that $c(r, \phi) = c(r)$ and $k(\theta, \phi) = k(\theta)$. Therefore ψ and Φ are given by

$$\left. \begin{aligned} \psi &= 3 \left(-\frac{\Lambda}{3} \Psi^{-1/3} + \Psi^{2/3}\right)^{-3} = k(\theta) \left(1 - \frac{1}{g^2}\right)^{-3/2} \\ \Phi &= 9 \left(\frac{\Lambda}{3} \Psi^{-2/3} + 2\Psi^{1/3}\right)^{-3} = c(r) \sin^{-3} \theta \end{aligned} \right\}. \quad (66)$$

Equations (66) yield the following two conditions which must be satisfied

$$\left. \begin{aligned} -\frac{\Lambda}{3} \Psi^{-1/3} + \Psi^{2/3} &= k(\theta) \sqrt{1 - \frac{1}{g^2}} \\ \frac{\Lambda}{3} \Psi^{-2/3} + 2\Psi^{1/3} &= c(r) \sin \theta \end{aligned} \right\}. \quad (67)$$

It appears not possible to satisfy both conditions in (67) unless $\Lambda = 0$. Setting $\Lambda = 0$, then we have the following consistency condition

$$\left. \begin{aligned} [c(r) \sin \theta]^2 &= k(\theta) \sqrt{1 - \frac{1}{g^2}} \longrightarrow c(r) = \left(1 - \frac{1}{g^2}\right)^{1/4} \\ k(\theta) &= \sin^2 \theta \end{aligned} \right\}. \quad (68)$$

Substituting (68) back into (67), we obtain

$$\Psi = \Psi(r, \theta) = \left(1 - \frac{1}{g^2}\right)^{3/4} \sin^3 \theta. \quad (69)$$

Using the magnetic field for the configuration chosen, which is the same as for the previous permutation $\vec{\lambda}_{(1)}$, then (6) yields a 3-metric

$$(h_{ij})_{\vec{\lambda}_{(2)}} = -\frac{1}{2} \left(1 - \frac{1}{g^2}\right)^{-3/4} \sin^{-3} \theta \times \begin{pmatrix} \frac{4\left(\frac{d}{dr}g^{-1}\right)^2}{1 - \frac{1}{g^2}} & 0 & 0 \\ 0 & 1 - \frac{1}{g^2} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2}\right) \sin^2 \theta \end{pmatrix}.$$

This particular permutation of eigenvalues is allowed only for $\Lambda = 0$.

§6.1. Hodge duality condition for $\lambda_{(2)}$ for $\Lambda = 0$. The initial value constraints imply the existence of a spatial 3-metric $(h_{ij})_{\vec{\lambda}_{(2)}}$. We must enforce the Hodge duality condition as a consistency condition, and examine the implications with respect to the Birkhoff theorem. From the Gauss' law constraint we can read off from (69) that

$$\Psi = \left(1 - \frac{1}{g^2}\right)^{3/4} \sin^3 \theta. \quad (70)$$

So upon implementation of the Hodge duality condition, which requires equality of (6) with (8), the 3-metric must satisfy the condition

$$(h_{ij})_{\Lambda=0} = -\frac{1}{2} \sin^{-3} \theta \times \begin{pmatrix} \left(4 \frac{d}{dr} g^{-1}\right)^2 \left(1 - \frac{1}{g^2}\right)^{-7/4} & 0 & 0 \\ 0 & \left(1 - \frac{1}{g^2}\right)^{-1/4} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2}\right)^{-1/4} \sin^2 \theta \end{pmatrix} =$$

$$= -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\ 0 & \frac{(1-\frac{1}{g^2})}{(f'/g)'(f'/g')} & 0 \\ 0 & 0 & \frac{(1-\frac{1}{g^2})}{(f'/g)'(f'/g')} \sin^2 \theta \end{pmatrix}.$$

Consistency of the conformal factor fixes the lapse function as

$$N^2 = \frac{1}{2} \sin^{-3} \theta. \quad (71)$$

The remaining consistency conditions are on g_{rr} , namely

$$4 \frac{g'}{g^2} \left(1 - \frac{1}{g^2}\right)^{-7/4} = \left(\frac{g'}{f'}\right)^2 \longrightarrow f' = \frac{1}{2} g^2 \left(1 - \frac{1}{g^2}\right)^{7/8}, \quad (72)$$

as well as on $g_{\theta\theta}$

$$\left(1 - \frac{1}{g^2}\right)^{1/4} = \frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)'(f'/g')} \longrightarrow \left(\frac{f'}{g}\right)' f' = g' \left(1 - \frac{1}{g^2}\right)^{3/4}. \quad (73)$$

Putting the result of (72) into (73) leads to the condition

$$g' \left(1 - \frac{1}{g^2}\right)^{7/8} \left[1 + \frac{7}{4g^2} \left(1 - \frac{1}{g^2}\right)^{-1} - \frac{4}{g^2} \left(1 - \frac{1}{g^2}\right)^{3/4}\right] = 0. \quad (74)$$

The solution to (74) is $g' = 0$, which means that g is a numerical constant given by the roots of the term in brackets. This is a seventh degree polynomial, which we will not attempt to solve in this paper. Note for g constant that $g_{rr} = 0$. If any of the roots of the polynomial are real, then they would yield the following metric

$$ds^2 = -\frac{1}{2} \sin^{-3} \theta \left[dt^2 + k_2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (75)$$

The resulting metric is conformal to a 2-sphere radius $\sqrt{k_2}$, where $g = (1 - k_2^4)^{1/2}$ is any one of the seven roots of (75). The metric resulting from $\vec{\lambda}_{(2)}$ is degenerate since $g_{rr} = 0$, and also not spherically symmetric on account of the θ -dependent conformal factor. The interpretation is that Birkhoff theorem still holds and does not apply to (75), which constitutes a new General Relativity solution.

§7. Third permutation of eigenvalues $\vec{\lambda}_{(3)}$. For the third permutation of eigenvalues $\vec{\lambda}_{(3)}$ we have $\vec{\varphi} = (\varphi, \varphi, \varphi_1)$, which leads to the Gauss' law constraint equations (31)

$$\mathbf{v}_3 \{\ln \Phi\} = 3C_{[12]}, \quad \mathbf{v}_1 \{\ln \psi\} = 3C_{23}, \quad \mathbf{v}_2 \{\ln \psi\} = -3C_{13}. \quad (76)$$

The first equation from (76) is given by

$$\mathbf{v}_3\{\ln \Phi\} = 3C_{[12]} \longrightarrow \left(\frac{d}{dr} g^{-1}\right) \frac{\partial \ln \Phi}{\partial \phi} = 0, \quad (77)$$

which implies that $\Phi = \Phi(r, \theta)$ is at this stage an arbitrary function of two variables. The second equation of (76) is given by

$$\begin{aligned} \mathbf{v}_1\{\ln \psi\} &= 3C_{23} \longrightarrow \\ &\longrightarrow -\sin \theta \left(1 - \frac{1}{g^2}\right) \frac{\partial \ln \psi}{\partial r} = \frac{3}{2} \sin \theta \frac{\partial}{\partial r} \left(1 - \frac{1}{g^2}\right), \end{aligned} \quad (78)$$

which integrates to

$$\psi = k(\theta) \left(1 - \frac{1}{g^2}\right)^{-3/2}. \quad (79)$$

This is consistent with the results from (77), since there can be no ϕ dependence. The third equation of (76) is given by

$$\mathbf{v}_2\{\ln \psi\} = -3C_{13} \longrightarrow \frac{\partial}{\partial r} \left(\frac{\sin \theta}{g}\right) \frac{\partial \ln \psi}{\partial \theta} = -3 \frac{\partial}{\partial r} \left(\frac{\cos \theta}{g}\right), \quad (80)$$

which integrates to

$$\psi = c(r) \sin^{-3} \theta, \quad (81)$$

where c is at this stage an arbitrary function. From (77) $\Phi = \Phi(r, \theta)$ can be an arbitrary function of r and θ , and hence we are free to determine this dependence entirely from ψ . Consistency of (79) with (81) implies that

$$\psi = -\frac{\Lambda}{3} \Psi^{-1/3} + \Psi^{2/3} = \sin^{-3} \theta \left(1 - \frac{1}{g^2}\right)^{-3/2}. \quad (82)$$

Unlike the case for $\vec{\lambda}_{(2)}$ we are allowed to have a nonzero Λ in equation (82), since there is no longer a constraint on the functional dependence of Φ . Therefore we are free to solve equation (82) for Ψ , which enables us to fix $\Phi = \Phi(\psi)$. Equation (82) is a cubic polynomial equation for the quantity $\Psi^{1/3}$, which can be solved in closed form (see e.g. Appendix A for the derivation)

$$\left. \begin{aligned} \Psi &= 2 \sqrt{-\frac{\psi}{3}} \sin \left\{ \frac{1}{3} \arcsin \left[\frac{\sqrt{3}\Lambda}{2} (-\psi)^{-3/2} \right] \right\}^3 \\ \psi &= \sin^{-3} \theta \left(1 - \frac{1}{g^2}\right)^{-3/2} \end{aligned} \right\}. \quad (83)$$

For the purposes of constructing a 3-metric we will be content with the $\Lambda = 0$ case which follows from (82), yielding

$$\Psi = \Psi(r, \theta) = (\sin \theta)^{-9/2} \left(1 - \frac{1}{g^2}\right)^{-9/4}. \quad (84)$$

Using the previous configuration, equation (84) yields a 3-metric

$$(h_{ij})_{\bar{\lambda}_{(3)}} = \frac{1}{2} \left(1 - \frac{1}{g^2}\right)^{9/4} \sin^{9/2} \theta \begin{pmatrix} \frac{4\left(\frac{d}{dr}g^{-1}\right)^2}{1 - \frac{1}{g^2}} & 0 & 0 \\ 0 & 1 - \frac{1}{g^2} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2}\right) \sin^2 \theta \end{pmatrix}.$$

§7.1. Hodge duality condition for $\lambda_{(3)}$ for $\Lambda = 0$. The initial value constraints imply the existence of a spatial 3-metric $(h_{ij})_{\bar{\lambda}_{(3)}}$. We must enforce the Hodge duality condition as a consistency condition, and examine the implications with respect to Birkhoff's theorem. From the Gauss' law constraint we can read off from (84) that

$$\Psi = \left(1 - \frac{1}{g^2}\right)^{3/4} \sin^3 \theta. \quad (85)$$

So upon implementation of the Hodge duality condition, then the 3-metric must satisfy the condition $(h_{ij})_{\text{Constraints}} = (h_{ij})_{\text{Hodge}}$, given by

$$\begin{aligned} (h_{ij})_{\bar{\lambda}_{(3)}} &= \frac{1}{2} \sin^{9/2} \theta \times \\ &\times \begin{pmatrix} \left(4 \frac{d}{dr} g^{-1}\right)^2 \left(1 - \frac{1}{g^2}\right)^{5/4} & 0 & 0 \\ 0 & \left(1 - \frac{1}{g^2}\right)^{13/4} & 0 \\ 0 & 0 & \left(1 - \frac{1}{g^2}\right)^{13/4} \sin^2 \theta \end{pmatrix} = \\ &= -N^2 \begin{pmatrix} (g'/f')^2 & 0 & 0 \\ 0 & \frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)'} (f'/g') & 0 \\ 0 & 0 & \frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)'} (f'/g') \sin^2 \theta \end{pmatrix}. \end{aligned}$$

Consistency of the conformal factor fixes the lapse function as

$$N^2 = \sin^{9/2} \theta. \quad (86)$$

The remaining consistency conditions are on g_{rr} , namely

$$4 \frac{g'}{g^2} \left(1 - \frac{1}{g^2}\right)^{5/4} = \left(\frac{g'}{f'}\right)^2 \rightarrow f' = \frac{1}{2} g^2 \left(1 - \frac{1}{g^2}\right)^{-5/8}, \quad (87)$$

as well as on $g_{\theta\theta}$

$$\left(1 - \frac{1}{g^2}\right)^{13/4} = \frac{\left(1 - \frac{1}{g^2}\right)}{(f'/g)'(f'/g')} \rightarrow \left(\frac{f'}{g}\right)' f' = g' \left(1 - \frac{1}{g^2}\right)^{-9/4}. \quad (88)$$

Putting the result of (87) into (88) leads to the condition

$$g' \left(1 - \frac{1}{g^2}\right)^{-5/8} \left(1 - \frac{37}{8g^2} + \frac{37}{8g^4}\right) = 0. \quad (89)$$

The solution to (89) is $g'=0$, which means that g is a constant given by the roots of the quartic polynomial in brackets. The solution is

$$g = \pm \sqrt{\frac{37}{16} \pm \frac{1}{8} \sqrt{\frac{185}{2}}}. \quad (90)$$

There are four roots, each of which corresponds to a 2-sphere

$$ds^2 = -\frac{1}{2} \sin^{9/2} \theta \left[dt^2 + k_3 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (91)$$

The resulting metric is conformal to a 2-sphere of radius $\sqrt{k_3}$, determined by any of the four roots (90). In direct analogy with the case from $\vec{\lambda}_{(2)}$, the solutions corresponding to $\vec{\lambda}_{(3)}$ are also degenerate and not spherically symmetric. Hence Birkhoff's theorem still holds and $\vec{\lambda}_{(1)}$ yields the unique static spherically symmetric vacuum solution.

§8. Conclusion. In this paper we have constructed some solutions to the Einstein equations using the instanton representation method. We have applied this method to spacetimes of Petrov Type D, producing some known solutions. We first constructed the Schwarzschild blackhole solution from a particular permutation $\vec{\lambda}_{(1)}$ of the eigenvalues of Ψ_{ae} solving the initial value constraints, by implementation of the Hodge duality condition. This was done to establish the validity of the method for a simple well-known case. Then using the remaining eigenvalue permutations $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$, we constructed additional solutions which might be not as well-known, and perhaps even new. This would on the surface suggest that the instanton representation method be rendered inadmissible, since Birkhoff's theorem implies that any additional solutions besides the Schwarzschild solution must not exist. However,

upon further analysis we have shown that the Hodge duality condition applied to the $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ metrics imposed stringent restrictions on their form. These restrictions led to two new solutions for which Birkhoff's theorem does not apply. The metrics for $\vec{\lambda}_{(2)}$ and $\vec{\lambda}_{(3)}$ became conformally related to 2-spheres of fixed radius determined by the roots of certain polynomial equations. Since the conformal factor depends on θ , then these metrics are not spherically symmetric in the usual sense. This, combined with the observation that the metrics are degenerate, leads us to conclude that the instanton representation method as applied in this paper is fully consistent with Birkhoff's theorem, and also is indeed capable of producing General Relativity solutions. Our main results have been the validation of the instanton representation method for the Schwarzschild case, and as well the construction of two solutions (75) and (91) which to the present author's knowledge appear to be new. Having established the validity of the instanton representation method for a special situation governing the Petrov Type D case as a testing ground, we are now ready to apply the method to the construction of more general solutions.

Appendix A. Roots of the cubic polynomial in trigonometric form. We would like to solve the cubic equation

$$z^3 + pz = q, \quad (92)$$

Many techniques for solving the cubic involve complicated radicals, which introduce complex numbers which are not needed when the roots are real-valued. We prefer the trigonometric method, which avoids such complications. Define a transformation

$$z = u \sin \theta. \quad (93)$$

Substitution of (93) into (92) yields

$$\sin^3 \theta + \frac{p}{u^2} \sin \theta = \frac{q}{u^3}. \quad (94)$$

Comparison of (94) with the trigonometric identity

$$\sin^3 \theta - \frac{3}{4} \sin \theta = -\frac{1}{4} \sin(3\theta) \quad (95)$$

enables one to make the identifications

$$\frac{p}{u^2} = -\frac{3}{4}, \quad \frac{q}{u^3} = -\frac{1}{4} \sin(3\theta). \quad (96)$$

This implies that

$$u = \frac{2}{\sqrt{3}} (-p)^{1/2}, \quad \sin(3\theta) = -\frac{3\sqrt{3}}{2} \frac{q}{(-p)^{3/2}}. \quad (97)$$

We can now solve (97) for θ

$$\theta = \frac{1}{3} \arcsin \left[-\frac{3\sqrt{3}}{2} \frac{q}{(-p)^{3/2}} \right] + \frac{2\pi m}{3}, \quad m = 0, 1, 2 \quad (98)$$

and in turn for z using (93). The solution is

$$z = \frac{1}{\sqrt{3}} (-p)^{1/2} T_{1/3}^m \left[-3\sqrt{3} q (-p)^{-3/2} \right], \quad (99)$$

where we have defined

$$T_{1/3}^m(t) = 2 \sin \left[-\frac{1}{3} \arcsin \left(\frac{t}{2} \right) \right]. \quad (100)$$

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Instanton Representation of Plebanski Gravity. Consistency of the Initial Value Constraints under Time Evolution

Eyo Eyo Ita III*

Abstract: The instanton representation of Plebanski gravity provides as equations of motion a Hodge self-duality condition and a set of “generalized” Maxwell’s equations, subject to gravitational degrees of freedom encoded in the initial value constraints of General Relativity. The main result of the present paper will be to prove that this constraint surface is preserved under time evolution. We carry this out not using the usual Dirac procedure, but rather the Lagrangian equations of motion themselves. Finally, we provide a comparison with the Ashtekar formulation to place these results into overall context.

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§1. Introduction. In [1] a new formulation of General Relativity has been presented, referred to as the instanton representation of Plebanski gravity. The basic dynamical variables are an $SO(3, C)$ gauge connection A_μ^a and a matrix Ψ_{ae} taking its values in two copies of $SO(3, C)$.[†] The consequences of the associated action I_{Inst} were determined via its equa-

*Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom. E-mails: eei20@cam.ac.uk; ita@usna.edu.

[†]Index labelling conventions for this paper are that symbols a, b, \dots from the beginning of the Latin alphabet denote internal $SO(3, C)$ indices while those from the middle i, j, k, \dots denote spatial indices. Both of these sets of indices take values 1, 2 and 3. The Greek symbols μ, ν, \dots refer to spacetime indices which take values 0, 1, 2, 3.

tions of motion, which hinge crucially on weak equalities implied by the initial value constraints. For these consequences to be self-consistent, the constraint surface must be preserved for all time by the evolution equations. The present paper will show that this is indeed the case. Due to the necessity to avoid some technical difficulties, we will not use the usual Dirac formulation for totally constrained systems [2]. In fact we will not make use of Poisson brackets or of any canonical structure implied by I_{Inst} . Rather, we will deduce the time evolution of the initial value constraints directly from the Lagrangian equations of motion of I_{Inst} .

The organization of this paper is as follows. §2 provides some background on the relation between I_{Inst} and the Ashtekar formulation. There is a common notion that these theories are the same within their common domain of definition. §2 argues that this is not the case, which sets the stage for the present paper. §3 and §4 present I_{Inst} as a stand-alone action and derive the time evolution of the basic variables. §5, §6 and §7 demonstrate that the nondynamical equations, referred to as the diffeomorphism, Gauss' law and Hamiltonian constraints, evolve into combinations of the same constraint set. The result is that the time derivatives of these constraints are weakly equal to zero with no additional constraints generated on the system. While we do not use the usual Dirac procedure in this paper, the result is still that I_{Inst} is in a sense Dirac-consistent. We will make this inference clearer by comparison with the Ashtekar formulation in §8. On a final note, the terms *diffeomorphism* and *Gauss' law* constraints are used loosely in this paper, in that we have not specified what transformations of the basic variables these constraints generate. The use of these terms will be primarily for notational purposes, due to their counterparts which appear in the Ashtekar formalism.

§2. Background: Relation of the instanton representation to the Ashtekar formalism. The action for the instanton representation can be written in the following 3+1 decomposed form [1]

$$I_{\text{Inst}} = \int dt \int_{\Sigma} d^3x \left[\Psi_{ae} B_e^i \dot{A}_i^a + A_0^a B_e^i D_i \{ \Psi_{ae} \} - \epsilon_{ijk} N^i B_a^j B_e^k \Psi_{ae} - \right. \\ \left. - iN \sqrt{\det \| B \|} \sqrt{\det \| \Psi \|} (\Lambda + \text{tr } \Psi^{-1}) \right], \quad (1)$$

where D_i is the $\text{SO}(3, \mathbb{C})$ covariant derivative, whose action on $\text{SO}(3, \mathbb{C})$ -valued 3-vectors v_a is given by

$$D_i v_a = \partial_i v_a + f_{abc} A_i^b v_c \quad (2)$$

with structure constants $f_{abc} = \epsilon_{abc}$. The phase space variables are a spatial $\text{SO}(3, \mathbb{C})$ connection A_i^a with magnetic field B_a^i and a matrix $\Psi_{ae} \in \text{SO}(3, \mathbb{C}) \otimes \text{SO}(3, \mathbb{C})$, and the quantities (A_0^a, N, N^i) are nondynamical fields. One would like to compute the Hamiltonian dynamics of (1) using phase space variables $\Omega_{\text{Inst}} = (\Psi_{ae}, A_i^a)$ as the fundamental fields. But the phase space of (1) is noncanonical since its symplectic two form,

$$\begin{aligned} \Omega_{\text{Inst}} &= \delta\theta_{\text{Inst}} = \delta \left(\int_{\Sigma} d^3x \Psi_{ae} B_e^i \delta A_i^a \right) = \\ &= \int_{\Sigma} d^3x B_e^i \delta \Psi_{ae} \wedge \delta A_i^a + \int_{\Sigma} d^3x \Psi_{ae} \epsilon^{ijk} D_j (\delta A_k^e) \wedge \delta A_i^a, \end{aligned} \quad (3)$$

is not closed owing to the presence of the second term on the right hand side of (3). The equations of motion for (A_0^a, N, N^i) define a constraint surface on Ω_{Inst} , which as a necessary condition for self-consistency must be shown to be preserved under time evolution.

The initial stages of the Dirac procedure for constrained systems [2] applied to (1) imply that the momentum canonically conjugate to A_i^a yields the primary constraint

$$\Pi_a^i = \frac{\delta I_{\text{Inst}}}{\delta \dot{A}_i^a} = \Psi_{ae} B_e^i, \quad (4)$$

where $\det \|B\|$ and $\det \|\Psi\|$ are nonzero. Then making the identification $\tilde{\sigma}_a^i = \Pi_a^i$ and upon substitution of (4) into (1), one obtains the action

$$I_{\text{Ash}} = \int dt \int_{\Sigma} d^3x \left[\tilde{\sigma}_a^i \dot{A}_i^a + A_0^a G_a - N^i H_i - \frac{i}{2} \underline{N} H \right], \quad (5)$$

where (G_a, N^i, N) are the Gauss' law, vector and Hamiltonian constraints given by

$$G_a = D_i \tilde{\sigma}_a^i, \quad H_i = \epsilon_{ijk} \tilde{\sigma}_a^j B_a^k, \quad H = \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \left(\frac{\Lambda}{3} \tilde{\sigma}_c^k + B_c^k \right). \quad (6)$$

Equation (5) is the action for the Ashtekar formulation of General Relativity [3, 4] defined on the phase space $\Omega_{\text{Ash}} = (\tilde{\sigma}_a^i, A_i^a)$, where $\tilde{\sigma}_a^i$ is the densitized triad. The auxiliary fields $(A_0^a, N^i, \underline{N})$ are $\text{SO}(3, \mathbb{C})$ rotation angle A_0^a , the shift vector N^i and the densitized lapse function $\underline{N} = N(\det \|\tilde{\sigma}\|)^{-1/2}$. From (5) one reads off the symplectic two form Ω_{Ash} given by

$$\Omega_{\text{Ash}} = \int_{\Sigma} d^3x \delta \tilde{\sigma}_a^i \wedge \delta A_i^a = \delta \left(\int_{\Sigma} d^3x \tilde{\sigma}_a^i \delta A_i^a \right) = \delta \theta_{\text{Ash}}, \quad (7)$$

which is the exact functional variation of the canonical one form θ_{Ash} .

This implies the following Poisson brackets between any two phase space functions f and g for fundamental variables Ω_{Ash} defined on three-dimensional spatial hypersurfaces Σ

$$\{f, g\} = \int_{\Sigma} d^3x \left(\frac{\delta f}{\delta \tilde{\sigma}_a^i(x)} \frac{\delta g}{\delta A_i^a(x)} - \frac{\delta g}{\delta \tilde{\sigma}_a^i(x)} \frac{\delta f}{\delta A_i^a(x)} \right). \quad (8)$$

Since equation (8) is of canonical form, it is straightforward to compute the constraints algebra and the Hamilton's equations of motion for (5). The constraints algebra for (6) based on these Poisson brackets is

$$\left. \begin{aligned} \{\vec{H}[\vec{N}], \vec{H}[\vec{M}]\} &= H_k [N^i \partial^k M_i - M^i \partial^k N_i] \\ \{\vec{H}[N], G_a[\theta^a]\} &= G_a [N^i \partial_i \theta^a] \\ \{G_a[\theta^a], G_b[\lambda^b]\} &= G_a [f_{bc}^a \theta^b \lambda^c] \\ \{H(\underline{N}), \vec{H}[\vec{N}]\} &= H [N^i \partial_i \underline{N}] \\ \{H(\underline{N}), G_a(\theta^a)\} &= 0 \\ [H(\underline{N}), H(\underline{M})] &= H_i [(N \partial_j M - M \partial_j N) H^{ij}] \end{aligned} \right\}, \quad (9)$$

with structure functions $H^{ij} = \tilde{\sigma}_a^i \tilde{\sigma}_a^j$, which is first class due to closure. Therefore the algebra (9) is consistent in the Dirac sense.

Following the step-by-step Dirac procedure, one would be led naively to the conclusion that (1), shown in [1] to describe General Relativity for certain Petrov types, for $\det\|B\| \neq 0$ and $\det\|\Psi\| \neq 0$ is the same theory as (5) which also describes General Relativity. One might then infer, on account of (4), the Dirac-consistency of (1). In this paper we will probe beyond the surface and show that (1) and (5) are indeed different versions of General Relativity. Certainly as a minimum, one can regard (1) as a noncanonical version of (5) which is canonical.

As a first step via the standard Hamiltonian approach, one should compute the Hamiltonian dynamics of (1) using Poisson brackets constructed from the inverse of the symplectic matrix derivable from (3), without making use of (4). However the implementation of these Poisson brackets in practice presently appears to be unclear, and will require some additional research.* To substantiate the claim that (1) is

*The fundamental Poisson brackets of (1) are noncanonical and have been computed in Appendix A. The present difficulty lies specifically in the interpretation of the sequence of the action of spatial derivatives on the phase space variables when one considers the full theory. We will therefore relegate as a direction of future research the computation of the associated constraints algebra.

at some level fundamentally different from (5) while at the same time being self-consistent, we must therefore find an alternate means for verifying consistency of the constraints defined on $\Omega_{\text{Inst}} = (\Psi_{ae}, A_i^a)$ under time evolution. Our method will be to use the Lagrangian equations of motion of (1) as the starting point. In this way, we will avoid the necessity to define a canonical structure and Poisson brackets for (1), which appear from Appendix A to be relatively complicated.

§3. Instanton representation of Plebanski gravity. After an integration of parts with discarding of boundary terms, using $F_{0i}^a = \dot{A}_i^a - D_i A_0^a$ for the temporal curvature components, the starting action for the instanton representation of Plebanski gravity (1) can be written as [1]

$$I_{\text{Inst}} = \int dt \int_{\Sigma} d^3x \Psi_{ae} B_e^k (F_{0k}^a + \epsilon_{kjm} B_a^j N^m) - iN \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}), \quad (10)$$

where $N^\mu = (N, N^i)$ are the lapse function and shift vector from the metric of General Relativity, and Λ is the cosmological constant. The basic fields are Ψ_{ae} and A_i^a , and the action (10) is defined only on configurations for which $\det\|B\| \neq 0$ and $\det\|\Psi\| \neq 0$.* In the Dirac procedure one refers to N^μ as nondynamical fields, since their velocities do not appear in the action. While the velocity $\dot{\Psi}_{ae}$ also does not appear, it is important to distinguish this field from N^μ since the action (10), unlike the case for N^μ , is nonlinear in Ψ_{ae} .†

The equation of motion for the shift vector N^i , the analogue of Hamilton's equation for its conjugate momentum $\Pi_{\bar{N}}$, is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N^i} = \epsilon_{ijk} B_a^j B_e^k \Psi_{ae} = (\det\|B\|) (B^{-1})_i^d \psi_d \sim 0, \quad (11)$$

where $\psi_d = \epsilon_{dae} \Psi_{ae}$ is the antisymmetric part of Ψ_{ae} . This is equivalent to the diffeomorphism constraint H_i owing to the nondegeneracy of B_a^i , and we will often use H_i and ψ_d interchangeably in this paper. The equation of motion for the lapse function N , the analogue of Hamilton's equation for its conjugate momentum Π_N , is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N} = \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) = 0. \quad (12)$$

*The latter case limits the application of our results to spacetimes of Petrov Types I, D and O (see e.g. [6] and [7]).

†Additionally, since Ψ_{ae} multiplies the velocity of another field, then according to the instanton representation it should accurately be regarded more-so as an intrinsic part of the canonical structure than as a nondynamical field.

Nondegeneracy of Ψ_{ae} and of the magnetic field B_e^i implies that on-shell, the following relation must be satisfied

$$\Lambda + \text{tr} \Psi^{-1} = 0, \quad (13)$$

which we will similarly treat as being synonymous with the Hamiltonian constraint (12). The equation of motion for Ψ_{ae} is

$$\begin{aligned} \frac{\delta I_{\text{Inst}}}{\delta \Psi_{ae}} &= B_e^k F_{0k}^a + \epsilon_{kjm} B_e^k B_a^j N^m + \\ &+ iN \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Psi^{-1} \Psi^{-1})^{ea} \sim 0, \end{aligned} \quad (14)$$

up to a term proportional to (13) which we have set weakly equal to zero. One could attempt to define a momentum conjugate to Ψ_{ae} , for which (14) would be the associated Hamilton's equation of motion. But since Ψ_{ae} forms part of the canonical structure of (10), then our interpretation is that this is not necessary.*

The equation of motion for the connection A_μ^a is given by

$$\begin{aligned} \frac{\delta I_{\text{Inst}}}{\delta A_\mu^a} &\sim \epsilon^{\mu\sigma\nu\rho} D_\sigma (\Psi_{ae} F_{\nu\rho}^e) - \frac{i}{2} \delta_i^\mu \bar{D}_{da}^{ji} \left[4\epsilon_{mjk} N^m B_e^k \Psi_{[de]} + \right. \\ &\left. + N (B^{-1})_j^d \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) \right] \sim 0, \end{aligned} \quad (15)$$

where we have defined

$$\left. \begin{aligned} \bar{D}_{ea}^{ji}(x, y) &\equiv \frac{\delta B_e^j(y)}{\delta A_i^a(x)} = \epsilon^{jki} (-\delta_{ae} \partial_k + f_{eda} A_k^d) \delta^{(3)}(x, y) \\ \bar{D}_{ea}^{0i} &\equiv 0 \end{aligned} \right\}. \quad (16)$$

The terms in large square brackets in (15) vanish weakly, since they are proportional to the constraints (11) and (13) and their spatial derivatives. Hence we can regard (15) as being synonymous with

$$\epsilon^{\mu\sigma\nu\rho} D_\sigma (\Psi_{ae} F_{\nu\rho}^e) \sim 0. \quad (17)$$

In an abuse of notation, we will treat (14) and (17) as strong equalities in this paper. This will be justified once we have completed the demonstration that the constraint surface defined collectively by (11), (12) and the Gauss's law constraint contained in (17) is indeed preserved under time evolution. As a note prior to proceeding we will often make

*This is because (14) contains a velocity \dot{A}_k^a within F_{0k}^a and will therefore be regarded as an evolution equation rather than a constraint. This is in stark contrast with (11) and (12), which are genuine constraint equations due to the absence of any velocities.

the following identification derived in [1]

$$N \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} \equiv \sqrt{-g} \quad (18)$$

as a shorthand notation, to avoid cluttering many of the derivations which follow in the present paper.

§3.1. Internal consistency of the equations of motion. Prior to embarking upon the issue of consistency of time evolution of the initial value constraints, we will check for internal consistency of I_{Inst} , which entails probing of the physical content implied by (17) and (14). First, equation (17) can be decomposed into its spatial and temporal parts as

$$D_i(\Psi_{bf} B_f^i) = 0, \quad D_0(\Psi_{bf} B_f^i) = \epsilon^{ijk} D_j(\Psi_{bf} F_{0k}^f). \quad (19)$$

The first equation of (19) is the Gauss' law constraint of a SO(3) Yang–Mills theory, when one makes the identification of $\Psi_{bf} B_f^i \sim E_b^i$ with the Yang–Mills electric field. The Maxwell equations for U(1) gauge theory with sources (ρ, \vec{J}) , in units where $c=1$, are given by

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \dot{B} = -\vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{E} = \rho, \quad \dot{E} = -\vec{J} + \vec{\nabla} \times \vec{B}. \quad (20)$$

Equations (19) can be seen as a generalization of the first two equations of (20) to SO(3) nonabelian gauge theory in flat space when:

- 1) One identifies $F_{0k}^f \equiv E_k^f$ with the SO(3) generalization of the electric field \vec{E} , and
- 2) One chooses $\Psi_{ae} = k \delta_{ae}$ for some numerical constant k .

When $\rho=0$ and $\vec{J}=0$, then one has the vacuum theory and equations (20) are invariant under the transformation

$$(\vec{E}, \vec{B}) \longrightarrow (-\vec{B}, \vec{E}). \quad (21)$$

Then the second pair of equations of (20) become implied by the first pair. This is the condition that the Abelian curvature $F_{\mu\nu}$, where $F_{0i} = E_i$ and $\epsilon_{ijk} F_{jk} = B_i$, is Hodge self-dual with respect to the metric of a conformally flat spacetime. But equations (19) for more general Ψ_{ae} encode gravitational degrees of freedom, which as shown in [1] generalizes the concept of self-duality to more general spacetimes solving the Einstein equations. Let us first attempt to derive the analogue for (19) of the second pair of equations in (20) in the vacuum case. Acting on the first equation of (19) with D_0 yields

$$D_0 D_i(\Psi_{bf} B_f^i) = D_i D_0(\Psi_{bf} B_f^i) + [D_0, D_i](\Psi_{bf} B_f^i) = 0. \quad (22)$$

Substituting the second equation of (19) into the first term on the right hand side of (22) and using the definition of temporal curvature as the commutator of covariant derivatives on the second term we have

$$\begin{aligned} D_i \left[\epsilon^{ijk} D_j (\Psi_{bf} F_{0k}^f) \right] + f_{bcd} F_{0i}^c \Psi_{df} B_f^i &= \\ &= f_{bcd} (B_c^k F_{0k}^f + B_f^k F_{0k}^c) \Psi_{df} = 0, \end{aligned} \quad (23)$$

where we have also used the spatial part of the commutator $\epsilon^{ijk} D_i D_j v_a = f_{abc} B_b^k v_c$. Note that the right hand side of (23) is symmetric in f and c , and also forms the symmetric part of the left hand side of (14)

$$B_f^i F_{0i}^b + i\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{fb} + \epsilon_{ijk} B_f^i B_b^j N^k = 0, \quad (24)$$

re-written here for completeness. To make progress from (23), we will substitute (24) into (23). This causes the last term of (24) to drop out due to antisymmetry, which leaves us with

$$\begin{aligned} -i\sqrt{-g} f_{bcd} \left[\Psi_{df} (\Psi^{-1} \Psi^{-1})^{fc} + \Psi_{df} (\Psi^{-1} \Psi^{-1})^{fc} \right] &= \\ &= -2i\sqrt{-g} f_{bcd} \Psi_{dc}^{-1}. \end{aligned} \quad (25)$$

The equations are consistent only if (25) vanishes, which is the requirement that $\Psi_{ae} = \Psi_{ea}$ be symmetric. This of course is the requirement that the diffeomorphism constraint (11) be satisfied. So the analogue of the second pair of (20) in the vacuum case must be encoded within the requirement that $\Psi_{ae} = \Psi_{ea}$ be symmetric.

§4. The time evolution equations. We must now verify that the initial value constraints are preserved under time evolution defined by the equations of motion (14) and (15). Since the temporal part of (19) is already a constraint, then the only equality required is the Hodge duality condition

$$B_f^k F_{0k}^b + i\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{fb} + \epsilon_{ijk} N^i B_b^j B_f^k = 0, \quad (26)$$

and the spatial part of (19)

$$\epsilon^{ijk} D_j (\Psi_{ae} F_{0k}^e) = D_0 (\Psi_{ae} B_e^i). \quad (27)$$

Since the initial value constraints were used to obtain the second line of (26) from (10), then we must verify that these constraints are preserved under time evolution as a requirement of consistency. Using $F_{0i}^b = \dot{A}_i^b - D_i A_0^b$ and defining

$$\sqrt{-g} (B^{-1})_i^f (\Psi^{-1} \Psi^{-1})^{fb} + \epsilon_{mnk} N^m B_b^n \equiv iH_k^b, \quad (28)$$

then equation (26) can be written as a time evolution equation for the connection A_i^a . Note that this is not the same thing as a constraint equation, as noted previously,

$$F_{0i}^b = -iH_i^b \longrightarrow \dot{A}_i^b = D_i A_0^b - iH_i^b. \quad (29)$$

From equation (29) we can obtain the following time evolution equation for the magnetic field B_a^i , given by

$$\begin{aligned} \dot{B}_e^i &= \epsilon^{ijk} D_j \dot{A}_k^e = \epsilon^{ijk} D_j (D_k A_0^e - iH_k^e) = \\ &= f_{ebc} B_b^i A_0^c - i\epsilon^{ijk} D_j H_k^e = -\delta_{\vec{\theta}} B_e^i - i\epsilon^{ijk} D_j H_k^e, \end{aligned} \quad (30)$$

which will be useful. On the first term on the right hand side of (30) we have used the definition of the curvature as the commutator of covariant derivatives. The notation $\delta_{\vec{\theta}}$ in (30) suggests that that B_e^i transforms as a $\text{SO}(3, \mathbb{C})$ vector under gauge transformations parametrized by $\theta^b \equiv A_0^b$. Since we have not defined the canonical structure of I_{Inst} , then $\delta_{\vec{\theta}}$ as used in (30) and in (33) should at this stage be regarded simply as a shorthand notation.

We will now apply the Leibnitz rule in conjunction with the definition of the temporal covariant derivatives to (27) to determine the equation governing the time evolution of Ψ_{ae} . This is given by

$$D_0(\Psi_{ae} B_e^i) = B_e^i \dot{\Psi}_{ae} + \Psi_{ae} \dot{B}_e^i + f_{abc} A_0^b (\Psi_{ce} B_e^i) = \epsilon^{ijk} D_j (\Psi_{ae} F_{0k}^e). \quad (31)$$

Substituting (30) and (29) into both sides of (31), we have

$$\begin{aligned} B_e^i \dot{\Psi}_{ae} + \Psi_{ae} (f_{ebc} B_b^i A_0^c - i\epsilon^{ijk} D_j H_k^e) + f_{abc} A_0^b (\Psi_{ce} B_e^i) = \\ = -i\epsilon^{ijk} D_j (\Psi_{ae} H_k^e). \end{aligned} \quad (32)$$

In what follows, it will be convenient to use the following transformation properties for Ψ_{ae} as A_i^a under $\text{SO}(3, \mathbb{C})$ gauge transformations*

$$\left. \begin{aligned} \delta_{\vec{\theta}} \Psi_{ae} &= (f_{abc} \Psi_{ce} + f_{ebc} \Psi_{ac}) A_0^b \\ \delta_{\vec{\theta}} A_i^a &= -D_i A_0^a \\ \delta_{\vec{\theta}} B_e^i &= -f_{ebc} B_b^i A_0^c \end{aligned} \right\}. \quad (33)$$

Then using (33), the time evolution equations for the phase space variables Ω_{Inst} can be written in the following compact form

$$\dot{A}_i^b = -\delta_{\vec{\theta}} A_i^b - iH_i^b, \quad \dot{\Psi}_{ae} = -\delta_{\vec{\theta}} \Psi_{ae} - i\epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) H_k^f. \quad (34)$$

*Note that these are based purely on the transformation properties of a $\text{SO}(3, \mathbb{C})$ gauge connection and of a second-rank $\text{SO}(3, \mathbb{C})$ tensor, which hold irrespective of any canonical formalism.

We have determined the evolution equations for Ψ_{ae} and A_i^a directly from I_{Inst} . Recall that we have not used Poisson brackets, and have assumed that the constraints G_a , H_i and H vanish weakly. Therefore the first order of business will be to check for the preservation of the initial value constraints under the time evolution generated by (34). This means that we must check that the time evolution of the diffeomorphism, Gauss' law and Hamiltonian constraints are combinations of terms proportional to the same set of constraints and their spatial derivatives, and terms which vanish when the constraints hold.*

These constraints are given by

$$\left. \begin{aligned} \mathbf{w}_e \{ \Psi_{ae} \} &= 0 \\ (\det \| B \|) (B^{-1})^d{}_i \psi_d &= 0 \\ \sqrt{\det \| B \|} \sqrt{\det \| \Psi \|} (\Lambda + \text{tr} \Psi^{-1}) &= 0 \end{aligned} \right\}, \quad (35)$$

where $\det \| B \| \neq 0$ and $\det \| \Psi \| \neq 0$. We will occasionally make the identification

$$N \sqrt{\det \| B \|} \sqrt{\det \| \Psi \|} \equiv \sqrt{-g} \quad (36)$$

for a shorthand notation. Additionally, the following definitions are provided for the vector fields appearing in the Gauss constraint

$$\mathbf{w}_e = B_e^i D_i, \quad \mathbf{v}_e = B_e^i \partial_i, \quad (37)$$

where D_i is the $\text{SO}(3, \mathbb{C})$ covariant derivative with respect to the connection A_i^a . Recall that equations (35) are precisely the equations of motion for the auxiliary fields A_0^a , N^i and N in (1).

§5. Consistency of the diffeomorphism constraint under time evolution. The diffeomorphism constraint is directly proportional to $\psi_d = \epsilon_{dae} \Psi_{ae}$, the antisymmetric part of Ψ_{ae} . So to establish the consistency condition for this constraint, it suffices to show that the antisymmetric part of the second equation of (34) vanishes weakly. This is given by

$$\epsilon_{dae} \dot{\Psi}_{ae} = -\delta_{\vec{\theta}} (\epsilon_{dae} \Psi_{ae}) - i \epsilon_{dae} \epsilon^{ijk} (B^{-1})^e{}_i (D_j \Psi_{af}) H_k^f, \quad (38)$$

which splits into two terms. Using (33), one finds that the first term on the right hand side of (38) is given by

$$-\epsilon_{dae} \delta_{\vec{\theta}} \Psi_{ae} = -\epsilon_{dae} (f_{abc} \Psi_{ce} + \Psi_{ac} f_{ebc}) A_0^b. \quad (39)$$

*This includes any nonlinear function of linear order or higher in the constraints, a situation which involves the diffeomorphism constraint.

In what follows and in various other places in this paper, we will use the fact that the $SO(3)$ structure constants f_{abc} are numerically the same as the three-dimensional epsilon symbol ϵ_{abc} . So the following identities hold

$$\epsilon_{abc}\epsilon_{fec} = \delta_{af}\delta_{be} - \delta_{ae}\delta_{bf}, \quad \epsilon_{abc}\epsilon_{ebc} = 2\delta_{ae}, \quad \epsilon_{abc}\epsilon_{abc} = 6. \quad (40)$$

Using (40), then (39) is given by

$$\begin{aligned} -\epsilon_{dae}\delta_{\bar{\theta}}\Psi_{ae} &= -\left[(\delta_{eb}\delta_{dc} - \delta_{ec}\delta_{bd})\Psi_{ce} + (\delta_{db}\delta_{ac} - \delta_{dc}\delta_{ab})\Psi_{ac}\right]A_0^b = \\ &= -(\Psi_{db} - \delta_{bd}\text{tr}\Psi + \delta_{db}\text{tr}\Psi - \Psi_{bd})A_0^b = 2\Psi_{[bd]}A_0^b = -\epsilon_{dbh}A_0^b\psi_h, \end{aligned} \quad (41)$$

which is proportional to the diffeomorphism constraint. The second term on the right hand side of (38) has two contributions due to H_k^f as defined in (28), and the first contribution reduces to

$$\begin{aligned} -i\epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})(H_{(1)})_k^f &= \\ &= -i\epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})\sqrt{-g}(B^{-1})_k^g(\Psi^{-1}\Psi^{-1})^{gf}. \end{aligned} \quad (42)$$

Using the definition of the determinant of nondegenerate 3×3 matrices

$$\epsilon^{ijk}\epsilon_{abc}(B^{-1})_j^b(B^{-1})_k^c = B_a^i(\det\|B\|)^{-1}, \quad (43)$$

then (42) further simplifies to

$$\begin{aligned} i\epsilon_{dae}(\det\|B\|)^{-1}\epsilon^{egh}(\Psi^{-1}\Psi^{-1})^{gf}B_h^jD_j\Psi_{af} &= \\ &= i(\det\|B\|)^{-1}(\Psi^{-1}\Psi^{-1})^{gf}(\delta_a^g\delta_a^h - \delta_a^g\delta_a^h)\mathbf{v}_d\{\Psi_{af}\} = \\ &= i(\det\|B\|)^{-1}(\Psi^{-1}\Psi^{-1})^{gf}(\delta_a^g\mathbf{v}_a\{\Psi_{af}\} - \mathbf{v}_d\{\Psi_{gf}\}) = \\ &= i(\det\|B\|)^{-1}\left[(\Psi^{-1}\Psi^{-1})^{df}G_f + \mathbf{v}_d\{\Lambda + \text{tr}\Psi^{-1}\}\right]. \end{aligned} \quad (44)$$

The first term on the final right hand side of (44) is proportional to the Gauss' law constraint and the second term to the derivative of a term proportional to the Hamiltonian constraint.* The second contribution to the second term of (38) is given by

$$\begin{aligned} \epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})(H_{(2)})_k^f &= \\ &= \epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})\epsilon_{mnk}N^mB_f^n = \\ &= \epsilon_{dac}(\delta_m^i\delta_n^j - \delta_n^i\delta_m^j)(B^{-1})_i^e(D_j\Psi_{af})N^mB_f^n, \end{aligned} \quad (45)$$

*We have added in a term Λ , which can be regarded as a constant of integration with respect to the spatial derivatives from \mathbf{v}_d .

where we have used the analogue of the first identity of (40) for spatial indices. Then (45) further simplifies to

$$\begin{aligned} \epsilon_{dae} N^i (B^{-1})_i^e \mathbf{v}_f \{ \Psi_{af} \} - N^j D_j (\epsilon_{dae} \Psi_{ae}) = \\ = \epsilon_{dae} N^i (B^{-1})_i^e G_a - N^j D_j \psi_d. \end{aligned} \quad (46)$$

The result is that the time evolution of the diffeomorphism constraint is directly proportional to

$$\begin{aligned} \dot{\psi}_d = \left[i (\det \| B \|)^{-1} (\Psi^{-1} \Psi^{-1})^{da} + \epsilon_{dae} N^i (B^{-1})_i^e \right] G_a + \\ + (A_0^b \epsilon_{bdh} - \delta_{dh} N^j D_j) \psi_h + i (\det \| B \|)^{-1} \mathbf{v}_d \{ (-g)^{-1/2} H \}, \end{aligned} \quad (47)$$

which is a linear combination of terms proportional to the constraints (35) and their spatial derivatives. The result is that the diffeomorphism constraint $H_i = 0$ is consistent with respect to the Hamiltonian evolution generated by the equations (34). So it remains to verify consistency of the Gauss' law and the Hamiltonian constraints G_a and H .

§6. Consistency of the Gauss constraint under time evolution.

Having verified the consistency of the diffeomorphism constraint under time evolution, we now move on to the Gauss constraint. Application of the Leibniz rule to the first equation of (35) yields

$$\dot{G}_a = \dot{B}_e^i D_i \Psi_{ae} + B_e^i D_i \dot{\Psi}_{ae} + B_e^i (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) \dot{A}_i^b. \quad (48)$$

Upon substitution of (30) and (34) into (48), we have

$$\begin{aligned} \dot{G}_a = (-\delta_{\bar{\theta}} B_e^i - i \epsilon^{ijk} D_j H_k^e) D_i \Psi_{ae} + \\ + B_e^m D_m \left[-\delta_{\bar{\theta}} \Psi_{ae} - i \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) H_k^f \right] + \\ + B_e^i (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) (-\delta_{\bar{\theta}} A_i^b - i H_i^b). \end{aligned} \quad (49)$$

Using the Leibniz rule to re-combine the $\delta_{\bar{\theta}}$ terms of (49), we have

$$\begin{aligned} \dot{G}_a = -\delta_{\bar{\theta}} G_a - i \epsilon^{ijk} \left\{ (D_j H_k^e) D_i \Psi_{ae} + \right. \\ \left. + B_e^m D_m \left[(B^{-1})_i^e (D_j \Psi_{af}) H_k^f \right] \right\} - i (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) B_e^i H_i^b. \end{aligned} \quad (50)$$

The requirement of consistency is that we must show that the right hand side of (50) vanishes weakly. First, we will show that the third term on the right hand side of (50) vanishes up to terms of linear order and higher in the diffeomorphism constraint. This term, up to an

insignificant numerical factor, has two contributions. The first contribution is

$$\begin{aligned} & (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}) B_e^i (H_{(1)})_i^b = \\ & = \sqrt{-g} (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}) (\Psi^{-1}\Psi^{-1})^{eb} = \\ & = \sqrt{-g} [f_{abf}(\Psi^{-1})^{fb} + f_{ebg}(\Psi^{-1}\Psi^{-1})^{eb}\Psi_{ag}] \sim \delta_a^{(1)}(\vec{\psi}) \sim 0, \end{aligned} \quad (51)$$

which is directly proportional to a nonlinear function of first order in ψ_d which is proportional to the diffeomorphism constraint. The second contribution to the third term on the right hand side of (50) is

$$\begin{aligned} & (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}) B_e^i (H_{(2)})_i^b = \\ & = (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}) \epsilon_{kmn} N^k B_e^m B_b^n = \\ & = (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag}) (\det\|B\|) N^k (B^{-1})_k^d \epsilon_{deb}. \end{aligned} \quad (52)$$

We can now apply the epsilon identity (40) to (52), using the fact that $f_{abc} = \epsilon_{abc}$. This yields

$$\begin{aligned} & (\det\|B\|) N^k (B^{-1})_k^d [(\delta_{fd}\delta_{ae} - \delta_{fe}\delta_{ad})\Psi_{fe} + 2\delta_{dg}\Psi_{ag}] = \\ & = (\det\|B\|) N^k (B^{-1})_k^d (\Psi_{da} - \delta_{ad}\text{tr}\Psi + 2\Psi_{ad}) \equiv \delta_a^{(2)}(\vec{N}), \end{aligned} \quad (53)$$

which does not vanish, and neither is it expressible as a constraint. For the Gauss' law constraint to be consistent under time evolution, a necessary condition is that this $\delta_a^{(2)}(\vec{N})$ term must be exactly cancelled by another term arising from the variation.

Let us expand the terms in (50) associated with the square brackets. This is given, applying the Leibniz rule to the second term, by

$$\begin{aligned} & \epsilon^{ijk} (D_j H_k^e) (D_i \Psi_{ae}) + \epsilon^{ijk} B_e^m D_m [(B^{-1})_i^e (D_j \Psi_{ae}) H_k^f] = \\ & = \epsilon^{ijk} (D_j H_k^e) (D_i \Psi_{ae}) - \epsilon^{ijk} B_e^m (B^{-1})_n^e (D_m B_g^n) (B^{-1})_i^g \times \\ & \times (D_j \Psi_{af}) H_k^f + \epsilon^{mjk} (D_m D_j \Psi_{af}) H_k^f + \epsilon^{mjk} (D_j \Psi_{af}) (D_m H_k^f). \end{aligned} \quad (54)$$

The first and last terms on the right hand side of (54) cancel, which can be seen by relabelling of indices. Upon application of the definition of curvature as the commutator of covariant derivatives to the third term, then (54) reduces to

$$- \epsilon^{ijk} (D_n B_g^n) (B^{-1})_i^g (D_j \Psi_{af}) H_k^f + H_k^f B_b^k (f_{abc}\Psi_{cf} + f_{fbc}\Psi_{ac}). \quad (55)$$

The first term of (55) vanishes on account of the Bianchi identity and

the second term contains two contributions which we must evaluate. The first contribution is given by

$$\begin{aligned} (H_{(2)})_k^f B_b^k (f_{abc} \Psi_{cf} + f_{fbc} \Psi_{ac}) &= \\ &= (\det \|B\|) N^k (B^{-1})_k^d \epsilon_{dbf} (f_{abc} \Psi_{cf} + f_{fbc} \Psi_{ac}). \end{aligned} \quad (56)$$

Applying (40), then (56) simplifies to

$$\begin{aligned} (\det \|B\|) N^k (B^{-1})_k^d \left[(\delta_{da} \delta_{fc} - \delta_{dc} \delta_{fa}) \Psi_{cf} - 2 \delta_{dc} \Psi_{ac} \right] &= \\ = (\det \|B\|) N^k (B^{-1})_k^d (\delta_{da} \text{tr} \Psi - \Psi_{da} - 2 \Psi_{ad}) &= -\delta_a^{(2)}(\vec{N}), \end{aligned} \quad (57)$$

with $\delta_a^{(2)}(\vec{N})$ as given in (51). So putting the results of (54), (55) and (57) into (50), we have

$$\begin{aligned} \dot{G}_a &= -\delta_{\vec{\theta}} G_a + \delta_a^{(2)}(\vec{N}) + \delta_a^{(1)}(\vec{\psi}) + \delta_a^{(1)}(\vec{\psi}) - \delta_a^{(2)}(\vec{N}) = \\ &= -\delta_{\vec{\theta}} G_a + 2 \delta^{(1)}(\vec{\psi}), \end{aligned} \quad (58)$$

whence the $\delta^{(2)}(\vec{\psi})$ terms have cancelled out. The velocity of the Gauss' law constraint is a linear combination of the Gauss constraint with terms of the diffeomorphism constraint of linear order and higher. Hence the time evolution of the Gauss' law constraint is consistent in the sense that we have defined, since $\delta^{(1)}(\vec{\psi})$ vanishes for $\psi_d = 0$.

§7. Consistency of the Hamiltonian constraint under time evolution. The time derivative of the Hamiltonian constraint, the third equation of (35), is given by

$$\dot{H} = \left[\frac{d}{dt} \left(\sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} \right) \right] (\Lambda + \text{tr} \Psi^{-1}) + \frac{\sqrt{-g}}{N} \frac{d}{dt} (\Lambda + \text{tr} \Psi^{-1}) \quad (59)$$

which has split up into two terms. The first term is directly proportional to the Hamiltonian constraint, therefore it is already consistent. We will nevertheless expand it using (30) and (34)

$$\begin{aligned} \frac{1}{2} \left[(B^{-1})_i^d \dot{B}_d^i + (\Psi^{-1})^{ae} \dot{\Psi}_{ae} \right] \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) &= \\ = \frac{1}{2} \left\{ (B^{-1})_i^d (-\delta_{\vec{\theta}} B_d^i - i \epsilon^{ijk} D_j H_k^d) + \right. & \\ \left. + (\Psi^{-1})^{ae} \left[-\delta_{\vec{\theta}} \Psi_{ae} - i \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) \right] H_k^f \right\} H. \end{aligned} \quad (60)$$

We will be content to compute the $\delta_{\vec{\theta}}$ terms of (60). These are

$$(B^{-1})_i^d \delta_{\vec{\theta}} B_d^i = (B^{-1})_i^d f_{dbf} B_b^i A_0^f = \delta_{db} f_{dbf} A_0^f = 0 \quad (61)$$

on account of antisymmetry of the structure constants, and

$$(\Psi^{-1})^{ea} \delta_{\vec{\theta}} \Psi_{ae} = (\Psi^{-1})^{ea} (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) A_0^b = 0, \quad (62)$$

also due to antisymmetry of the structure constants. We have shown that the first term on the right hand side of (59) is consistent with respect to time evolution. To verify consistency of the Hamiltonian constraint under time evolution, it remains to show that the second term is weakly equal to zero. It suffices to show this just for the second term, in brackets, of (59)

$$\begin{aligned} \frac{d}{dt} (\Lambda + \text{tr} \Psi^{-1}) &= -(\Psi^{-1} \Psi^{-1})^{fe} \dot{\Psi}_{ef} = \\ &= (\Psi^{-1} \Psi^{-1})^{ae} \left[\delta_{\vec{\theta}} \Psi_{ae} - i \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) H_k^f \right], \end{aligned} \quad (63)$$

where we have used (34). Equation (63) has split up into two terms, of which the first term is

$$\begin{aligned} (\Psi^{-1} \Psi^{-1})^{ea} \delta_{\vec{\theta}} \Psi_{ae} &= (\Psi^{-1} \Psi^{-1})^{ea} (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) A_0^b = \\ &= \left[f_{abf} (\Psi^{-1})^{fa} + f_{ebg} (\Psi^{-1})^{eg} \right] A_0^b = m(\vec{\psi}) \sim 0 \end{aligned} \quad (64)$$

which vanishes weakly since it is a nonlinear function of at least linear order in ψ_d . The second term of (63) splits into two terms which we must evaluate. The first contribution is proportional to

$$\begin{aligned} (\Psi^{-1} \Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) (H_{(1)})_k^f &= \\ = \sqrt{-g} (\Psi^{-1} \Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) (B^{-1})_k^d (\Psi^{-1} \Psi^{-1})^{df}. \end{aligned} \quad (65)$$

Proceeding from (65) and using (43) to simplify the magnetic field contributions, we have

$$\begin{aligned} -\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{ea} (\Psi^{-1} \Psi^{-1})^{df} (\det \|B\|)^{-1} \epsilon^{edg} B_g^j D_j \Psi_{af} &= \\ = -\sqrt{-g} (\det \|B\|)^{-1} \epsilon^{edg} (\Psi^{-1} \Psi^{-1})^{ea} (\Psi^{-1} \Psi^{-1})^{df} \times \\ \times \mathbf{v}_g \{ \Psi_{af} \} \equiv \mathbf{v} \{ \vec{\psi} \} \end{aligned} \quad (66)$$

for some vector field \mathbf{v} . We have used the fact that the term in (66) quartic in Ψ^{-1} is antisymmetric in a and f due to the epsilon symbol. Hence Ψ_{af} as acted upon by \mathbf{v}_g can appear only in an antisymmetric combination, and is therefore proportional to the diffeomorphism constraint ψ_d whose spatial derivatives weakly vanish. Therefore (66) presents a consistent contribution to the time evolution of H , which leaves remaining the second contribution to the second term of (63). This term is propor-

tional to

$$\begin{aligned}
& (\Psi^{-1}\Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) (H_{(2)})_k^f = \\
& = (\Psi^{-1}\Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) \epsilon_{mnk} N^m B_f^n = \\
& = (\delta_m^i \delta_n^j - \delta_n^i \delta_m^j) (B^{-1})_i^e N^m B_f^n (\Psi^{-1}\Psi^{-1})^{ea} (D_j \Psi_{af}) \quad (67)
\end{aligned}$$

where we have applied the epsilon identity. Proceeding from the right hand side of (67), we have

$$\begin{aligned}
& \left[N^i (B^{-1})_i^e B_f^j - \delta_{ef} N^j \right] (\Psi^{-1}\Psi^{-1})^{ea} (D_j \Psi_{af}) = \\
& = (-g)^{-1/2} N^i H_i^a \mathbf{v}_f \{ \Psi_{af} \} - (\Psi^{-1}\Psi^{-1})^{fa} (N^j D_j \Psi_{af}) = \\
& = (-1)^{-1/2} N^i H_i^a G_a - N^j D_j (\Lambda + \text{tr} \Psi^{-1}). \quad (68)
\end{aligned}$$

The first term on the final right hand side of (68) is proportional to the Gauss' law constraint, and the second term is proportional to the derivative of the Hamiltonian constraint, since $N^i D_i = N^i \partial_i$ on scalars. To obtain this second term we have added in Λ which becomes annihilated by ∂_j . Substituting (64), (66) and (68) into (63), then we have

$$\dot{H} = \sim \hat{O}(\vec{\psi}) + (-g)^{-1/2} N^i H_i^a G_a + \hat{T} [(-g)^{-1/2} H], \quad (69)$$

where \hat{O} and \hat{T} are operators consisting of spatial derivatives acting to the right and c numbers. The time derivative of the Hamiltonian constraint is a linear combination of the Gauss' law and Hamiltonian constraints and its spatial derivatives, plus terms of linear order and higher in the diffeomorphism constraint and its spatial derivatives. Hence we have shown that the Hamiltonian constraint is consistent under time evolution.

§8. Recapitulation and discussion. The most important aspect of consistency required for any totally constrained system is that the constraint surface be preserved under time evolution for all time. If upon taking the time derivative of a constraint one obtains a quantity which does not vanish on-shell, then this introduces additional constraints on the system which must similarly be verified to be consistent with the existing constraints. One must proceed in this manner until a self-consistent system of constraints is obtained. Hopefully, one is then left with a system which still contains nontrivial dynamics. In the case of the instanton representation, we have performed this test on all of the constraints arising from the action.

The final equations governing the time evolution of the initial value

constraints are given weakly by

$$\left. \begin{aligned} \dot{\psi}_d &= \left[i (\det \|B\|)^{-1} (\Psi^{-1} \Psi^{-1})^{da} + \epsilon_{dae} N^i (B^{-1})_i^e \right] G_a + \\ &\quad + (A_0^b \epsilon_{bdh} - \delta_{dh} N^j D_j) \psi_h + i (\det \|B\|)^{-1} \mathbf{v}_d \{ \Lambda + \text{tr } \Psi^{-1} \} \\ \dot{G}_a &= -f_{abc} A_0^b G_c + \delta_a^{(1)}(\vec{\psi}) \\ \dot{H} &= \left[-\frac{i}{2} \epsilon^{ijk} (B^{-1})_i^d (D_j H_k^d) + \right. \\ &\quad \left. + \epsilon^{ijk} (B^{-1})_i^e (\Psi^{-1})^{ae} (D_j \Psi_{af}) H_k^f - N^j \partial_j \right] (\Lambda + \text{tr } \Psi^{-1}) + \\ &\quad + (-g)^{-1/2} N^i H_i^a G_a - \sqrt{-g} (\det \|B\|)^{-1} \times \\ &\quad \times \epsilon^{edg} (\Psi^{-2} \Psi^{-1})^{ea} (\Psi^{-1} \Psi^{-1})^{df} \mathbf{v}_g \{ \epsilon_{afh} \psi_h \} + m(\vec{\psi}) \end{aligned} \right\}. \quad (70)$$

Equations (70) show that all constraints derivable from the the action (10) are preserved under time evolution, since their time derivatives yield linear combinations of the same set of constraints and their spatial derivatives, with no additional constraints. In spite of the fact that we have defined neither the canonical structure of (1) nor any Poisson brackets, this is tantamount to the Dirac consistency of (1).

Equations (70) can be written schematically in the following form

$$\left. \begin{aligned} \dot{\vec{H}} &\sim \vec{H} + \vec{G} + H \\ \dot{\vec{G}} &\sim \vec{G} + \Phi(\vec{H}) \\ \dot{H} &\sim H + \vec{G} + \Phi(\vec{H}) \end{aligned} \right\}, \quad (71)$$

where Φ is some nonlinear function of the diffeomorphism constraint \vec{H} , which is of at least first order in \vec{H} . In the Hamiltonian formulation of a theory, one identifies time derivatives of a variable f with $\dot{f} = \{f, \mathbf{H}\}$, the Poisson brackets of f with a Hamiltonian \mathbf{H} . So while we have not defined Poisson brackets, equation (71) implies the existence of Poisson brackets associated to some Hamiltonian \mathbf{H}_{Inst} for the action (10), with

$$\left. \begin{aligned} \{\vec{H}, \mathbf{H}_{\text{Inst}}\} &\sim \vec{H} + \vec{G} + H \\ \{\vec{G}, \mathbf{H}_{\text{Inst}}\} &\sim \vec{G} + \Phi(\vec{H}) \\ \{H, \mathbf{H}_{\text{Inst}}\} &\sim H + \Phi(\vec{H}) + \vec{G} \end{aligned} \right\}. \quad (72)$$

So the main result of this paper has been to demonstrate that the instanton representation of Plebanski gravity forms a consistent system,

in the sense that the constraint surface is preserved under time evolution. As a direction of future research we will compute the algebra of constraints for (10) directly from its Poisson brackets. Nevertheless it will be useful for the present paper to think of equations (70) in the Dirac context, mainly for comparison with alternate formulations of General Relativity. This will bring us back to the Ashtekar variables.

Let us revisit (9) for each constraint with the total Hamiltonian \mathbf{H}_{Ash} and compare with (72). This is given schematically by

$$\left. \begin{aligned} \{\vec{H}, \mathbf{H}_{\text{Ash}}\} &\sim \vec{H} + \vec{G} + H \\ \{\vec{G}, \mathbf{H}_{\text{Ash}}\} &\sim \vec{G} + \vec{H} \\ \{H, \mathbf{H}_{\text{Ash}}\} &\sim H + \vec{H} \end{aligned} \right\}. \quad (73)$$

Comparison of (73) with (72) shows an essentially similar structure for the top two lines involving \vec{H} and \vec{G} (we regard the linearity versus nonlinearity of the diffeomorphism constraints on the right hand side as a dissimilarity, albeit a minor dissimilarity). But there is a marked dissimilarity with respect to the Hamiltonian constraint H . Note that there is a Gauss' law constraint appearing in the right hand side of the last line of (72) whereas there is no such constraint on the corresponding right hand side of (73). This means that while the Hamiltonian constraint is gauge-invariant under $\text{SO}(3, \mathbb{C})$ gauge-transformations as implied by (9) and (73), this is not the case in (72). This means that the action (10), which as shown in [1] describes General Relativity for Petrov Types I, D and O, suggests a different role for the Gauss' law and Hamiltonian constraints than the action (5), which also describes General Relativity. The conclusion is therefore that I_{Inst} and I_{Ash} at some level must correspond to genuinely different descriptions of General Relativity, a feature which would have been missed had we applied the step-by-step Dirac procedure.*

On a final note, there is a common misconception that I_{Inst} is the same action as a certain action leading to the CDJ pure spin connection formulation of [5], or should fall under the CDJ formalism. Additionally, we would like to dispell any notion that the pure spin connection action $I_{\text{CDJ}} = I_{\text{CDJ}}[\eta, A]$ or its antecedent $I_1 = I_1[\Psi, A]$ are directly related to I_{Inst} . They are related in the sense that $I_{\text{CDJ}}, I_1 \subset I_{\text{Inst}}$, but the converse is not true for the reasons shown in [1], which we will not repeat here.

*The Dirac procedure naively applied to I_{Inst} would lead one directly to I_{Ash} via (4), which might suggest superficially that these two theories are the same. However, as the results of this paper show, I_{Inst} is a stand-alone action with an algebraic structure different from I_{Ash} .

The Ashtekar action I_{Ash} has been shown in [8] to be the 3+1 decomposition of I_{CDJ} for Petrov Types I, D and O. We have shown in §2 that I_{Inst} as well exhibits this feature. However, this is not the case on the noncanonical phase space $\Omega_{\text{Inst}} = (\Psi_{ae}, A_i^a)$, which the present paper has demonstrated.

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Radial Distance on a Stationary Frame in a Homogeneous and Isotropic Universe

Robert C. Fletcher*

Abstract: This paper presents a physical distance to all radial events in a homogeneous and isotropic universe as a transform from Friedman-Lemaître-Robertson-Walker (FLRW) coordinates, the model that solves the Einstein field equation for an ideal fluid. Any well behaved transform is also a solution. The problem is relating the coordinates of the transform to observables. In the present case the objective is to find T, R on a stationary frame that has the R be a physical observable for all distances. We do this by working backwards, assuming the form of the metric that we desire, with some undetermined coefficients. These coefficients are then related to the partial derivatives of the transform. The transformed coordinates T, R are found by the integration of partial differential equations in the FLRW variables. We show that dR has the same units as the radial differential of the FLRW metric, which makes it observable. We develop a criterion for how close the transformed T comes to an observable time. Close to the space origin at the present time, T also becomes physical, so that the stationary acceleration becomes Newtonian. We show that a galactic point on a R, T plot starts close to the space origin at the beginning, moves out to a physical distance and finite time where it can release light that will be seen at the origin at the present time. Lastly, because the observable R has a finite limit at a finite T for $t=0$ where the galactic velocity approaches the light speed, we speculate that the Universe filled with an ideal fluid as seen on clocks and rulers on the stationary frame has a finite extent like that of an expanding empty universe, beyond which are no galaxies and no space.

*Bell Telephone Laboratories (retired), Murray Hill, New Jersey, USA. Mailing address (for correspondence): 1000 Oak Hills Way, Salt Lake City, UT 84108, USA. E-mail: robert.c.fletcher@utah.edu.

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§1. Introduction. The FLRW metric [1, 2] metric is derived for a homogeneous and isotropic universe. With the assumption of a stress-energy tensor for an ideal liquid, the solution of the Einstein Field Equation gives the FLRW model of the Universe [3–5] used in this paper. The FLRW variables (t, χ, θ, ϕ) are interpreted as co-moving coordinates with dt being a physical (observable) time, and $a(t)d\chi$ being a physical radial distance, and $a(t)$ being the scale factor of the FLRW universe. We visualize these galactic points representing galaxies as moving away from an observer (us) at the origin at a velocity measured by distance and time on our stationary frame. We can measure their velocity by the red shift of the lines of their spectrum, but their distance is more difficult. The proper distance $a(t)\chi$, the distance to χ at constant t , does not have a Minkowski metric with t (except for small χ) required for physical coordinates. Weinberg describes a sequence of physical distance measures, running from parallax measurements for nearby stars to standard candle measurements for more distant sources [5]. These all accurately represent the observables as derived from the FLRW coordi-

nates, but are not our stationary distance; they differ from each other and from proper distance for large red shifts.

This paper presents a physical distance on a stationary frame to all radial events in a homogeneous and isotropic universe as a transform from FLRW coordinates. Any well behaved transform from FLRW coordinates is also a solution of the field equation because tensor equations are invariant to transforms. The problem is relating the coordinates of the transform to observables. In the present case we would like the observables to be the time on clocks and the distance on rulers on the stationary frame attached to the spatial origin, but this is only possible close to the origin. The next best objective is to find T, R for all distances that become these observables close to the origin, and then have the R be a physical observable for all distances.

The key to showing that the distance is universal is to show that it is rigid with the same constant units as the FLRW radial differential distance on galactic points, which units are assumed to be physical. We outline here how we will do this. We will look for a transform to radial coordinates T, R from radial FLRW coordinates t, χ that are attached to the FLRW origin $\chi=0$. We can call T, R stationary because the velocity of the galactic point is zero at $\chi=0$.

To find the transform of t, χ to T, R we will start with a metric with dT, dR that is spherically symmetric in space, and find the coefficients of dT, dR by putting constraints on them and then integrating the resultant dT, dR , subject to the boundary conditions at $\chi=0$. The constraints we will impose is that the points of R be motionless with respect to each other (i.e., rigid), that the the line element ds of dT, dR be the same as FLRW, that its metric (the line element expressed in coordinate differentials) have no cross products $dT dR$ (i.e., diagonal), that the metric in dT, dR become Minkowski close to the origin, and that dR have the same units for all distances as the radial differential of the FLRW line element, $a(t)d\chi$. The resulting paths of galactic points and photons in the stationary coordinates closely resembles the known transform to T, R from t, χ for an expanding empty universe [8]. These have a Minkowski metric for all distances, and have universal physical coordinates T, R .

§2. Assumptions. We will use the word “line element” to represent the invariant ds and the word “metric” for the ds of a particular set of coordinates, and limit ourselves to the time and radial distance of spherical space symmetry (no transverse motion). The analysis depends on the following assumptions:

Assumption 1: The FLRW metric is a valid representation of a homogeneous and isotropic universe, whose coordinates (χ, θ, ϕ, t) are co-moving with the galactic points (representing smoothed out galaxies), and whose universe scale factor is $a(t)$.

Definition: “Physical” coordinates in time or distance over some interval will be defined as those having a linear relationship to the readings on a standard clock or a standard ruler, respectively (a particular case of observables). We call them physical because it describes coordinates on the rigid frame close to and attached to the point at the origin $\chi=0$.

In the limit of small intervals on an inertial frame, if physical time represents clocks at the location represented by the physical distance, according to General Relativity, their differentials will have a Minkowski metric. This definition includes the Schwarzschild coordinates of time and distance; but the Schwarzschild metric is not Minkowski because we represent time in its metric to be measured remotely from the distance. Note that $d\chi$ is not physical, but dt is, by our definition. Our objective is to find a *physical* distance for all radial events in the Universe.

Assumption 2: The radial coordinate differentials dt and $a(t)d\chi$ of FLRW are assumed physical.

This is plausible because they have the Minkowski metric in these dimensions and are assumed to be on adjacent inertial frames. dt will represent the time increment on galactic clocks that keep time like a standard clock, and $a(t)d\chi$ will represent the physical distance between adjacent galactic points separated by $d\chi$ as measured by light signals, i.e., a round trip light time would be $2a(t)\frac{d\chi}{c}$, the same method used to synchronize clocks on a rigid frame or to determine radar distance.

Definition: The coordinates $x^\mu(R, \theta, \phi, T)$ are transforms from FLRW with spherically symmetric space coordinates whose space origins are the same as FLRW, so $\chi=0$ and $V=0$ at $R=0$. We will therefore call them stationary coordinates.

Like all well behaved transforms, they satisfy the Einstein field equations, but we need to relate them to observables. The physical velocity with respect to a galactic point t, χ of a point on R will be $V = a(t)\left(\frac{\partial\chi}{\partial t}\right)_R$. For this radial point R of x^μ , we can find contravariant velocity vectors $U^\mu = \frac{dx^\mu}{ds}$ and acceleration vectors $A^\mu = \frac{DU^\mu}{ds}$ whose components will transform the same as dx^μ . R is rigid because the radial component of U^μ in stationary coordinates is $\left(\frac{\partial R}{\partial T}\right)_R \frac{dT}{ds} \equiv 0$, so the points of R are motionless with respect to each other.

Assumption 3: R will measure physical distance on the stationary frame if its differential is physical for all distances.

Without this assumption, R might represent a measurement whose increments would sometimes be smaller and sometimes larger than a standard ruler, and therefore not qualify for a physical distance measure on a stationary frame.

Assumption 4: The unit of dR is the same unit as that of the radial differential distance of the FLRW metric $a(t)d\chi$ if $(\frac{\partial R}{\partial \chi})_t = (\frac{a \partial x}{\partial R})_T$. This is the mathematical relation that shows that two observers are using the same distance units if each measures the other's unit to be the same as the other measures his [6].

Assumption 5: The known solution of the field equations of General Relativity for the FLRW universe gives $a(t)$.

Because of the properties of Riemann tensors, any transform from FLRW will also be a solution. This allows evaluation of T, R of an event for a given universe energy density.

§3. Procedure for finding stationary coordinates using the velocity V . We assume that the concentrated lumps of matter, like stars and galaxies, can be averaged to the extent that the universe matter can be considered continuous, and that the surroundings of every point in space can be assumed isotropic and the same for every point.

By embedding a maximally symmetric (i.e., isotropic and homogeneous) three dimensional sphere, with space dimensions r, θ , and ϕ , in a four dimension space which includes time t , one can obtain a differential line element ds [5, p. 403] such that

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (1)$$

where

$$r = \begin{cases} \sin \chi, & k = 1 \\ \chi, & k = 0 \\ \sinh \chi, & k = -1 \end{cases} \quad (2)$$

while k is a spatial curvature determinant to indicate a closed, flat, or open universe, respectively, and

$$d\chi^2 \equiv \frac{dr^2}{1 - kr^2}, \quad (3)$$

where $a(t)$ is the cosmic scale factor multiplying the three dimensional spatial sphere, and the differential radial distance is $a(t)d\chi$.

The resulting equation for the differential line element becomes the FLRW metric:

$$ds^2 = c^2 dt^2 - a(t)^2 [d\chi^2 + r^2 d\omega^2], \quad (4)$$

where $d\omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. For radial motions this metric becomes Minkowski in form with a differential of physical radius of $a(t) d\chi$.

We would now like to find radial transforms that will hold for all values of the FLRW coordinates and have a Minkowski metric for small distances from the origin as required if dT and dR are to be the time and distance on our stationary frame. The most general line element for a time-dependent spatially spherically symmetric (i.e., isotropic) line element [5] is

$$ds^2 = c^2 A^2 dT^2 - B^2 dR^2 - 2cCdTdR - F^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (5)$$

where A , B , C , and F are implicit functions of T and R , but explicit functions of t and χ .

We would like T and R to locate the same radial events as t and χ , so $T = T(t, \chi)$ and $R = R(t, \chi)$. Therefore

$$\left. \begin{aligned} dT &= T_t dt + T_\chi d\chi = \frac{1}{c} T_t d\hat{t} + T_\chi d\chi \\ dR &= R_t dt + R_\chi d\chi = \frac{1}{c} R_t d\hat{t} + R_\chi d\chi \end{aligned} \right\}, \quad (6)$$

where the subscripts indicate partial derivatives with respect to the subscript variable and $d\hat{t} = cdt$.

We will look for transformed coordinates which have their origins on the same galactic point as $\chi = 0$, so $R = 0$ when $\chi = 0$, where there will be no motion between them, and where T is t , since the time on clocks attached to every galactic point is t , including the origin. We will keep the same angular coordinates for the transform and make $F = ar$ to correspond to the FLRW metric, but will find only radial transforms where the angular differentials are zero.

Let us consider a radial point at R . When measured from the FLRW system, it will be moving at a velocity given by

$$V = a(t) \left(\frac{\partial \chi}{\partial t} \right)_R \equiv c\hat{V}, \quad (7)$$

This velocity will be the key variable that will enable us to obtain radial transforms of the full radial coordinates.

We will first use this to find the components in the FLRW coordi-

nates $(\chi, 0, 0, \hat{t})$ for the contravariant velocity vector U^μ of the point at R . To get the time component, we divide (194) by $d\hat{t}^2$ with $d\omega = 0$ to obtain

$$\left(\frac{ds}{d\hat{t}}\right)^2 = 1 - a(t)^2 \left(\frac{d\chi}{d\hat{t}}\right)^2 = (1 - \hat{V}^2) \equiv \frac{1}{\gamma^2}. \quad (8)$$

To get the spatial component, we use the chain rule applied to (25) and (26):

$$\frac{d\chi}{ds} = \frac{d\chi}{d\hat{t}} \frac{d\hat{t}}{ds} = \frac{\hat{V}}{a} \gamma. \quad (9)$$

Thus, the FLRW components of U^μ are

$$U^\mu = \left(\frac{\gamma \hat{V}}{a}, 0, 0, \gamma\right). \quad (10)$$

with the time component listed last.

The transformed coordinates are (R, θ, ϕ, T) . To get U^μ in these coordinates, we make the spatial components be zero as required for it to be a vector velocity of the point R (see the definitions under Assumption 3). This assumption means that a test particle attached to the radial coordinate will feel a force caused by the gravitational field, but will be constrained not to move relative to the coordinate. Alternatively, a co-located free particle at rest relative to the radial point will be accelerated by the same force, but will thereafter not stay co-located. This means that the radial point is not inertial, except when close to the origin.

The time component of the transformed vector is $\frac{dT}{ds} = \frac{1}{cA}$. This makes the vector U^μ in the transformed coordinates be

$$U^\mu = \left(0, 0, 0, \frac{1}{cA}\right). \quad (11)$$

Since U^μ is to be contravariant, its components will transform the same as dT, dR in (1):

$$\left. \begin{aligned} \frac{1}{cA} &= \frac{1}{c} T_t \gamma + \frac{1}{a} T_\chi \gamma \hat{V} \\ 0 &= \frac{1}{c} R_t \gamma + \frac{1}{a} R_\chi \gamma \hat{V} \end{aligned} \right\}. \quad (12)$$

Manipulating the second line of (30) gives

$$\hat{V} = -\frac{a R_t}{c R_\chi}. \quad (13)$$

If we invert (1), we get

$$\left. \begin{aligned} d\hat{t} &= \frac{1}{D} (R_\chi dT - T_\chi dR) \\ d\chi &= \frac{1}{D} \left(-\frac{1}{c} R_t dT + \frac{1}{c} T_t dR \right) \end{aligned} \right\}, \quad (14)$$

where

$$D = \frac{1}{c} T_t R_\chi - \frac{1}{c} R_t T_\chi = \frac{1}{c} T_t R_\chi \left(1 + \hat{V} \frac{cT_\chi}{aT_t} \right), \quad (15)$$

using (31).

We can enter $d\hat{t}$ and $d\chi$ of (32) into the FLRW metric (194). One way to make the line element ds to be the same as FLRW is to equate these coefficients individually with those of (24):

$$A^2 = \frac{1}{T_t^2} \left[\frac{1 - \hat{V}^2}{\left(1 + \hat{V} \frac{cT_\chi}{aT_t}\right)^2} \right], \quad (16)$$

$$B^2 = \frac{a^2}{R_\chi^2} \left[\frac{1 - \left(\frac{cT_\chi}{aT_t}\right)^2}{\left(1 + \hat{V} \frac{cT_\chi}{aT_t}\right)^2} \right], \quad (17)$$

$$C = -\frac{a}{T_t R_\chi} \left[\frac{\hat{V} + \frac{cT_\chi}{aT_t}}{\left(1 + \hat{V} \frac{cT_\chi}{aT_t}\right)^2} \right]. \quad (18)$$

If we put $ds = 0$ in (24), we obtain a coordinate velocity of light v_p :

$$\frac{v_p}{c} = \left(\frac{\partial R}{c \partial T} \right)_s = -\frac{C}{B^2} \pm \sqrt{\left(\frac{C}{B^2} \right)^2 + \frac{A^2}{B^2}}. \quad (19)$$

The equations for A , B , and v_p simplify for a diagonal metric ($C = 0$). From (36) we get

$$\frac{cT_\chi}{aT_t} = -\hat{V}. \quad (20)$$

So rigidity gives a relation of dR to V (31), and diagonality gives a relation of dT to V (38). Thence, equations (34), (35), and (37) become

$$A = \frac{\gamma}{T_t} = \frac{t_T}{\gamma}, \quad (21)$$

$$B = \frac{a\gamma}{R_\chi} = \frac{a\chi_R}{\gamma}, \quad (22)$$

$$\frac{v_p}{c} = \frac{A}{B}, \quad (23)$$

where we have used (32) with $C = 0$ to obtain the inverse partials. This metric becomes the physical Minkowski (\hat{M}) when $A \rightarrow 1$, $B \rightarrow 1$, $C \rightarrow 0$ and $ar \rightarrow R$. Thus, the coordinate light speed for the stationary coordinates starts at c for $R = 0$ where $T_t = \gamma = \frac{R_x}{a} = 1$, and increases by the ratio $\frac{A}{B}$ as R increases.

We can say something about the physicality of the coordinates with the use of criteria developed by Bernal et al. [6]. They developed a theory of fundamental units based on the postulate that two observers will be using the same units of measure when each measures the other's differential units at the same space-time point compared to their own and finds these reciprocal units to be equal. Thus, even if $A \neq 1$, dR will be physical if $C = 0$ and $B = 1$ because then $\frac{R_x}{a} = a\chi_R = \gamma$ (40), and dR is physical because it uses the same measure of distance as $ad\chi$, which we assume is physical. Then R uses physical units when dR is integrated indefinitely out to the visible horizon.

At this point we would like to examine quantitatively how far from the \hat{M} metric our stationary metric is allowed to be in order for its coordinates to reasonably represent physical measurements. We can consider the coefficients A , B , and C one at a time departing from their value in the \hat{M} metric. Thus, let us consider the physical distance case $B = 1$, $C = 0$ and examine the possible departure of the time rate in the transform from that physically measured. Then, from equations (39): $T_t = \frac{\gamma}{A}$, $t_T = \gamma A$. Thus, $1 - A$ represents a fractional increase from γ in the transformed time rate T_t , and thus the fractional increase from the physical T_t of an inertial rod at rest with the stationary coordinates at that point. We can make a contour of constant A on our world map to give a limit for a desired physicality of the stationary time.

§4. Stationary physical distance using a diagonal metric. For diagonal coordinates with physical dR for all t and χ , $B = 1$, so (35) becomes

$$R_\chi = a\gamma. \quad (24)$$

By integration we find

$$R = a \int_0^\chi \gamma \partial\chi_t, \quad (25)$$

where the subscript on $\partial\chi_t$ represents integration of χ at constant t . Partial differentiation with respect to t gives

$$R_t = c\dot{a} \int_0^\chi \gamma \partial\chi_t + a \int_0^\chi \gamma_t \partial\chi_t, \quad (26)$$

where the dot represents differentiation by cdt . We can then find \hat{V} from (31), (73), and (75) as

$$\hat{V} = -\frac{R_t}{c\gamma} = -\frac{1}{c\gamma} \left[c\dot{a} \int_0^\chi \gamma \partial_{\chi t} + a \int_0^\chi \gamma_t \partial_{\chi t} \right]. \quad (27)$$

This is an integral equation for \hat{V} . It can be converted into a partial differential equation by multiplying both sides by γ and partial differentiating by χ :

$$\gamma^2 \left[\hat{V}_x + \frac{a}{c} \hat{V} \hat{V}_t \right] = -\dot{a} = -\frac{1}{c} \frac{da}{dt}. \quad (28)$$

General solutions of (77) for $\hat{V}(t, \chi)$ are found in the Appendix. $T(t, \chi)$, and $R(t, \chi)$ are also found there by their dependence on $\hat{V}(t, \chi)$. This will complete our search for a transform from FLRW that will satisfy the General Relativity field equation with a physical R to all radial events.

§5. Interpretation. We can use these solutions to show on a R, T plot the paths of galaxies ($\chi = \text{constant}$) and photons ($ds = 0$). We will use Ω as defined by Peebles [3] (see Appendix D). Thus $\Omega = 1$ is a flat universe ($k = 0$) with no cosmological constant ($\Lambda = 0$). These are shown in Figs. 1 and 2. An approximate upper limit of physicality is shown by the dashed line for $A = 1.05$. Below this line, T is close to physical. R vs. T at $t = 0$ provides a horizon, where the visible universe has a finite physical distance, but where T is non-physical.

Fig. 1 coordinates are both physical, the physical distance R to a galactic point at χ characterized by the red shift $z = -1 + \left(\frac{t_0}{t}\right)^{2/3}$ and the time at the origin (or on any galactic point) $\frac{t}{t_0}$. The light we see now at the origin originates from a galactic point when its path crosses the light path, where the physical distance is as shown. Notice that the light comes monotonically towards us even from the farthest galactic point, but its coordinate speed slows down the farther it is away in these coordinates where the clocks measuring t are on moving galactic points whereas the rulers measuring R are on the stationary frame. Thus, $\left(\frac{a\partial\chi}{\partial t}\right)_s = c$ when clocks and rulers are on the same frame, but $\left(\frac{\partial R}{\partial t}\right)_s$ is a coordinate light speed not equal to c (except close to the origin where t approaches T) because R and t are on different frames. This is much like the coordinate light speed of the Schwarzschild coordinates [5] where clocks and distances are at different locations.

Although the distance to the origin in Fig. 2 is physical, the co-located time coordinate T is not, but comes close to it near the space

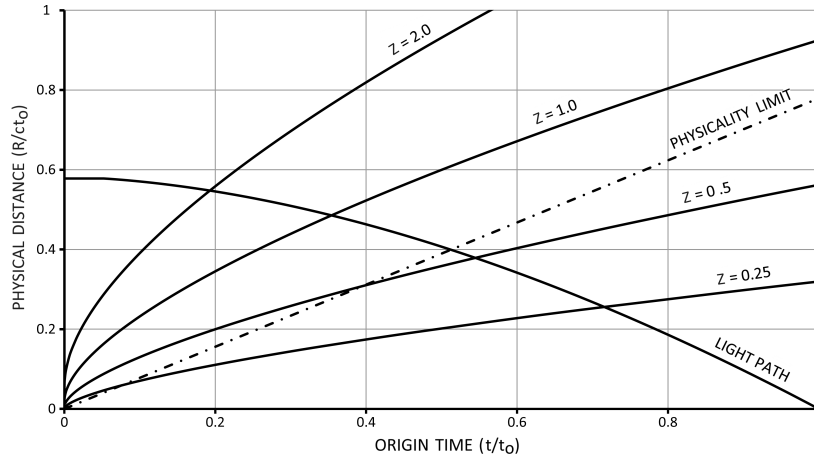


Fig. 1: Physical distance R/c_0t_0 for a flat universe ($\Omega = 1$) vs. physical time at the origin t/t_0 (or galactic time). The paths of galactic points representing galaxies are characterized by their red shift z . The transformed time (not shown) is close to physical only below the PHYSICALITY LIMIT line where $A = 1.05$. Light comes towards the origin along the LIGHT PATH from all radial points of the universe, travelling slower than its present speed in these coordinates where the clocks are on different frames from the rulers (like the Schwarzschild coordinates). The galactic paths show the expanding universe in physical coordinates, some travelling faster than the light speed in these coordinates.

origin. For distances and times below the physicality line $A = 1.05$, T_t is less than the physical time by only 5% on a co-located stationary inertial frame.

As such, the physical interpretation seems very clear. A galactic point seems to start at $R = 0$, $T = 0$ and moves out to the time it emits its light that can be later seen at the origin at $T = t_0$. The farthest galactic points travel out the fastest and release their light the earliest, but even for the most remote galactic point, the distance never gets greater than $0.58 c_0t_0$, which can be considered to be the size of the visible universe when the farthest light was emitted.

Near the horizon, some oddities occur in this interpretation; when $t \rightarrow 0$, T comes close but is not zero for most galactic paths, but also is not physical at that point. R uses physical rulers at that point, but also is not quite zero. This discrepancy becomes the largest for the farthest galactic points. The lack of physicality near the horizon is presumably what causes the transform to show non-zero times and distances at $t = 0$,

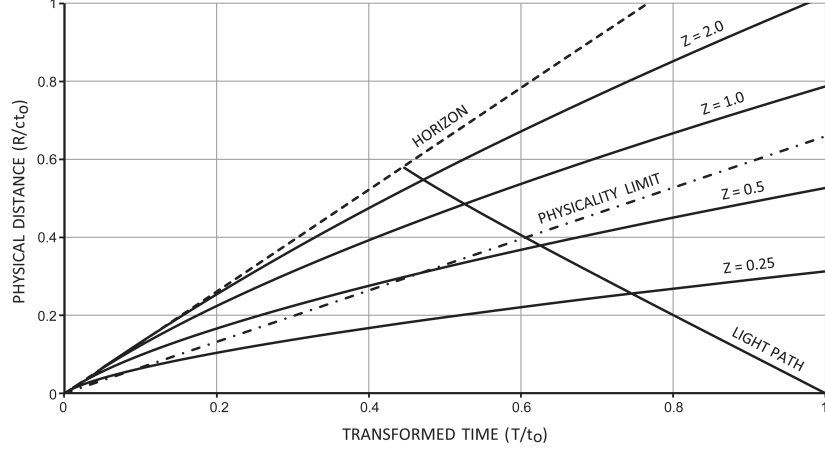


Fig. 2: Physical distance ($R/c_0 t_0$) for a flat universe ($\Omega = 1$) vs. the stationary local time T/t_0 , which is close to physical below the PHYSICALITY LIMIT line ($A = 1.05$). A galactic point starts close to the physical origin at $T = 0$ and travels out to where it may emit light when it crosses the LIGHT PATH that will travel close to the physical light speed and be visible later at the physical origin at $T = t_0$. The line labelled HORIZON represents singularities ($t = 0$ and $\gamma = \infty$), beyond which there are no galactic points, and no space.

$\gamma = \infty$ and the farthest galactic points to go close to the stationary coordinate light speed which approaches $1.3c$ near the horizon. Of course, if there were to be a different stationary transform with constant inertial light speed that had a rigid physical distance, it might have a different universe radius, but this seems unlikely. (A possible alternative with different assumptions is explored in [7].)

The above interpretation is strengthened by examining the physical coordinates of an expanding empty FLRW universe, a solution that was first published by Robertson with $a = ct$ [8]:

$$R = ct \sinh \chi, \quad T = t \cosh \chi. \quad (29)$$

Robertson showed that these transformed coordinates obeyed the Minkowski metric for all t, χ , as is required of physical coordinates in empty space. This solution is plotted in Fig. 3 with the same format as Fig. 2. The paths are very similar except that all the paths are straight with no universe density to curve them, and the galactic paths do not have a gap at the singularity at $t = 0$. Notice that along the light path that $\chi_p = \ln(\frac{t_0}{t})$, and $R_p = \frac{1}{2} ct_0 \left(1 - \frac{t^2}{t_0^2}\right)$ so that at $t = 0$, the extent of the

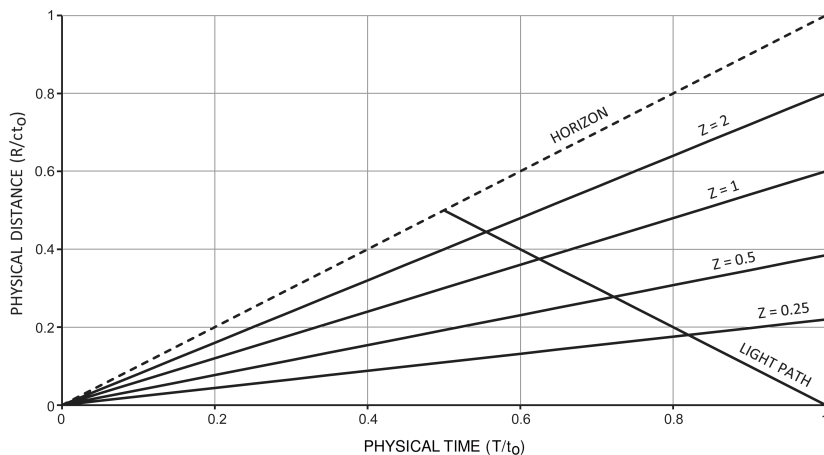


Fig. 3: Physical distance (R/ct_0) for the empty expanding universe ($\Omega = 0$, $\Omega_r = 1$) plotted against the transformed time (T/t_0) on clocks attached at R for various galaxy paths (labelled by their red shift z) and for the light path that arrives at the origin at $T = t_0$. The horizon is the locus of points where $t = 0$. All lines are straight and physical, since there is no space curvature. The remotest galactic point travels from the origin at $T = 0$ out to $\frac{1}{2} ct_0$ at the light speed c . There is no space outside the horizon to contain any galactic points.

visible universe is $R_p = \frac{1}{2} ct_0$. In terms of the clocks at the origin that read t , light seems to come monotonically towards us from this distance, the coordinate light speed $\left(\frac{\partial R}{\partial t}\right)_s$ slowing to zero at $t = 0$.

It's natural to wonder what is outside the horizon. In this empty universe, all galactic points are within the horizon. Outside the horizon not only are there no galactic points, there is *no space* when viewed from a stationary frame. Like a Fitzgerald contraction all differential radial distances shrink to zero at the horizon as the galactic points approach the light speed. From one of those points, it would have a different stationary frame and so would see a finite space in its vicinity and would see a different horizon.

For a finite universe energy density, many higher values of χ are not included inside the horizon. All of this is clouded by the non-physicality of T close to the horizon. But it would appear that all galactic matter starts outward from $R = 0$ at $T = 0$ travelling close to the light speed near the horizon, the largest χ travelling the fastest. As each χ_1 is slowed down by the inward gravitational pull of the mass of galaxies inside it, it departs from the horizon after a time $T_1(\chi_1)$ at a distance

$R_1(\chi_1)$, where it can emit light that will be the earliest light visible at the origin at time $T = t_1$, so $\chi_1 = 3\left(\frac{t_1}{t_0}\right)^{1/3}$ for $\Omega = 1$, see (102). Those with a $\chi > \chi_1$ will not be visible at the origin at $T = t_1$, because they must be inside the horizon to emit light, presumably because of the singularities in γ and in energy density along the horizon. Thus, only a limited number of galactic points can be visible at the origin at the present time, those with $t_1 \leq t_0$.

This is a different picture from a universe that is full of galactic points beyond a visible horizon, but are invisible only because there is not enough time to see them. I am suggesting that like the empty universe, there are no galactic points and no space beyond the horizon when viewed from a stationary frame.

§6. Newtonian gravity for flat space ($\Omega = 1$) close to the origin. In Appendix C, the acceleration vector A^μ of a point on R is found in FLRW coordinates and in stationary coordinates. The latter is solved for a flat universe ($k = 0$) with no dark energy $\Omega = 1$ in normalized coordinates ($\frac{\chi a_0}{ct_0} \rightarrow x$, $\frac{t}{t_0} \rightarrow t$, see Appendix C.2). For small $u = \frac{x}{t^{1/3}}$, the gravitational acceleration g goes to zero as $-\frac{2u}{9t} = -\frac{2R}{9}$. Since small u is the region with physical coordinates, it is interesting to express g and R in unnormalized coordinates:

$$-g \rightarrow \frac{2}{9} \frac{c}{t_0} \frac{R}{ct_0} = \frac{4\pi\rho_0 GR}{3} = \frac{GM_0}{R^2}, \quad (30)$$

where we have used $\rho = \rho_0\left(\frac{a_0}{a}\right)^3$ in (86) and M_0 as the present universe mass inside the radius R . Thus, the gravitational force in the stationary coordinates used here near the origin is the same as Newtonian gravity, as one might expect if both T and R are physical near the origin and if $V \ll c$.

§7. Conclusion. We have found a transform to stationary coordinates from the FLRW coordinates for radial events that has a physical radial distance to each event. Although the time of the transform is not physical for all events, it is physical close to the space origin. When the physical radius is plotted against the physical time at the origin, the coordinate light speed $\frac{dR}{dt}$ slows down the farther out in the Universe it is observed, much like the coordinate light speed of Schwarzschild coordinates. But, since the time and distance of the stationary coordinates are both local on the same frame, the light speed in these coordinates remains fairly constant out to the far reaches of the Universe. Near the origin in these coordinates, the gravitational acceleration becomes New-

tonian for a flat universe ($\Omega=1$). The stationary coordinates indicate that galactic points originate at $T=0$, move out to a finite distance in a finite time when it releases light that we can see at the present time. For the most distant galactic points the distance at emission is about half of the so-called time-of-flight distance $c(t_0-t_e)$. An expanding horizon occurs in the stationary coordinates where the radial velocity of the galactic points approaches the light speed, outside of which there are no galactic points, and no space. At the time of the release of the earliest light that we could see today, the horizon was at $R=0.58ct_0$ in the stationary coordinates for a flat universe ($\Omega=1$) with no dark energy.

Appendix A. Stationary coordinates for any $a(t)$

A.1. Solution for \hat{V} . Equation (77) can be solved as a standard initial-value problem. Let $W \equiv -\hat{V}$. Then (77) becomes

$$W_\chi - \frac{a}{c} WW_t = \frac{1}{c} \frac{da}{dt} (1 - W^2). \quad (31)$$

Define a characteristic for $W(t, \chi)$ by

$$\left(\frac{\partial t}{\partial \chi} \right)_c = -\frac{a}{c} W \quad (32)$$

so

$$\left(\frac{\partial W}{\partial \chi} \right)_c = \frac{1}{c} \frac{da}{dt} (1 - W^2) \quad (33)$$

(the subscript c here indicates differentiation along the characteristic). If we divide (80) by (79) we get

$$\left(\frac{\partial W}{\partial t} \right)_c = -\frac{1}{a} \frac{da}{dt} \frac{(1 - W^2)}{W}. \quad (34)$$

This can be rearranged to give

$$\frac{W(\partial W)_c}{W^2 - 1} = \frac{(\partial a)_c}{a}. \quad (35)$$

This can be integrated along the characteristic with the boundary condition at $\chi = 0$ that $W=0$ and $a = a_c$:

$$1 - W^2 = \frac{a^2}{a_c^2} = \frac{1}{\gamma^2}. \quad (36)$$

This value for W (assumed positive for expanding universe) can be inserted into (79) to give

$$\left(\frac{\partial t}{\partial \chi}\right)_c = -\frac{a}{c} \sqrt{1 - \frac{a^2}{a_c^2}}. \quad (37)$$

We can convert this to a differential equation for a by noting that $c(\partial t)_c = (\partial \dot{t})_c = \frac{1}{\dot{a}}(\partial a)_c$

$$\left(\frac{\partial a}{\partial \chi}\right)_c = -a \dot{a} \sqrt{1 - \frac{a^2}{a_c^2}}. \quad (38)$$

The \dot{a} can be found as a function of a from the well known solution of the General Relativity field equation for the FLRW universe [3]:

$$\dot{a} = a \sqrt{\frac{8\pi G}{3c^2} \rho - \frac{k}{a^2} + \frac{\Lambda}{3}}, \quad (39)$$

where ρ is the rest mass density of the ideal liquid assumed for the universe that can be found as a function of a (see Appendix D); G is the gravitation constant; k is the constant in the FLRW metric (192); and Λ is the cosmological constant possibly representing the dark energy of the universe.

Equation (85) can be integrated along the characteristic with constant α_c , starting with $\alpha = \alpha_c$ at $\chi = 0$. This will give $\chi = X(\alpha, \alpha_c)$. This can be inverted to obtain $\alpha_c(\alpha, \chi)$. When this is inserted into (83), we have a solution to (78) for $W(\alpha, \chi)$. Integration of (86) gives $a(t)$ and thus W as a function of t, χ .

A.2. Obtaining T, R from \hat{V} . Equations (25), (31), and (38) show that

$$W = -\frac{a}{c} \left(\frac{\partial \chi}{\partial t}\right)_R = \frac{a}{c} \frac{R_t}{R_\chi} = \frac{c}{a} \frac{T_\chi}{T_t} \quad (40)$$

so

$$T_\chi - \frac{a}{c} W T_t = 0. \quad (41)$$

Thus T has the same characteristic as W (79), so that $\left(\frac{\partial T}{\partial \chi}\right)_c = 0$, and T is constant along this characteristic:

$$T(t, \chi) = T(t_c, 0) = t_c \equiv t[\alpha_c(t, \chi)], \quad (42)$$

where $t(\alpha)$ is given by the integration of (86) and $\alpha_c[\alpha(t), \chi]$ is found by inverting the integration of (86). This gives us the solution for $T(t, \chi)$

and A

$$A = \frac{\gamma}{T_t} = \frac{a_c}{a} \left(\frac{\partial t}{\partial t_c} \right)_\chi = \frac{a_c}{a} \frac{\frac{da_c}{dt_c}}{\frac{da}{dt}} \left(\frac{\partial a}{\partial a_c} \right)_\chi. \quad (43)$$

The solution for R can be obtained by integrating (74), using γ from (83) and $a_c(t, \chi)$ from (86):

$$R(t, \chi) = a \int_0^\chi \gamma \partial \eta_t = \int_0^\chi a_c(t, \eta) \partial \eta_t. \quad (44)$$

Alternatively, for ease of numerical integration we would like to integrate dR along the same characteristic as T and W . This can be obtained from the partial differential equation

$$\left(\frac{\partial R}{\partial \chi} \right)_c = R_\chi + R_t \left(\frac{\partial t}{\partial \chi} \right)_c. \quad (45)$$

If we insert into equation (45) the values for these three quantities from (73), (88), and (79), we get

$$\left(\frac{\partial R}{\partial \chi} \right)_c = \gamma a + \frac{cW}{a} \gamma a \left(-\frac{cW}{a} \right) = \frac{a}{\gamma} = \frac{a^2}{a_c}. \quad (46)$$

Appendix B. Similarity solutions for flat universe ($\Omega = 1$). But I have found a simpler integration of (77) for the special case of $\Omega = 1$ where the General Relativity solution is $a = a_0 \left(\frac{t}{t_0} \right)^{2/3}$ [3]. To simplify notation, let us normalize: $\frac{t}{t_0} \rightarrow t$, $\frac{a}{a_0} = \alpha = t^{2/3}$, and $\frac{\chi a_0}{ct_0} \rightarrow x$, $\frac{T}{t_0} \rightarrow T$, $\frac{R}{ct_0} \rightarrow R$, and let $W = -\hat{V}$.

B.1. Ordinary differential equation for \hat{V} or $-W$. Equation (77) then becomes

$$W_x - t^{2/3} W W_t = \frac{2}{3} t^{-1/3} (1 - W^2). \quad (47)$$

This can be converted into an ordinary differential equation by letting

$$u \equiv \frac{x}{t^{1/3}} \quad (48)$$

so that (95) becomes

$$W' \left(1 + \frac{uW}{3} \right) = \frac{2}{3} (1 - W^2), \quad (49)$$

where the prime denotes differentiation by u .

B.2. Ordinary differential equation for T and R . Similarly we can find ordinary differential equations for T and R by defining:

$$\frac{T}{t} \equiv q(u), \quad (50)$$

and

$$\frac{R}{t} \equiv s(u), \quad (51)$$

where $q(u)$ and $s(u)$, from (38) and (31), are given by the coupled ordinary differential equations:

$$q' \left(1 + \frac{uW}{3} \right) = qW, \quad (52)$$

and

$$s \left(W + \frac{u}{3} \right) = s. \quad (53)$$

It is useful to find that $q = \gamma^{3/2}$, $s = \gamma \frac{u+3W}{3}$, $s' = \gamma$, $A = \frac{\gamma}{Tt} = \frac{1}{\gamma} \left(1 + \frac{uW}{3} \right) = \frac{v_p}{c}$; so $T = t\gamma^{3/2}$, and $R = \frac{1}{3}t\gamma(u+3W)$.

For small values of u , $W = \frac{2u}{3}$, $q = 1 + \frac{u^2}{3}$, $s = u$, $\frac{v_p}{c} = 1 + O(W^4)$, and $R = t^{2/3}x = ax$. The light speed v_p measured on T, R remains close to c out to large R . We also note that $T_t \rightarrow 1 + \frac{W^2}{4}$, which is slower than a Lorentz requirement of $T_t \rightarrow \gamma \rightarrow 1 + \frac{W^2}{2}$.

As $t \rightarrow 0$, $u \rightarrow \infty$, $\gamma \rightarrow \kappa u^2$, $W \rightarrow 1 - \frac{1}{2\kappa^2 u^4}$, $q \rightarrow \kappa^{3/2} u^3$, and $s \rightarrow \frac{\kappa u^3}{3}$. The quantities T and R both remain finite at this limit with $T \rightarrow \kappa^{3/2} x^3$, and $R \rightarrow \frac{\kappa x^3}{3}$ with $\frac{T}{R} \rightarrow 3\kappa^{1/2}$. The κ is difficult to determine from the numerical integration because of the singularity at large u , but my integrater gives $\kappa = 0.0646$. The fact that T does not go to zero when t goes to zero results from equating T with t at $t=1$ and not at $t=0$.

The distance R and time T can be found from the numerical integration of the coupled ordinary differential equations. The paths of galactic points are those for constant x . The path photons have taken reaching the origin at t_1 is found by calculating x_p vs. t and using the transform to T, R . Thus, for $\Omega = 1$,

$$x_p = \int_{t_1}^t \frac{c}{a} dt = 3(t_1^{1/3} - t^{1/3}). \quad (54)$$

For light arriving now, $t_1 = 1$, the value of u_p becomes

$$u_p = 3 \left(\frac{1}{t^{1/3}} - 1 \right). \quad (55)$$

Note that $x_p \rightarrow 3$ as $t \rightarrow 0$ at the beginning of the path for photons. This makes the horizon at the earliest time of release of light visible today be $0.581ct_0$.

Appendix C. Gravitational field

C.1. Gravitational field in the FLRW and stationary coordinates. We wish to find the components of the radial acceleration of a test particle located at R in the stationary system. We will do this by calculating the FLRW components of the acceleration vector and find the transformed components by using the known diagonal transforms. For the FLRW components, we will use the metric

$$ds^2 = dt^2 - a^2 d\chi^2 - a^2 r^2 d\theta^2 - a^2 r^2 \sin^2 \theta d\phi^2. \quad (56)$$

Let

$$x^1 = \chi, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = \hat{t} = ct, \quad (57)$$

and the corresponding metric coefficients become

$$g_{44} = 1, \quad g_{11} = -a^2, \quad g_{22} = -a^2 r^2, \quad g_{33} = -a^2 r^2 \sin^2 \theta. \quad (58)$$

For any metric, the acceleration vector for a test particle is

$$A^\lambda = \frac{dU^\lambda}{ds} + \Gamma_{\mu\nu}^\lambda U^\mu U^\nu, \quad (59)$$

where the Γ 's are the affine connections and U^λ is the velocity vector of the test particle. In our case the test particle is at the point R on the transformed coordinate, but not attached to the frame so that it can acquire an acceleration. Instantaneously, it will have the same velocity as the point on the transformed coordinate, and its velocity and acceleration vectors will therefore transform the same as the point (30).

We will be considering accelerations only in the radial direction so that we need find affine connections only for indices 1, 4. The only non-zero partial derivative with these indices is

$$\frac{\partial g_{11}}{\partial x^4} = -2a\dot{a}. \quad (60)$$

The general expression for an affine connection for a diagonal metric is

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2g_{\lambda\lambda}} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right). \quad (61)$$

The only three non-zero affine connections with 1, 4 indices are

$$\Gamma_{11}^4 = a\dot{a}, \quad \Gamma_{41}^1 = \Gamma_{14}^1 = \frac{\dot{a}}{a}. \quad (62)$$

The acceleration vector in FLRW coordinates of our test particle moving at the same velocity as a point on the transformed frame becomes

$$\left. \begin{aligned} A^{\hat{t}} &= \frac{dU^4}{ds} + \Gamma_{11}^4 U^1 U^1 \\ A^x &= \frac{dU^1}{ds} + \Gamma_{41}^1 (U^4 U^1 + U^1 U^4) \end{aligned} \right\}. \quad (63)$$

Using U^4 and U^1 in (29), we find

$$\left. \begin{aligned} A^{\hat{t}} &= \gamma \left(\frac{\partial \gamma}{\partial \hat{t}} \right)_R + a \dot{a} \frac{\gamma^2 \hat{V}^2}{a^2} = \gamma^4 \hat{V} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \gamma^2 \hat{V}^2 \\ A^x &= \gamma \left[\frac{\partial}{\partial \hat{t}} \left(\frac{\gamma \hat{V}}{a} \right) \right]_R + 2 \frac{\dot{a}}{a} \frac{\gamma^2 \hat{V}}{a} = \frac{\gamma^4}{a} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a^2} \gamma^2 \hat{V} \end{aligned} \right\}. \quad (64)$$

Since the acceleration vector of the test particle at R in the stationary coordinates will be orthogonal to the velocity vector, it becomes

$$\left. \begin{aligned} A^T &= 0 \\ A^R &= \frac{DU^R}{ds} \equiv -\frac{g}{c^2} \end{aligned} \right\}. \quad (65)$$

Here A^R is the acceleration of a point on the R axis, so the gravitational field affecting objects like the galactic points is the negative of this. g is defined so that mg is the force acting on an object whose mass is m . Close to the origin, $g = \frac{d^2 R}{dT^2}$, the normal acceleration. Since the vector A^λ will transform like dT, dR (1):

$$A^R = \frac{1}{c} R_t A^{\hat{t}} + R_x A^x \quad (66)$$

so that

$$-\frac{g}{c^2} = \left[\gamma^4 \hat{V} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \gamma^2 \hat{V}^2 \right] \frac{1}{c} R_t + \left[\frac{\gamma^4}{a} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a^2} \gamma^2 \hat{V} \right] R_x. \quad (67)$$

With the use of (31), this can be simplified to

$$-\frac{g}{c^2} = \frac{R_x}{a} \left[\gamma^2 \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \hat{V} \right]. \quad (68)$$

The acceleration g can be thought of as the gravitational field caused by the mass of the surrounding galactic points, which balances to zero at

the origin, where the frame is inertial, but goes to infinity at the horizon. It is the field which is slowing down the galactic points (for $\Lambda = 0$).

C.2. Gravitational force for flat universe ($\Omega = 1$). If we insert the values of V , R , and $\frac{a}{c}$ in Appendix C, (128), we obtain

$$g = -\frac{s'}{t} \left[\gamma^2 W' \left(\frac{u}{3} + W \right) - \frac{2W}{3} \right] \quad (69)$$

where g has the units $\frac{ca}{t_0}$. The insertion of W' and s' into (129) gives

$$-g = \frac{2}{9\gamma t} \frac{u}{1 + \frac{uW}{3}}. \quad (70)$$

Appendix D. Gravitational field Peebles has shown a convenient way to represent equation 86 (see [3, p.312]), the solution of the field equation for the FLRW metric in a universe filled with an ideal liquid. He defines

$$\Omega \equiv \rho_0 \frac{8\pi G}{3c^2 H_0^2}, \quad (71)$$

and

$$\Omega_r \equiv \frac{-k}{H_0^2 a_0^2}, \quad (72)$$

and

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}. \quad (73)$$

Here ρ_0 , H_0 , and a_0 are the energy density, Hubble constant, and universe scale factor, respectively, at the present time t_0 . For very small a there will also be radiation energy term $\Omega_R \approx 2 \times 10^{-5}$ [3].

Let $\alpha = \frac{a}{a_0}$. It is found by the solution of the ordinary differential equation:

$$\frac{1}{\alpha} \frac{d\alpha}{dt} \equiv H = H_0 E(\alpha), \quad (74)$$

where the normalized Hubble ratio E is

$$E(\alpha) = \sqrt{\frac{\Omega_R}{\alpha^4} + \frac{\Omega}{\alpha^3} + \frac{\Omega_r}{\alpha^2} + \Omega_\Lambda}. \quad (75)$$

The Ω s are defined so that

$$\Omega_R + \Omega + \Omega_r + \Omega_\Lambda = 1. \quad (76)$$

At $t = t_0$: $\alpha = 1$ and $E = 1$.

The cosmic time t measured from the beginning of the FLRW universe becomes

$$cH_0 t = \int_0^\alpha \frac{1}{\alpha E} d\alpha. \quad (77)$$

For a flat universe with $\Omega = 1$ and $\Omega_R = \Omega_r = \Omega_\Lambda = 0$:

$$\alpha = \left(\frac{t}{t_0}\right)^{2/3}, \quad t_0 = \frac{2}{3cH_0}. \quad (78)$$

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Cosmological Mass-Defect — A New Effect of General Relativity

Dmitri Rabounski

Abstract: This study targets the change of mass of a mass-bearing particle with the distance travelled in the space of the main (principal) cosmological metrics. The mass-defect is obtained due to a new method of deduction: by solving the scalar geodesic equation (equation of energy) of the particle. This equation manifests three factors affecting the particle's mass: gravitation, non-holonomy, and deformation of space. In the space of Schwarzschild's mass-point metric, the obtained solution coincides with the well-known gravitational mass-defect whose magnitude increases toward the gravitating body. No mass-defect has been found in the rotating space of Gödel's metric, and in the space filled with a homogeneous distribution of ideal liquid and physical vacuum (Einstein's metric). The other obtained solutions manifest a mass-defect of another sort than that in the mass-point metric: its magnitude increases with distance from the observer, so that manifests itself at cosmologically large distances travelled by the particle. This effect has been found in the space of Schwarzschild's metric of a sphere of incompressible liquid, in the space of a sphere filled with physical vacuum (de Sitter's metric), and in the deforming spaces of Friedmann's metric (empty or filled with ideal liquid and physical vacuum). Herein, we refer to this effect as the cosmological mass-defect. It has never been considered prior to the present study: it is a new effect of the General Theory of Relativity.

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§1. Problem statement. In 2008, I presented my theory of the cosmological Hubble redshift [1]. According to the theory, the Hubble redshift was explained as the energy loss of photons with distance due to the work done against the field of global non-holonomy (rotation) of the isotropic space, which is the home of photons*. I arrived at this conclusion after solving the scalar geodesic equation (equation of energy) of a photon travelling in a static (non-deforming) universe. The calculation matched the observed Hubble law, including its non-linearity.

My idea now is that, in analogy to photons, we could as well consider mass-bearing particles.

Let's compare the isotropic and non-isotropic geodesic equations, which are the equations of motion of particles. According to the chronometrically invariant formalism, which was introduced in 1944 by Abraham Zelmanov [3–5], any four-dimensional quantity is observed as its projections onto the time line and three-dimensional spatial section of the observer[†]. The projected (chronometrically invariant) equations for non-isotropic geodesics have the form [3–5]

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = 0, \quad (1.1)$$

$$\frac{d(mv^i)}{d\tau} - mF^i + 2m(D_k^i + A_{k.}^i)v^k + m\Delta_{nk}^i v^n v^k = 0, \quad (1.2)$$

while the projected equations for isotropic geodesics are

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_i c^i + \frac{\omega}{c^2} D_{ik} c^i c^k = 0, \quad (1.3)$$

$$\frac{d(\omega c^i)}{d\tau} - \omega F^i + 2\omega(D_k^i + A_{k.}^i)c^k + \omega\Delta_{nk}^i c^n c^k = 0. \quad (1.4)$$

Thus, according to the chronometrically invariant equations of motion, the factors affecting the particles are: the gravitational inertial force F_i , the angular velocity A_{ik} of the rotation of space due to its non-holonomy, the deformation D_{ik} of space, and the non-uniformity of space (expressed by the Christoffel symbols Δ_{jk}^i).

*The four-dimensional pseudo-Riemannian space (space-time) consists of two segregate regions: the non-isotropic space, which is the home of mass-bearing particles, and the isotropic space inhabited by massless light-like particles (photons). The isotropic space rotates with the velocity of light under the conditions of both Special Relativity and General Relativity, due to the sign-alternating property of the space-time metric. See [2] for details.

[†]Chronometric invariance means that the projected (chronometrically invariant) quantities and equations are invariant along the spatial section of the observer.

As is seen, the non-isotropic geodesic equations have the same form as the isotropic ones. Only the sublight velocity v^i and the relativistic mass m are used instead of the light velocity c^i and the frequency ω of a photon. Therefore, the factors of gravitation, non-holonomity, and deformation, presented in the scalar geodesic equation, should change the mass of a moving mass-bearing particle with distance just as they change the frequency of a photon.

Relativistic mass change due to the field of gravitation of a massive body (the space of Schwarzschild's mass-point metric) is a textbook effect of General Relativity, well verified by experiments. It is regularly deduced from the conservation of energy of a mass-bearing particle in the stationary field of gravitation [6, §88]. However, this method of deduction can only be used in stationary fields [6, §88], wherein gravitation is the sole factor affecting the particle.

In contrast, the new method of deduction of the relativistic mass change with distance I propose herein — through integrating the scalar geodesic equation, based on the chronometrically invariant formalism, — is universal. This is because the scalar geodesic equation contains all three factors changing the mass of a moving mass-bearing particle with distance (these are gravitation, non-holonomity, and deformation), and these factors are presented in their general form, without any limitations. Therefore the suggested method of deduction can equally be applied to calculating the relativistic mass change with distance travelled by the particle in any particular space metric known due to the General Theory of Relativity.

In the next paragraphs of this paper, we will apply the suggested method of deduction to the main (principal) cosmological metrics. As a result, we will see how a mass-bearing particle changes its mass with the distance travelled in most of these spaces, including “cosmologically large” distances where the relativistic mass change thus becomes *cosmological mass-defect*.

§2. The chronometrically invariant formalism in brief. Before we solve the geodesic equations in chronometrically invariant form, we need to have a necessary amount of definitions of those quantities specifying the equations. According to the chronometrically invariant formalism [3–5], these are: the chr.inv.-vector of the gravitational inertial force F_i , the chr.inv.-tensor of the angular velocity of the rotation of space A_{ik} due to its non-holonomity (non-orthogonality of the time lines to the three-dimensional spatial section), the chr.inv.-tensor of the deformation of space D_{ik} , and the chr.inv.-Christoffel symbols Δ_{jk}^i (they

manifest the non-uniformity of space)

$$F_i = \frac{1}{\sqrt{g_{00}}} \left(\frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \quad \sqrt{g_{00}} = 1 - \frac{w}{c^2}, \quad (2.1)$$

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} (F_i v_k - F_k v_i), \quad (2.2)$$

$$D_{ik} = \frac{1}{2} \frac{{}^* \partial h_{ik}}{\partial t}, \quad D^{ik} = -\frac{1}{2} \frac{{}^* \partial h^{ik}}{\partial t}, \quad D = h^{ik} D_{ik} = \frac{{}^* \partial \ln \sqrt{h}}{\partial t}, \quad (2.3)$$

$$\Delta_{jk}^i = h^{im} \Delta_{jk,m} = \frac{1}{2} h^{im} \left(\frac{{}^* \partial h_{jm}}{\partial x^k} + \frac{{}^* \partial h_{km}}{\partial x^j} - \frac{{}^* \partial h_{jk}}{\partial x^m} \right). \quad (2.4)$$

They are expressed through the chr.inv.-differential operators

$$\frac{{}^* \partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}, \quad \frac{{}^* \partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} v_i \frac{{}^* \partial}{\partial t}, \quad (2.5)$$

as well as the gravitational potential w , the linear velocity v_i of space rotation due to the respective non-holonomy, and also the chr.inv.-metric tensor h_{ik} , which are determined as

$$w = c^2 (1 - \sqrt{g_{00}}), \quad v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}, \quad (2.6)$$

$$h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k, \quad h^{ik} = -g^{ik}, \quad h_k^i = \delta_k^i, \quad (2.7)$$

while the derivation parameter of the equations is the physical observable time

$$d\tau = \sqrt{g_{00}} dt - \frac{1}{c^2} v_i dx^i. \quad (2.8)$$

This is enough. We now have all the necessary equipment to solve the geodesic equations in chronometrically invariant form.

§3. Local mass-defect in the space of a mass-point (Schwarzschild's mass-point metric). This is an empty space*, wherein a spherical massive island of matter is located, thus producing a spherically symmetric field of gravitation (curvature). The massive island is

*In the General Theory of Relativity, we say that a space is empty if it is free of distributed matter — substance or fields, described by the right-hand side of Einstein's equations, — except for the field of gravitation, which is the same as the field of the space curvature described by the left-hand side of the equations.

approximated as a mass-point at distances much larger than its radius. The metric of such a space was introduced in 1916 by Karl Schwarzschild [7]. In the spherical three-dimensional coordinates $x^1 = r$, $x^2 = \varphi$, $x^3 = \theta$, the metric has the form

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (3.1)$$

where r is the distance from the mass-island of the mass M , $r_g = \frac{2GM}{c^2}$ is the corresponding gravitational radius of the mass, and G is the world-constant of gravitation. As is seen from the metric, such a space is free of rotation and deformation. Only the field of gravitation affects mass-bearing particles therein.

Differentiating the gravitational potential $w = c^2(1 - \sqrt{g_{00}})$ with respect to x^i , we obtain

$$F_i = \frac{1}{\sqrt{g_{00}}} \frac{\partial w}{\partial x^i} = -\frac{c^2}{2g_{00}} \frac{\partial g_{00}}{\partial x^i}, \quad (3.2)$$

wherein, according to the metric (3.1), we should readily substitute

$$g_{00} = 1 - \frac{r_g}{r}. \quad (3.3)$$

Thus the gravitational inertial force (2.1) in the space of Schwarzschild's mass-point metric has the following nonzero components

$$F_1 = -\frac{c^2 r_g}{2r^2} \frac{1}{1 - \frac{r_g}{r}}, \quad F^1 = -\frac{c^2 r_g}{2r^2} \quad (3.4)$$

which, if the mass-island is not a collapsar ($r \gg r_g$), are

$$F_1 = F^1 = -\frac{GM}{r^2}. \quad (3.5)$$

As a result, the scalar geodesic equation for a mass-bearing particle (1.1) takes the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_1 v^1 = 0, \quad (3.6)$$

where $v^1 = \frac{dr}{d\tau}$. This equation transforms into $\frac{dm}{m} = \frac{1}{c^2} F_1 dr$, thus we obtain the equation $d \ln m = -\frac{GM}{c^2} \frac{dr}{r^2}$. It solves, obviously, as

$$m = m_0 e^{\frac{GM}{c^2 r}} \simeq m_0 \left(1 + \frac{GM}{c^2 r}\right). \quad (3.7)$$

According to the solution, a spacecraft's mass measured on the surface of the Earth ($M = 6.0 \times 10^{27}$ gram, $r = 6.4 \times 10^8$ cm) will be greater than its mass measured at the distance of the Moon ($r = 3.0 \times 10^{10}$ cm) by a value of $1.5 \times 10^{-11} m_0$ due to the greater magnitude of the gravitational potential near the Earth.

This mass-defect is a local phenomenon: it decreases with distance from the source of the field, thus becoming negligible at “cosmologically large” distances even in the case of such massive sources of gravitation as the galaxies. This is not a cosmological effect, in other words.

It is known as the gravitational mass-defect in the Schwarzschild mass-point field, which is just one of the basic effects of the General Theory of Relativity. The reason why I speak of this well-known effect herein is that this method of deduction — through integrating the scalar geodesic equation, based on the chronometrically invariant formalism, — differs from the regular deduction [6, §88], derived from the conservation of energy of a particle travelling in a stationary field of gravitation.

§4. Local mass-defect in the space of an electrically charged mass-point (Reissner-Nordström's metric). Due to the suggested new method of deduction — through integrating the scalar geodesic equation, based on the chronometrically invariant formalism, — we can now calculate mass-defect in the space of Reissner-Nordström's metric. This is a space analogous to the space of the mass-point metric with the only difference being that the spherical massive island of matter is electrically charged: in this case, the massive island is the source of both the gravitational field (the field of the space curvature) and the electromagnetic field. Therefore such a space is not empty but filled with a spherically symmetric electromagnetic field (distributed matter). Such a space has a metric which appears as an actual extension of Schwarzschild's mass-point metric (3.1). The metric was first introduced in 1916 by Hans Reissner [8] then, independently, in 1918 by Gunnar Nordström [9]. It has the form

$$ds^2 = \left(1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (4.1)$$

where r is the distance from the charged mass-island, $r_g = \frac{2GM}{c^2}$ is the corresponding gravitational radius, M is its mass, G is the constant of gravitation, $r_q^2 = \frac{Gq^2}{4\pi\epsilon_0 c^4}$, where q is the corresponding electric charge, and $\frac{1}{4\pi\epsilon_0}$ is Coulomb's force constant. As is seen from the metric, such a space is free of rotation and deformation. The gravitational inertial

force is, in this case, determined by both Newton's force and Coulomb's force according the component g_{00} of the metric (4.1) which is

$$g_{00} = 1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}, \quad (4.2)$$

thus we obtain

$$F_1 = - \frac{c^2}{2 \left(1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}\right)} \left(\frac{r_g}{r^2} - \frac{2r_q^2}{r^3} \right), \quad (4.3)$$

$$F^1 = - - \frac{c^2}{2} \left(\frac{r_g}{r^2} - \frac{2r_q^2}{r^3} \right). \quad (4.4)$$

If the massive island is not a collapsar ($r \gg r_g$), and it bears a weak electric charge ($r \gg r_q$), we have

$$F_1 = F^1 = - \frac{c^2}{2} \left(\frac{r_g}{r^2} - \frac{2r_q^2}{r^3} \right) = - \frac{GM}{r^2} + \frac{Gq^2}{4\pi\epsilon_0 c^2} \frac{1}{r^3}. \quad (4.5)$$

Thus, the scalar geodesic equation for a mass-bearing particle (1.1) takes the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_1 v^1 = 0, \quad (4.6)$$

where $v^1 = \frac{dr}{d\tau}$. It transforms into $d \ln m = \left(- \frac{GM}{c^2 r^2} + \frac{Gq^2}{4\pi\epsilon_0 c^4} \frac{1}{r^3} \right) dr$, which solves, obviously, as

$$m = m_0 e^{\frac{GM}{c^2 r} - \frac{1}{2r^2} \frac{Gq^2}{4\pi\epsilon_0 c^4}} \simeq m_0 \left(1 + \frac{GM}{c^2 r} - \frac{1}{2r^2} \frac{Gq^2}{4\pi\epsilon_0 c^4} \right). \quad (4.7)$$

As is seen from the solution, we should expect a mass-defect to be observed in the space of Reissner-Nordström's metric. Its magnitude is that of the mass-defect of the mass-point metric (the second term in the solution) with a second-order correction — the mass-defect due to the electromagnetic field of the massive island (the third term). The magnitude of the correction decreases with distance from the source of the field (a charged spherical massive island) even faster than the mass-defect due to the field of gravitation of the massive island. Therefore, the mass-defect in the space of Reissner-Nordström's metric we have obtained here is a local phenomenon, not a cosmological effect.

Note that this is the first case, where a mass-defect is predicted due to the presence of the electromagnetic field. Such an effect was not considered in the General Theory of Relativity prior to the present study.

A note concerning two other primary extensions of Schwarzschild's mass-point metric. Kerr's metric describes the space of a rotating mass-point. It was introduced in 1963 by Roy P. Kerr [10] then transformed into suitable coordinates by Robert H. Boyer and Richard W. Lindquist [11]. The Kerr-Newman metric was introduced in 1965 by Ezra T. Newman [12,13]. It describes the space of a rotating, electrically charged mass-point. These metrics are deduced in the vicinity of the point-like source of the field: they do not contain the distribution function of the rotational velocity with distance from the source. As a result, when taking into account the geodesic equations to be integrated in the space of any one of the rotating mass-point metrics, we should introduce the functions on our own behalf. This is not good at all: our choice of the functions, based on our understanding of the space rotation, can be true or false. We therefore omit calculation of mass-defect in the space of a rotating mass-point (Kerr's metric), and in the space of a rotating, electrically charged mass-point (the Kerr-Newman metric).

§5. No mass-defect present in the rotating space with self-closed time-like geodesics (Gödel's metric). This space metric was introduced in 1949 by Kurt Gödel [14], in order to find a possibility of time travel (realized through self-closed time-like geodesics). Gödel's metric, as was shown by himself [14], satisfies Einstein's equations where the right-hand side contains the energy-momentum tensor of dust and also the λ -term. This means that such a space is not empty, but filled with dust and physical vacuum (λ -field). Also, it rotates so that time-like geodesics are self-closed therein. Gödel's metric in its original notation, given in his primary publication [14], is

$$ds^2 = a^2 \left[(d\tilde{x}^0)^2 + 2e^{\tilde{x}^1} d\tilde{x}^0 d\tilde{x}^2 - (d\tilde{x}^1)^2 + \frac{e^{2\tilde{x}^1}}{2} (d\tilde{x}^2)^2 - (d\tilde{x}^3)^2 \right], \quad (5.1)$$

where $a = \text{const} > 0$ [cm] is a constant of the space, determined through Einstein's equations as $\lambda = -\frac{1}{2a^2} = -\frac{4\pi G\rho}{c^2}$ so that $a^2 = \frac{c^2}{8\pi G\rho} = -\frac{1}{2\lambda}$, and ρ is the dust density. Gödel's metric in its original notation (5.1) is expressed through the dimensionless Cartesian coordinates $d\tilde{x}^0 = \frac{1}{a} dx^0$, $d\tilde{x}^1 = \frac{1}{a} dx^1$, $d\tilde{x}^2 = \frac{1}{a} dx^2$, $d\tilde{x}^3 = \frac{1}{a} dx^3$, which emphasize the meaning of the world-constant a of such a space. Also, this is a constant-curvature space wherein the curvature radius is $R = \frac{1}{a^2} = \text{const} > 0$.

We now move to the regular Cartesian coordinates $ad\tilde{x}^0 = dx^0 = cdt$, $ad\tilde{x}^1 = dx^1$, $ad\tilde{x}^2 = dx^2$, $ad\tilde{x}^3 = dx^3$, which are more suitable for the calculation of the components of the fundamental metric tensor, thus

manifesting the forces acting in the space better. As a result, we obtain Gödel's metric in the form

$$ds^2 = c^2 dt^2 + 2e^{\frac{x^1}{a}} c dt dx^2 - (dx^1)^2 + \frac{e^{\frac{2x^1}{a}}}{2} (dx^2)^2 - (dx^3)^2. \quad (5.2)$$

As is seen from this form of Gödel's metric,

$$g_{00} = 1, \quad g_{02} = e^{\frac{x^1}{a}}, \quad g_{01} = g_{03} = 0, \quad (5.3)$$

thus implying that such a space is free of gravitation, but rotates with a three-dimensional linear velocity v_i (determined by g_{0i}) whose only nonzero component is v_2 . The velocity v_2 (actually, the component g_{02}) manifests the cosine of the angle of inclination of the line of time $x^0 = ct$ to the spatial axis $x^2 = y$. Therefore the lines of time are non-orthogonal to the spatial axis at each single point of a Gödel space, owing to which local time-like geodesics are the elements of big circles (self-closing time-like geodesics) therein. The nonzero v_2 also means that the shift of the whole three-dimensional space along the axis draws a big circle. This velocity, according to the definition of v_i (2.6) provided by the chronometrically invariant formalism, is

$$v_2 = -ce^{\frac{x^1}{a}}, \quad (5.4)$$

which, obviously, does not depend on time. Therefore, in the space of Gödel's metric, the second (inertial) term of the gravitational inertial force F_i (2.1) is zero as well as the first (gravitational) term. The metric is also free of deformation: the spatial components g_{ik} of the fundamental metric tensor do not depend on time therein.

As a result, we see that no one of the factors changing the mass of a mass-bearing particle according to the scalar geodesic equation (whose factors are gravitation, non-holonomy, and deformation of space) is present in the space of Gödel's metric. We therefore conclude that mass-bearing particles do not achieve mass-defect with the distance travelled in a Gödel universe.

§6. Cosmological mass-defect in the space of Schwarzschild's metric of a sphere of incompressible liquid. This is the internal space of a sphere filled, homogeneously, with an incompressible liquid. The preliminary form metric of such a space was introduced in 1916 by Karl Schwarzschild [15]. He however limited himself to the assumption that the three-dimensional components of the fundamental metric tensor should not possess breaking (discontinuity). The general form of this metric, which is free of this geometric limitation, was deduced in 2009

by Larissa Borissova: see formula (3.55) in [16], or (1.1) in [17]. It is

$$ds^2 = \frac{1}{4} \left(3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)^2 c^2 dt^2 - \frac{dr^2}{1 - \frac{\varkappa \rho_0 r^2}{3}} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (6.1)$$

where $\varkappa = \frac{8\pi G}{c^2}$ is Einstein's gravitational constant, $\rho_0 = \frac{M}{V} = \frac{3M}{4\pi a^3}$ is the density of the liquid, a is the sphere's radius, and r is the radial coordinate from the central point of the sphere. The metric manifests that such a space is free of rotation and deformation. Only gravitation affects mass-bearing particles therein. It is determined by

$$g_{00} = \frac{1}{4} \left(3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)^2. \quad (6.2)$$

Respectively, the gravitational inertial force (2.1) in the space of the generalized Schwarzschild metric of a sphere of incompressible liquid has the following nonzero components

$$F_1 = - \frac{c^2 \varkappa \rho_0 r}{3 \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \left(3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)}, \quad (6.3)$$

$$F^1 = - \frac{c^2 \varkappa \rho_0 r \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}}{3 \left(3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)}, \quad (6.4)$$

while the remaining components of the force are zero, because, as is seen from the metric (6.1), the component g_{00} , which determines the force, is only dependent on the radial coordinate $x^1 = r$.

Thus the scalar geodesic equation for a mass-bearing particle (1.1) takes the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_1 v^1 = 0, \quad (6.5)$$

where $v^1 = \frac{dr}{d\tau}$, while F_1 is determined by (6.3). This equation transforms, obviously, into $d \ln m = \frac{1}{c^2} F_1 dr$, thus

$$d \ln m = - \frac{\varkappa \rho_0 r}{3 \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}} \frac{dr}{\left(3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)}. \quad (6.6)$$

Meanwhile,

$$d \left(3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right) = \frac{\varkappa \rho_0 r}{3} \frac{dr}{\sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}}, \quad (6.7)$$

therefore the initial equation transforms into

$$d \ln m = - d \ln \left(3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right), \quad (6.8)$$

which solves as

$$m = m_0 \frac{3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - 1}{3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}}. \quad (6.9)$$

Because the world-density is quite small, $\rho_0 \approx 10^{-29}$ gram/cm³ or even less than it, and Einstein's gravitational constant is very small as well, $\varkappa = \frac{8\pi G}{c^2} = 1.862 \times 10^{-27}$ cm/gram, the obtained solution (6.9) at distances much smaller than the radius of such a universe ($r \ll a$), takes the simplified form

$$m = m_0 \left(1 - \frac{\varkappa \rho_0 r^2}{12} \right). \quad (6.10)$$

As such, mass-defect in a spherical universe filled with incompressible liquid is negative. The magnitude of the negative mass-defect increases with distance from the observer, eventually taking the ultimately high numerical value at the event horizon. Hence, this is definitely a true instance of cosmological effects. We will therefore further refer to this effect as the *cosmological mass-defect*.

In other words, the more distant an object we observe in such a universe is, the less is its observed mass in comparison to its real rest-mass measured near this object.

If our Universe would be a sphere of incompressible liquid, the mass-defect would be negligible within our Galaxy "Milky Way" (because ρ_0 and \varkappa are very small). However, it would become essential at distances of even the closest galaxies: an object located as distant as the Andromeda Galaxy ($r \simeq 780 \times 10^3$ pc $\simeq 2.4 \times 10^{24}$ cm) would have a negative cosmological mass-defect equal, according to the linearized solution (6.10), to $\frac{\varkappa \rho_0 r^2}{12} \approx 10^{-8}$ of its true rest-mass m_0 .

At the ultimate large distance in such a universe, which is the event horizon $r = a$, the obtained solution (6.9) manifests the ultimately high

mass-defect

$$m = m_0 \frac{3 \sqrt{1 - \frac{\kappa \rho_0 a^2}{3}} - 1}{2 \sqrt{1 - \frac{\kappa \rho_0 a^2}{3}}}. \quad (6.11)$$

§7. Cosmological mass-defect in the space of a sphere filled with physical vacuum (de Sitter's metric). Such a space was first considered in 1917 by Willem de Sitter [18,19]. It contains no substance, but is filled with a spherically symmetric distribution of physical vacuum (λ -field). Its curvature is constant at each point: this is a constant-curvature space. Its metric, introduced by de Sitter, is

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (7.1)$$

which contains the λ -term of Einstein's equations. Such a space is as well free of rotation and deformation, while gravitation is only determined by the λ -term

$$g_{00} = 1 - \frac{\lambda r^2}{3}. \quad (7.2)$$

Respectively, the sole nonzero components of the gravitational inertial force (2.1) in such a space are

$$F_1 = \frac{\lambda c^2}{3} \frac{r}{1 - \frac{\lambda r^2}{3}}, \quad F^1 = \frac{\lambda c^2}{3} r, \quad (7.3)$$

while the remaining ones are zero: the component g_{00} , which determines gravitation, in de Sitter's metric (7.1) is dependent only on the radial coordinate $x^1=r$. This is a non-Newtonian gravitational force which is due to the λ -field (physical vacuum). Its magnitude increases with distance: if $\lambda < 0$, this is a force of attraction, if $\lambda > 0$ this is a force of repulsion.

Thus the scalar geodesic equation for a mass-bearing particle (1.1) in this case has the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_1 v^1 = 0, \quad (7.4)$$

where $v^1 = \frac{dr}{d\tau}$, with F_i determined by (7.3). It transforms, obviously, into $d \ln m = \frac{1}{c^2} F_1 dr$, which is

$$d \ln m = \frac{\lambda r}{3} \frac{dr}{1 - \frac{\lambda r^2}{3}}. \quad (7.5)$$

Because

$$d \ln \left(1 - \frac{\lambda r^2}{3} \right) = -\frac{2\lambda r}{3} \frac{dr}{1 - \frac{\lambda r^2}{3}}, \quad (7.6)$$

the initial equation takes the form

$$d \ln m = -\frac{1}{2} d \ln \left(1 - \frac{\lambda r^2}{3} \right), \quad (7.7)$$

which solves as

$$m = \frac{m_0}{\sqrt{1 - \frac{\lambda r^2}{3}}}. \quad (7.8)$$

Because, according to astronomical estimates, the λ -term is quite small as $\lambda \leq 10^{-56} \text{ cm}^{-2}$, at small distances this solution becomes

$$m = m_0 \left(1 + \frac{\lambda r^2}{6} \right). \quad (7.9)$$

As is seen from the obtained solution, a positive mass-defect should be observed in a de Sitter universe: the more distant the observed object therein is, the greater is its observed mass in comparison to its real rest-mass measured near the object. The magnitude of this effect increases with distance with respect to the object under observation. In other words, this is another *cosmological mass-defect*.

For instance, suppose our Universe to be a de Sitter world. Consider an object, which is located at the distance of the Andromeda Galaxy ($r \simeq 780 \times 10^3 \text{ pc} \simeq 2.4 \times 10^{24} \text{ cm}$). In this case, with $\lambda \leq 10^{-56} \text{ cm}^{-2}$ and according to the linearized solution (7.9), the mass of this object registered in our observation should be greater than its true rest-mass m_0 for a value of $\frac{\lambda r^2}{6} \leq 10^{-8}$. However, at the event horizon $r \approx 10^{28} \text{ cm}$, which is the ultimately large distance observed in our Universe according to the newest data of observational astronomy, the magnitude of the mass-defect, according to the obtained exact solution (7.8), is expected to be very high, even approaching infinity.

Therefore, the one of experimenta crucis answering the question “is our Universe a de Sitter world or not?” would be a substantially high positive mass-defect of distant galaxies and quasars.

§8. No mass-defect present in the space of a sphere filled with ideal liquid and physical vacuum (Einstein’s metric).

This cosmological solution was introduced by Albert Einstein in his famous presentation [20], held on February 8, 1917, wherein he introduced relativistic cosmology. This solution implies a closed spherical space, which is

filled with homogeneous and isotropic distribution of ideal (non-viscous) liquid and physical vacuum (λ -field). It was not the first of the exact solutions of Einstein's equations, found by the relativists, but the *first cosmological model* — this metric was suggested (by Einstein) as the most suitable model of the Universe as a whole, answering the data of observational astronomy known in those years. The metric of such a space, known also as Einstein's metric, has the form

$$ds^2 = c^2 dt^2 - \frac{dr^2}{1 - \lambda r^2} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (8.1)$$

which is similar to de Sitter's metric (7.1), with only the difference being that Einstein's metric has $g_{00} = 1$ and there is no numerical coefficient $\frac{1}{3}$ in the denominator of g_{11} . Herein $\lambda = \frac{4\pi G\rho}{c^2}$, i.e. the cosmological λ -term has the opposite sign compared to that of Gödel's metric.

As is seen, in Einstein's metric,

$$g_{00} = 1, \quad g_{01} = g_{02} = g_{03} = 0, \quad (8.2)$$

thus implying that such a space is free of gravitation and rotation. It is also not deforming: the three-dimensional components g_{ik} do not depend on time therein. So, the metric contains no one of the factors changing the mass of a mass-bearing particle according to the scalar geodesic equation. This means that mass-bearing particles do not achieve mass-defect with the distance travelled in the space of Einstein's metric.

§9. Cosmological mass-defect in the deforming spaces of Friedmann's metric. This space metric was introduced in 1922 by Alexander Friedmann as a class of non-stationary solutions to Einstein's equations aimed at generalizing the static homogeneous, and isotropic cosmological model suggested in 1917 by Einstein. Spaces of Friedmann's metric can be empty, or filled with a homogeneous and isotropic distribution of ideal (non-viscous) liquid in common with physical vacuum (λ -field), or filled with one of the media. In a particular case, it can be dust. This is because the energy-momentum tensor of ideal liquid transforms into the energy-momentum tensor of dust by removing the term containing pressure (in this sense, dust behaves as pressureless ideal liquid).

Friedmann's metric in the spherical three-dimensional coordinates has the form

$$ds^2 = c^2 dt^2 - R^2 \left[\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right], \quad (9.1)$$

where $R = R(t)$ is the curvature radius of the space, while $\kappa = 0, \pm 1$ is the curvature factor*. In the case of $\kappa = -1$, the four-dimensional space curvature is negative: this manifests an open three-dimensional space of the hyperbolic type. The case of $\kappa = 0$ yields zero curvature (flat three-dimensional space). If $\kappa = +1$, the four-dimensional curvature is positive, giving a closed three-dimensional space of the elliptic type.

The non-static cosmological models with $\kappa = +1$ and $\kappa = -1$ were considered in 1922 by Friedmann in his primary publication [21] wherein he pioneered non-stationary solutions of Einstein's equations, then in 1924, in his second (last) paper [22]. However, the most popular among the cosmologists is the generalized formulation of Friedmann's metric, which contains all three cases $\kappa = 0, \pm 1$ of the space curvature as in (9.1). It was first considered in 1925 by Georges Lemaître [23, 24], who did not specify κ , then in 1929 by Howard Percy Robertson [25], and in 1937 by Arthur Geoffrey Walker [26]. Friedmann's metric in its generalized form (9.1) containing $\kappa = 0, \pm 1$ is also conventionally known as the Friedmann-Lemaître-Robertson-Walker metric.

A short note about the dimensionless radial coordinate r used in Friedmann's metric (9.1). In a deforming (expanding or compressing) space, the regular coordinates change their scales with time. In particular, if the space deforms as any expanding or compressing spherical space, the regular radial coordinate will change its scale. To remove this problem, Friedmann's metric is regularly expressed through a "homogeneous" radial coordinate r as in (9.1)[†]. It comes as the regular radial coordinate (circumference measured on the hypersphere), which is then divided by the curvature radius whose scale changes with time accordingly. As a result, the homogeneous radial coordinate r ("reduced" circumference) does not change its scale with time during expansion or compression of the space.

Let's have a look at Friedmann's metric (9.1). We see that

$$g_{00} = 1, \quad g_{0i} = 0, \quad g_{ik} = g_{ik}(t), \quad (9.2)$$

hence, such a space is free of gravitation and rotation, while its three-dimensional subspace deforms. Therefore, the scalar geodesic equation

*This form of Friedmann's metric, containing the curvature factor κ , was introduced due to the independent studies conducted by Lemaître [23, 24] and Robertson [25], following Friedmann's death in 1925.

[†]Sometimes, Cartesian coordinates are more reasonable for the purpose of calculation. In this case, Friedmann's metric is expressed through the "homogeneous" Cartesian coordinates, which are derived in the same way from the regular Cartesian coordinates, and which are also dimensionless. See Zelmanov's book on cosmology [4] and his paper [5], for instance.

(1.1) for a mass-bearing particle which travels in the space of Friedmann's metric (we assume that it travels along the radial coordinate r with respect to the observer) takes the form

$$\frac{dm}{d\tau} + \frac{m}{c^2} D_{11} v^1 v^1 = 0, \quad (9.3)$$

where $v^1 = \frac{dr}{d\tau}$ [sec⁻¹], while only the space deformation along the radial coordinate, which is D_{11} , affects the mass of the particle during its motion. According to Friedmann's metric, $d\tau = dt$ due to $g_{00} = 1$ and $g_{0i} = 0$. Thus the scalar geodesic equation (9.3) transforms into

$$d \ln m = -\frac{1}{c^2} D_{11} \dot{r}^2 dt. \quad (9.4)$$

Unfortunately, this equation, (9.4), cannot be solved alone, as well as the scalar geodesic equation in any deforming space: the deformation term of the equation contains the velocity of the particle which is unknown and is determined by the space metric. We find the velocity from the vectorial geodesic equation (1.2), which for a mass-bearing particle travelling in the radial direction r in the space of Friedmann's metric (9.1) takes the form

$$\frac{dv^1}{d\tau} + \frac{1}{m} \frac{dm}{d\tau} v^1 + 2D_1^1 v^1 + \Delta_{11}^1 v^1 v^1 = 0. \quad (9.5)$$

To remove m from the vectorial geodesic equation (9.5), we make a substitution of the scalar equation (9.3). We obtain a second-order differential equation with respect to r , which has the form

$$\ddot{r} + 2D_1^1 \dot{r} + \Delta_{11}^1 \dot{r}^2 - \frac{1}{c^2} D_{11} \dot{r}^3 = 0. \quad (9.6)$$

According to the definitions of D_{ik} (2.3) and Δ_{ik}^i (2.4), we calculate D_{11} , D_1^1 , and Δ_{11}^1 in the space of Friedmann's metric. To do it, we use the components of the chr.inv.-metric tensor h_{ik} (2.7) calculated according to Friedmann's metric (9.1). After some algebra, we obtain

$$h_{11} = \frac{R^2}{1 - \kappa r^2}, \quad h_{22} = R^2 r^2, \quad h_{33} = R^2 r^2 \sin^2 \theta, \quad (9.7)$$

$$h = \det \|h_{ik}\| = h_{11} h_{22} h_{33} = \frac{R^6 r^4 \sin^2 \theta}{1 - \kappa r^2}, \quad (9.8)$$

$$h^{11} = \frac{1 - \kappa r^2}{R^2}, \quad h^{22} = \frac{1}{R^2 r^2}, \quad h^{33} = \frac{1}{R^2 r^2 \sin^2 \theta}. \quad (9.9)$$

As a result, we obtain, in the general case of an arbitrary space of Friedmann's metric,

$$D_{11} = \frac{R}{1 - \kappa r^2} \frac{\partial R}{\partial t} = \frac{R \dot{R}}{1 - \kappa r^2}, \quad D_1^1 = \frac{\dot{R}}{R}, \quad D = \frac{3\dot{R}}{R}, \quad (9.10)$$

$$\Delta_{11}^1 = \frac{\kappa r}{1 - \kappa r^2}, \quad (9.11)$$

thus our equation (9.6) takes the form

$$\ddot{r} + \frac{2\dot{R}}{R} \dot{r} + \frac{\kappa r}{1 - \kappa r^2} \dot{r}^2 - \frac{R\dot{R}}{c^2(1 - \kappa r^2)} \dot{r}^3 = 0. \quad (9.12)$$

This equation is non-solvable being considered in the general form as here. To solve this equation, we should simplify it by assuming particular forms of the functions κ and $R = R(t)$.

The curvature factor κ can be chosen very easily: with $\kappa = 0$ we have a deforming flat universe, $\kappa = +1$ describes a deforming closed universe, while $\kappa = -1$ means a deforming open universe.

The curvature radius as a function of time, $R = R(t)$, appears due to that fact that the space deforms. This function can be found through the tensor of the space deformation D_{ik} , whose trace

$$D = h^{ik} D_{ik} = \frac{{}^* \partial \ln \sqrt{h}}{\partial t} = \frac{1}{\sqrt{h}} \frac{{}^* \partial \sqrt{h}}{\partial t} = \frac{1}{V} \frac{{}^* \partial V}{\partial t} \quad (9.13)$$

yields the speed of relative deformation (expansion or compression) of the volume of the space element [4, 5]. The volume of a space element, which plays the key rôle in the formula, is calculated as follows. A parallelepiped built on the vectors $r_{(1)}^i, r_{(2)}^i, \dots, r_{(n)}^i$ in an n -dimensional Euclidean space has its volume calculated as $V = \pm \det \|r_{(n)}^i\| = \pm |r_{(n)}^i|$. We thus have an invariant $V^2 = |r_{(n)}^i| |r_{(m)i}| = |r_{(n)}^i| |h_{ik} r_{(m)}^k| = |h_{ik} r_{(n)}^i r_{(m)}^k|$, where $h_{ik} \equiv -g_{ik}$ according to Euclidean geometry. Thus, we obtain $(dV)^2 = |h_{ik} dx_{(n)}^i dx_{(m)}^k| = |h_{ik}| |dx_{(n)}^i| |dx_{(m)}^k| = h |dx_{(n)}^i| |dx_{(m)}^k|$. Finally, we see that the volume of a differentially small element of an Euclidean space is calculated as $dV = \sqrt{h} |dx_{(n)}^i|$. Extending this method into a Riemannian space such as the basic space (space-time) of the General Theory of Relativity, we obtain $dV = \sqrt{-g} |dx_{(\nu)}^\alpha|$. In particular, the volume of a three-dimensional (spatial) differentially small element therein is $dV = \sqrt{h} |dx_{(n)}^i|$, or, if the parallelepiped's edges meet the (curved) spatial coordinate axes, $dV = \sqrt{h} dx^1 dx^2 dx^3$. The total volume of an extended space element is a result of integration of dV along all three spatial coordinates. Thus, in an arbitrary three-dimensional space, which

is a subspace of the entire space-time, we obtain

$$D = \frac{{}^*\partial \ln \sqrt{h}}{\partial t} = \frac{1}{\sqrt{h}} \frac{{}^*\partial \sqrt{h}}{\partial t} = \frac{1}{V} \frac{{}^*\partial V}{\partial t} = \gamma \frac{1}{a} \frac{{}^*\partial a}{\partial t} = \gamma \frac{u}{a}, \quad (9.14)$$

where a is the radius of the extended volume ($V \sim a^3$), u is the linear velocity of its deformation (positive in the case of expansion, and negative in the case of compression), and $\gamma = \text{const}$ is the shape factor of the space ($\gamma = 3$ in the homogeneous isotropic models [4, 5]).

Taking this formula into account, I would like to introduce two main types of the corresponding space deformation, and two respective types of the function $R = R(t)$. They are as follows.

A constant-deformation (homotachydioncotic) universe. Each single volume V of such a universe, including its total volume and differential volumes, undergoes equal relative changes with time*

$$D = \frac{1}{V} \frac{{}^*\partial V}{\partial t} = \gamma \frac{u}{a} = \text{const}. \quad (9.15)$$

If such a universe expands, the linear velocity of the expansion increases with time. This is an accelerated expanding universe. In contrast, if such a universe compresses, the linear velocity of its compression decreases with time: this is a decelerated compressing universe.

In spaces of Friedmann's metric, $D = \frac{3\dot{R}}{R}$ (9.10). Once $\frac{\dot{R}}{R} = A = \text{const}$ that means $D = \text{const}$, we have $\frac{1}{R} dR = A dt$ that means $d \ln R = A dt$. As a result, denoting $R_0 = a_0$, we obtain that

$$R = a_0 e^{At}, \quad \dot{R} = a_0 A e^{At} \quad (9.16)$$

in this case. Substituting the solutions into the general formulae (9.10), we obtain that, in a constant deformation Friedmann universe,

$$D = \frac{3\dot{R}}{R} = 3A = \text{const}, \quad (9.17)$$

$$D_{11} = \frac{R\dot{R}}{1 - \kappa r^2} = \frac{a_0^2 A e^{2At}}{1 - \kappa r^2}, \quad (9.18)$$

$$D_1^1 = \frac{\dot{R}}{R} = A = \text{const}. \quad (9.19)$$

*I refer to this kind of universe as *homotachydioncotic* (ομοταχυδιογκωτικό). This term originates from *homotachydioncosis* — ομοταχυδιόγκωσης — volume expansion with a constant speed, from *όμο* which is the first part of *όμοιος* (omeos) — the same, *ταχύτητα* — speed, *διόγκωση* — volume expansion, while compression can be considered as negative expansion.

A constant speed deforming (homotachydiastolic) universe.

Such a universe deforms with a constant linear velocity* $u = \frac{\partial a}{\partial t} = \text{const.}$ As a result, the radius of any volume element changes linearly with time $a = a_0 + ut$ (the sign of u is positive in an expanding universe, and negative in the case of compression). Thus, relative change of such a volume is expressed, according to the general formula (9.14), as

$$D = \gamma \frac{u}{a_0 + ut} \simeq \gamma \frac{u}{a_0} \left(1 - \frac{ut}{a_0}\right). \quad (9.20)$$

We see that deformation of such a universe decreases with time in the case of expansion, and increases with time if it compresses.

With $D = \frac{\gamma u}{a_0 + ut}$ (9.20), because $D = \frac{3\dot{R}}{R}$ in spaces of Friedmann's metric, we arrive at the simplest equation $\frac{3}{R} dR = \frac{\gamma u}{a_0 + ut} dt$. It obviously solves, in the Friedmann case ($\gamma = 3$), as $R = a_0 + ut$. Thus we obtain

$$R = a_0 + ut, \quad \dot{R} = u. \quad (9.21)$$

As a result, substituting the solutions into the general formulae (9.10), we obtain, in a constant-speed deforming Friedmann universe,

$$D = \frac{3\dot{R}}{R} = \frac{3u}{a_0 + ut}, \quad (9.22)$$

$$D_{11} = \frac{R\dot{R}}{1 - \kappa r^2} = \frac{(a_0 + ut)u}{1 - \kappa r^2}, \quad (9.23)$$

$$D_1^1 = \frac{\dot{R}}{R} = \frac{u}{a_0 + ut}. \quad (9.24)$$

In reality, space expands or compresses as a whole so that its volume undergoes equal relative changes with time. Therefore, if our Universe really deforms — expands or compresses — it is a space of the homotachydioncotic (constant deformation) kind. Therefore, we will further consider a constant-deformation Friedmann universe as follows.

Consider the vectorial geodesic equation (9.12) in the simplest case of Friedmann universe, wherein $\kappa = 0$. This is a flat three-dimensional space which expands or compresses due to the four-dimensional curvature which, having a radius R , is nonzero. In such a Friedmann universe

*I refer to this kind of universe as *homotachydiastolic* (ομοταχυδιαστολικός). Its origin is *homotachydiastoli* — ομοταχυδιαστολή — linear expansion with a constant speed, from *όμο* which is the first part of *όμοιος* — the same, *ταχύτητα* — speed, and *διαστολή* — linear expansion (compression is the same as negative expansion).

($\kappa = 0$, $D = 3A = \text{const}$), while taking into account that under the condition of constant deformation we have $R = a_0 e^{At}$ and $\dot{R} = a_0 A e^{At}$ (9.16), the vectorial geodesic equation (9.12) takes the most simplified form

$$\ddot{r} - \frac{a_0^2 A e^{2At}}{c^2} \dot{r}^3 + 2A\dot{r} = 0. \quad (9.25)$$

Let's introduce a new variable $\dot{r} \equiv p$. Thus $\ddot{r} = \frac{dr}{dt} \frac{dp}{dr} = pp'$, where $p' = \frac{dp}{dr}$. Thus re-write the initially equation (9.25) with the new variable. We obtain

$$pp' - \frac{a_0^2 A e^{2At}}{c^2} p^3 + 2Ap = 0. \quad (9.26)$$

Assuming that $p \neq 0$, we reduce this equation by p . We obtain

$$p' - \frac{a_0^2 A e^{2At}}{c^2} p^2 + 2A = 0. \quad (9.27)$$

By introducing the denotations $a = -\frac{a_0^2 A e^{2At}}{c^2}$ and $b = -2A$ we transform this equation into the form

$$p' + ap^2 = b. \quad (9.28)$$

This is Riccati's equation: see Kamke [27], Part III, Chapter I, §1.23. We assume a natural condition that $ab = \frac{2a_0^2 A^2 e^{2At}}{c^2} > 0$. The solution of Riccati's equation under $ab > 0$, and with the initially conditions $\xi \equiv r(t_0)$ and $\eta \equiv \dot{r}_0 = \dot{r}(t_0)$, is

$$\dot{r} = p = \frac{\dot{r}_0 \sqrt{ab} + b \tanh \sqrt{ab} (r - r_0)}{\sqrt{ab} + a \dot{r}_0 \tanh \sqrt{ab} (r - r_0)}, \quad (9.29)$$

where we immediately assume $r(t_0) = 0$ and $\dot{r}_0 = \dot{r}(t_0) = 0$, then extend the variables a and b according to our denotations. We obtain

$$\dot{r} = \frac{br \tanh \sqrt{ab}}{\sqrt{ab}} = \frac{\sqrt{2} cr}{a_0 e^{At}} \tanh \frac{\sqrt{2} a_0 A e^{At}}{c}. \quad (9.30)$$

Let's now substitute this solution into the initial scalar geodesic equation (9.4). We obtain

$$d \ln m = -2Ar^2 \tanh^2 \left(\frac{\sqrt{2} a_0 A e^{At}}{c} \right) dt, \quad (9.31)$$

thus we arrive at an integral which has the form

$$\ln m = -2A \int r^2 \tanh^2 \left(\frac{\sqrt{2} a_0 A e^{At}}{c} \right) dt + B, \quad B = \text{const}. \quad (9.32)$$

This integral is non-solvable. We can only qualitatively study it. So... the solution should have the following form:

$$m = m_0 e^{-2A \int r^2 \tanh^2\left(\frac{\sqrt{2} a_0 A e^{At}}{c}\right) dt} \quad (9.33)$$

We see that, in an expanding Friedmann universe ($A > 0$), the particle's mass m decreases, exponentially, with the distance travelled by it. In a compressing Friedmann universe ($A < 0$), the mass increases, exponentially, according to the travelled distance. In any case, the magnitude of the mass-defect increases with distance from the object under observation. So, this is another instance of *cosmological mass-effect*.

So, we have obtained that cosmological mass-defect should clearly manifest in the space of even the simplest Friedmann metric. Experimental verification of this theoretical conclusion should manifest whether, after all, we live in a Friedmann universe or not.

The vectorial geodesic equation (9.12) with $\kappa = +1$ or $\kappa = -1$ is much more complicated than the most simplified equation (9.25) we have considered in the case of $\kappa = 0$. It leads to integrals which are not only non-solvable by exact methods, but also hard-to-analyze in the general form (without simplification). Therefore, I see two practical ways of considering cosmological mass-defect in the closed and open Friedmann universes ($\kappa = \pm 1$, respectively). First, the consideration of a very particular case of such a universe, with many simplifications and artificially determined functions. Second, the application of computer-aided numerical methods. Anyhow, these allusions are beyond the scope of this principal study.

§10. Conclusions. As is well-known, mass-defect due to the field of gravitation is regularly attributed to the generally covariant formalism, which gives a deduction of it through the conservation of the energy of a particle moving in a stationary field of gravitation [6, §88]. In other words, this well-known effect is regularly considered per se.

In contrast, the chronometrically invariant formalism manifests the gravitational mass-defect as one instance in the row of similar effects, which can be deduced as a result of integrating the scalar geodesic equation (equation of energy) of a mass-bearing particle. This new method of deduction has been suggested herein. It is not limited to the very particular case of the Schwarzschild mass-point field as is the case of the aforementioned old method. The new method can be applied to

a particle travelling in the space of any metric theoretically conceivable due to the General Theory of Relativity.

Herein, we have successfully applied this new method of deduction to the main (principal) cosmological metrics.

In the space of Schwarzschild's mass-point metric, the obtained solution coincides with the known gravitational mass-defect [6, §88] whose magnitude increases toward the gravitating body. A similar effect has been found in the space of an electrically charged mass-point (Reissner-Nordström's metric), with the difference being that there is a mass-defect due to both the gravitational and electromagnetic fields. The presence of an electromagnetic field in the mass of a particle was never considered in this fashion prior to the present study.

No mass-defect has been found in the rotating space of Gödel's metric, and in the space filled with a homogeneous distribution of ideal liquid and physical vacuum (Einstein's metric). This means that a mass-bearing particle does not achieve an add-on to its mass with the distance travelled in a Gödel universe or in an Einstein universe.

The other obtained solutions manifest a mass-defect of another sort than that in the case of the mass-point metric. Its magnitude increases with the distance travelled by the particle. Thus this mass-defect manifests itself at cosmologically large distances travelled by the particle. We therefore refer to it as the *cosmological mass-defect*.

According to the calculations presented in this study, cosmological mass-defect has been found in the space of Schwarzschild's metric of a sphere of incompressible liquid, in the space of a sphere filled with physical vacuum (de Sitter's metric), and in the deforming spaces of Friedmann's metric (empty or filled with ideal liquid and physical vacuum). In other words, a mass-bearing particle travelling in each of these spaces changes its mass according to the travelled distance.

The origin of this effect is the presence of gravitation, non-holonomy, and deformation of the space wherein the particle travels (if at least one of the factors is presented in the space): these are only three factors affecting the mass of a mass-bearing particle according to the scalar geodesic equation. In other words, a particle which travels in the field gains an additional mass due to the field's work accelerating the particle, or it loses its own mass due to the work against the field (depending on the condition in the particular space).

All these results have been obtained only due to the chronometrically invariant formalism, which has led us to the new method of deduction through integrating the scalar geodesic equation (equation of energy) of a mass-bearing particle.

Note that cosmological mass-defect — an add-on to the mass of a particle according to the travelled distance — has never been considered prior to the present study. It is, therefore, a new effect predicted due to the General Theory of Relativity.

A next step should logically be the calculation of the frequency shift of a photon according to the distance travelled by it. At first glance, this problem could be resolved very easily due to the similarity of the geodesic equations for mass-bearing particles and massless (light-like) particles (photons). However, this is not a trivial task. This is because massless particles travel in the isotropic space (home of the trajectories of light), which is strictly non-holonomic so that the lines of time meet the three-dimensional coordinate lines therein (hence the isotropic space rotates as a whole in each its point with the velocity of light). Therefore, all problems concerning massless (light-like) particles should be considered only by taking the strict non-holonomic condition of the isotropic space into account. I will focus on this problem, and on the calculation of the frequency shift of a photon according to the travelled distance, in the next paper (under preparation).

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The EGR Field Quantization

Patrick Marquet*

Abstract: In this paper, we show that the EGR curvature tensor can be quantized according to the procedure set forth by André Lichnérowicz which relies on the definition of tensor propagators. This quantization is here successfully applied to a space-time with constant curvature defined in the framework of the EGR Theory. Having then extended the initial Einstein space, it implies ipso facto the existence of a generalized cosmological constant which thereby finds here a full physical justification.

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*Postal address: 7, rue du 11 nov, 94350 Villiers/Marne, Paris, France. E-mail: patrick.marquet6@wanadoo.fr. Tel: (33) 1-49-30-33-42.

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Introduction. General Relativity and quantum field theories are still the greatest achievements of present-time physics. Although the second part of our last century has seen some significant progresses, quantization rules in a curved space (space-time) background remain a never-ending unfinished story.

To date, it seems that André Lichnérowicz remains the pioneer who first succeeded in applying the regular commutation rules to the gravitational field in a constant-curvature space. Following the standard procedure applied to the electromagnetic field in the Minkowski space, Lichnérowicz formally showed that the *varied* Riemannian curvature tensor can also be quantized in the particular case of an Einstein space with constant curvature. This essential work was published in three communications to the French Academy of Sciences [1–3]. Those were lectured at the Collège de France in Paris, during the year 1958–1959.

The quantization rules, which were formulated by Lichnérowicz, state that:

- a) The gravitational field is entirely described by the Riemann curvature tensor;
- b) By strict analogy with the electromagnetic field, the *varied* curvature tensor can be adequately quantized in the Minkowski space and by continuity in constant-curvature space.

In a curved space-time, the adopted procedure requires the use of *tensor propagators* associated with second-order differential operators (Lichnérowicz [4]). Such propagators are based on the concept of *displacement bi-tensors* and are analogous to the Green functions introduced by Bryce de Witt et al., during the same period [5]. In this paper, we will only restrict our study to the related general definitions and we invite the reader to the referred bibliography for deeper mathematical analysis.

In the EGR framework (Marquet [6]), the EGR field equations always retain a true background persistent field tensor $(t^{ab})_{\text{field}}$ that *super-*

sedes the ill-defined energy-momentum pseudo-tensor of a mass gravitational field t^{ab} required to satisfy the conservation law within the Riemannian physics. In the absence of substance (source-free field equations), the persistent field can be formally merged into a *generalized cosmological term*, thus allowing the definition of a *EGR Einstein space*.

With this preparation, we are able to extend here the procedure already developed in the Riemannian framework. The quantization rules (commutators) are next applied to a *massless varied* EGR field tensor defined within the EGR Einstein space, and by doing so, the inferred EGR second-order curvature tensor becomes symmetric. As a result, all existing differential operations still hold, and a similar commutator for the varied EGR 4th-rank tensor can be derived in the EGR constant-curvature space.

Chapter 1. Some Topics within EGR Theory

§1.1. The EGR manifold

§1.1.1. The EGR field equations

We briefly recall here our previous results needed for the clarity of this paper.

On the EGR manifold M , are defined the components of the EGR curvature tensor

$$(R^{a\cdots})_{\text{EGR}} = \partial_c \Gamma_{bf}^a - \partial_f \Gamma_{bc}^a + \Gamma_{dc}^a \Gamma_{bf}^d - \Gamma_{df}^a \Gamma_{bc}^d$$

with the EGR semi-affine connection

$$\Gamma_{ab}^d = \{^d_{ab}\} + (\Gamma_{ab}^d)_J, \quad (1.1)$$

where $\{^d_{ab}\}$ are the regular Christoffel symbols and

$$(\Gamma_{ab}^d)_J = \frac{1}{6} (\delta_a^d J_b + \delta_b^d J_a - 3g_{ab} J^d).$$

As to the physical interpretation of the vector J^a , one can refer to the explanation given in the earlier publication [7].

The EGR covariant derivative denoted hereinafter by D or $'$, applies to the metric as

$$D_a g_{bc} = \partial_a g_{bc} - \Gamma_{ba}^f g_{fc} - \Gamma_{ca}^f g_{bf} = \frac{1}{3} (J_c g_{ab} + J_b g_{ac} - J_a g_{bc}). \quad (1.2)$$

The second-order curvature tensor

$$(R_{bc})_{\text{EGR}} = \partial_a \Gamma_{bc}^a - \partial_c \Gamma_{ba}^a + \Gamma_{bc}^d \Gamma_{da}^a - \Gamma_{ba}^d \Gamma_{dc}^a \quad (1.3)$$

reveals its *non-symmetric* property*

$$(R_{ab})_{\text{EGR}} = R_{ab} - \frac{1}{2} \left(g_{ab} \nabla_e J^e + \frac{1}{3} J_a J_b \right) + \frac{1}{6} (\partial_a J_b - \partial_b J_a) \quad (1.4)$$

and leads to the *EGR Einstein tensor*

$$(G_{ab})_{\text{EGR}} = (R_{ab})_{\text{EGR}} - \frac{1}{2} \left(g_{ab} R_{\text{EGR}} - \frac{2}{3} J_{ab} \right) \quad (1.5)$$

with the *EGR curvature scalar*

$$R_{\text{EGR}} = R - \frac{1}{3} \left(\nabla_e J^e + \frac{1}{2} J^2 \right). \quad (1.6)$$

The EGR theory allows for a *vacuum persistent field* to pre-exist, which appears in the *source-free EGR field equations*

$$(G_{ab})_{\text{EGR}} = (R_{(ab)})_{\text{EGR}} - \frac{1}{2} \left(g_{ab} R_{\text{EGR}} - \frac{2}{3} J_{ab} \right) = \varkappa (t_{ab})_{\text{EGR}}, \quad (1.7)$$

where $\varkappa = \frac{8\pi\mathfrak{G}}{c^4}$ is the Einstein constant and \mathfrak{G} is the Newton constant.

When a massive (anti-symmetric) tensor $T_{ab}(\rho)$ is present on the right-hand side, we have the EGR field equations

$$(G_{ab})_{\text{EGR}} = \varkappa [T_{ab}(\rho) + (t_{ab})_{\text{EGR}}]. \quad (1.8)$$

In the EGR theory, the mass density ρ is now increased by its own gravity field precisely due to the continuity of the persistent field $(t_{ab})_{\text{EGR}}$ (Marquet [8]). The EGR formulation is therefore a theory which is capable of describing a dynamical entity (massive particle together with its gravity field), that follows a geodesic distinct from the Riemannian geodesic. Accordingly, the isotropic vectors on M are slightly modified, as we will see below.

§1.1.2. The EGR line element

On the manifold M, the *isotropic conoids* as they are defined in the Riemannian picture, do not exactly coincide with the EGR representation, because the EGR line-element slightly deviates from the standard Einstein geodesic invariant.

*We denote covariant derivative on the Riemannian manifold V_4 by ∇_a or ; while keeping denotation D_a for covariant derivative on M.

Indeed, consider the vector \mathbf{l} whose square is given by

$$\mathbf{l}^2 = g_{ab} A^a A^b. \quad (1.9)$$

Along an infinitesimal closed path, this vector will now *vary* when parallel transported according to

$$\begin{aligned} d\mathbf{l}^2 &= d(g_{ab}) A^a A^b + g_{ab} (dA^a)^\parallel A^b + g_{ab} A^a (dA^b)^\parallel = \\ &= (dg_{ab} - \Gamma_{ad}^c dx^d g_{cb} - \Gamma_{bd}^c dx^d g_{ac}) A^a A^b \end{aligned} \quad (1.10)$$

since

$$(dA^a)^\parallel = -\Gamma_{id}^a A^i dx^d \quad (1.11)$$

with the EGR semi-affine connection defined above (1.1).

From the general definition of the covariant derivative of the metric tensor (1.2)

$$D_d g_{ab} = \partial_d g_{ab} - \Gamma_{ad}^c g_{cb} - \Gamma_{bd}^c g_{ac} \quad (1.12)$$

we write the differential as

$$D g_{ab} = dg_{ab} - (\Gamma_{ad}^c g_{cb} - \Gamma_{bd}^c g_{ac}) dx^d \quad (1.13)$$

so, we have

$$d\mathbf{l}^2 = (D g_{ab}) A^a A^b, \quad (1.14)$$

$$d\mathbf{l}^2 = (D_d g_{ab}) A^a A^b dx^d. \quad (1.15)$$

The EGR line-element includes a small correction due to the Riemannian invariant ds^2

$$(ds^2)_{\text{EGR}} = ds^2 + d(ds^2). \quad (1.16)$$

Therefore, we have

$$d(ds^2) = d(g_{ab} dx^a dx^b). \quad (1.17)$$

Taking then into account (1.13), we find

$$d(ds^2) = (\partial_d g_{ab} - \Gamma_{ad,b} - \Gamma_{bd,a}) dx^a dx^b dx^d, \quad (1.18)$$

or

$$d(ds^2) = (D_d g_{ab}) dx^a dx^b dx^d \quad (1.19)$$

having noted that

$$\Gamma_{ab,i} = g_{id} \Gamma_{ab}^d. \quad (1.20)$$

Eventually

$$d(ds^2) = (D g_{ab}) dx^a dx^b \quad (1.21)$$

with $Dg_{ab} = \frac{1}{3}(J_c g_{ab} + J_b g_{ac} - J_a g_{bc}) dx^c$.

Hence, the EGR line-element is simply expressed by

$$(ds^2)_{\text{EGR}} = (g_{ab} + Dg_{ab}) dx^a dx^b, \quad (1.22)$$

which naturally reduces to the Riemannian (invariant) interval ds^2 when the covariant derivative of the metric tensor g_{ab} vanishes (i.e. in the case where $J_a = 0$).

The form of the second term (correction) is legitimate since it must exhibit the metric covariant variation that corresponds to the parallel transported variable vector, in contrast to Riemannian geometry. Thus, the EGR conoids C_{EGR}^\pm , which will be used hereinafter, do not exactly coincide with the Riemannian conoids C^\pm .

§1.2. The constant-curvature space in the EGR Theory

§1.2.1. Definitions

In the Riemannian framework, it is well known that the four-dimensional space-time metric with constant curvature is

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}) \quad (1.23)$$

with

$$K = \frac{R}{12}, \quad (1.24)$$

where R is the *constant* curvature scalar. If $K = \frac{\lambda}{3}$, where λ is the cosmological constant, the constant-curvature Riemannian manifold V_4 is the so-called *Einstein space* (see, for instance, the explanation given by L. Borissova and D. Rabounski [9], formulae 5.33–5.34). In this case, one writes

$$G_{ab} = R_{ab} = \lambda g_{ab}. \quad (1.25)$$

§1.2.2. The Einstein space in the EGR representation

In the EGR formulation, as we have seen,

$$R_{\text{EGR}} = R - \frac{1}{3} \left(\nabla_e J^e + \frac{1}{2} J^2 \right),$$

and while keeping R constant, we see that when $J^a = \text{const}$, we are guaranteed that R_{EGR} is also constant. With this choice, inspection shows that the symmetries of the EGR curvature tensors are identical to the Riemannian ones, and that the EGR second-order curvature tensor $(R_{ab})_{\text{EGR}}$ is now symmetric.

If we wish to define the EGR equivalent to the Einstein space, we must take into account the energy-momentum tensor of the persistent background field, which now reduces to the symmetric expression

$$\frac{1}{\varkappa} g_{ab} (R^{cd} R_{cd})_{\text{EGR}}. \quad (1.26)$$

In the EGR field equations when substance is absent, this term becomes purely geometric

$$g_{ab} (R^{cd} R_{cd})_{\text{EGR}}. \quad (1.27)$$

From the EGR Einstein tensor (1.5), we can then infer the new EGR second-order curvature tensor by grouping all remaining terms into the right hand side of the field equations (1.7), and we find the *symmetric EGR second rank curvature tensor* as

$$(R_{ab})_{\text{EGR}} = g_{ab} \lambda_{\text{EGR}} \quad (1.28)$$

with

$$\lambda_{\text{EGR}} = 3K_{\text{EGR}} = -\frac{1}{4} \left[\frac{1}{2} \left(R - \frac{1}{6} J^2 \right) - (R^{cd} R_{cd})_{\text{EGR}} \right], \quad (1.29)$$

where the last term of the right-hand side is assumed to be nearly constant. This equation, (1.28), can be considered as representing the EGR formulation of the classical Einstein space.

This result closely matches our earlier statement where the *prevailing term* $\frac{1}{6} g_{ab} J^2$ (see [6], formula 3.25) was regarded as generalizing the regular Riemannian term $g_{ab} \lambda$, when the persistent field is discarded.

This derivation allows one to emphasize the arbitrary introduction of the long-debated *cosmological term* λg_{ab} within the Riemannian physics, whereas the EGR theory provides a natural justification for its mere existence.

We will thus simply define the EGR space-time metric of a constant curvature K as

$$(R_{abcd})_{\text{EGR}} = K_{\text{EGR}} (g_{ac} g_{bd} - g_{ad} g_{bc}). \quad (1.30)$$

Chapter 2. Theory of Varied Fields

§2.1. Linear differential operations in the EGR framework

§2.1.1. Definitions

The varied field theory, as put forward by Lichnérowicz [10], relies on the infinitesimal finite variation of the metric tensor g_{ab} which defines a new tensor

$$\delta g_{ab} = h_{ab}, \quad (2.1)$$

$$\delta g^{ab} = -g^{ac}g^{bd}h_{cd} = -h^{ab}. \quad (2.2)$$

This fundamentally differs from the regular *linearized gravitation theory* which is based on the slight deviation

$$g_{ab} = \eta_{ab} + h_{ab}, \quad h_{ab} \ll 1,$$

and where h_{ab} is (loosely) regarded as a tensor defined in a *flat background space-time*. By contrast, the variation (2.1) takes place in the chosen manifold, and it determines the varied connections and curvature tensors which retain the same properties as their generic quantities. By doing so, the corresponding finite variations can adequately fit in the quantization process. Before detailing those derivations, we will need first to define some differential operations.

§2.1.2. The generalized EGR Laplacian

On the oriented manifold M of class C^{h+1} (always equipped with the metric g_{ab}), we consider the second-order linear differential operator Δ_{EGR} on the p -tensors, such that

$$(\Delta_{\text{EGR}} T)_{a_1 \dots a_p} = -D^b D_b T_{a_1 \dots a_p} = -g^{bc} D_b D_c T_{a_1 \dots a_p}. \quad (2.3)$$

This operator transforms any C^{k+2} tensor (where $0 \leq k \leq h-2$) into a C^k tensor.

Let now d_{EGR} and δ_{EGR} denote, respectively, the EGR exterior differential, and the EGR co-differential operators acting on forms.

The EGR differential operator d_{EGR} is built as the anti-symmetrized EGR covariant derivative and it generalizes the Riemannian curl operator

$$(d_{\text{EGR}} F)_{abc} = D_a F_{bc} + D_c F_{ab} + D_b F_{ca}.$$

The EGR co-differential operator δ_{EGR} generalizes the Riemannian divergence operator

$$(\delta_{\text{EGR}} F)_b = D^a F_{ab}.$$

In the case of *anti-symmetric tensors* we make use of the EGR Laplacian in the sense of *Georges de Rham*:

$$\Delta_{\text{EGR}} \mathbf{T} = (d_{\text{EGR}} \delta_{\text{EGR}} + \delta_{\text{EGR}} d_{\text{EGR}}) \mathbf{T}. \quad (2.4)$$

Then Δ_{EGR} commutes with d_{EGR} and δ_{EGR} , since $d_{\text{EGR}}^2 = \delta_{\text{EGR}}^2 = 0$. Explicitly, the Laplacian Δ_{EGR} is expressed with covariant derivatives

as follows

$$\begin{aligned} (\Delta_{\text{EGR}} T)_{a_1 \dots a_p} &= -D^c D_c T_{a_1 \dots a_p} + \frac{1}{(p-1)!} \varepsilon_{a_1 \dots a_p}^{db_2 \dots b_p} R_{de} T_{b_2 \dots b_p}^e - \\ &- \frac{1}{(p-2)!} \varepsilon_{a_1 \dots a_p}^{deb_2 \dots b_p} R_{dcef} T_{b_2 \dots b_p}^{cf}, \end{aligned} \quad (2.5)$$

where $\varepsilon_{a_1 \dots a_p}^{db_1 \dots b_2}$ and $\varepsilon_{a_1 \dots a_p}^{deb_2 \dots b_p}$ are the generalized Kronecker tensors which take on the numerical values:

- +1, if all indices $a_1 \dots a_p$ are distinct, and if the substitution s which makes $b_1 \dots$ to $a_1 \dots$ is pair;
- 1, if indices $a_1 \dots a_p$ are all distinct for odd s ;
- or 0, for all other cases.

Generally speaking, if we denote by C a linear operator field on tensors, and B a linear application field on the same tensors, we define the EGR differential operator

$$L\mathbf{T} = \Delta_{\text{EGR}} \mathbf{T} + B^b D_b \mathbf{T} + C\mathbf{T} \quad (2.6)$$

which transforms any tensor of class C^{k+2} into a tensor of class C^k .

The *adjoint* differential operator is thus defined by

$$L^* \mathbf{V} = \Delta_{\text{EGR}} \mathbf{V} - D_b B^{*b} \mathbf{V} + C^* \mathbf{V}$$

with

$$B^{*b} = -B^b, \quad C^* = C - D_b B^b.$$

The EGR Laplacian of an arbitrary tensor \mathbf{T} defined by (2.5) has the following properties:

- a) It is self-adjoint;
- b) It commutes with all contractions and with all index transpositions.

Furthermore, if \mathbf{T} has zero covariant derivative,

$$\Delta_{\text{EGR}}(\mathbf{T} \otimes \mathbf{V}) = \mathbf{T} \otimes \Delta_{\text{EGR}} \mathbf{V}$$

for any 2-tensor \mathbf{T} and vector \mathbf{A} , we have

$$\delta \Delta_{\text{EGR}} \mathbf{T} = \Delta_{\text{EGR}} \delta \mathbf{T}, \quad D \Delta_{\text{EGR}} \mathbf{A} = \Delta_{\text{EGR}} D \mathbf{A}.$$

Therefore, for an anti-symmetric tensor \mathbf{T} of rank 2, we have

$$\begin{aligned} (\Delta_{\text{EGR}} T)_{ab} &= -D^c D_c T_{ab} + (R_a^d)_{\text{EGR}} T_{db} + \\ &+ (R_b^d)_{\text{EGR}} T_{ad} - 2(R_{acbe})_{\text{EGR}} T^{ce}. \end{aligned} \quad (2.7)$$

This last relation will be useful in discussing further results related to symmetric propagators.

§2.2. EGR curvature tensor variations

§2.2.1. The EGR curvature 4th-rank tensor variation

Let us consider the EGR manifold that reduces to the EGR Einstein equations

$$(R_{ab})_{\text{EGR}} = g_{ab} \lambda_{\text{EGR}} \quad (2.8)$$

with the following obvious constraint

$$\delta(R_{ab})_{\text{EGR}} = \lambda_{\text{EGR}} h_{ab}. \quad (2.9)$$

We are now going to evaluate the corresponding variations of the general connection

$$\delta\Gamma_{ab}^c = W_{ab}^c \quad (2.10)$$

so, we first calculate

$$\delta V_{abd} = \frac{1}{2} (\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}), \quad (2.11)$$

$$\delta V_{abd} = \frac{1}{2} (D_a h_{bd} + D_b h_{ad} - D_d h_{ab}) + h_{de} \Gamma_{ab}^e, \quad (2.12)$$

hence

$$W_{ab}^c = \delta g^{cd} V_{abd} + \frac{1}{2} (D_a h_b^c + D_b h_a^c - D^c h_{ab}) + h_e^c \Gamma_{ab}^e, \quad (2.13)$$

that is

$$W_{ab}^c = \frac{1}{2} (D_a h_b^c + D_b h_a^c - D^c h_{ab}) + h_e^c \Gamma_{ab}^e - h_d^c \Gamma_{ab}^d.$$

Eventually, we find

$$W_{ab}^c = \frac{1}{2} (D_a h_b^c + D_b h_a^c - D^c h_{ab}). \quad (2.14)$$

Now setting $W_{cab} = g_{cd} W_{ab}^d$, we have

$$W_{cab} = \frac{1}{2} (D_a h_{bc} + D_b h_{ac} - D_c h_{ab}). \quad (2.15)$$

The variation of the EGR tensor components (2.9) expressed with the tensor W_{cab} is then given by

$$\delta(R_{bcf}^a)_{\text{EGR}} = D_c W_{bf}^a - D_f W_{bc}^a. \quad (2.16)$$

Let us now evaluate the varied tensors

$$\delta(R_{abcf})_{\text{EGR}} = H_{abcf}, \quad \delta(R^{abcf})_{\text{EGR}} = H^{abcf}. \quad (2.17)$$

Since we are here in the EGR Einstein space-time picture, inspection shows that all Riemannian symmetries are also satisfied by the EGR tensor $(R_{abcf})_{\text{EGR}}$ and equally hold for H_{abcf} , i.e.

$$H_{abcf} = H_{bacf} = -H_{abfc} = H_{cfab}. \quad (2.18)$$

If Ξ denotes the summation after circular permutation on indices, this tensor also satisfies the identity

$$\Xi H_{abcf} = 0 \quad (2.19)$$

and the Bianchi identity

$$\Xi D_e H_{abcf} = 0. \quad (2.20)$$

Let B_{abcf} be an arbitrary tensor of the 4th rank; we introduce the denotation ${}^0\Sigma$ which acts on B_{abcf} as

$${}^0\Sigma B_{abcf} = B_{abcf} + B_{bfac} - B_{bcfa} - B_{afbc}.$$

§2.2.2. Relation of the tensor h_{ab} with R_{abcd}

We first evaluate the tensor

$$M_{abfg}(\underline{h}) = \delta(R_{abfg}) \quad (2.21)$$

which verifies (2.18–2.19), and where $\underline{h} \Leftrightarrow h_{ab}$.

Let us then introduce the symmetric tensor $P_{hkcd}(\underline{h})$:

$${}^0\Sigma D_d D_k h_{eh} = D_d D_k h_{he} + D_h D_e h_{kd} - D_e D_d h_{hk} - D_k D_e h_{hd}. \quad (2.22)$$

We can show that

$$2M_{abfg}(\underline{h}) = -P_{abfg} - h_{ac}(R_{\cdot bfg}^{c\cdot\cdot})_{\text{EGR}} - h_{bc}(R_{a\cdot fg}^{\cdot c\cdot\cdot})_{\text{EGR}} \quad (2.23)$$

so that we can infer the components of another tensor Q_{abfg}

$$Q_{abfg}(\underline{h}) = -P_{abfg} - h_{fc}(R_{ab\cdot g}^{\cdot\cdot\cdot c})_{\text{EGR}} - h_{gc}(R_{abf\cdot}^{\cdot\cdot\cdot c})_{\text{EGR}} \quad (2.24)$$

uniquely expressed as a function of both the h_{ab} and the EGR curvature tensor $(R_{abcd})_{\text{EGR}}$.

After a lengthy tedious calculation, one finds

$$Q_{abfg}(\underline{h}) = M_{abfg}(\underline{h}) + g_{ac}g_{bh}g_{fk}g_{ge}\delta(R^{chke})_{\text{EGR}}(\underline{h}). \quad (2.25)$$

The quantity $Q_{abfg}(\underline{h})$ will play a major role in view of quantizing the EGR gravitational field.

We will also need to evaluate $Q_{abfg}(\mathcal{DA})$ with $\underline{h} = \mathcal{DA}$, where the *extended Lie derivative* of the metric tensor with respect to the arbitrary vector \mathbf{A} is given according to Marquet [11]:

$$(\mathcal{DA})_{ab} = A_{b;a} + A_{a;b} + g_{ab} (g^{ik} D_c g_{ik} A^c). \quad (2.26)$$

Let us denote by \mathcal{L} the *extended Lie derivative operator* and we write

$$\mathcal{DA} = \mathcal{L}_A \mathbf{g}. \quad (2.27)$$

Respectively, $M_{abfg}(\mathcal{DA})$ can be shown to be the EGR Lie derivative of the vector $(R_{abfg})_{\text{EGR}}$ with respect to the vector \mathbf{A} . We have

$$M_{abfg}(\mathcal{DA}) = \mathcal{L}_A (R_{abfg})_{\text{EGR}} = A^d D_d (R_{abfg})_{\text{EGR}} + {}^0\Sigma D_a A_d (R_{\cdot bfg}^{d\cdot\cdot\cdot})_{\text{EGR}}.$$

Hence, for the tensor Q_{abfg} (2.25), we have

$$Q_{abfg}(\mathcal{DA}) = 2A^d D_d (R_{abfg})_{\text{EGR}} + {}^0\Sigma (D_a A_d - D_d A_a) (R_{\cdot bfg}^{d\cdot\cdot\cdot})_{\text{EGR}} \quad (2.28)$$

and after a further tedious calculation, we obtain

$$\begin{aligned} D_k Q_{abfg}(\underline{h}) &= 2\delta D_k (R_{abfg})_{\text{EGR}} + \\ &+ {}^0\Sigma (D_a h_{dk} - D_d h_{ak}) (R_{\cdot bfg}^{d\cdot\cdot\cdot})_{\text{EGR}} - {}^0\Sigma h_{ad} D_k (R_{\cdot bfg}^{d\cdot\cdot\cdot})_{\text{EGR}}. \end{aligned} \quad (2.29)$$

§2.2.3. Second-order curvature tensor variation

The relevant variation of the EGR tensor $(R_{ab})_{\text{EGR}}$ is

$$\delta(R_{bf})_{\text{EGR}} = D_d W_{bf}^d - D_f W_{db}^d. \quad (2.30)$$

Taking account of (2.15), one may write

$$2\delta(R_{ab})_{\text{EGR}} = g^{de} D_d (D_a h_{be} + D_b h_{ae} - D_e h_{ab}) - D_a D_b h, \quad (2.31)$$

where we set

$$h = g^{de} h_{de}.$$

Considering the *Ricci identity* within the EGR framework, applied to the tensor h^{ab}

$$h_{b',ea}^e - h_{b',ae}^e = (R_{ad})_{\text{EGR}} h_b^d - h^{ed} (R_{aebd})_{\text{EGR}},$$

one deduces for (2.31):

$$\begin{aligned} 2\delta(R_{ab})_{\text{EGR}} &= -D^e D_e h_{ab} + (R_a^d)_{\text{EGR}} h_{db} + (R_b^d)_{\text{EGR}} h_{ad} - \\ &- 2(R_{aebd})_{\text{EGR}} h^{ed} + (D_a D_e h_b^e + D_b D_e h_a^e - D_a D_b h). \end{aligned} \quad (2.32)$$

And, together with formula (2.26), this leads to

$$2\delta(R_{ab})_{\text{EGR}} = \Delta_{\text{EGR}} h_{ab} + (\mathcal{D}\mathbf{k})_{ab}, \quad (2.33)$$

where the vector $\mathbf{k}(\underline{h})$ has components

$$k_a(\underline{h}) = D_d h_a^d - \frac{1}{2} D_a h. \quad (2.34)$$

From (2.33), the contraction yields

$$g^{de} \delta(R_{de})_{\text{EGR}} = \frac{1}{2} \Delta_{\text{EGR}} h + D_a k^a(\underline{h}). \quad (2.35)$$

For the *EGR Einstein space* (1.28), eventually holds the following relation

$$D^a \left[\delta(R_{ab})_{\text{EGR}} - \frac{1}{2} g_{ab} g^{ed} \delta(R_{ed})_{\text{EGR}} \right] = \lambda_{\text{EGR}} k_b(\underline{h}) \quad (2.36)$$

which could be formally derived from the variation of the conservation identity of the EGR Einstein tensor (1.5) reduced here to its *symmetric* version. This is an important result, as (2.36) precisely matches the equivalent Riemannian relation derived by Lichnérowicz.

Such an equivalence lends strong support to the EGR theory, thus appearing as a legitimate generalization of the classical General Relativity in the varied field formulation.

With the EGR *symmetric* second-order curvature tensor variation still being bound to the condition

$$\delta(R_{ab})_{\text{EGR}} = \lambda_{\text{EGR}} h_{ab}, \quad (2.37)$$

inspection shows that this equation is invariant upon the EGR gauge transformation

$$\underline{h}' \rightarrow \underline{h} + \mathcal{D}\mathbf{A}, \quad (2.38)$$

where \mathbf{A} is, as usual, an arbitrary infinitesimal vector. This is certainly true, provided the vector J_a is constant, which is indeed the case according to (2.37).

LEMMA (LICHNÉROWICZ)

For the *EGR Einstein space*, we have

$$(\Delta_{\text{EGR}} - 2\lambda_{\text{EGR}}) A_b = - (D^a D_a A_b + \lambda_{\text{EGR}} A_b).$$

As a result of Lichnérowicz' lemma, a formal calculation leads to

$$(\Delta_{\text{EGR}} - 2\lambda_{\text{EGR}}) \mathbf{A} = \mathbf{k}(\underline{h})$$

so, with the constraint $\mathbf{k}(\underline{h}) = 0$ which is the initial condition, we have

$$(\Delta_{\text{EGR}} - 2\lambda_{\text{EGR}}) \mathbf{A} = 0,$$

and we then eventually obtain the field equations for \underline{h} which take the form

$$(\Delta_{\text{EGR}} - 2\lambda_{\text{EGR}}) \underline{h} = 0. \quad (2.39)$$

Chapter 3. Quantizing Varied Fields

§3.1. Tensor propagators

§3.1.1. Displacement bi-tensors

Tensor propagators have been introduced in order to generalize the scalar propagator on a curved manifold. Indeed, in an Euclidian space, the quantum field theory makes an intensive use of Fourier's transform. In a curved space-time, this transform no longer applies and therefore an alternate theory developed by Lichnérowicz, can be adequately substituted, which is based on the so-called concept of *displacement bi-tensors*.

On the differentiable manifold M , we consider a point x' located in the neighbourhood of another point x . Along the EGR geodesic connecting x' to x , can be defined a *displacement* which represents a *canonical isomorphism* (base-independence) of the space T_x at x tangent to the manifold onto the tangent space $T_{x'}$ at x' .

The free bases $e_a(x)$ and $e_{c'}(x')$ are attached to those neighbourhoods. The relevant isomorphism therefore defines a *bi-tensor* denoted by \mathbf{t} which is named *displacement tensor* and whose components are labeled t'_a .

For further analysis and subsequent properties, it is useful to refer to our earlier publication [12].

In the foregoing, we will restrict our study to *massless fields* only.

§3.1.2. Elementary kernels and propagators

In the most general manner, the definition of any commutator requires the analytic description of the isotropic EGR conoids (see §1.1.2).

For this, always on the manifold M , we denote by $(C_{x'})_{\text{EGR}}$ the characteristic EGR conoid with apex x' and wherefrom are generated the EGR geodesics.

This regular point x' belongs to the compact subset Ω , *neighbourhood homeomorphic* to the *Euclidean open ball*, that is, the tangent vector space T_x at x' .

Herein the subset Ω exhibits three regions: *future* I^+ of x' , *past* I^- of x , and *elsewhere*. The first two regions characterize *two* temporal domains $(C_{x'}^\pm)_{\text{EGR}}$, (compact sets), which correspond to the subdivision of $(C_{x'})_{\text{EGR}}$ in two half conoids, one oriented towards the future, the other towards the past.

With the following considerations being purely *local*, it can be shown that there exist two p -tensors satisfying

$$L_x^* \mathbf{E}^{(p)\pm}(x, x') = \delta^{(p)}(x, x'). \quad (3.1)$$

The $\mathbf{E}^{(p)\pm}(x, x')$ are two elementary solutions called *elementary kernels* of L on $\Omega \times \Omega$, and which, for each x' , do have their supports respectively in $I^+(x')$ and $I^-(x')$.

One may then define in $(C_{x'})_{\text{EGR}}$ the *EGR p -tensor*

$$\mathbf{E}^{(p)}(x, x') = \mathbf{E}^{(p)+}(x, x') - \mathbf{E}^{(p)-}(x, x') \quad (3.2)$$

which is by definition the tensor propagator associated with the operator L . In the Minkowski space, the scalar propagator $\mathbf{E}^{(0)}$ reduces to the *Jordan-Pauli propagator* denoted by \mathbf{D} .

§3.1.3. Propagators associated with the operator $\Delta_{\text{EGR}} + \mu$

Letting μ be a constant, the operator $\Delta_{\text{EGR}} + \mu$ acts on anti-symmetric tensors of rank p .

Anti-symmetrizing the *kernels* $\mathbf{E}^{(p)\pm}$, we obtain *two* unique solutions $\mathbf{G}^{(p)\pm}$ (p -forms), which satisfy for each x' and x , the partial derivative equations

$$[(\Delta_x)_{\text{EGR}} + \mu] \mathbf{G}^{(p)\pm}(x, x') = \delta^{(p)}(x, x'), \quad p = 0, 1, \dots, n$$

with support respectively *in* and *on* $C_{x'}^+$ and $C_{x'}^-$.

Likewise, for each x , these kernels define *two* solutions near x' within Ω

$$[(\Delta_{x'})_{\text{EGR}} + \mu] \mathbf{G}^{(p)\pm}(x', x) = \delta_{(p)}(x', x).$$

The difference

$$\mathbf{G}^{(p)}(x, x') = \mathbf{G}^{(p)+}(x', x) - \mathbf{G}^{(p)-}(x', x) \quad (3.3)$$

defines the *anti-symmetric propagator* associated with the operator $\Delta_{\text{EGR}} + \mu$, which is a solution of

$$[(\Delta_x)_{\text{EGR}} + \mu] \mathbf{G}^{(p)}(x, x') = 0. \quad (3.4)$$

By symmetrizing the elementary kernels $\mathbf{E}^{(2)\pm}(x', x)$ (limited to order 2) related to our operator, one notes the emergence of *two symmetric kernels* $\mathbf{K}^\pm(x', x)$ which are symmetric 2-tensors satisfying at x

$$[(\Delta_x)_{\text{EGR}} + \mu] \mathbf{K}^\pm(x', x) = \mathbf{t} \delta(x, x') \quad (3.5)$$

with \mathbf{t} having the components

$$t_{abc'd'} = t_{ac'} t_{bd'} + t_{ad'} t_{bc'} , \quad (3.6)$$

which means that the symmetrization operation was applied to the 2-tensor. We therefore call the *symmetric propagator* related to $\Delta_{\text{EGR}} + \mu$ the symmetric 2-tensor defined by

$$\mathbf{K}(x, x') = \mathbf{K}^+(x', x) - \mathbf{K}^-(x', x) . \quad (3.7)$$

§3.2. Commutation rules

§3.2.1. Electromagnetic field in the Minkowski space

The potential 1-form \mathbf{A} induces an electromagnetic field \mathbf{F} according to the equations

$$\mathbf{F} = d\mathbf{A} , \quad d\mathbf{A} = A_b \wedge dx^b . \quad (3.8)$$

which are invariant under the gauge transformation

$$A_b \longrightarrow A'_b = A_b + \partial_b U .$$

Classically, with our notations used so far, we express the *commutator* for the *potential* in the form (see [13], formula 11.27)

$$[\mathbf{A}(x), \mathbf{A}(x')] = -\frac{\hbar}{i} \{ \mathbf{tD}(x, x') \} , \quad (3.9)$$

where $\hbar = \frac{h}{2\pi}$, the mass term is characterized by $\mu = 0$, and the Jordan-Pauli propagator \mathbf{D} is related to the regular Laplace operator which satisfies the following conditions

$$\Delta A = 0 , \quad \delta A = 0 . \quad (3.10)$$

Taking this result into account, the commutator (3.9) is written

$$[\mathbf{A}(x), \mathbf{A}(x')] = -\frac{\hbar}{i} \{ \mathbf{G}^{(1)}(x, x') \} \quad (3.11)$$

which leads, for the electromagnetic field \mathbf{F} , to the commutator

$$[\mathbf{F}(x), \mathbf{F}(x')] = -\frac{\hbar}{i} \{ d_x d_{x'} G^{(1)}(x, x') \} . \quad (3.12)$$

Short inspection shows that this commutator is compatible with the regular Maxwell equations

$$d\mathbf{F} = 0, \quad \delta\mathbf{F} = 0, \quad (3.13)$$

and $\delta\mathbf{A} = 0$ once some initial conditions have been applied.

§3.2.2. Commutator for the varied EGR second-order curvature tensor

By a strict analogy, we have the evident correspondence

$$\begin{aligned} \Delta_{\text{EGR}} \underline{h} = 0 &\longrightarrow \mathbf{A} = 0, \\ \mathbf{k}(\underline{h}) = 0 &\longrightarrow \delta\mathbf{A} = 0, \end{aligned}$$

so, we are led to adopt the commutator for \underline{h}

$$[\underline{h}(x), \underline{h}(x')] = \frac{\mathfrak{G}\hbar}{i c^2} \{ \mathbf{K}(x, x') - \mathbf{g}(x) \mathbf{g}(x') \mathbf{G}^{(0)}(x, x') \}, \quad (3.14)$$

where the propagators are related to the operator $(\Delta_{\text{EGR}} - 2\lambda_{\text{EGR}})$, and $\mathbf{g} = g_{ab} dx^a \otimes dx^b$.

§3.3. Quantization in the constant-curvature space

§3.3.1. Commutator for higher-order fields

In the Minkowski space with metric tensor η_{ab} , we use here a system of orthonormal basis. It is interesting to evaluate the commutator (3.14) as applied to the field \underline{h} , for the tensor H_{abcd} (2.17) which verifies the equations (2.18–2.19).

The commutator for H_{abcd} is classically given by

$$\begin{aligned} [H_{abcd}(x), H_{efgh}(x')] = & \\ = \frac{\mathfrak{G}\hbar}{4i c^2} \{ & ({}^0\Sigma \eta_{bf} \partial_e \partial_a) ({}^0\Sigma \eta_{dh} \partial_g \partial_e) + ({}^0\Sigma \eta_{df} \partial_e \partial_c) ({}^0\Sigma \eta_{bh} \partial_g \partial_a) - \\ & - ({}^0\Sigma \eta_{fh} \partial_e \partial_g) ({}^0\Sigma \eta_{bd} \partial_c \partial_a) \} \mathbf{D}_{(0)}(x, x'). \end{aligned} \quad (3.15)$$

In an arbitrary basis system and after changing the indices, a lengthy calculation shows that the term in the brackets can be split up into two following parts. The first part is

$${}^0\Sigma D_{g'} D_{e'} {}^0\Sigma D_c D_a [t_{bf'} t_{dh'} + t_{df'} t_{bh'}] \mathbf{D}_{(0)}, \quad (3.16)$$

i.e.

$$\mathbf{Q}(x') \mathbf{Q}(x) \mathbf{K}(x, x'), \quad (3.17)$$

where Q_x is the operator Q defined above, acting on the tensors of rank 2 and which is defined at the point x .

The second part is

$${}^0\Sigma D_{g'} D_{e'} {}^0\Sigma D_c D_a (g_{f'h'} g_{bd}) \mathbf{D}_{(0)}(x, x'), \quad (3.18)$$

i.e.

$$Q_{x'} Q_x \mathbf{g}(x) \mathbf{g}(x') \mathbf{D}_{(0)}. \quad (3.19)$$

Eventually, we obtain for an arbitrary basis

$$[\mathbf{H}(x), \mathbf{H}(x')] = \frac{\mathfrak{G}\hbar}{4ic^2} Q_{x'} Q_x \{ \mathbf{K}(x, x') - \mathbf{g}(x) \mathbf{g}(x') \mathbf{D}_{(0)}(x, x') \}, \quad (3.20)$$

where $\mathbf{D}_{(0)}$ and \mathbf{K} are respectively the scalar and symmetric propagators of rank 2 associated with the operator Δ_{EGR} .

§3.3.2. The EGR constant-curvature space

We now consider a curved manifold specialized to the EGR space with a constant curvature as defined in (1.30), in which case we make use of the results of §2.2.

From the derived relation (2.28), one infers

$$Q(\mathcal{D}\mathbf{A}) = 0 \quad (3.21)$$

for any vector \mathbf{A} . Moreover, from (2.29), for any symmetric tensor \underline{h} , we have

$$\Xi D_k Q_{abfg}(\underline{h}) = 0. \quad (3.22)$$

Consider the commutator (3.14): using the operators Q_x and $Q_{x'}$ and taking account of (3.21), we get

$$[Q\underline{h}(x), Q\underline{h}(x')] = \frac{\mathfrak{G}\hbar}{ic^2} Q_x Q_{x'} \{ \mathbf{K}(x, x') - \mathbf{g}(x) \mathbf{g}(x') \mathbf{G}^{(0)} \}. \quad (3.23)$$

Setting

$$H_{abfg}(\underline{h}) = \frac{1}{2} Q_{abfg}(\underline{h})$$

which has the same properties, we obtain the EGR commutator

$$[\mathbf{H}(x), \mathbf{H}(x')] = \frac{\mathfrak{G}\hbar}{4ic^2} Q_x Q_{x'} \{ \mathbf{K}(x, x') - \mathbf{g}(x) \mathbf{g}(x') \mathbf{G}^{(0)} \}, \quad (3.24)$$

which is formally the extension of the commutator (3.20) established for any arbitrary basis, in the Minkowski space.

Thus, the theory elaborated for the Minkowski space has been successfully generalized to the EGR constant-curvature space.

Conclusion. In the previous exposition, we have only sketched the full theory of Lichnérowicz which actually thoroughly covers the massive field commutators among which the *Fierz commutators* (spin 2 fields) are formally generalized to the Riemannian Einstein spaces.

Generally speaking, the definition of commutators leads to a physical description of the quantized *varied* gravitational field represented by a 4th-rank tensor.

The important work of Lichnérowicz has proven essential for the initial knowledge of this Riemannian quantization technique even if it is restricted to a constant-curvature space.

Performing a similar derivation within the extended Einstein space explicitly shows that the EGR field 4th-rank tensor, when *varied*, fits in the same quantization pattern.

In addition, the EGR Einstein space necessarily implies the natural existence of a generalized cosmological constant which is arbitrarily introduced in the Riemannian framework.

This natural constant, however, remains a particular case, since in the EGR theory, such a cosmological term is variable as it is intrinsically part of the relevant geometry inherent to the theory.

All these tend once more to confirm that the extended General Relativity — the EGR theory suggested in [6] — is a viable model that offers and justifies broad new perspectives in physics.

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On a $c(t)$ -Modified Friedmann-Lemaître-Robertson-Walker Universe

Robert C. Fletcher*

Abstract: This paper presents a compelling argument for the physical light speed in the homogeneous and isotropic Friedman-Lemaître-Robertson-Walker (FLRW) universe to vary with the cosmic time coordinate t of FLRW. It will be variable when the radial co-moving differential coordinate of FLRW is interpreted as physical and therefore transformable by a Lorentz transform locally to differentials of stationary physical coordinates. Because the FLRW differential radial distance has a time varying coefficient $a(t)$, in the limit of a zero radial distance the light speed $c(t)$ becomes time varying, proportional to the square root of the derivative of $a(t)$. Since we assume homogeneity of space, this derived $c(t)$ is the physical light speed for all events in the FLRW universe. This impacts the interpretation of astronomical observations of distant phenomena that are sensitive to light speed. In particular, it will modify the dark energy used to explain the apparent universe acceleration. A transform from FLRW is shown to have a physical radius out to all radial events in the visible universe. This shows a finite horizon beyond which there are no galaxies and no space. The General Relativity (GR) field equation to determine $a(t)$ and $c(t)$ is maintained by using a variable gravitational constant and rest mass that keeps constant the gravitational and particle rest energies. This keeps constant the proportionality constant between the GR tensors of the field equation and conserves the stress-energy tensor of the ideal fluid used in the FLRW GR field equation. In the same way all of Special and General Relativity can be extended to include a variable light speed.

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*Bell Telephone Laboratories (retired), Murray Hill, New Jersey, USA. Mailing address (for correspondence): 1000 Oak Hills Way, Salt Lake City, UT 84108, USA. E-mail: robert.c.fletcher@utah.edu.

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§1. Introduction. Here I will use a significantly different approach than other attempts in the literature to investigate a variable speed of light. Those mostly tried to find a new cosmology to provide alternatives to inflation in order to resolve horizon and flatness problems [1–5]. In common with those approaches, the present approach is a major departure from the prevailing paradigm that the speed of light is constant. However, my calculation of a variable light speed $c(t)$ seems to be consistent with being interpreted as physical in the FLRW universe, but is a variable function of the chronometrically invariant observable constant light speed c , dependent on the specific conditions in this universe compared with the more general universes treated by A. Zelmanov [6].

I derive a variable light speed using the same assumptions used for almost a century, except that I allow for a variable light speed: 1) that light speed (even though variable) is independent of the velocity of the observers, 2) that the universe is homogeneous and isotropic, and 3) that the radial FLRW differential variables derivable for this universe represent physical time and distance. The first assumption leads to a Lorentz transform between moving observers, extended to allow a variable light speed; the second leads to the FLRW metric [7–9] that allows for a variable light speed; and the third allows us to locally apply the extended Lorentz transform from the FLRW time dt and radial distance $a(t)d\chi$ to the stationary time dT and distance dR . We show that this requires a variable physical light speed to be

$$c(t) \propto \sqrt{\frac{da}{dt}}$$

in order to be consistent with the time varying distance differential of FLRW. This is done by expanding the physical time and distance along a stationary rod in a power series of the FLRW co-moving coordinate χ and extrapolating to zero χ . We assume (fourth assumption) that the Lorentz transform remains valid from the origin out to at least the lowest power χ and therefore the lowest power of the velocity between the two frames. This derivation is fairly simple and covers only the first 8 pages of this paper. The remainder of the paper addresses the reasonableness and implications of this derivation.

We find two different systems of full radially transformed coordinates from FLRW, good for all distances, whose differentials close to the origin have a Minkowski metric. These transforms all have the same variable light speed at the origin as the power series expansion, a universality that I find persuasive.

For a homogeneous universe, since the origin can be placed on any

galactic point, this means that this variable physical light speed enters all our physical laws throughout the universe. In particular it is possible that standard candles like the supernovae Ia [10–13] and galactic clusters [15] are dimmed by the right amount by higher light speeds to provide an alternate to dark energy to explain the apparent acceleration of the universe.

To maintain unchanged the field equation of General Relativity, we assume the gravitational “constant” G to be time varying, but keep constant the proportionality function between the GR tensors of the field equation. This is done by assuming the particle rest energy and the Newton gravitational energy to be constant. This also conserves the stress-energy tensor of an ideal fluid used in the GR field equation for an FLRW universe.

We can express the gravitational field in transformed stationary coordinates using Riemannian geometry. In the region near the origin for a flat universe this field increases linearly with distance just like the Newtonian field for a spherical distribution of uniform mass density.

A surprise bonus from this endeavor is that one of the radial transforms has a physical distance to all parts of the universe. Even though three rigid accelerated axes are inadequate to describe three-dimensional motion, it is apparently possible to find one rigid axis to measure radial distance, at least for a homogeneous FLRW universe, although the transformed time on this axis becomes non-physical at large distances. This shows that in the coordinates of the rigid frame attached to the origin that the universe is contained within an expanding spherical shell outside of which there are no galactic points and no space.

I also outline in the Appendix how not only the Lorentz transform, but all of the vectors and tensors of Special Relativity can be extended to include a variable light speed so they can be used in the standard field equation of General Relativity.

§2. The derivation of $c(t)$

§2.1. Assumptions. Only four assumptions are needed for the derivation of $c(t)$. The first three are the same assumptions for Special Relativity and for the universe that have been made for almost a century. What is new is the allowance for the possibility that the physical light speed is variable. We will use “line element” to describe the invariant ds and “metric” to describe the particular differential coordinates that equal ds . We will be considering only radial motion in a spherically symmetric universe.

Assumption I. The physical light speed is the same for all co-located observers who may be moving at various velocities in an accelerating field.

From this we can derive the extended Lorentz transform (\hat{L}) between such observers, even when the light speed and velocities are variable (Appendix C.2). Each observer will have an extended Minkowski metric (\hat{M}).

Assumption II. The universe is isotropic and homogeneous in space.

From this we can derive an extended FLRW metric (Appendix C.7) that allows for a variable light speed $c(t)$, where t is the physical time on the co-moving galactic points of the FLRW solution. This derivation depends only on the assumed symmetry and not on the general relativistic field equation. χ is a co-moving radial coordinate with which a galactic point (representing a galaxy) stays constant. $a(t)$ is a universe scale factor that multiplies $d\chi$ in the metric.

Definition: “Physical” coordinates in time or distance over some interval will be defined as those that have a linear relationship to the readings on a co-located standard clock or a standard ruler, respectively. We call them physical because it describes coordinates on the rigid frame for an observer at the origin $\chi = 0$. In principle if a standard clock or ruler were at the location indicated by the physical coordinate, the coordinate would be observable. In the limit of small intervals on an inertial frame, if physical time represents clocks at the location represented by the physical distance, according to General Relativity, their differentials will have an \hat{M} metric (see Appendix C). Physical velocity is the ratio of the differential physical distance to the differential physical time, when both are located at the same space point.

Assumption III. The FLRW time and radial differentials dt and $a(t)d\chi$ are physical.

This is a usual assumption. It is reasonable since the radial motion of the FLRW metric is \hat{M} in these differentials. With this assumption the radial physical light speed is $a(\frac{\partial\chi}{\partial t})_s$, and the physical radial velocity V of a moving object, labeled R , located at t, χ is $a(\frac{\partial\chi}{\partial t})_R$.

Definition: We will use AP (almost physical) to describe spherically symmetric coordinate systems $x^\mu(R, \theta, \phi, T)$ that are transforms from the FLRW coordinates with a radial metric that approaches \hat{M} as χ approaches zero. We will attach the AP space origin to the same galactic point as $\chi = 0$, so at this point there is no motion

between them. Thus we can call the AP coordinates stationary. Since their differentials have a \hat{M} metric close to the origin, they can be \hat{L} transformed from the physical coordinates dt and $a(t)d\chi$. For a point on R , they will have contravariant vectors for velocity $U^\mu = \frac{dx^\mu}{ds}$ and acceleration $A^\mu = \frac{DU^\mu}{ds}$, whose components transform like the coordinate differentials. R is rigid in a mathematical sense because the radial component of U^μ in stationary coordinates is $(\frac{\partial R}{\partial T})_R(\frac{dT}{ds}) \equiv 0$, so the points of R are motionless with respect to each other. (It is rigid in its mathematical properties, but not in its acoustical properties). It will be helpful in finding AP transforms if we further require the AP metric be diagonal (zero coefficient of $dT dR$).

Definition: We define a generalized Hubble ratio as $H(\hat{t}) = \frac{\dot{a}}{a}$, where the dot is the $\hat{d}\hat{t} = c(t)dt$ derivative, see formula (196).

Assumption IV. The Lorentz transform between FLRW and AP radial coordinates is valid for the partial differentials of T and R from the origin out to at least the lowest power of the velocity between them.

Without this substantial assumption, a constant light speed would be allowed [17].

With these assumptions and definitions, we will show that the light speed is variable and proportional to $\dot{a} = aH$ (or equivalently to $\sqrt{\frac{da}{dt}}$) by two different procedures:

- 1) Integrate \hat{L} transformed physical differentials dT, dR in a power series in χ (see §2.2);
- 2) Find full rigid diagonal radial AP transforms T, R for all t, χ (see §2.3).

Each of these has the same variable light speed $c(t)$ in the limit of $\chi \rightarrow 0$. The first shows this for any and all AP transforms for an expansion of T_t that is internally consistent to the second power of χ as required for Lorentz to be applicable. The second shows this for a large number of full radial AP transforms which have an \hat{M} metric close to the origin. Thus, the first is a completeness proof that if there are such transforms, they must have this $c(t)$, and the second is an existence proof that there are such transforms with $c(t)$ at the origin, and that the expansion of the first is further justified for being internally consistent to the second power of χ .

Additional assumptions are needed to apply this variable physical light speed to physical laws. We will use the following:

Assumption V. We assume the Bernal criteria [16] that two observers will be using the same units of measure when each measures the other's differential units at the same space-time point compared to their own and finds these cross measurements to be equal.

We will find a radial AP transform (T, R) called physical distance coordinates whose differential dR is physical for all distances by virtue of this assumption (see §2.3 and Appendix A.2).

Assumption VI. We assume that the Einstein field equations can be maintained unchanged for $c(t)$ by assuming a gravitation “constant” that varies as $c(t)^4$. This keeps constant the proportionality function between the GR field tensors, see formula (176).

The effect of $c(t)$ is introduced by an extended metric and an extended conserved stress-energy tensor (Appendix C.5). The extended FLRW metric solves the extended GR field equation for an ideal fluid. A well-behaved transform will also be a solution since Riemann tensors are invariant to transforms. The solution allows us to calculate $a(t)$ and $c(t)$ and galactic and photon paths on the AP frame for a homogeneous and isotropic universe with a variable light speed (see §3 and §4).

Assumption VII. We assume that the molecular spectra of particles are constant.

Thus, we keep constant the fine structure constant and Rydberg frequency by making the vacuum electric and magnetic ‘constants’ vary inversely with $c(t)$. This also allows us to redefine electro-magnetic field vectors to maintain Maxwell's equations (Appendix C.6).

§2.2. Variable light speed $c(t)$ required for a transform that is Lorentz close to the origin.

§2.2.1. Extended Lorentz transform from galactic points to the stationary inertial frame using the velocity V between them. We will consider only radial world lines with physical coordinates T and R on the AP inertial frame. We would like these to describe the same events as the FLRW coordinates t and χ (Appendix C.7), so $T = T(t, \chi)$ and $R = R(t, \chi)$ with $R = 0$ at $\chi = 0$. So

$$\left. \begin{aligned} dT &= T_t dt + T_\chi d\chi = \frac{1}{c} T_t d\hat{t} + T_\chi d\chi \\ dR &= R_t dt + R_\chi d\chi = \frac{1}{c} R_t d\hat{t} + R_\chi d\chi \end{aligned} \right\}, \quad (1)$$

where the subscripts indicate partial derivatives with respect to the subscript variable, and where we use $d\hat{t} = c(t) dt$ (see Appendix C). We will

find $T = T(t, \chi)$ and $R = R(t, \chi)$ by integrating the differentials of the Lorentz transform for a short distance. We assume the \hat{M} metric applies to physical differential times and distances of limited size anywhere and anytime. The FLRW metric in (194) has a radial Minkowski-like metric with $dT^* \rightarrow dt$ and $dR^* \rightarrow a d\chi$ that we have assumed are physical. If a point on the AP frame is moving at a radial velocity $V(t, \chi)$ when measured with the FLRW coordinates, the \hat{L} transform of $dt, a d\chi$ to dT, dR for a radial path keeps the line element ds invariant (154):

$$\left. \begin{aligned} dT &= \gamma(t, \chi) \left(dt - \frac{V(t, \chi)}{c(t)^2} a(t) d\chi \right) \\ dR &= \gamma(t, \chi) (-V(t, \chi) dt + a(t) d\chi) \end{aligned} \right\}. \quad (2)$$

If we compare (2) with (1), we get

$$T_t = \gamma, \quad (3)$$

$$T_\chi = -\gamma a \frac{V}{c^2} = -\gamma a \hat{V} c, \quad (4)$$

$$R_t = -\gamma V = -\gamma c \hat{V}, \quad (5)$$

$$R_\chi = \gamma a, \quad (6)$$

where for simplification we have introduced $\hat{V} \equiv \frac{V}{c}$. These relations are exact for differentials as $\chi \rightarrow 0$, and therefore are approximately correct when the differentials are integrated for small χ at constant t . We can rearrange the two expressions for \hat{V} to give

$$\hat{V} = -\frac{a R_t}{c R_\chi} = -\frac{c T_\chi}{a T_t}. \quad (7)$$

With (3), (6), and (7) this gives two relations each for dT and dR in terms of \hat{V} . When we integrate these partial differential equations, we integrate dT, dR along the R frame but integrate the $dt, d\chi$ along a radial connection between the co-moving galactic points χ . Because the radial differential changes with time, $V(t, \chi)$ changes with time and distance. We will find this combination requires $c(t)$ to vary with t in a determined way, at least for the short distance from the origin where a power series is valid. When there is no acceleration, and V is a function only of χ in an expanding universe, $c(t)$ will be constant (see Appendix A.5).

§2.2.2. Power series in χ determines $c(t)$. To obtain $T(t, \chi)$ and $R(t, \chi)$ near the origin, we need to integrate the differentials dT and dR

for small χ . We will do this by expanding these physical coordinates in a power series in χ out to the lowest power that will give a non-trivial $c(t)$ in the limit of zero χ . We will use the two relations for dR to determine the expansion coefficients of R and \hat{V} , then use the resultant expansion of \hat{V} in the two relations for dT to expand T and determine the requirement for $c(t)$.

Since \hat{V} will vanish at the origin (see definitions in §2.1), the constant in the power series for \hat{V} is zero; so let

$$-\hat{V} = w_1(t)\chi + w_2(t)\chi^2 + w_3(t)\chi^3 + O(\chi^4) \dots \quad (8)$$

where the $w_i(t)$ are unknown functions to be determined. From $R_\chi = a\gamma$ (6) we get

$$\begin{aligned} R_\chi &= a \left(1 + \frac{1}{2} \hat{V}^2 + \frac{3}{8} \hat{V}^4 + \dots \right) = \\ &= a \left(1 + \frac{1}{2} w_1^2 \chi^2 + w_1 w_2 \chi^3 + O(\chi^4) \right) \dots \end{aligned} \quad (9)$$

If we integrate (9) at constant t , noting that R vanishes at $\chi=0$ (see definitions in §1), we obtain

$$R = a\chi + \frac{1}{6} a w_1^2 \chi^3 + \frac{1}{4} a w_1 w_2 \chi^4 + O(\chi^5) \dots \quad (10)$$

Herein $R(t, \chi)$ is the physical differential dR summed over all the galactic points up to χ , and is thus the physical distance to χ at time t . The first term of (10) is the “proper” distance to which all measurements of distance reduce close to the origin [9].

Partial differentiation of (10) by t at constant χ gives

$$R_t = c\dot{a}\chi + \frac{1}{6} \chi^3 \frac{d}{dt}(a w_1^2) + \frac{1}{4} \chi^4 \frac{d}{dt}(a w_1 w_2) + O(\chi^5) \dots, \quad (11)$$

where the dot represents the derivative with respect to \hat{t} . We can then find \hat{V} from equations (7), (9), and (11):

$$-\hat{V} = \frac{aR_t}{cR_\chi} = \dot{a}\chi + f(t)\chi^3 + O(\chi^4) \dots, \quad (12)$$

where

$$f(t) = -\frac{1}{2} w_1^2 \dot{a} + \frac{1}{6c} \frac{d}{dt}(a w_1^2). \quad (13)$$

By comparison of (12) with (8), we see that $w_1 = \dot{a}$, $w_2 = 0$, and $w_3 = f(t)$.

We will now use this expression for \hat{V} to find two relations for T_t . The first comes from $T_t = \gamma$ (3):

$$T_t = 1 + \frac{1}{2} \hat{V}^2 = 1 + \frac{1}{2} \dot{a}^2 \chi^2 + O(\chi^4) \dots, \quad (14)$$

Even though dT and dt are both measured on standard physical clocks, we note that the galactic clocks t measured at constant χ run slower than the AP clocks T as they move away from the origin (dilation). When measured at constant R , the AP clocks run slower, $t_T = 1 + \frac{V^2}{2}$, in accordance with the Lorentz transform. Neither of these apply if we don't carry out the power series to the second power of χ . Of course the distance contraction is also consistent with Lorentz, $\frac{R_x}{a} = a\chi_R = 1 + \frac{\hat{V}^2}{2}$ (9).

We can find an expression for T_χ , using (7), (14), and (12):

$$T_\chi = -\frac{a}{c} T_t \hat{V} = \frac{a}{c} \left[1 + \frac{1}{2} \dot{a}^2 \chi^2 + O(\chi^4) \dots \right] \times \\ \times \left[\dot{a} \chi + f(t) \chi^3 + O(\chi^4) \dots \right], \quad (15)$$

and multiplying the brackets gives

$$T_\chi = \frac{a}{c} \left[\dot{a} \chi + \frac{1}{2} \dot{a}^3 \chi^3 + f(t) \chi^3 + O(\chi^4) \dots \right]. \quad (16)$$

By integration with χ at constant t with $T = t$ at $\chi = 0$ we find

$$T = t + \frac{1}{2} \frac{a \dot{a}}{c} \chi^2 + O(\chi^4) \dots \quad (17)$$

If we partially differentiate (17) by t , we get a second expression for T_t :

$$T_t = 1 + \frac{1}{2} \chi^2 \frac{d}{dt} \left(\frac{a \dot{a}}{c} \right) + O(\chi^4) \dots \quad (18)$$

If the transform is to have a Lorentz transform close to the origin, we must have the two expressions for T_t (equations 14 and 18) agree to at least the 2nd power of χ (i.e., the second power of \hat{V}). This leads to a differential equation that determines a variable $c(t)$ given by

$$\dot{a}^2 = \frac{d}{dt} \left(\frac{a \dot{a}}{c(t)} \right). \quad (19)$$

Also mathematically, when we regard $c(t)$ as a variable to be determined by the limiting process of $\chi \rightarrow 0$, we must keep the term in χ^2

since it is the lowest term that determines $c(t)$, which we have therefore called non-trivial. (In the previous publication [17] the author showed that a transform with physical distance can be found for a constant light speed that leads to a $T_t \propto \frac{\hat{V}^2}{4}$, and therefore is not consistent with a Lorentz transform and is valid for only a smaller range of physicality.)

To get an explicit expression for $c(t)$, multiply (19) by a , change the variable dt to $da = \dot{a}c(t)dt$ to yield

$$\frac{da}{a} = \frac{c}{a\dot{a}} d\left(\frac{a\dot{a}}{c}\right). \quad (20)$$

One can see that $c \propto \dot{a}$ is a solution, so

$$\frac{c(t)}{c_0} = \frac{\dot{a}(\hat{t})}{\dot{a}(t_0)} = \alpha E, \quad (21)$$

where α is the normalized scale factor

$$\alpha \equiv \frac{a}{a_0}, \quad (22)$$

and E is the normalized Hubble ratio $H(\hat{t})$

$$E \equiv \frac{H}{H_0} = \frac{1}{H_0} \frac{\dot{a}}{a}. \quad (23)$$

The subscript 0 denotes the value at $t=t_0$, the present time. We can take c_0 to be unity, so that $c(t)$ would be measured in units of c_0 , but for most equations in this paper I will retain c_0 for clarity. The field equation (see §3) will enable us to evaluate α and E and thus $c(t)$.

§2.3. Variable light speed $c(t)$ derived from radial AP transforms (defined in §2.1).

§2.3.1. Procedure for finding radial AP transforms using the velocity V . We would now like to find radial AP transforms that will hold for all values of the FLRW coordinates and reduce to the physical coordinates for small distances from the origin. The most general line element for a time dependent spherically symmetric (i.e., isotropic) line element (see Weinberg [9, p. 335]) is

$$ds^2 = c^2 A^2 dT^2 - B^2 dR^2 - 2cCdTdR - F^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (24)$$

where A , B , C , and F are implicit function of T and R , but explicit functions of t and χ . We are using the same notation for time and distance as we did for the physical coordinates, but understand that they

may be physical only for small distances from the origin. We have included the physical light speed $c(t)$ in the definition of the coefficient of dT .

We will look for transformed coordinates which have their origins on the same galactic point as $\chi=0$, so $R=0$ when $\chi=0$, where there will be no motion between them, $\hat{V}=0$, and where T is t , since the time on clocks attached to every galactic point is t , including the origin. We will use the same angular coordinates as FLRW and make $F=ar$ to correspond to the FLRW metric, but will find only radial transforms where the angular differentials are zero. Of course, full four dimensional transforms to time and three rigid axes have not been found, nor are they required to determine $c(t)$. They have only to meet the requirement of becoming \hat{L} close to the origin. By definition radial AP transforms do this.

Then R and T will be functions of only t and χ : $T=T(t, \chi)$ and $R=R(t, \chi)$, and we will still have (1). Let us consider a radial point at R in the AP system. When measured from the FLRW system $(\chi, 0, 0, \hat{t})$, it will be moving at a velocity given by

$$V = a(t) \left(\frac{\partial \chi}{\partial t} \right)_R = c \hat{V}. \quad (25)$$

This velocity will be the key variable that will enable us to obtain radial AP transforms of the full radial coordinates. We will now find the components of the contravariant velocity vector $U^{\hat{t}} = \frac{d\hat{t}}{ds}$ of a point on the R axis in both the FLRW coordinates and the AP coordinates. To get the time component in FLRW coordinates $\chi, \theta, \phi, \hat{t}$ we divide the FLRW metric (197) by $d\hat{t}^2$ with $d\omega=0$ to obtain

$$\left(\frac{ds}{d\hat{t}} \right)^2 = 1 - a(t)^2 \left(\frac{d\chi}{d\hat{t}} \right)^2 = (1 - \hat{V}^2) \equiv \frac{1}{\gamma^2}. \quad (26)$$

To get the spatial component, we use the chain rule applied to formulae (25) and (26):

$$\frac{d\chi}{ds} = \frac{d\chi}{d\hat{t}} \frac{d\hat{t}}{ds} = \frac{\hat{V}}{a} \gamma. \quad (27)$$

For AP coordinates R, θ, ϕ, T , the radial component of the contravariant velocity vector is zero (see definitions under Assumption III). The point is not moving in those coordinates; that is, the radial component is rigid. This means that a test particle attached to the radial coordinate will feel a force caused by the gravitational field, but will be constrained not to move relative to the coordinate. Alternatively, a co-located free particle at rest relative to the radial point will be acceler-

ated, but will thereafter not stay co-located.

The AP time component of the velocity vector is $\frac{dT}{ds} = \frac{1}{cA}$. This makes the vector $U^\mu = \frac{dx^\mu}{ds}$ in the AP coordinates

$$U^\mu = \left(0, 0, 0, \frac{1}{cA} \right) \quad (28)$$

and in the FLRW coordinates

$$U^\mu = \left(\frac{\gamma \hat{V}}{a}, 0, 0, \gamma \right). \quad (29)$$

To make it contravariant, its components must transform the same as dT, dR in (1):

$$\left. \begin{aligned} \frac{1}{cA} &= \frac{1}{c} T_t \gamma + \frac{1}{a} T_\chi \gamma \hat{V} \\ 0 &= \frac{1}{c} R_t \gamma + \frac{1}{a} R_\chi \gamma \hat{V} \end{aligned} \right\}. \quad (30)$$

Manipulating the second line of (30) gives

$$\hat{V} = -\frac{a R_t}{c R_\chi}. \quad (31)$$

If we invert (1), we get

$$\left. \begin{aligned} d\hat{t} &= \frac{1}{D} (R_\chi dT - T_\chi dR) \\ d\chi &= \frac{1}{D} \left(-\frac{1}{c} R_t dT + \frac{1}{c} T_t dR \right) \end{aligned} \right\}, \quad (32)$$

where

$$D = \frac{1}{c} T_t R_\chi - \frac{1}{c} R_t T_\chi = \frac{1}{c} T_t R_\chi \left(1 + \hat{V} \frac{c T_\chi}{a T_t} \right) \quad (33)$$

using (31). We can enter $d\hat{t}$ and $d\chi$ of (32) into the FLRW metric (194) to obtain coefficients of dT^2 , dR^2 and $dTdR$. One way to make ds^2 invariant is to equate these coefficients to those of (24):

$$A^2 = \frac{1}{T_t^2} \frac{1 - \hat{V}^2}{\left(1 + \hat{V} \frac{c T_\chi}{a T_t} \right)^2}, \quad (34)$$

$$B^2 = \frac{a^2}{R_\chi^2} \frac{1 - \left(\frac{c T_\chi}{a T_t} \right)^2}{\left(1 + \hat{V} \frac{c T_\chi}{a T_t} \right)^2}, \quad (35)$$

and

$$C = -\frac{a}{T_t R_\chi} \frac{\hat{V} + \frac{cT_\chi}{aT_t}}{\left(1 + \hat{V} \frac{cT_\chi}{aT_t}\right)^2}. \quad (36)$$

If we put $ds = 0$ in (24), we obtain a coordinate velocity of light v_p :

$$\frac{v_p}{c} = \left(\frac{\partial R}{\partial \hat{T}}\right)_S = -\frac{C}{B^2} \pm \sqrt{\left(\frac{C}{B^2}\right)^2 + \frac{A^2}{B^2}}. \quad (37)$$

We need to remember that the $c(t)$ in these equations is the physical light speed assumed for the FLRW metric.

The equations for A , B , and v_p simplify for a diagonal metric ($C = 0$). Then (36) becomes

$$\frac{cT_\chi}{aT_t} = -\hat{V} \quad (38)$$

and (34), (35), and (37) become

$$A = \frac{\gamma}{T_t} = \frac{t_T}{\gamma}, \quad (39)$$

$$B = \frac{a\gamma}{R_\chi} = \frac{a\chi_R}{\gamma}, \quad (40)$$

$$\frac{v_p}{c} = \frac{A}{B}, \quad (41)$$

where we have used (32) with $C = 0$ to obtain the inverse partials.

Thus, rigidity gives us a relation of dR to \hat{V} (31), and diagonalization gives us a relation of dT to \hat{V} (38). If we find $\hat{V}(t, \chi)$, we can find $R(t, \chi)$ and $T(t, \chi)$ by partial integration.

This metric becomes \hat{M} when $A \rightarrow 1$, $B \rightarrow 1$, $C \rightarrow 0$ and $ar \rightarrow R$, and we get the relations in formulae (3–6) so that the transformed metric becomes \hat{M} in four dimensions. The light speed for AP coordinates differs from that of the FLRW coordinates as R increases from zero by the ratio $\frac{A}{B}$.

Even when the full physicality conditions are not met, we can say something about the physicality of the coordinates with the use of criteria (Assumption V) developed by Bernal et al. [16]. They developed a theory of fundamental units based on the postulate that two observers will be using the same units of measure when each measures the other's differential units at the same space-time point compared to their own and finds these cross measurements to be equal. Thus, if $A, B, C = A, 1, 0$, dR will be physical because $\frac{R_\chi}{a} = a\chi_R = \gamma$ (40) and

dR uses the same measure of distance as $ad\chi$, which FLRW assumes is physical. (Of course, the converse is not true; if this equality does not hold for dT , it may still be physical, but the clocks may be running slower due to gravitational time shifts; e.g., see equation 112). Similarly, if $A, B, C = 1, B, 0$, then the AP transform will have physical time.

At this point we would like to examine quantitatively how far from the \hat{M} metric our transformed metric is allowed to be in order for its coordinates to reasonably represent physical measurements. We can consider the coefficients A , B , and C one at a time departing from their value in the \hat{M} metric. For example, let us consider the physical distance case $B = 1$, $C = 0$ and examine the possible departure of the time rate in the transform from that physically measured. Then, from (39): $T_t = \frac{\gamma}{A}$, $t_T = \gamma A$. Thus, $1 - A$ represents a fractional increase from γ in the transformed time rate T_t and dT , and thus the fractional increase from physical of an inertial rod at that point. We can make a contour of constant A on our world map to give a limit for a desired physicality of the transform.

§2.3.2. Diagonal radial AP transformed coordinates have physical $c(t)$ close to the origin. We show in Appendix A that there exist an infinite number of radial AP transformed coordinate systems which satisfy the \hat{M} requirements close to the origin. Appendix A.1 derives diagonal transforms ($C = 0$) using physical time ($A = 1$) for all physical times T . These all independently show that the light speed becomes $c(t) \propto \dot{a}$ for small distances, where the transforms become Lorentz.

Appendix A.2 shows the diagonal transforms for physical distance ($B = 1$, $C = 0$) for all physical distances R . To integrate the PDEs for this transform, we need to use the GR field equation (FE). Because the equations in Appendix A.1 and Appendix A.2 are different from each other, they show, as we would expect, that it is not possible to have diagonal transforms with physical R and physical T simultaneously for all values of t, χ (except for an empty universe).

At all distances for $A = 1$, $C = 0$, the AP time T can be measured on AP physical clocks, but the AP distance R cannot be measured on physical rulers for all distances. For $B = 1$, $C = 0$, the AP distance R can be measured by physical rulers on a AP frame for all distances, but the AP time T cannot be measured by physical clocks (except for small R). We can calculate an acceleration (Appendix B) for a flat universe that is zero at the origin, and increases with distance; the physical distance R acts like you might expect for a rigid ruler on whom the surrounding

masses balance their gravitational force to zero at the origin, but develop an inward pull as the distance increases.

Appendix A.3 describes similarity solutions for both types for a flat universe ($\Omega = 1$). These solutions are very useful to display alternatives. For the physical distance transform, when we use the FE for a constant light speed [17], we get a transform that does not have the Lorentz dependence on \hat{V} . When we use the FE that allows a varying light speed, this yields a transform that has the Lorentz dependence on \hat{V} if, and only if, we use the same light speed $c(t)$ as for the power series expansion and the physical time transform. This self-consistency indicates that we are using the correct FE and the correct $c(t)$.

To summarize, we have shown that every transform that has a variation of $T_t = 1 + \frac{\hat{V}^2}{2}$, as required by a Lorentz transform close to the origin, has $c(t) \propto \sqrt{\frac{da}{dt}}$. If we do not require this variation of T_t , it is possible to find a physical distance transform with a constant c [17], although its physicality goes a much shorter distance into the universe. However, it is not possible to find a diagonal physical time transform with constant c (see Appendix A.1). Although there is no requirement that there be such a transform nor that the physical distance transform have a large range of physicality, the derived $c(t)$ has an attractive universality that can be made consistent with Special and General Relativity (see §3 and Appendix C).

§3. Extension of General Relativity to incorporate $c(t)$. We can accommodate the variable light speed $c(t)$ in the field equation of General Relativity for FLRW by allowing the gravitation “constant” G to be time varying so as to keep constant the proportionality function between the GR tensors (176). We avoid taking derivatives of $c(t)$ by using \hat{t} , where $d\hat{t} = c(t)dt$. The dependence on real time t is found by transforming the resultant solution back to t from \hat{t} . This is described in Appendix C.8. This enables us to calculate $a(t)$, $c(t)$, and trajectories in the time and distance of AP coordinates.

In (202) and (203) of the Appendix are the two significant field equations of the extended GR applied to an ideal fluid:

$$\frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} - \Lambda = \frac{8\pi G}{c^4} \rho c^2, \quad (42)$$

and

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = -\frac{8\pi G}{c^4} p, \quad (43)$$

where the dots represent derivatives with respect to \hat{t} . Following Peebles

(see [22, p. 312]), we define

$$\Omega \equiv \rho_0 \frac{8\pi G_0}{3c_0^2 H_0^2} \quad (44)$$

and

$$\Omega_r \equiv -\frac{k}{H_0^2 a_0^2} \quad (45)$$

and

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}. \quad (46)$$

For very small a there will also be radiation energy density which will not be considered in this paper.

The normalized Hubble ratio E in (23) is determined by (42):

$$\frac{1}{H_0} \frac{\dot{a}}{a} = E = \sqrt{\frac{\Omega}{\alpha^3} + \frac{\Omega_r}{\alpha^2} + \Omega_\Lambda}, \quad (47)$$

which allows us to evaluate $\frac{c(t)}{c_0} = \alpha E$. The Ω s are defined so that

$$\Omega + \Omega_r + \Omega_\Lambda = 1. \quad (48)$$

At $t = t_0$: $\alpha = 1$, $E = 1$, and $\frac{c}{c_0} = 1$.

The cosmic time t measured from the beginning of the FLRW universe (Big Bang) becomes

$$c_0 H_0 t = \int_0^\alpha \frac{c_0 d\alpha}{c \alpha E} = \int_0^\alpha \frac{d\alpha}{\alpha^2 E^2}. \quad (49)$$

For a flat universe with $\Omega = 1$ and $\Omega_r = \Omega_\Lambda = 0$:

$$\left. \begin{aligned} c_0 H_0 t &= \frac{\alpha^2}{2} \\ c_0 H_0 t_0 &= \frac{1}{2} \\ \alpha &= \left(\frac{t}{t_0}\right)^{1/2} \end{aligned} \right\}, \quad (50)$$

$$\frac{c}{c_0} = \alpha E = \alpha^{-1/2} = \left(\frac{t_0}{t}\right)^{1/4}. \quad (51)$$

For other densities with $\Omega_r = 1 - \Omega$, $\Omega_\Lambda = 0$,

$$c_0 H_0 t = \frac{\Omega}{(1 - \Omega)^2} [y - \ln(1 + y)], \quad (52)$$

where

$$y = \frac{1 - \Omega}{\Omega} \alpha. \quad (53)$$

There is no periodicity of α with t for $\Omega > 1$. The higher density decreases the time $t_0 \rightarrow \ln \frac{\Omega}{c_0 H_0} \Omega$, and the universe scale factor α continues to expand, asymptotically approaching a maximum at $\frac{\Omega}{(\Omega-1)}$. As $\Omega \rightarrow 0$, $a \rightarrow t$, $c \rightarrow c_0$, the universe becomes Minkowski (see Appendix A.5).

For experiments attempting to measure the variation of the light speed at the present time, the derivative of $c(t)$ (equation 21 with $\frac{\Omega_R}{\alpha^4} \ll 1$) will be more useful:

$$\frac{1}{c_0 H_0} \left(\frac{1}{c} \frac{dc}{dt} \right)_{t=t_0} = 1 - \frac{3}{2} \Omega - \Omega_r = -\frac{\Omega}{2} + \Omega_\Lambda. \quad (54)$$

Notice that this fraction is negative when matter dominates, and goes from zero at zero density to $-\frac{1}{2}$ at the critical universe density. A vacuum energy density opposes the gravitational effect of matter; when it dominates, the slope is an increasing function of time.

§4. Paths of galactic points and received light. Because there is a special interest in having a physical description for distance in the universe, we display the physical distance transforms. The physical distance results for flat space (Appendix A.3) are shown in Figs. 1 and 2. Here we have used the field equations with the generalized time (see §3) to derive the equations for $a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2}$ and $c(t) = c_0 \left(\frac{t_0}{t}\right)^{1/4}$.

Fig. 1 plots distance R against the time at the origin (cosmic time t) for galaxies (constant χ) and for incoming light reaching the origin at $\frac{t}{t_0} = 1$. The galactic paths are labeled with their red shift z , determined by the time t of the intersection of the photon path with the galactic path $z = -1 + \frac{c}{\alpha} = -1 + \left(\frac{t_0}{t}\right)^{3/4}$, assuming the frequency of the emitted light does not change with $c(t)$. Notice that light comes monotonically towards the origin from all galactic points. This photon path has a slope of $c_0 = 1$ close to the origin where the distance R and time t are both physical, but decreases as the distance increases and the time decreases, different from $c(t)$.

Although the distance uses physical rulers, the coordinate system as a whole may not be physical for times shorter than some limit. A reasonable limit (see §2.3.2) might be $A = 0.95$, $R = 2.3 \left(\frac{t_0}{t}\right)^{3/4}$, shown by the heavy dotted line in the figures. Together with $B = 1$, $C = 0$ for these physical distance coordinates, this shows that the assumption that T and R represent a physical AP coordinate system inside this limit is very good, with distances accurately represented, and time rates T_t within

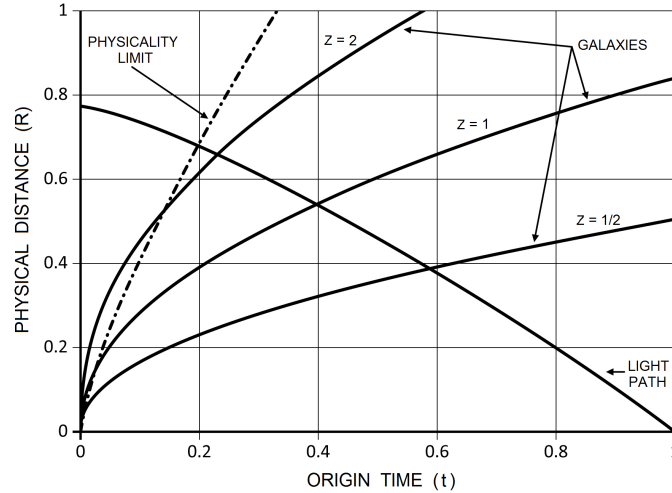


Fig. 1: Physical distances ($\frac{R}{c_0 t_0}$) for $\Omega = 1$ plotted against the normalized time on clocks at the origin ($\frac{t}{t_0}$) for various galaxy paths (labeled by their red shift z) and for the light path which the photons take after emission by any galaxy that arrives at the origin at $\frac{t}{t_0} = 1$. Notice that the slope of this light path close to the origin is $c_0 = 1$, where $V^2 \ll c^2$. The light path starts at the far horizon at $t = 0$, traveling monotonically towards the origin, but slower than its present speed in these non-local coordinates (like the Schwarzschild coordinates). The galactic paths show the expanding universe in physical coordinates, some traveling faster than the light speed in these non-local coordinates. The dotted line shows the approximate upper limit of physicality, where both R and the transformed time T are physical.

5% of physical measurements on adjacent inertial rods.

Fig. 2 plots these distances vs the transform time T at R . At the emission of the photons, T is finite (even for $t = 0$), presumably the transformed time it takes for the galactic point to get out to the point of emission. At $\frac{T}{t_0} = 1$ the slope of the light path is $c_0 = 1$, and at the physicality limit $\frac{T}{t_0} = 0.40$ the slope is only 5% less than $c(t) = 1.50$. At the intersection of this physicality limit with the photon path that arrives at the origin at t_0 , the time $\frac{t}{t_0} = 0.2$ and the red shift $z = 2.4$. Thus, if we have a flat universe with $\Omega = 1$, the last 80% of the universe history out to a z of 2.4 can be treated with physical coordinates T and R . This z is as large as any of the supernova Ia whose measurements have suggested an accelerating universe. It extends out into the universe much farther than a similar transform for a constant light speed [17] that

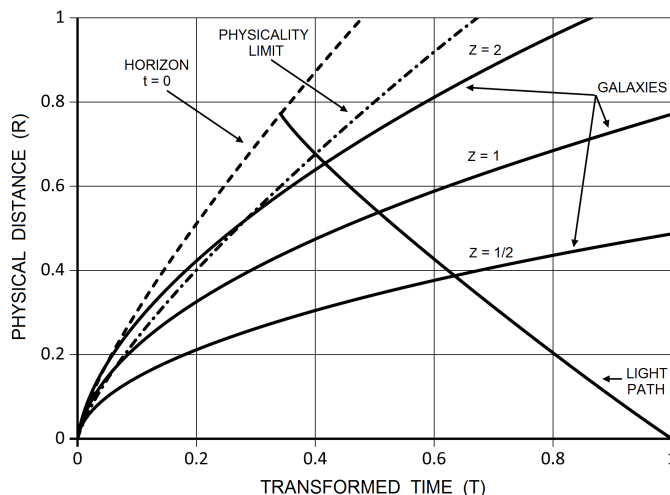


Fig. 2: Physical distances ($\frac{R}{c_0 t_0}$) for $\Omega = 1$ vs the transformed time ($\frac{T}{t_0}$) on clocks attached at R for various galaxy paths (labeled by their red shift z) and for the light path that arrives at the origin at $T = t_0 = 0.75 \left(\frac{1}{c_0 H_0}\right)$. The horizon is the locus of points where $t = 0, \gamma = \infty$. The heavy dotted line shows the approximate upper limit of physicality for the transformed coordinates ($A = 0.95$, that is $< 5\%$ error in physical time rate T_t). The slope of the light path is very close to $c(t)$ out to the limit of physicality. Light is emitted at finite T allowing transformed time for galactic points to move out from $R = 0$ before emitting their light we can see.

extends only out to a red shift of $z = 0.5$.

When the velocity of the points of the physical distance approaches the light speed when viewed from FLRW, the physical distance shows a Fitzgerald-like contraction so that it reaches a finite limit at the horizon ($t = 0$), beyond which there are no galactic points and no space. This is true for all universe densities including an empty universe. (It is also true for a constant light speed [17].)

I have included three additional figures, also using the extended field equation of §3 (and Appendix C.8). Fig. 3 is for a density of $\Omega = \frac{1}{2}$ (Appendix A.2), which has paths intermediate between $\Omega = 1$ and $\Omega = 0$. Fig. 4 shows the effect of dark energy (Appendix A.2 for $\Omega_\Lambda = \frac{3}{4}$), where all the curves tend to have inflection points when the dark energy becomes dominant. The empty universe ($\Omega = 0$ in Appendix A.1.4) shown in Fig. 5 is physical for all space-time, undistorted by gravitational curvature; galactic points and light travel in straight lines. It is very

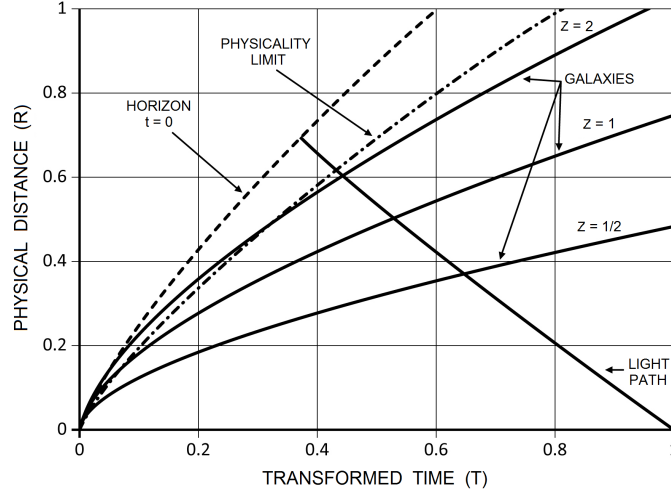


Fig. 3: Physical distance ($\frac{R}{c_0 t_0}$) for lower density universe ($\Omega = \frac{1}{2}$, $\Omega_r = \frac{1}{2}$) plotted against the transformed time ($\frac{T}{t_0}$) on clocks attached at R for various galaxy paths (labeled by their red shift z) and for the light path that arrives at the origin at $T = t_0 = 0.767 \left(\frac{1}{c_0 H_0} \right)$. The horizon ($t=0$) and the physicality line ($A=0.96$) occur at later times and shorter distances than for a flat universe (Fig. 2), but not as much as for the empty universe (Fig. 5). Similarly, the light path is straighter than Fig. 2, but not as straight as Fig. 5.

similar to Figs. 2–4 in that it demonstrates a finite horizon, beyond which there are no galactic points and no space. Figs. 1–2 are from the numerically integrated similarity solution, Figs. 3–4 are from the numerically integrated initial value solution, and Fig. 5 is an analytic function solution [27]. These illustrate complete coverage of $0 \leq \Omega \leq 1$.

§5. Underlying physics. Our objective of transforming the FLRW into the physical variables of the AP frame is the same as Zelmanov's chronometric invariants [6] that project events onto observable coordinates. The AP transforms, of course, do not have the generality of Zelmanov's chronometric invariants. Somehow, the variable light speed $c(t)$ considered as physical according to my definition must be a variable function of his invariant constant observable light speed. The present paper allows for the possibility of a variable light speed and then derives a relation for it for the FLRW universe. The GR field equation can be maintained unchanged to calculate value for $c(t)$ as a function of the universe energy density and curvature by assuming the gravitational

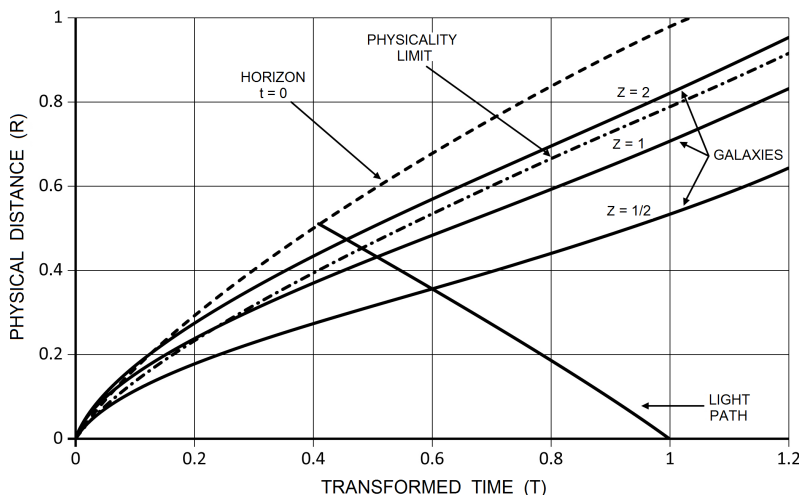


Fig. 4: Physical distance ($\frac{R}{c_0 t_0}$) vs transformed time ($\frac{T}{t_0}$) for dark energy ($\Omega = \frac{1}{4}$, $\Omega_\Lambda = \frac{3}{4}$). The present time $t_0 = 0.407 \left(\frac{1}{c_0 H'_0} \right)$. Notice the inflection points on all curves where the dark energy density becomes larger than the matter density.

constant G to be proportional to $c(t)^4$.

It really should not surprise us that the universe has a variable light speed. It is well known that an observer accelerated relative to an inertial observer measures a variable light speed depending on the acceleration (see [21, p. 173]).

The effect of gravitational potential on light speed is also demonstrated by the Schwarzschild coordinates, where the coordinate light speed as well as the time on clocks are changed by the gravitational potential at a distance from a central mass.

In the FLRW universe there are clearly gravitational forces caused by the energy density of the universe. These cause the expansion of the universe to be slowed down (or speeded up if dark energy predominates) shown by the change in the FLRW scale factor $\dot{a}(t)$. The case we have considered differs from either of the first two. We have examined a rigid radial rod whose gravitational force as felt by an observer attached to the rod increases with distance along the rod. The light speed v_L measured by such an observer stays within 5% of $c(t)$ while the latter changes by a factor of 1.5 (for $\Omega = 1$) out to the physicality limit. The variation is not directly caused by the acceleration $\frac{dV}{dt}$, but mostly by

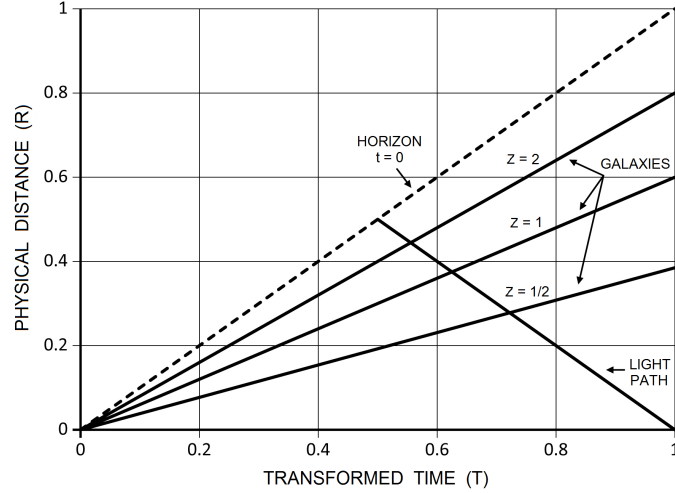


Fig. 5: Physical distance $(\frac{R}{c_0 t_0})$ for the empty expanding universe ($\Omega = 0$, $\Omega_r = 1$) plotted against the transformed time $(\frac{T}{t_0})$ on clocks attached at R for various galaxy paths (labeled by their red shift z) and for the light path that arrives at the origin at $T = t_0$. The horizon is the locus of points where $t = 0$. All lines are straight and physical, since there is no space curvature, and the light speed is $c(t) = c_0$. The remotest galactic point travels from the origin at $T = 0$ out to $\frac{c_0 t_0}{2}$ at the light speed c_0 .

the change in a , which in turn is affected by the gravitational forces. An alternate way to view the light speed variation is to recognize that the FLRW metric has already accounted for both the gravitational forces and the light speed variation when it satisfies the GR field equation, which therefore relates the two.

In Appendix C for FLRW we show that for a flat universe ($\Omega = 1$) with the presently derived variable light speed, there is a gravitational field g in the physicality region that increases linearly with distance from the origin. If we insert into (105) the mass of the universe inside the radius R , $M_0 = 4\pi\rho_0 \frac{R^3}{3}$ at time t_0 , we obtain

$$g = -\frac{G_0 M_0}{R^2} \quad (55)$$

the Newtonian expression for gravitational field at a radius R inside a sphere of uniform density. This is another indication that the T, R coordinates are obeying Special Relativity laws near the origin because an accelerated particle in the rest frame of SR has the Newtonian ac-

celeration [23]. Note that $g < 0$ indicates an inward pull on the galactic points towards the origin of the AP axis, which we can interpret as the cause for the universe expansion to slow down (for $\Lambda = 0$).

Thus, just as the assumption of homogeneity requires the universe to be either expanding or contracting, it seems to require the physical light speed to depend on this rate of expansion or contraction.

§6. To observe $c(t)$. The most straight forward way to observe $c(t)$ is to find a way to directly measure the light speed or the atomic spectra wavelengths with the same precision and stability that we can now measure spectra frequency. A fractional change in speed or wavelength should be 6×10^{-17} in 100 secs or 2×10^{-11} in a year if $\frac{c}{c_0} = \left(\frac{t_0}{t}\right)^{1/4}$. With this much sensitivity, however, an observation would have to separate out the possible effects on light speed of the gravitational forces of local masses like the Earth, the Moon, and the Sun.

The variable light speed $c(t)$ might affect all of distant observations. For instance in the measurement of supernova Ia [10–13] it will affect the measurement of acceleration of galaxies. Thus, if the luminosity of the super novae Ia decreases with increasing c , this would decrease the implied acceleration and the dark energy density. For a flat universe $\Omega = 1$ the apparent distance d_L is given by

$$d_L = \frac{2}{H_0} (1+z) \left[1 - (1+z)^{-1/3} \right] \sqrt{L(c)}, \quad (56)$$

where L is the fraction by which the luminosity is changed by $\frac{c(t)}{c_0}$. Note that if H'_0 is the reciprocal of the measured slope of d_L vs z for small z , then $H'_0 = \frac{3H_0}{2}$ and $c_0 t_0 = \frac{1}{2H_0} = \frac{3}{4H'_0}$. In general, $H'_0 = H_0 \left(1 + \frac{\Omega}{2} - \Omega_\Lambda\right)$.

For the observations to be entirely explained by $c(t)$ instead of dark energy would require the luminosity to vary as $c^{-5} = (1+z)^{-5/3}$ for a flat universe without dark energy. This is the c dependence of a radiating atom; that is the radiation of a dipole ea_B , where $a_B \propto \frac{\epsilon_0}{mc^2} \propto \frac{1}{c(t)}$ is the Bohr orbit and we have assumed $\epsilon_0 c(t)$, and $mc^2(t)$ to be constant. All atom and ion radiation should have this same dependence on c . Since all the light is presumably from excited atoms or ions, this seems to be a credible alternative to dark energy as the source for the apparent accelerating universe. Measurements at large z [14] should be able to distinguish between dark energy and $c(t)$ dimming. Recently, this apparent acceleration has been confirmed using distant large clusters as a standard candle [15]. This light is also likely to come from excited atoms or ions. Even if $c(t)$ does not explain all the apparent acceleration, the

calibration of the standard candles and therefore the amount of dark energy will be affected. Other astronomical observations that might be affected by $c(t)$ are cosmic background radiation, gravitational lensing, and dynamical estimates of galactic cluster masses.

Unfortunately, the $c(t)$ calculated herein solves neither the flatness nor the horizon problem without inflation: The flatness problem changes little because the Hubble ratio has a similar dependence on the universe scale factor $a(t)$. The horizon problem remains because $c(t)$ enters both the transverse speed of light and the radial speed of galactic points. At the time of the release of the CBR photons, without inflation light could have traveled laterally only $\theta = \int_0^t \frac{c \partial t_\chi}{ra}$, where $\chi = \int_t^{t_0} \frac{cdt}{a}$. For $\Omega = 1$, $z = 3,000$, $\theta = z^{-1/3} = 0.07$ radians, or 4 degrees, and so galactic points could not have interacted separated by more than this angle.

§7. Conclusion. From the cosmological principle of spatial homogeneity and isotropy we can obtain the FLRW metric, which allows a variable light speed, that describes a universe of inertial frames attached to expanding galactic points with FLRW differential co-moving coordinate times the scale factor $a(t)$ interpreted as a physical differential distance. The FLRW metric is Minkowski-like in its radial derivative. Locally, SR applies, so a AP rigid frame attached to the origin has a Minkowski metric. Thus, for a radial world line we can use a Lorentz transform from FLRW to the AP frame that keeps the two Minkowski world line elements invariant in order to obtain time and distance coordinates to describe radial movement in the universe close to the origin. Because the FLRW metric has a time varying coefficient multiplying the space differential, this produces a velocity between the galactic points and the AP frame that is a function of time and distance. If the Lorentz transform is to remain valid out from the origin to the lowest power of this velocity, a consistent limiting process to zero distance from the origin requires a variable light speed $c(t) \propto \sqrt{\frac{da}{dt}}$, the square root of the rate of change of the scale factor of the FLRW universe.

By homogeneity, the origin can be placed on any galactic point, so that this variable light speed enters physical laws throughout the universe.

We extend the field equation by allowing the gravitational “constant” and the rest masses of particles to vary in such a way as to keep constant the rest mass energy and the Newtonian gravitational energy. We have shown that this results in a constant relating the tensors of the field equation, like the field equation with a non-variable light speed.

This yields a new function of cosmological time for the scale factor of the FLRW universe and thus values for $c(t)$. These enable the calculation of physical distance vs physical time for galactic and light paths in the universe.

Although three orthogonal rigid axes are inadequate to describe three-dimensional motion in accelerating fields, it is possible to describe one dimensional motion on a single axis. We have done this for the FLRW universe by finding radial AP transforms from FLRW for all distances whose differentials remain close to SR Minkowski with this same variable light speed out to a red shift of 2 for a flat universe.

I have shown that the physical coordinates on the AP frame near the origin have a gravitational field for a flat universe that increases linearly with radius just like the Newtonian field for a spherical distribution of uniform mass density. Like Schwarzschild, a gravitational red shift is predicted for a distant AP light source observed at the origin of the FLRW universe.

To summarize, I am persuaded that the physical light speed throughout the FLRW universe is proportional to $\sqrt{\frac{da}{dt}}$ because:

- 1) Based on usual assumptions, in the limit of zero distance from the origin a radial Lorentz transform from FLRW to a AP rigid frame requires it;
- 2) All radial AP transforms from FLRW coordinates that I have investigated that have a Lorentz transform from FLRW near the origin have this same variable light speed;
- 3) We can use an extended Einstein field equation to calculate the transformed distance vs time for galactic points and light that behave in a physically sensible way;
- 4) The transformed gravitational field in the physicality region for a flat universe is Newtonian for a spherical distribution of uniform mass density and can be considered the cause of the deceleration of the universe (when dark energy can be neglected);
- 5) The AP transform extends out into space much farther than for a constant light speed.

Just as the assumption of homogeneity requires the universe to be either expanding or contracting, it seems to require the light speed to depend on this rate of expansion or contraction under the influence of gravity.

One of the radial AP transforms from FLRW has a distance coordinate that remains physical for all distances. We can interpret this to

be a global reference distance (used in Figs. 1–5), although the time of this transform becomes unphysical at large distances.

Some other physical “constants” that depend on the light speed must also be changing with cosmic time. I have suggested some constraints on this variability: 1) retaining the conservation of the stress-energy tensor, including keeping constant the rest mass energy, the gravitational energy, and the Schwarzschild radius, and 2) keeping frequency of atomic spectra constant, which means the fine structure constant, and the Rydberg frequency. These still make possible the geometrization of relativity with an adaptation of vectors and tensors such as the energy-momentum vector, the stress-energy tensor, and the electromagnetic field tensor.

This $c(t)$ should be observable by direct measurement of light speed or spectral wavelength if they could be measured to the same precision as frequency, and if the possible effects on light speed of the gravitational forces of nearby masses like the Earth, the Moon, and the Sun could be isolated. It should have an impact on understanding distant cosmic observations. Perhaps it will provide an alternative to dark energy to explain the apparent acceleration of galaxies via supernova Ia. Analysis of cosmic background radiation, gravitational lensing, and dynamical estimates of galactic cluster masses could also be affected. But the recognition of this $c(t)$ does not solve the flatness nor horizon problems without inflation.

I have outlined in Appendix C how a variable light speed can be included in an extended Special and General Relativity by keeping constant the rest energy of particles and the energy of Newtonian gravity acting between them.

Appendix A. AP (almost physical) coordinates with diagonal metrics

A.1. AP coordinates with physical time

A.1.1. Partial differential equation for $\hat{V} = V/c(t)$. We will be considering radial AP transforms for diagonal coordinates that (36) makes

$$\hat{V} = -\frac{cT_\chi}{aT_t}. \quad (57)$$

For diagonal coordinates with physical time at all t and χ , $A=1$. Thus, (34) becomes

$$T_t = \gamma. \quad (58)$$

This automatically guarantees the Lorentz time dilation $(\frac{\partial T}{\partial t})_R = \frac{1}{t_T} = \frac{1}{\gamma}$ (32). We need only find a transform for which $B \rightarrow 1$ close to the origin to make it AP.

We proceed by finding a differential equation with \hat{V} as the only dependent variable. Thus, we write a formula for T , using (58) and (57):

$$T = t + \int_0^\chi T_\chi \partial\chi_t = t + \int_0^\chi \left(-\frac{a}{c} \gamma \hat{V}\right) \partial\chi_t, \quad (59)$$

where we have used the boundary condition that at $\chi=0$, $T=t$, and the symbol $\partial\chi_t$ signifies integration with χ at constant t . It can be partially differentiated with respect to t (giving γ) and then with respect to χ and with the use of (25), noting that $d\gamma = \gamma^3 \hat{V} d\hat{V}$ and $1 + \hat{V}^2 \gamma^2 = \gamma^2$, we obtain a PDE for \hat{V} :

$$\hat{V}_t + \hat{V}_\chi \left(\frac{\partial\chi}{\partial t}\right)_R = \left(\frac{\partial\hat{V}}{\partial t}\right)_R = -\hat{V} \left(1 - \hat{V}^2\right) \frac{c}{a} \frac{d}{dt} \left(\frac{a}{c}\right). \quad (60)$$

A.1.2. The general solution for \hat{V} , R , and T for all a . Equation (60) can be rewritten as

$$\frac{\partial\hat{V}_R}{\hat{V}(1 - \hat{V}^2)} = -\frac{\partial(\frac{a}{c})_R}{\frac{a}{c}}, \quad (61)$$

where the subscript on the partial differential indicates the variable to be held constant. This can be integrated with an integration constant $\ln \kappa$. Since the integration is done at constant R , then $\kappa = \kappa(R)$, and inversely, $R = R(\kappa)$. Integrating (61), we get

$$\hat{V} = -\frac{\kappa}{\sqrt{\frac{a^2}{c^2} + \kappa^2}}, \quad (62)$$

where the sign of κ will be positive for an expanding universe, where the χ points will stream out radially past a point at R .

At this point, R is an unknown function of κ . The various possible coordinate systems which solve our PDEs are characterized, in large part, by the function $R(\kappa)$. But for all, in order for \hat{V} to vanish when $R = 0$ (see definitions in §1), κ must also; so always

$$\kappa(0) = 0. \quad (63)$$

We note that as long as $\kappa(R)$ remains finite, \hat{V} goes to -1 , and V goes to $-c(t)$, for $a(t) = 0$, i.e. for $t = 0$, the horizon.

Let us now look at lines of constant $\kappa(R)$, i.e. constant R , in t, χ space. Equation (25) can be integrated for χ with use of (62) at constant

κ to give the following:

$$\chi(t, \kappa) = \int_t^\infty \frac{c \kappa \partial s_\kappa}{a(s) \sqrt{\frac{a^2}{c^2} + \kappa^2}}. \quad (64)$$

For an expanding universe, we have set the upper limit at ∞ , because we expect that if R is kept constant the galactic point χ that will be passing any given R will eventually approach zero as FLRW time t approaches infinity.

At this point, we have obtained $\hat{V} = \hat{V}^*(t, \kappa)$ from (62) and have also obtained the function $\chi(t, \kappa)$. We can in principle invert (64) to obtain κ in terms of t and χ : $\kappa = K(t, \chi)$. This gives us the velocity function $\hat{V}(t, \chi) = \hat{V}^*(t, K(t, \chi))$. If the function $R(\kappa)$ were known, we would then also have $R(t, \chi) = R(K(t, \chi))$.

The time $T(t, \chi)$ can be found by noting from (57) that

$$T_\chi = -\frac{a \hat{V}}{c} T_t = -\frac{a \hat{V}}{c} \gamma = \kappa. \quad (65)$$

By substituting (65) into (59), and integrating over κ instead of χ by dividing the integrand by the partial derivative of (64) with respect to κ , we find an expression for $T(t, \chi)$:

$$T(t, \chi) = t + \int_t^\infty \left(1 - \frac{1}{\sqrt{1 + \frac{c^2 \kappa^2}{a^2}}} \right) \partial s_\kappa, \quad (66)$$

where κ is put equal to $K(t, \chi)$ after integration at constant κ in order to get $T(t, \chi)$.

This completes the solution. Since $\kappa(R)$ can be any function that vanishes at the origin, there thus exist an infinite number of solutions for our transformed coordinates with $A = 1$, $C = 0$.

A.1.3. Independent determination of $c(t)$. To determine physicality, we will next find $\frac{1}{R_\chi}$ close to the origin. $R_\chi = \frac{\kappa_\chi}{\kappa'(R)}$ can be written in an inverted form by taking the derivative of (64) with respect to κ at constant t :

$$\frac{1}{R_\chi} = \kappa' \left[\left(\frac{\partial K}{\partial \chi} \right)_t \right]^{-1} = \kappa' \left(\frac{\partial \chi}{\partial \kappa} \right)_t = \kappa'(R) \int_t^\infty \frac{c^2 \partial s_\kappa}{a^2 \left(1 + \frac{c^2 \kappa^2}{a^2} \right)^{3/2}}. \quad (67)$$

By (41), the light speed is given by

$$v_L = \frac{cA}{B} = \frac{c\gamma a}{R_\chi} = c\gamma a \kappa'(R) \int_t^\infty \frac{c^2 \partial s_\kappa}{a^2 \left(1 + \frac{c^2 \kappa^2}{a^2} \right)^{3/2}}. \quad (68)$$

It was (68) that gave me the first indication that the light speed could be variable, and that it was the same near the origin where $\kappa(0) = 0$ for all $\kappa(R)$, which would be a requirement that it was indeed the physical light speed.

To be physical $B = \frac{\gamma a}{R_\chi} \rightarrow 1$ as R approaches 0. Putting $\gamma = 1$, $\kappa(0) = 0$, and $R_\chi = a$ in (67), and changing the integration variable from t to $a(t)$ gives

$$\frac{1}{a} = \kappa'(0) \int_a^\infty \frac{c da}{a^2 \dot{a}}, \quad (69)$$

where the dot indicates differentiation by \hat{t} , and $\kappa'(0)$ is a constant to be determined by $c(t_0) = 1$. Note that the integral of (69) is independent of the functional form of $\kappa(R)$, and is therefore the same for all $\kappa(R)$. It was (67) that gave me the first indication that the light speed ($\frac{cR_\chi}{\gamma a}$) was variable, and that it was the same near the origin for all $\kappa(R)$.

Equation (69) is an integral equation for $c(t)$. By differentiation of both sides of (69) by a , we can obtain

$$c(t) = \frac{1}{\kappa'(0)} \dot{a}, \quad (70)$$

which, as we should expect, is the same $c(t)$ of (21) we showed for all physical coordinate systems for $\kappa'(0) = \frac{\dot{a}(t_0)}{c_0} = \frac{a_0 H_0}{c_0}$. This independent derivation of $c(t)$ confirms the validity of carrying the series expansion to second order since these complete transforms give the same $c(t)$.

Notice that we have found this solution and the value for $c(t)$ without using the GR field equation nor any assumption about the variation of rest mass m and gravitational constant G .

A.1.4. Zero density universe $\Omega = 0$. It is interesting to consider the limiting case of a zero density universe: $\Omega = 0$, $\Omega_r = 1$, $a_0 H_0 = 1$ (45). Equation (21) makes $c = 1$. Equation (47) makes $\dot{a} = H_0$ for all t, χ . Integrating gives $a = t$. Equation (64) gives $\chi = \cosh^{-1} \frac{t}{\kappa}$, or $\kappa = = K(t, \chi) = t \sinh \chi$. We can then find from (62) that $V(t, \chi) = -\tanh \chi$ and from (26) that $\gamma = \cosh \chi$ so that

$$c = 1 = \frac{R_\chi}{\gamma a} = \frac{dR}{d\kappa} \frac{K_\chi}{\gamma t} = \frac{dR}{d\kappa}. \quad (71)$$

Thus the physicality condition is met for all R with $R = K$ and $A = 1$, $B = 1$, so that the complete transform with (66) becomes

$$R = t \sinh \chi, \quad T = t \cosh \chi. \quad (72)$$

These coordinates have been known ever since Robertson [27] showed that this transformation from the FLRW co-moving coordinates at zero density obeyed the Minkowski metric. What is new is that this solution was derived from the equations we obtained for our physical time transforms with $A=1$. It can also be obtained from the physical distance transforms ($B=1$) since equations (60) and (77) for \hat{V} become identical with $\hat{V}_t=0$ and $\frac{a}{c}=a=t$. It is the only known rigid physical coordinate system for all times and distances in a homogeneous and isotropic universe. In Fig. 5, R is plotted vs T to show how similar it is to the physicality region of Figs. 2–4.

A.2. AP coordinates with physical distance

A.2.1. Partial differential equation for \hat{V} . For diagonal coordinates with physical dR for all t and χ , $B=1$, so (35) becomes

$$R_\chi = a\gamma. \quad (73)$$

By integration we find

$$R = a \int_0^\chi \gamma \partial\chi_t, \quad (74)$$

and partial differentiation with respect to t gives

$$R_t = c\dot{a} \int_0^\chi \gamma \partial\chi_t + a \int_0^\chi \gamma_t \partial\chi_t. \quad (75)$$

We can then find \hat{V} from (31), (73), and (74) as

$$\hat{V} = -\frac{R_t}{c\gamma} = -\frac{1}{c\gamma} \left(c\dot{a} \int_0^\chi \gamma \partial\chi_t + a \int_0^\chi \gamma_t \partial\chi_t \right). \quad (76)$$

This is an integral equation for \hat{V} . It can be converted into a partial differential equation by multiplying both sides by γ and partial differentiating by χ :

$$\gamma^2 \left(\hat{V}_x + \frac{a}{c} \hat{V}_t \hat{V}_t \right) = -\dot{a} = -\frac{1}{c} \frac{da}{dt}. \quad (77)$$

Note that this is substantially different from the (60) for \hat{V} that we obtained for physical time. This means that it is not possible to find diagonal transforms with both physical time and physical distance for all values of t and χ (except for $\Omega=0$). It is possible to have either one or the other be physical at all t and χ with the other being physical only close to the origin.

A.2.2. General solution for \hat{V} . Equation (77) can be solved as a standard initial-value problem. Let $W = -\hat{V}$. Equation (77) becomes

$$W_\chi - \frac{a}{c} WW_t = \frac{1}{c} \frac{da}{dt} (1 - W^2). \quad (78)$$

Define a characteristic for $W(t, \chi)$ by

$$\left(\frac{\partial t}{\partial \chi} \right)_c = -\frac{a}{c} W \quad (79)$$

so

$$\left(\frac{\partial W}{\partial \chi} \right)_c = \frac{1}{c} \frac{da}{dt} (1 - W^2) \quad (80)$$

(the subscript c here indicates differentiation along the characteristic). If we divide (80) by (79) we get

$$\left(\frac{\partial W}{\partial t} \right)_c = -\frac{1}{a} \frac{da}{dt} \frac{(1 - W^2)}{W}. \quad (81)$$

This can be rearranged to give

$$\frac{W(\partial W)_c}{W^2 - 1} = \frac{(\partial a)_c}{a}. \quad (82)$$

This can be integrated along the characteristic with the boundary condition at $\chi = 0$ that $W = 0$ and $a = a_c$:

$$1 - W^2 = \frac{a^2}{a_c^2} = \frac{1}{\gamma^2}. \quad (83)$$

This value for W (assumed positive for expanding universe) can be inserted into (79) to give

$$\left(\frac{\partial t}{\partial \chi} \right)_c = -\frac{a}{c} \sqrt{1 - \frac{a^2}{a_c^2}}. \quad (84)$$

We can convert this to a differential equation for a by noting that $c dt_c = d\hat{t}_c = \frac{1}{\alpha} da_c$

$$\left(\frac{\partial a}{\partial \chi} \right)_c = -a \dot{a} \sqrt{1 - \frac{a^2}{a_c^2}}. \quad (85)$$

We can provide an integrand containing functions of only α by using the GR relation for \dot{a} in (47), which does not assume that $\dot{a} \propto c$. Equa-

tion (85) then becomes

$$\left(\frac{\partial\alpha}{\partial\chi}\right)_c = -a_0 H_0 \alpha^2 E(\alpha) \sqrt{1 - \frac{\alpha^2}{\alpha_c^2}}. \quad (86)$$

This can be integrated along the characteristic with constant α_c , starting with $\alpha = \alpha_c$ at $\chi = 0$. This will give $\chi = X(\alpha, \alpha_c)$. This can be inverted to obtain $\alpha_c(\alpha, \chi)$. When this is inserted into (83), we have a solution to (78) for $W(\alpha, \chi)$.

I will now assume that $c \propto \dot{\alpha}$, then later show numerically that this makes $A \rightarrow 1$ as $R \rightarrow 0$ to prove physicality. (For $\Omega = 1$ in §A.3.1, $c \propto \dot{\alpha}$ is shown explicitly.) Then $W(t, \chi)$ can be found from $W(\alpha, \chi)$ by using $\frac{c}{c_0} = \alpha E(\alpha)$ in (21) to get $t(\alpha)$:

$$t = \int_0^\alpha \frac{d\alpha}{c\dot{\alpha}} = \frac{1}{c_0 H_0} \int_0^\alpha \frac{d\alpha}{\alpha^2 E^2}. \quad (87)$$

A.2.3. Obtaining T, R from \hat{V} . Equations (25), (31), and (57) show that

$$W = -\frac{a}{c} \left(\frac{\partial\chi}{\partial t}\right)_R = \frac{a}{c} \frac{R_t}{R_\chi} = \frac{c}{a} \frac{T_\chi}{T_t} \quad (88)$$

so

$$T_\chi - \frac{a}{c} W T_t = 0. \quad (89)$$

Thus T has the same characteristic as W (79), so that $(\frac{\partial T}{\partial \chi})_c = 0$, and T is constant along this characteristic:

$$T(t, \chi) = T(t_c, 0) = t_c \equiv t(\alpha_c(t, \chi)), \quad (90)$$

where $t(\alpha)$ is given in (87) and $\alpha_c(\alpha(t), \chi)$ is found by inverting the integration of (86). This gives us the solution for $T(t, \chi)$ and A

$$A = \frac{\gamma}{T_t} = \frac{a_c}{a} \left(\frac{\partial t}{\partial t_c}\right)_\chi = \frac{a_c}{a} \frac{\frac{da_c}{dt_c}}{\frac{da}{dt}} \left(\frac{\partial a}{\partial a_c}\right)_\chi. \quad (91)$$

The solution for R can be obtained by integrating (74), using γ from (83) and $a_c(t, \chi)$ from (86):

$$R(t, \chi) = a \int_0^\chi \gamma \partial \eta_t = \int_0^\chi a_c(t, \eta) \partial \eta_t. \quad (92)$$

Alternatively, for ease of numerical integration we would like to integrate dR along the same characteristic as T and W . This can be obtain-

ed from the PDE

$$\left(\frac{\partial R}{\partial \chi}\right)_c = R_\chi + R_t \left(\frac{\partial t}{\partial \chi}\right)_c. \tag{93}$$

If we insert the values for these three quantities from equations (73), (88), and (79), we get

$$\left(\frac{\partial R}{\partial \chi}\right)_c = \gamma a + \frac{cW}{a} \gamma a \left(-\frac{aW}{c}\right) = \frac{a}{\gamma} = \frac{a^2}{a_c}. \tag{94}$$

It is interesting that this solution for the physical distance coordinates (PD) is unique for each $a(t)$, whereas for the physical time coordinates (PT), there are an infinite number of solutions. This is because to obtain a solution for PD, we had to provide an additional relation, viz, for \dot{a} (86), whereas for PT no additional relation was needed. Possibly we could use the same relation in PT to make $\kappa(R) \propto R$ as for the similarity solution for a flat universe (see §A.3.2). This would make PT unique as well, but I haven't been able to show this.

A.3. Similarity solutions for flat universe, $\Omega = 1$ I have found similarity integrations for the special case of $\Omega = 1$ where the GR solution is $a = a_0(\frac{t}{t_0})^{1/2}$ and $c = c_0(\frac{t_0}{t})^{1/4}$ (see §3). To simplify notation let us normalize time to $\frac{t}{t_0} \rightarrow t$, $\frac{a}{a_0} \rightarrow a$, and $\chi \frac{a_0}{c_0 t_0} \rightarrow x$, $\frac{T}{t_0} \rightarrow T$, $\frac{R}{c_0 t_0} \rightarrow R$, and let $W = -\hat{V}$.

A.3.1. Physical distance. Equation (77) then becomes

$$W_x - t^{3/4} W W_t = \frac{1}{2} t^{-1/4} (1 - W^2). \tag{95}$$

This can be converted into an ordinary differential equation (ODE) by letting

$$u = \frac{x}{t^{1/4}} \tag{96}$$

so that (95) becomes

$$W' \left(1 + \frac{uW}{4}\right) = \frac{1}{2} (1 - W^2), \tag{97}$$

where the prime denotes differentiation by u .

Similarly we can find ODE's for T and R by defining:

$$\frac{T}{t} \equiv q(u), \tag{98}$$

and

$$\frac{R}{t^{3/4}} \equiv s(u), \tag{99}$$

where $q(u)$ and $s(u)$, from equations (57) and (31), are given by the coupled ODE's:

$$q' \left(1 + \frac{uW}{4} \right) = qW, \quad (100)$$

and

$$s' \left(W + \frac{u}{4} \right) = \frac{3}{4} s. \quad (101)$$

It is useful to find that $q = \gamma^2$, $s' = \gamma$, $s = \gamma \frac{u+4W}{3}$, and $A = \frac{\gamma}{T_t} = \frac{1+\frac{uW}{4}}{\gamma} = \frac{v_p}{c}$; so $T = \gamma^2 t$, and $R = t^{3/4} \gamma \frac{u+4W}{3}$.

For small values of u , $W = \frac{u}{2}$, $q = 1 + \frac{u^2}{8}$, $s = u$, $\frac{v_p}{c} = 1 + O(W^4)$, and $R = t^{1/2} x = ax$. The light speed v_p measured on AP remains close to that measured on FLRW out to large R . We also note that $T_t \rightarrow 1 + \frac{W^2}{2}$, confirming that these coordinates have physical time close to the origin, justifying $c(t) = t^{-1/4}$.

An alternate approach would be to start with $c(t)$ unknown, but of the form $c = t^{-b}$. Then the GR field equation (42) will give $\alpha = t^d$, where $d = \frac{2}{3}(1 - b)$. Equation (95) then becomes $W'(2 + uWd) = 2d(1 - W^2)$, where the independent variable is $u = \frac{x}{t^{d/2}}$. For $T = tq$ this will make $q'(2 + uWd) = 2qW$ and $T_t = q - \frac{udq'}{2}$. For small u , $q \rightarrow 1 + \frac{d}{2}u^2$, $W \rightarrow ud$, and $T_t \rightarrow 1 + \frac{(1/d-1)}{2}W^2$. To be Lorentz $T_t \rightarrow 1 + \frac{W^2}{2}$ so that $d = \frac{1}{2}$ and $b = \frac{1}{4}$, confirming that $c \propto \sqrt{\frac{da}{dt}}$.

For constant light speed, $b = 0$, $d = \frac{2}{3}$ and $T_t \rightarrow 1 + \frac{W^2}{4}$, slower than Lorentz as found by the author in [17]. This has implications for the use of the GR field equation. We can't integrate the physical distance transforms without using the FE. When we use it for a constant light speed, we don't get the Lorentz transform for small \hat{V} . When we use it for an arbitrary varying light speed, we get the Lorentz transform when we use the same $c(t)$ as for the power series and for the physical distance transform. This self-consistency indicates that we are using the correct FE and the correct $c(t)$.

As $t \rightarrow 0$, $u \rightarrow \infty$, $\gamma \rightarrow \kappa u^2$, $W \rightarrow 1 - \frac{1}{2\kappa^2 u^4}$, $q \rightarrow \kappa^2 u^4$, and $s \rightarrow \kappa \frac{u^3}{3}$. T and R both remain finite at this limit with $T \rightarrow \kappa^2 x^4$, and $R \rightarrow \kappa \frac{x^3}{3}$, where $x_L \rightarrow 4$ at $t \rightarrow 0$. It is difficult to determine κ from the numerical integration because of the singularity at large u , but my integrater gives 0.0364. The fact that T does not go to zero when t goes to zero results from equating T with t at $t = 1$ and not at $t = 0$.

The distance R and time T can be found from the numerical integration of the coupled ODE's. The paths of galactic points are those

for constant x . The path photons have taken reaching the origin at t_1 is found by calculating x_p vs t and using the transform to T, R . Thus, for $\Omega = 1$

$$x_p = \int_{t_1}^t \frac{c}{a} dt = 4 (t_1^{1/4} - t^{1/4}). \quad (102)$$

For light arriving now, $t_1 = 1$, the value of u_p becomes

$$u_p = 4 \left(\frac{1}{t^{1/4}} - 1 \right) = 4(c - 1), \quad (103)$$

where we inserted $c = \frac{1}{t^{1/4}}$ to obtain the relation of c to u_p .

Galactic and photon paths are shown in Figs. 1 and 2. An approximate upper limit of physicality is shown by the heavy dotted line: $A = 0.953$, $\frac{W^2}{2} = 0.253$, $u = 2.0$, $R = 2.30 t^{3/4} = 1.35 T^{3/4}$. At $t = 0$, R vs T provides a non-physical horizon: $R_h = 1.747 T_h^{3/4}$.

It is also interesting to calculate the acceleration g . If we insert the values of V , R , and $\frac{a}{c}$ in (128), we obtain

$$-g = \frac{1}{8\gamma t} \left(\frac{u}{1 + \frac{uW}{4}} \right), \quad (104)$$

where the units of g are $\frac{c_0}{t_0}$. For small u close to $t = t_0$, g goes to zero as $-\frac{u}{8}$.

Since small u is the region with physical coordinates, it is interesting to express g in unnormalized coordinates:

$$-g = \frac{1}{8} \frac{c_0}{t_0} \frac{R}{c_0 t_0} = G_0 \rho_0 \frac{4\pi}{3} R, \quad (105)$$

where we have used $\Omega = 1$ in (44). For small t , $-g$ goes to ∞ as $\frac{1}{2\gamma t^{5/4}} = 1.2 t^{-3/4}$ along the light path. At the physicality limit, $-g = 0.69 \frac{c_0}{t_0} = 12.5 \times 10^{-9}$ m/sec².

The g can be obtained from a gravitational potential using $g = -\frac{d\phi}{dR}$, which for close distances is:

$$\phi = G_0 \rho_0 \frac{2\pi}{3} R^2 = \frac{R^2}{16 t_0^2} = c_0^2 \frac{u^2}{16}. \quad (106)$$

The slope of the light path in Fig. 1, a coordinate velocity of light, can be shown in normalized units for this incoming light path to be

$$v_L = \left(\frac{dR}{dt} \right)_L = - \left(1 + \frac{u}{4} \right) \left(\frac{1-W}{1+W} \right)^{1/2}. \quad (107)$$

For small u ,

$$v_L \longrightarrow 1 - \frac{u}{4}. \quad (108)$$

For the outgoing light path

$$v_L = \left(\frac{dR}{dt} \right)_L = \left(1 - \frac{u}{4} \right) \left(\frac{1+W}{1-W} \right)^{1/2}. \quad (109)$$

For small u ,

$$v_L \longrightarrow 1 + \frac{u}{4}, \quad (110)$$

Thus, the coordinate light speed has a different u dependence on R for incoming and outgoing light paths because the slope is dependent on t , not R . This differs from the Schwarzschild solution that has the same R dependence of the coordinate light speed for both directions of the light path.

The observed light at the origin ν that is emitted from a AP source at R as ν_0 is also smaller than the same light emitted at the origin:

$$\frac{\nu}{\nu_0} = \left(\frac{\partial T}{\partial t} \right)_R = \frac{1}{\gamma A} = q \frac{1-W^2}{1+\frac{uW}{4}}. \quad (111)$$

Close to the origin it is:

$$\frac{\nu}{\nu_0} = 1 - \frac{\hat{V}^2}{2} = 1 - \frac{u^2}{8} = 1 - \frac{2\phi}{c_0^2}. \quad (112)$$

This, of course, is the same as a dilation effect for a collocated galactic point at R that shows up as a gravitation red shift at the origin due to homogeneity of t .

A.3.2. Physical time. There is also a similarity solution for physical time, $A=1$, for $\Omega=1$. With the same normalizations as above, using (60), the ODE for W is

$$W' \left(W + \frac{u}{4} \right) = \frac{3}{4} W (1 - W^2) \quad (113)$$

with the ODE's for (T, R, q, s, x_p) . This is the same as physical distance for small u , but differs numerically at large u . Useful relations for physical time are obtained from the general solution in Appendix A.1: $R=2\kappa$, $q=\gamma\left(1+\frac{uW}{4}\right)$, $s=2\gamma W$, and $B=\gamma\frac{u+4W}{3s}=\frac{2}{3}\left(1+\frac{u}{4W}\right)$. These can be used to find the gravitational field from (128):

$$g = -\frac{3}{2t^{5/4}} \frac{\gamma W^2}{u+4W}, \quad (114)$$

and the coordinate velocity for an incoming light path:

$$v_L = -\frac{3}{2} \left(\frac{1 + \frac{u}{4}}{1 + \frac{u}{4W}} \right) \left(\frac{1 - W}{1 + W} \right)^{1/2}. \quad (115)$$

Equations (114) and (115) approach the same values as physical distance for small u .

Appendix B. Gravitational field in the FLRW and AP coordinates

We wish to find the components of the radial acceleration of a test particle located at R in the AP transformed system. We will do this by calculating the FLRW components of the acceleration vector and find the transformed components by using the known diagonal transforms. For the FLRW components, we will use the metric

$$ds^2 = dt^2 - a^2 d\chi^2 - a^2 r^2 d\theta^2 - a^2 r^2 \sin^2 \theta d\phi^2. \quad (116)$$

Let

$$x^1 = \chi, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = \hat{t}, \quad (117)$$

and the corresponding metric coefficients become

$$g_{44} = 1, \quad g_{11} = -a^2, \quad g_{22} = -a^2 r^2, \quad g_{33} = -a^2 r^2 \sin^2 \theta. \quad (118)$$

For any metric, the acceleration vector for a test particle is

$$A^\lambda = \frac{dU^\lambda}{ds} + \Gamma_{\mu\nu}^\lambda U^\mu U^\nu, \quad (119)$$

where the Γ 's are the affine connections and U^λ is the velocity vector of the test particle. In our case the test particle is at the point R on the transformed coordinate, but not attached to the frame so that it can acquire an acceleration. Instantaneously, it will have the same velocity as the point on the transformed coordinate, and its velocity and acceleration vectors will therefore transform the same as the point (30).

We will be considering accelerations only in the radial direction so that we need find affine connections only for indices 1, 4. The only non-zero partial derivative with these indices is

$$\frac{\partial g_{11}}{\partial x^4} = -2a\dot{a}. \quad (120)$$

The general expression for an affine connection for a diagonal metric is

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2g_{\lambda\lambda}} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right). \quad (121)$$

The only three non-zero affine connections with 1,4 indices are

$$\Gamma_{11}^4 = a\dot{a}, \quad \Gamma_{41}^1 = \Gamma_{14}^1 = \frac{\dot{a}}{a}. \quad (122)$$

The acceleration vector in FLRW coordinates of our test particle moving at the same velocity as a point on the transformed frame becomes

$$\left. \begin{aligned} A^{\hat{t}} &= \frac{dU^4}{ds} + \Gamma_{11}^4 U^1 U^1 \\ A^x &= \frac{dU^1}{ds} + \Gamma_{41}^1 (U^4 U^1 + U^1 U^4) \end{aligned} \right\}. \quad (123)$$

Using U^4 and U^1 in (29) we find

$$\left. \begin{aligned} A^{\hat{t}} &= \gamma \left(\frac{\partial \gamma}{\partial \hat{t}} \right)_R + a\dot{a} \frac{\gamma^2 \hat{V}^2}{a^2} = \gamma^4 \hat{V} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \gamma^2 \hat{V}^2 \\ A^x &= \gamma \left(\frac{\partial}{\partial \hat{t}} \left(\frac{\gamma \hat{V}}{a} \right) \right)_R + 2 \frac{\dot{a}}{a} \frac{\gamma^2 \hat{V}}{a} = \frac{\gamma^4}{a} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a^2} \gamma^2 \hat{V} \end{aligned} \right\}. \quad (124)$$

Since the acceleration vector of the test particle at R in the transformed coordinates will be orthogonal to the velocity vector, it becomes

$$\left. \begin{aligned} A^T &= 0 \\ A^R &= \frac{DU^R}{ds} \equiv -\frac{g}{c^2} \end{aligned} \right\}, \quad (125)$$

where A^R is the acceleration of a point on the R axis (so the gravitational field affecting objects like the galactic points is the negative of this), and g is defined so that mg is the force acting on an object whose mass is m . For a range of time in which $c(t)$ is reasonably constant, $g = \frac{d^2 R}{dT^2}$, the normal acceleration. Since the vector A^λ will transform like dT, dR (1):

$$A^R = \frac{1}{c} R_t A^{\hat{t}} + R_x A^x \quad (126)$$

so that

$$\begin{aligned} -\frac{g}{c^2} &= \left[\gamma^4 \hat{V} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \gamma^2 \hat{V}^2 \right] \frac{1}{c} R_t + \\ &\quad + \left[\frac{\gamma^4}{a} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a^2} \gamma^2 \hat{V} \right] R_x. \end{aligned} \quad (127)$$

With the use of (31), this can be simplified to

$$-\frac{g}{c^2} = \frac{R_x}{a} \left[\gamma^2 \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \hat{V} \right]. \quad (128)$$

In terms of the normalized coordinates for a flat universe (Appendix A.3), this becomes

$$g = -\frac{s'}{t} \left[\gamma^2 W' \left(\frac{u}{4} + W \right) - \frac{W}{2} \right]. \quad (129)$$

The acceleration g can be thought of as the gravitational field caused by the mass of the surrounding galactic points, which balances to zero at the origin, where the frame is inertial, but goes to infinity at the horizon. It is the field which is slowing down the galactic points (for $\Lambda = 0$). It is also the field that can be thought of as causing the gravitational red shift (Appendix A.3).

Appendix C. Special and General Relativity extended to include a variable light speed

C.1. Introduction. The aim of this section is to outline a way that not only the Lorentz transform, but all of Special (SR) and General Relativity (GR) can be extended to allow a variable light speed with minimal changes from standard theory. The extended Lorentz transform for local coordinates is derived from the basic assumption of relativity that the light speed c is the same for all moving observers at the same space-time point even though the light speed and their relative velocity V may vary. To form SR vectors and tensors we use a differential construct $d\hat{T} = c dT$ from physical time T [4] and a dimensionless velocity $\hat{V} = \frac{V}{c}$. In addition we propose that the rest mass of a particle varies so as to keep its rest energy constant. This seems reasonable in order to eliminate the need for an external source or sink of energy for the rest mass. These assumptions simplify the construction of SR vectors and conserves the stress-energy tensor of an ideal fluid. For GR, we propose the standard GR Action, but use the extended stress-energy tensor and allow the gravitational constant G to vary with c . The variable light speed is introduced in the line element that determines the space-time curvature.

We will use the notation t for time when the light speed is $c(t)$, as it must be for a uniform and isotropic universe if it is to be variable. Then \hat{t} can be a transform from t : $\hat{t} = \int c(t) dt$. The GR curvature tensor is derived from a line element that typically has the time t appearing in

the combination of $c(t)dt$ that would require the tensor to contain the derivatives of c . The use of \hat{t} instead of t eliminates these derivatives without changing the relations of the components of the tensors, and also allows all the relations of curvilinear coordinates used for constant $c = 1$ to be retained. Then, from a solution with \hat{t} the observable physical t can be found with a transform from \hat{t} to t .

C.2. The extended Lorentz transform and Minkowski metric.

Let us consider two physical frames moving with respect to each other. The first frame (S) will have clocks and rulers whose readings we will represent by T and x . The second frame (S^*) will move in the x direction at a velocity of $V = \left(\frac{\partial x}{\partial T}\right)_{x^*}$ as measured by T and x and will have clocks and rulers whose coordinates we will represent by T^* and x^* . The velocity of the first frame will be V^* as measured by T^* and x^* . We assume that the light speed, even though variable, is the same as measured on both frames at the same space-time point. We also allow V to be variable.

In order for S to measure the small separation of points Δx^* on S^* , S^* sends two simultaneous ($\Delta T^* = 0$) signals as measured on its clocks, one at the beginning of Δx^* and the other at the end. S measures the space between the signals as Δx , but does not see these signals as simultaneous. The far end signal is delayed by ΔT over the near end signal for this reason. S measures Δx^* to be the distance Δx reduced by the distance that S^* has traveled in the time ΔT after S^* 's simultaneity ($\Delta T = 0$) with the near end, i.e., $\Delta x - V\Delta T$. Since we are looking for linear relationships, we assume that the S^* measure of Δx^* is proportional to the S measure:

$$\Delta x^* = \alpha (\Delta x - V\Delta T), \quad (130)$$

where we have allowed α and V to be varying, but approach a constant value for small Δ 's. We also assume that for the two Cartesian directions Δy and Δz perpendicular to the motion along x that the S^* and S coordinates are the same

$$\Delta y = \Delta y^*, \quad \Delta z = \Delta z^* \quad (131)$$

and that the time T does not depend on y or z . α will be determined from the assumption that the light speed is the same on all moving frames. We will adapt the analysis of Bergmann [26, p.33–36] to a variable light speed. Choosing the point of origin so that ΔT and ΔT^* vanish when Δx and Δx^* vanish, we expect that ΔT^* will be a linear

function of ΔT and Δx :

$$\Delta T^* = \gamma \Delta T + \zeta \Delta x, \quad (132)$$

where α , γ , and ζ are slowly varying functions that approach a constant for small Δ 's. We will now determine their values.

We assume that the light speed can be variable, but in small intervals of time and distance it will be almost constant. It will have the same values in S^* as in S at the same space-time point. For light moving in an arbitrary direction, each measures the light speed c as the change in distance divided by the change in time of its own coordinates:

$$\Delta x^2 + \Delta y^2 + \Delta z^2 = c^2 \Delta T^2, \quad (133)$$

$$\Delta x^{*2} + \Delta y^{*2} + \Delta z^{*2} = c^2 \Delta T^{*2}, \quad (134)$$

where we have chosen an origin where all the Δ 's vanish. By using (131) and (130) in (134), we can eliminate the starred items to get

$$\alpha^2 (\Delta x - V \Delta T)^2 + \Delta y^2 + \Delta z^2 = c^2 (\gamma \Delta T + \zeta \Delta x)^2. \quad (135)$$

We can rearrange the terms to obtain

$$\begin{aligned} (\alpha^2 - c^2 \zeta^2) \Delta x^2 - 2 (V \alpha^2 + c^2 \gamma \zeta) \Delta x \Delta T + \Delta y^2 + \Delta z^2 = \\ = (c^2 \gamma^2 - V^2 \alpha^2) \Delta T^2. \end{aligned} \quad (136)$$

If we compare this to (133) we get

$$c^2 \gamma^2 - V^2 \alpha^2 = c^2, \quad (137)$$

$$\alpha^2 - c^2 \zeta^2 = 1, \quad (138)$$

$$V \alpha^2 + c^2 \gamma \zeta = 0. \quad (139)$$

We can solve these three equations for the three unknowns α , γ , and ζ . We obtain the solutions:

$$\gamma^2 = \frac{1}{1 - \frac{V^2}{c^2}}, \quad (140)$$

$$\zeta = \frac{1 - \gamma^2}{\gamma V} = -\frac{\gamma V}{c^2}, \quad (141)$$

$$\alpha^2 = -\frac{c^2 \gamma \zeta}{V} = \gamma^2. \quad (142)$$

Thus in the differential limit of Δ 's going to zero, we write them as differentials, so the relation of differentials becomes

$$dT^* = \gamma \left(dT - \frac{V}{c^2} dx \right), \quad (143)$$

$$dx^* = \gamma (dx - V dT), \quad (144)$$

By inverting this we get

$$dT = \gamma \left(dT^* + \frac{V}{c^2} dx^* \right), \quad (145)$$

$$dx = \gamma (dx^* + V dT^*), \quad (146)$$

so $V^* = -V$ as you would expect.

This is the same as for a constant c , except here c has been allowed to vary.

We define a line element ds by the relation

$$ds^2 \equiv c^2 dT^2 - dx^2 - dy^2 - dz^2. \quad (147)$$

If we substitute (131), (145) and (146) into (147), the form is the same:

$$ds^2 = c^2 dT^{*2} - dx^{*2} - dy^{*2} - dz^{*2}. \quad (148)$$

That is, the extended world line is invariant in form to changes in coordinates on frames moving at different velocities. The line element is symmetric in the spatial coordinates, so it is valid for motion in any direction. In polar coordinates this becomes

$$ds^2 = c^2 dT^2 - dR^2 - R^2 d\theta^2 - R^2 \sin^2 \theta d\phi^2. \quad (149)$$

This is the Minkowski line element (\hat{M}) extended to allow for a variable light speed. Both \hat{L} and \hat{M} are valid in the two dimensions T and R even if the metrics of S^* and S did not have equal transverse differentials, but had no transverse events ($d\theta = d\phi = d\theta^* = d\phi^* = 0$).

Notice that if we divide (147) and (148) by c^2 the two equations still have identical forms, so that the differential time $d\tau \equiv \frac{ds}{c}$ is also invariant in form to \hat{L} transforms. Since $d\tau = dT$ for constant spatial coordinates, τ is the time on a clock moving with the frame.

This derivation has depended on a physical visualization so that we assume that differentials that represent physical time and radial distance must have a \hat{M} metric for their time and distance differentials

in at least two dimensions and an extended Lorentz transform \hat{L} to other collocated physical differentials of time and distance on a frame moving at a velocity V . We will call such differentials physical coordinates. Time and distance coordinates that do not have these relations will not be physical; one or the other may be physical, but not both unless they have a \hat{M} metric.

The extended Lorentz transform \hat{L} can be written in a symmetric form using $d\hat{T} \equiv cdT$ and $\hat{V} \equiv \frac{V}{c}$ with the velocity in the R direction as it will be in a homogeneous and isotropic (FLRW) universe:

$$d\hat{T}^* = \gamma \left(+d\hat{T} - \hat{V} dR \right), \quad (150)$$

$$dR^* = \gamma \left(-\hat{V} d\hat{T} + dR \right). \quad (151)$$

In general for a varying c , \hat{T} is not a transform from T alone, although, as we have shown in (150), we can use the construct $d\hat{T} = cdT$ to describe the \hat{L} transform. In a FLRW universe for events in the radial direction measured by the variables (t, χ) , if c is variable, it is a simple function of t since homogeneity in space makes it independent of χ . In this case \hat{t} is a transform from t alone (e.g., formula 196).

C.3. Extended SR particle kinematics using contravariant vectors. In this section I will outline the way vectors and tensors can be defined when the light speed is variable. In Cartesian coordinates, let $dx^1, dx^2, dx^3 = dx, dy, dz$, and $dx^4 = d\hat{T} = cdT$. The \hat{M} metric then becomes

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (152)$$

where $\eta_{\mu\nu} = (-1, -1, -1, +1)$ for $\mu = \nu$, and zero for $\mu \neq \nu$. The velocity \dot{x}^μ is $\frac{dx^\mu}{d\hat{T}} = \frac{V^\mu}{c}$ with $\dot{x}^4 = 1$. (The dot represents the derivative with respect to $d\hat{T}$.) The world velocity becomes

$$U^\mu = \frac{dx^\mu}{ds} = \gamma \dot{x}^\mu. \quad (153)$$

The quantities \dot{x}^μ and U^μ are therefore dimensionless. In order to make the rest mass energy constant, we define $\hat{m} = mc^2$ and the extended energy-momentum vector as

$$P^\mu = \hat{m}U^\mu = \hat{m}\gamma\dot{x}^\mu, \quad (154)$$

so that $P^4 = \hat{m}\gamma = E$, the particle energy. If p is the magnitude of the physical momentum (γmV), the EP vector magnitude is $E^2 - c^2p^2 = \hat{m}^2$. It has units of energy rather than momentum or mass.

The \hat{L} transform for the components of the EP vector is

$$E^* = \gamma (E - \hat{V}pc). \quad (155)$$

For photons, $\hat{m} = 0$, so $E = h\nu$ and $p = \frac{h}{\lambda} = \frac{h\nu}{c}$, and the \hat{L} transform is

$$\nu^* = \gamma\nu (1 - \hat{V}), \quad (156)$$

which is the familiar relativistic Doppler effect.

The force vector becomes

$$F^\mu = \frac{dP^\mu}{ds} = \hat{m}A^\mu = \hat{m} \frac{dU^\mu}{ds}. \quad (157)$$

The first three components $\frac{F^i}{\gamma}$ will be the force f^i felt by an object of mass m when the light speed is c (i represent the three spatial coordinates). In taking the derivative of P^i , we are implying that $mc \frac{d(\frac{\gamma V}{dT})}{dT}$ is more fundamental in determining the physical force than $m \frac{d(\frac{\gamma V}{dT})}{dT}$ when the light speed is variable. We can express the gravitation force in the usual way as mg^i , where $g^i = A^i \frac{c^2}{\gamma}$. Herein $\frac{c}{\gamma} F^4$ is the rate of work $f^i V^i$ required to change the rate of change of energy $\frac{d(\frac{\gamma \hat{m}}{dT})}{dT}$. All these world vectors are invariant to the \hat{L} transform and the \hat{M} line element. They become the usual vectors when c is constant.

C.4. Extended analytical mechanics. We will next show how the Euler-Langrange equations apply to extended particle kinematics [26]. For a mechanical system with conservative forces in $(n+1)$ -dimensional space whose differentials are $(dx^i, d\hat{T})$, the action S is

$$S = \int L_p ds. \quad (158)$$

Minimizing S gives relations for L_p , the particle Lagrangian. With no force acting, we will use

$$L_p = \hat{m} \sqrt{\eta_{\mu\nu} U^\mu U^\nu}, \quad (159)$$

so the momenta are

$$P_\mu = \frac{\partial L_p}{\partial U^\mu} = \frac{\hat{m} \eta_{\mu\nu} U^\nu}{\sqrt{\eta_{\mu\nu} U^\mu U^\nu}}. \quad (160)$$

The root in this equation has the value 1 which makes it possible to solve it for U^μ

$$U^\mu = \frac{\eta^{\mu\nu} P_\nu}{\hat{m} \sqrt{\eta_{\mu\nu} U^\nu U^\nu}} = \frac{P^\mu}{\hat{m}}, \quad (161)$$

consistent with (154).

So,

$$U^\mu P_\mu = \frac{\eta^{\mu\nu} P_\mu P_\nu}{\hat{m}}. \quad (162)$$

The Hamiltonian \hat{H} becomes

$$\hat{H} = -L_p + U^\mu P_\mu = -\sqrt{\eta^{\mu\nu} P_\mu P_\nu} + \frac{\eta^{\mu\nu} P_\mu P_\nu}{\hat{m}}. \quad (163)$$

Let $p \equiv \sqrt{\eta^{\mu\nu} P_\mu P_\nu} = \hat{m}$, so

$$\hat{H} = \frac{p^2}{\hat{m}} - p. \quad (164)$$

Thus \hat{H} vanishes, but its derivative with respect to P_μ does not:

$$U^\mu = \frac{\partial \hat{H}}{\partial P_\mu} = 2 \frac{\eta^{\mu\nu} P_\nu}{\hat{m}} - \frac{\eta^{\mu\nu} P_\nu}{p} = \frac{P^\mu}{\hat{m}}, \quad (165)$$

$$\frac{dP_\mu}{ds} = -\frac{\partial \hat{H}}{\partial x^\mu} = 0, \quad (166)$$

where P_μ is conserved since we have considered no force acting.

C.5. Extended stress-energy tensor for ideal fluid. An ideal fluid can be treated in a similar way. It is a collection of n particles per unit volume of mass m . We can form a rest energy density function $\hat{\rho} = n\hat{m}$. In this case, $\hat{\rho}$ is not constant because n is a function of time and distance. We will use t instead of T to indicate that we are initially limiting this analysis to a rest frame of FLRW attached to a galactic point where c is a function of t . This can be transformed to other frames by a \hat{L} transform. It turns out that $\hat{\rho}$ using $d\hat{t}$ and $u^\mu = \frac{V^\mu}{c(t)}$ has much the same properties as $\rho = nm$ using dt and V^μ with constant c .

The conservation law for particles in nonrelativistic terms for n flowing at a velocity $V^i = cu^i$ is

$$\frac{\partial n}{\partial \hat{t}} + n u^i_{,i} = 0, \quad (167)$$

where we have assumed that the differential of c with distance is zero. For the conservation of energy we must include the stress forces $t^{ij} dA_j$ operating on the area of the differential volume, like the pressure p where

$t^{ij} = p\delta^{ij}$. We can convert the area stress forces by Gauss' theorem to a volume change in momentum to give a total 3D energy flux of cP^i , where

$$P^i = \hat{\rho}u^i + u^j t^{ji}. \quad (168)$$

The conservation of the fluid rest energy ($u^i = 0$) then becomes

$$\frac{\partial \hat{\rho}}{\partial t} + \text{div}(cP^i) = 0, \quad (169)$$

or

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + P^i{}_{,i} = 0. \quad (170)$$

The Newtonian law linking the rate of change of the generalized velocity $u^i = \frac{dx^i}{dT}$ to the force per unit volume f^i in nonrelativistic terms can be written as

$$\hat{\rho} \frac{du^i}{d\hat{t}} = f^i. \quad (171)$$

We can follow through the steps in any of the standard texts [26] to obtain the generalized stress-energy tensor of an ideal fluid in its rest frame to be

$$T^{\mu\nu} = T_{\mu\nu} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & \hat{\rho} \end{pmatrix},$$

which is used in §C.8.

This can be generalized for a frame moving at a world velocity U^μ :

$$T^{\mu\nu} = (\hat{\rho} + p)U^\mu U^\nu - p\eta^{\mu\nu}. \quad (172)$$

One can see that this is the same tensor since in the rest frame of the fluid $\hat{T} = \hat{t}$, $U^i = 0$, $U^4 = 1$.

The divergence of the stress-energy tensor is the force per unit volume:

$$T^{\mu\nu}{}_{,\nu} = F^\mu. \quad (173)$$

The conservation of rest energy (eq170) can then be written:

$$F^\mu = T^{\mu\nu}{}_{,\nu} = 0. \quad (174)$$

C.6. Extended electromagnetic vectors and tensors. We will assume that the light speed that appears in electromagnetic theory

(E/M) is the same as appears in relativity theory. If it were not so, it would be a remarkable coincidence if they were the same today, but different at other times. The E/M light speed obeys the relation

$$c^2 = \frac{1}{\epsilon_0 \mu_0}, \quad (175)$$

where ϵ_0 and μ_0 are the electric and magnetic “constants” of free space, resp. If c is variable, then either ϵ_0 or μ_0 or both must vary.

Current measurements with atomic clocks [18,19] have achieved an accuracy that indicate the frequency of atomic spectra do not change with time. Of course, when measured on a frame moving at a different velocity or in a gravitational field, frequency does change. There are also astronomical indications of a variation in α_f [20], but these are much smaller than would occur if $c(t)$ changed as calculated here. On an inertial frame, this means that the fine structure constant α_f and the Rydberg constant $R_\infty c$ (expressed as a frequency) do not change with $c(t)$.

The fine structure constant α_f in SI units [24] is

$$\alpha_f = \frac{e^2}{4\pi\epsilon_0 \hbar c}, \quad (176)$$

and the Rydberg frequency is

$$R_\infty c = \alpha_f^2 \frac{m_e c^2}{4\pi \hbar} = \frac{e^4 m_e}{\epsilon_0^2 (4\pi \hbar)^3}. \quad (177)$$

Because α_f is dimensionless, the $4\pi\epsilon_0$ is often omitted in the fine structure constant since it is unity in Gaussian coordinates, but it is essential here if we are to consider a variable $c(t)$ for the universe.

For these to remain constant while keeping e , \hbar and mc^2 constant requires that

$$\epsilon_0(t) c(t) = \frac{1}{\mu_0(t) c(t)} \equiv k = \epsilon(t_0) c(t_0) = \text{constant}, \quad (178)$$

where $\frac{1}{k} = \sqrt{\frac{\mu_0}{\epsilon_0}}$, the impedance of free space. This assumption means that the electrostatic repulsion f between two electrons will vary:

$$f = -\frac{e^2}{\epsilon_0 R^2}. \quad (179)$$

Maxwell’s equations in 3D field vectors in the rest frame of FLRW

with a constant speed of light [25] are

$$\left. \begin{aligned} \operatorname{curl} E + \frac{\partial B}{\partial t} &= 0 \\ \operatorname{curl} H - \frac{\partial D}{\partial t} &= J \\ \operatorname{div} D &= \sigma \\ \operatorname{div} B &= 0 \\ \operatorname{div} J + \frac{\partial \sigma}{\partial t} &= 0 \end{aligned} \right\}. \quad (180)$$

Scalar ϕ and vector A potentials can be introduced such that

$$\left. \begin{aligned} B &= \operatorname{curl} A \\ E &= -\operatorname{grad} \phi - \frac{\partial A}{\partial t} \end{aligned} \right\}, \quad (181)$$

and the equation for the force on a particle with charge q , mass m , and velocity V is (see [26, p. 118])

$$m \frac{d(\gamma V)}{dt} = q(E + V \otimes B). \quad (182)$$

With the use of \hat{t} and (178) and with the relations $D = \epsilon_0 E$, $B = \mu_0 H$ for free space, these can be converted to exactly the same equations by replacing t by \hat{t} and by replacing the field variables by hat variables so that the partial time derivatives of hat variables do not include ϵ_0 , μ_0 , or c except in combinations equaling k , a constant. This is accomplished by the following: $\hat{B} = kB$, $\hat{H} = \frac{H}{c}$, $\hat{D} = D = \hat{E} = \epsilon_0 E$, $\hat{\sigma} = \sigma$, $\hat{J} = \frac{J}{c}$, $\hat{A} = kA$, $\hat{\phi} = \frac{c}{k}\phi$, and $\hat{q} = q$. Thus, with hat variables and \hat{t} , Maxwell's equations have only two fields \hat{E}, \hat{H} with no varying coefficients

$$\left. \begin{aligned} \operatorname{curl} \hat{E} + \frac{\partial \hat{H}}{\partial \hat{t}} &= 0 \\ \operatorname{curl} \hat{H} - \frac{\partial \hat{E}}{\partial \hat{t}} &= \hat{J} \\ \operatorname{div} \hat{E} &= \hat{\sigma} \\ \operatorname{div} \hat{H} &= 0 \\ \operatorname{div} \hat{J} + \frac{\partial \hat{\sigma}}{\partial \hat{t}} &= 0 \end{aligned} \right\}. \quad (183)$$

The potential equations become

$$\left. \begin{aligned} \hat{H} &= \text{curl } \hat{A} \\ \hat{E} &= -\text{grad } \hat{\phi} - \frac{\partial \hat{A}}{\partial \hat{t}} \end{aligned} \right\}. \quad (184)$$

Since they have no coefficients that vary with time, they are \hat{L} covariant to frames with $d\hat{T} = c(t)dT$ replacing $d\hat{t}$ just like the original Maxwell's equations. Thus, they are valid in every moving frame whose physical time is T .

With $\hat{V} = \frac{V}{c}$ and $\hat{m} = mc^2$ the pondermotive equation 182 becomes

$$\hat{m} \frac{d(\gamma \hat{V})}{d\hat{T}} = \frac{q}{\epsilon_0} (\hat{E} + \hat{V} \otimes \hat{H}). \quad (185)$$

These all become the usual expressions when the speed of light is constant $c = 1$.

E/M world vectors and tensors can be constructed in the usual way [26]. Thus, the extended potential vector is $\hat{\phi}_\mu = (\hat{A}_i, -\hat{\phi})$, the extended charge vector $\hat{\Gamma}^\mu = (\hat{J}^i, -\hat{\sigma})$, and the extended E/M field tensor is

$$\hat{F}_{\mu\nu} = \begin{pmatrix} 0 & -\hat{H}_3 & +\hat{H}_2 & -\hat{E}_1 \\ +\hat{H}_3 & 0 & -\hat{H}_1 & -\hat{E}_2 \\ -\hat{H}_2 & +\hat{H}_1 & 0 & -\hat{E}_3 \\ +\hat{E}_1 & +\hat{E}_2 & +\hat{E}_3 & 0 \end{pmatrix}.$$

The field tensor can be obtained from the curl of the potential vector

$$\hat{F}_{\mu\nu} = \hat{\phi}_{\mu,\nu} - \hat{\phi}_{\nu,\mu} \quad (186)$$

and Maxwell's equations become the divergence of the field tensor equaling the charge vector [26, p. 113]:

$$\hat{F}^{\mu\nu}{}_{,\nu} = -\hat{\Gamma}^\mu. \quad (187)$$

The pondermotive equation for a particle of charge q and mass m becomes a force vector equaling \hat{m} times an acceleration vector:

$$\frac{q}{\epsilon_0} \hat{F}_{\mu\nu} \hat{U}^\nu = -\hat{m} \eta_{\mu\nu} \frac{d\hat{U}^\nu}{ds}. \quad (188)$$

The SR stress-energy tensor is

$$T^{\mu\nu} = \frac{1}{\epsilon_0} \left(F_\lambda^\mu F^{\nu\lambda} - \frac{1}{4} \eta^{\mu\nu} F^{\mu\sigma} F_{\sigma\nu} \right). \quad (189)$$

For GR with curvilinear coordinates, the stress-energy tensor is

$$T^{\mu\nu} = \frac{1}{\epsilon_0} \left(F_{\lambda}^{\mu} F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\mu\sigma} F_{\sigma\nu} \right). \quad (190)$$

The dimensions of \hat{E} , \hat{H} , and $F^{\mu\nu}$ are electric charge per unit area, whereas for $T^{\mu\nu}$ it is energy per unit volume. Because of ϵ_0 , both the force and the energy density are dependent on $c(t)$ just like the force between two electrons (179).

C.7. The extended FLRW metric for a homogeneous and isotropic universe. We assume that the concentrated lumps of matter, like stars and galaxies, can be averaged to the extent that the universe matter can be considered continuous, and that the surroundings of every point in space can be assumed isotropic and the same for every point.

By embedding a maximally symmetric (i.e., isotropic and homogeneous) three-dimensional sphere, with space dimensions r , θ , and ϕ , in a four dimension space which includes time t , one can obtain a differential line element ds [9, p. 403] such that

$$ds^2 = g(t) dt^2 - f(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right), \quad (191)$$

where

$$r = \begin{cases} \sin \chi, & k = 1, \\ \chi, & k = 0, \\ \sinh \chi, & k = -1, \end{cases} \quad (192)$$

while k is a spatial curvature determinant to indicate a closed, flat, or open universe, resp., and

$$d\chi^2 \equiv \frac{dr^2}{1 - kr^2}. \quad (193)$$

We let $a(t) \equiv \sqrt{f(t)}$ be the cosmic scale factor multiplying the three-dimensional spatial sphere, so that the differential radial distance is $a(t) d\chi$.

The $g(t)$ has normally been taken as $g(t) = c^2 = \text{constant}$, so that c is the constant physical light speed and t is the physical time on each co-moving point of the embedded sphere. In both cases by physical, we mean that their value can represent measurements by physical means like standard clocks and rulers, or their technological equivalents. In order to accommodate the possibility of the light speed being a function

of time, we make $g(t) = c(t)^2$. The resulting equation for the differential line element becomes an extended FLRW metric:

$$ds^2 = c(t)^2 dt^2 - a(t)^2 (d\chi^2 + r^2 d\omega^2), \quad (194)$$

where $d\omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. For radial world lines this metric becomes Minkowski in form with a differential of physical radius of $a(t)d\chi$.

It will be convenient to introduce the time related quantity \hat{t} , which we will call a generalized cosmic time, defined by

$$\hat{t} \equiv \int_0^t c(t) dt, \quad (195)$$

$$t = \int_0^{\hat{t}} \frac{d\hat{t}}{\hat{c}(\hat{t})}, \quad (196)$$

where $\hat{c}(\hat{t}) = c(t)$, and where the lower limit is arbitrarily chosen as 0.

The line element then becomes

$$ds^2 = d\hat{t}^2 - a^2 (d\chi^2 + r^2 d\omega^2). \quad (197)$$

It should be emphasized that \hat{t} itself is a legitimate more general coordinate. \hat{t} plays the same role in the FLRW space with a variable $c(t)$ as does t for a FLRW space with constant c . The physical time t is a transform from it. \hat{t} and its transform to t allows for the physics to apply to a variable light speed.

C.8. Unchanged GR field equation for $c(t)$. We assume the standard action of GR without any non-standard additions that some have used to produce the variable light speed [4]. We allow the metric that determines the curvature tensor to introduce the varying light speed. This will create a relationship between the varying light speed and the components of the stress-energy tensor. In addition we use the Lagrangian L_{se} of the extended stress-energy tensor. In order to use the standard GR action, we assume that $\frac{G}{c^4} \equiv \hat{G} = \frac{G_0}{c_0^4}$ is constant. This is needed to keep constant the Newtonian energy $-\frac{Gm_1m_2}{R}$ when the rest energy of mass mc^2 is constant. We also assume that Λ is constant, possibly representing some kind of vacuum energy density:

$$S = \int \sqrt{-g} (R - 2\Lambda + 16\pi\hat{G}L_{se}) d^4\xi, \quad (198)$$

where R is the Ricci scalar for the metric

$$ds^2 = g_{\mu\nu} d\xi^\mu d\xi^\nu, \quad (199)$$

and g is the determinate of $g_{\mu\nu}$. Minimizing the variation of S with $g_{\mu\nu}$, we get the usual GR field equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \hat{G} T_{\mu\nu}, \quad (200)$$

where $G_{\mu\nu}$ is calculated for a particular metric using the usual Riemannian geometry.

C.9. GR for FLRW universe with $c(t)$. We will now apply this field equation to an ideal fluid of density ρ and pressure p in a homogeneous and isotropic universe for which the extended FLRW line element in the variables t, r, θ, ϕ is (194):

$$ds^2 = c(t)^2 dt^2 - a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right). \quad (201)$$

As ds is written in (201), the components of $G_{\mu\nu}$ will contain first and second derivatives of $c(t)$. In order to find a solution to the field equation we will transform the time variable t to $\xi^4 = \hat{t}$. This will not change the relation of $G_{\mu\nu}$ to $T_{\mu\nu}$, but will eliminate the derivatives of $c(t)$ in $G_{\mu\nu}$ and transform $G_{\mu\nu}$ to a known solution. t is still the observable, and \hat{t} is a non-physical transform from it. For a perfect fluid of pressure p and mass density ρ , we can define $\hat{\rho} \equiv \rho c^2$ so that $\hat{\rho}$ obeys the same conservation and acceleration laws using $d\hat{t}$ as does ρ using dt (see §C.5). We can then write the two significant field equations [21, p. 729] for $a(\hat{t})$ as

$$\frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} - \Lambda = 8\pi \hat{G} \hat{\rho}, \quad (202)$$

and

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = -8\pi \hat{G} p, \quad (203)$$

where the dots represent derivatives with respect to \hat{t} . All variables (including \hat{t} and a) are in standard units. Equation (202) can be solved to give a as a function of \hat{t} , ρ , k , and Λ . When we know $c(\hat{t})$, we can obtain the observables $a(t)$ and $c(t)$ by transforming \hat{t} back to t . Solutions of these equations are carried out in §3 for a particular $c(t)$.

We would now like to show that the proposed variation of $G(c)$ and $m(c)$ is internally consistent. Equation (202) with $\hat{G} = \frac{G}{c^4}$ and $\hat{\rho} = \rho c^2$ can be multiplied by $\frac{a^3}{3}$, differentiated, and subtracted from $\dot{a}a^2$ times (203) to give

$$\frac{d}{d\hat{t}} \left(\frac{G\rho a^3}{c^2} \right) = -\frac{3G}{c^4} \dot{a}a^2 p. \quad (204)$$

For small p ,

$$\frac{G\rho a^3}{c^2} = \text{constant}. \quad (205)$$

If the energy density consists of n particles per unit volume of mass m , so $\rho = nm$, then the conservation of particles requires na^3 be constant (for small velocities). This makes

$$\frac{Gm}{c^2} = \text{constant}. \quad (206)$$

This is consistent with our assumptions that $\frac{G}{c^4}$ and mc^2 are constant.

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