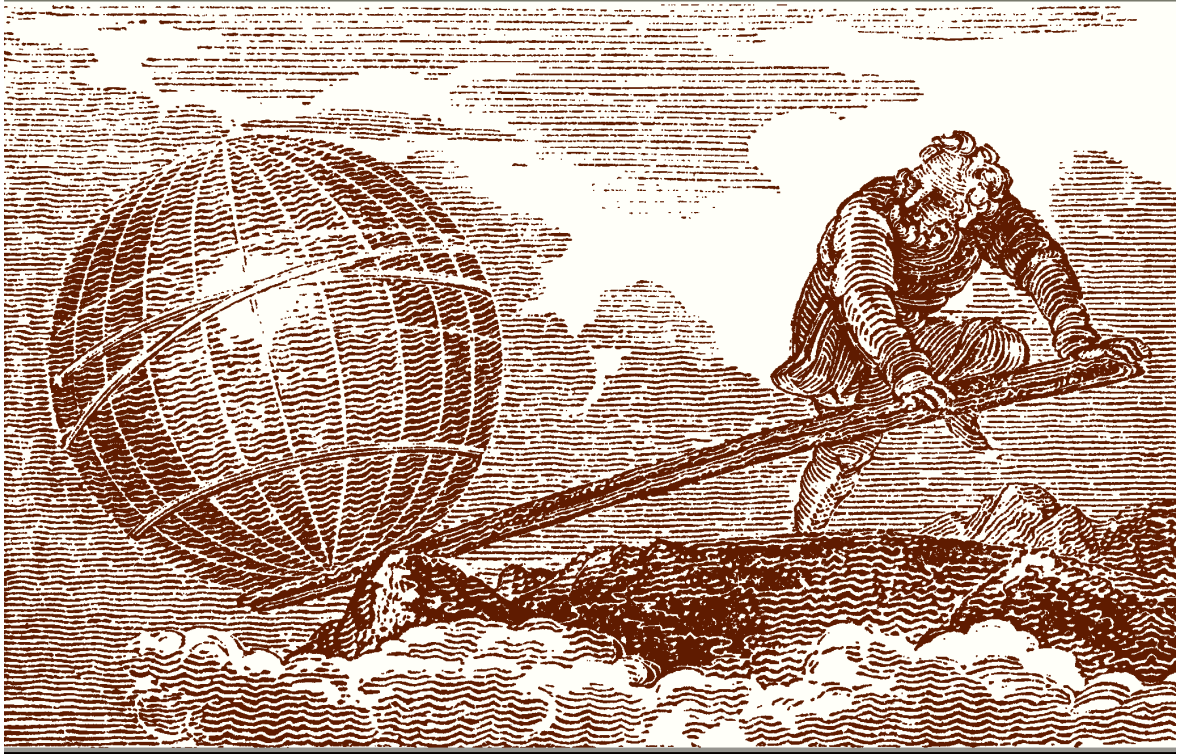


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# Non-Linear Cosmological Redshift: The Exact Theory According to General Relativity

Dmitri Rabounski

**Abstract:** A new method of calculation is applied to the frequency of a photon according to the travelled distance. It consists in solving the scalar geodesic equation (equation of energy) of the photon, and manifests gravitation, non-holonomy, and deformation of space as the intrinsic geometric factors affecting the photon's frequency. In the space of Schwarzschild's mass-point metric, the well-known gravitational redshift has been obtained. No frequency shift has been found in the rotating space of Gödel's metric, and in the space of Einstein's metric (a homogeneous distribution of ideal liquid and physical vacuum). The other obtained solutions manifest a cosmological effect: its magnitude increases with distance. The parabolic cosmological blueshift has been found in the space of a sphere of incompressible liquid (Schwarzschild's metric), and in the space of de Sitter's metric, which is a sphere filled with physical vacuum whose density is positive (it is a redshift, if the vacuum density is negative). The exponential cosmological redshift has been found in the expanding space of Friedmann's metric (empty or filled with ideal liquid and physical vacuum). This explains the accelerate expanding Universe. The redshift reaches  $z = e^\pi - 1 = 22.14$  at the event horizon. These results are obtained in a purely geometric way, without the use of the Doppler effect.

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**§1. Problem statement.** This is the second part of my research, which was started in the publication [1] where I introduced the *cosmological mass-defect* — a new predicted effect revealed according to the General Theory of Relativity. The essence of this effect is that a mass-bearing particle changes its relativistic mass according to the distance travelled by it. The magnitude of this effect can be either positive or negative, depending on the metric of that particular space (the kind of universe) wherein the particle travels.

As was shown, the cosmological mass-defect is obtained after integrating the scalar geodesic equation (equation of energy) of a mass-bearing particle. This equation determines the relativistic energy and mass of the particle at any distance (and moment of time) of its travel. When integrating the equation, the components of the fundamental metric tensor  $g_{\alpha\beta}$  are used according to the particular space metric (universe) under consideration. Thus the cosmological mass-defect can be calculated in each particular universe. Following this way, I showed that the cosmological mass-defect is present in most of the main (principal) cosmological models, and provided detailed calculation of its magnitude in each of these cases [1]. In the cosmological models, where this effect is present, the relativistic mass change becomes essential only at distances of the galaxies (“cosmologically large” distances).

This is the cosmological mass-defect in a nutshell. As was pointed out at the end of my previous paper [1], a logical continuation of this research would be solving the scalar geodesic equation of a massless particle (light-like particle, e.g. a photon). As a result, we should expect to obtain the frequency shift of the photon according to the travelled distance in each of the cosmological models.

Naturally, consider the geodesic equations. According to Zelmanov’s chronometrically invariant formalism [2–4], any four-dimensional (generally covariant) quantity is presented with its observable projections onto the line of time and the spatial section of an observer\*. This is as well true about the generally covariant geodesic equation. The time projection of it is the scalar geodesic equation (equation of energy). The spatial projection is the three-dimensional equation of motion of the particle. As was obtained by Zelmanov [2–4], the projected (chronomet-

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\*This formalism, known also as the theory of chronometric invariants, was introduced in 1944 by Abraham Zelmanov. It is originally given in his primary publications [2–4], while more details of the chronometrically invariant formalism can be found in the books [5, 6]. Chronometric invariance means that the quantity, which possesses this property, is invariant along the observer’s three-dimensional spatial section (which can be curved, inhomogeneous, deforming, rotating, etc.).

rically invariant) geodesic equations of a mass-bearing particle, whose relativistic mass is  $m$ , are

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i v^i + \frac{m}{c^2} D_{ik} v^i v^k = 0, \quad (1.1)$$

$$\frac{d(mv^i)}{d\tau} - m F^i + 2m (D_k^i + A_{k\cdot}^i) v^k + m \Delta_{nk}^i v^n v^k = 0, \quad (1.2)$$

while the projected geodesic equations of a massless (light-like) particle (we denote its relativistic frequency as  $\omega$ ) have the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i c^i + \frac{m}{c^2} D_{ik} c^i c^k = 0, \quad (1.3)$$

$$\frac{d(\omega c^i)}{d\tau} - \omega F^i + 2\omega (D_k^i + A_{k\cdot}^i) c^k + \omega \Delta_{nk}^i c^n c^k = 0, \quad (1.4)$$

where the derivation parameter  $d\tau = \sqrt{g_{00}} dt - \frac{1}{c^2} v_i dx^i$  is the physically observable (proper) time, which depends on the gravitational potential  $w = c^2 (1 - \sqrt{g_{00}})$  and the linear velocity  $v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}$  of the three-dimensional rotation of space. The factors affecting the particles are: the gravitational inertial force  $F_i$ , the angular velocity  $A_{ik}$  of the rotation of space due to its non-holonomy, the deformation  $D_{ik}$  of space, and the non-uniformity of space (the Christoffel symbols  $\Delta_{jk}^i$ ). Two factors of these affect the energy of the particle (according to the scalar geodesic equation, which is the equation of energy)

$$F_i = \frac{1}{\sqrt{g_{00}}} \left( \frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \quad \sqrt{g_{00}} = 1 - \frac{w}{c^2}, \quad (1.5)$$

$$D_{ik} = \frac{1}{2} \frac{\partial h_{ik}}{\partial t}, \quad D^{ik} = -\frac{1}{2} \frac{\partial h^{ik}}{\partial t}, \quad D = h^{ik} D_{ik} = \frac{\partial \ln \sqrt{h}}{\partial t}, \quad (1.6)$$

where  $\frac{\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}$ , and  $h_{ik}$  is the chr.inv.-metric tensor

$$h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k, \quad h^{ik} = -g^{ik}, \quad h_k^i = \delta_k^i. \quad (1.7)$$

As is seen, the geodesic equations of mass-bearing and massless particles have the same form. Only the sublight velocity  $v^i$  and the relativistic mass  $m$  are used for mass-bearing particles instead of the light velocity  $c^i$  and the frequency  $\omega$  of a photon.

It is natural then to suggest that we could solve the scalar geodesic equation of massless particles, which is equation (1.3), in analogy to, as I

solved in [1], the scalar geodesic equation of mass-bearing particles (1.1) with the components of the fundamental metric tensor  $g_{\alpha\beta}$  according to the respective space metrics (cosmological models).

However, at the end of my previous publication [1], I supposed that this is not a trivial task. My supposition was based on the fact that mass-bearing particles travel in the so-called non-isotropic region of space (space-time), which is the home of the sublight-speed and superluminal trajectories. Massless particles travel in the isotropic space, which is the home of the trajectories of light. The four-dimensional interval is zero everywhere in the isotropic space, while the interval of observable time  $d\tau$  and the three-dimensional observable interval  $d\sigma^2 = h_{ik} dx^i dx^k$  are nonzero and equal to each other

$$\left. \begin{aligned} ds^2 = c^2 d\tau^2 - d\sigma^2 = 0 \\ c^2 d\tau^2 = d\sigma^2 \neq 0 \end{aligned} \right\} \quad (1.8)$$

As a result, the isotropic space (space-time) is strict non-holonomic: the lines of time meet the three-dimensional coordinate lines at any point therein, and, hence, the isotropic space rotates as a whole at each of its points with the velocity of light (see [7,8] for detail). In terms of the language of algebra, the isotropic space condition, by equalizing the entire formula of  $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$  to zero, implies an additional relation among the particular components of  $g_{\alpha\beta}$  which thus can be transformed into each other in a certain way that does not violate the invariance of the metric as a whole\*. This additional condition should be taken into account when considering any problem in the isotropic space. As a result, the scalar equation of isotropic geodesics may have another solution than that obtained after integrating the scalar equation of non-isotropic geodesics. In other words, the frequency shift of photons may have another formulation than the relativistic mass change (mass-defect) of mass-bearing particles travelling in the same space (space-time).

This is why I initially split this study into two parts where, in the first part [1], the scalar geodesic equation of mass-bearing particles is solved, thus introducing the cosmological mass-defect.

However, after studying this problem in detail, I have arrived at another conclusion. Namely — the light-speed rotation, which is at-

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\*In particular, there should be a replacement among the components  $g_{00}$  and  $g_{0i}$ . In the case of Minkowski's space, which is the basic space-time of Special Relativity, this replacement means that the isotropic region of it should have the non-diagonal metric where  $g_{00} = 0$ ,  $g_{0i} = 1$ , and  $g_{11} = g_{22} = g_{33} = -1$ . Such isotropic metrics were studied in already the 1950's, by Alexei Petrov. See his *Einstein Spaces* [9]. For instance, §25 and the others therein.

tributed to the isotropic space (even in Minkowski's space, which is the basic space-time of Special Relativity) can be registered only by an observer who accompanies the isotropic space and photon. In other words, this is a light-like observer whose home is the isotropic space. A regular (sublight-speed) observer shall observe the isotropic space and all events in it according to the values of the fundamental metric tensor  $g_{\alpha\beta}$  which characterize his own (non-isotropic) space — home of “solid objects”. This is because, according to the theory of physically observable quantities, an observer should accompany his own reference body and coordinate grids spanned over it. As a result, the isotropic geodesic equations should be solved for a sublight-speed observer in the same way as the non-isotropic geodesic equations. In other words, when solving the scalar equation of isotropic geodesics, we should use the components of the fundamental metric tensor which are attributed to the home space (the coordinate grids and clocks) of “solid objects” which is the reference space of a regular observer.

I will give a complete explanation of this thesis later, in one of the chapters of the book on the cosmological mass-defect and the cosmological redshift (now — under preparation).

In the next paragraphs of this paper, after solving the scalar equation of isotropic geodesics in each of the main “cosmological” metrics, we will arrive at the formula of the frequency shift of a photon according to the travelled distance in each of the universes under consideration.

Actually, the method of integration and the solutions will have the same form as those for the cosmological mass-defect obtained for mass-bearing particles in my recent paper [1]. Therefore, to avoid repetition, I will omit some obvious formalities while simply referring to [1] wherein the calculations were explained with all necessary details.

**§2. Local redshift in the space of a mass-point (Schwarzschild's mass-point metric).** This is the metric of an empty space (in the sense that there is no distributed matter), wherein the field of gravitation and curvature is due to a spherical mass approximated as a mass-point at distances much larger than its radius. The metric, introduced in 1916 by Karl Schwarzschild [10], represented in the spherical three-dimensional coordinates  $x^1 = r$ ,  $x^2 = \varphi$ ,  $x^3 = \theta$ , has the form

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (2.1)$$

where  $r$  is the distance from the mass  $M$ ,  $r_g = \frac{2GM}{c^2}$  is the corresponding



gravitational radius of the mass, and  $G$  is the world-constant of gravitation.

This metric is free of rotation and deformation. The field of gravitation is the only factor affecting particles in the space. It is determined by  $g_{00}$  which is  $g_{00} = 1 - \frac{r_g}{r}$  according to the metric (2.1). Differentiating the gravitational potential  $w = c^2 (1 - \sqrt{g_{00}})$  according to the definition of the gravitational inertial force (1.5), then applying  $r \gg r_g$  (the field source is outside the state of gravitational collapse), we obtain the solely nonzero radial component of the force

$$F_1 = -\frac{c^2 r_g}{2r^2} = -\frac{GM}{r^2}. \quad (2.2)$$

In such a space, the scalar geodesic equation for a massless (light-like) particle (1.3), e.g. a photon, takes the form

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_1 c^1 = 0, \quad (2.3)$$

where  $c^1 = \frac{dr}{d\tau}$  is the observable velocity of light (massless particle). This equation transforms into  $\frac{d\omega}{\omega} = \frac{1}{c^2} F_1 dr$ , which is  $d \ln \omega = -\frac{GM}{c^2} \frac{dr}{r^2}$ . It has the solution

$$\omega = \omega_0 e^{\frac{GM}{c^2 r}} \simeq \omega_0 \left( 1 + \frac{GM}{c^2 r} \right). \quad (2.4)$$

This solution means that a photon emitted by a massive body, which is the field source, gains an additional energy due to the presence of the gravitational field. This phenomenon decreases with distance from the field source according to the formula (2.4). As a result, the photon's energy and frequency should decrease with the travelled distance: the photon's frequency should be redshifted when registered by an observer, who is distantly located from the field source. For instance, let a photon have a frequency  $\omega_0$  being emitted from the surface of a star, whose mass is  $M$  and whose radius is  $r_*$ . Then its frequency registered by an observer, who is located at a distance  $r$  from the star, is  $\omega < \omega_0$ . We then obtain, according to the formula (2.4), that the observed redshift of the photon has the magnitude

$$z = \frac{\omega_0 - \omega}{\omega} = e^{\frac{GM}{c^2 r_*} - \frac{GM}{c^2 r}} - 1 \simeq \frac{GM}{c^2 r_*} - \frac{GM}{c^2 r}. \quad (2.5)$$

Note that this is a local phenomenon, not a cosmological effect: its magnitude decreases with distance from the source of the field, very fast, so that it becomes actually zero at “cosmologically large” distances.

This is the gravitational redshift — the well-known effect of the General Theory of Relativity (first registered in the spectra of white dwarfs). I speak of this effect herein, because of the new method of derivation. Classically, it is derived from the conservation of energy of a photon travelling in a stationary gravitational field [11, §88]. The same classical method of derivation was also used by Zelmanov. On the other hand, our method of deduction, based on the integration of the scalar equation of isotropic geodesics, allows to represent this effect as something particular among the other similar effects which can be calculated for any other space metric (gravitational field).

**§3. Local redshift in the space of an electrically charged mass-point (Reissner-Nordström metric).** This is an extension of the mass-point metric, where the spherical massive island of matter is electrically charged. As a result, the space of the Reissner-Nordström metric is not empty but filled with a spherically symmetric electromagnetic field (distributed matter). This metric was first considered in 1916 by Hans Reissner [12] then, in 1918, by Gunnar Nordström [13]. It is

$$ds^2 = \left(1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad (3.1)$$

with the same denotations as those of the mass-point metric, while  $r_q^2 = \frac{Gq^2}{4\pi\epsilon_0 c^4}$ , where  $q$  is the corresponding electric charge, and  $\frac{1}{4\pi\epsilon_0}$  is Coulomb's force constant. This metric is as well free of rotation and deformation. The gravitational inertial force is determined, according to  $g_{00} = 1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}$ , by both Newton's force and Coulomb's force. Assuming that the source of the field is outside the state of gravitational collapse ( $r \gg r_g$ ), and that the electric field is weak ( $r \gg r_q$ ), we obtain

$$F_1 = -\frac{c^2}{2} \left( \frac{r_g}{r^2} - \frac{2r_q^2}{r^3} \right) = -\frac{GM}{r^2} + \frac{Gq^2}{4\pi\epsilon_0 c^2} \frac{1}{r^3}. \quad (3.2)$$

The scalar geodesic equation for a massless particle (1.3) takes the form

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_1 c^1 = 0, \quad (3.3)$$

which transforms into  $d \ln \omega = \left( -\frac{GM}{c^2 r^2} + \frac{Gq^2}{4\pi\epsilon_0 c^4} \frac{1}{r^3} \right) dr$ , and solves as

$$\omega = \omega_0 e^{\frac{GM}{c^2 r} - \frac{1}{2r^2} \frac{Gq^2}{4\pi\epsilon_0 c^4}} \simeq \omega_0 \left( 1 + \frac{GM}{c^2 r} - \frac{1}{2r^2} \frac{Gq^2}{4\pi\epsilon_0 c^4} \right). \quad (3.4)$$

This solution manifests that photons should also gain an additional energy in the field of an electrically charged massive body. But this additional energy is lesser than that gained from the gravitational field (the first term in our formula 3.4), owing to an energy loss due to the presence of the electric field (the negative second term in 3.4).

We observe no such an electrically charged massive body (like planets, stars, or galaxies) whose gravitational field would be weaker than its electromagnetic field. We therefore should conclude that the photon's frequency in the space of the Reissner-Nordström metric should be redshifted when registered by an observer who is located at a distance  $r$  from the field source. The redshift, according to our solution (3.4), should be

$$\begin{aligned} z = \frac{\omega_0 - \omega}{\omega} &= e^{\frac{GM}{c^2 r_*} - \frac{1}{2r_*^2} \frac{Gq^2}{4\pi\epsilon_0 c^4} - \frac{GM}{c^2 r} + \frac{1}{2r^2} \frac{Gq^2}{4\pi\epsilon_0 c^4}} - 1 \simeq \\ &\simeq \frac{GM}{c^2 r_*} - \frac{1}{2r_*^2} \frac{Gq^2}{4\pi\epsilon_0 c^4} - \frac{GM}{c^2 r} + \frac{1}{2r^2} \frac{Gq^2}{4\pi\epsilon_0 c^4}, \end{aligned} \quad (3.5)$$

where  $r_*$  is the radius of the field source. This redshift shall be lesser than the purely gravitational redshift (considered in §2).

The magnitude of the redshift decreases with distance from the field source. Therefore, the redshift in the space of the Reissner-Nordström metric is also a local phenomenon, not a cosmological effect.

Herein we have obtained that the frequency shift can be due to not only the field of gravitation, but also due to the electromagnetic field. Such an effect was not considered in the General Theory of Relativity prior to the present study.

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Following the same deduction, the frequency shift could also be calculated in two other primary extensions of the mass-point metric. The Kerr metric (introduced in 1963 by Roy P. Kerr [14, 15]) describes the space of a rotating mass-point. The Kerr-Newman metric (introduced in 1965 by Ezra T. Newman [16, 17]) describes the space of a rotating, electrically charged mass-point. However, there is a problem with the calculation. These metrics, determined in the vicinity of the mass-point (field source), do not contain the distribution function of the rotational velocity with distance from the source. Therefore, when integrating the geodesic equation, we should be enforced to introduce these functions on our own behalf (which can be true or false, depending on our understanding of the space rotation). We therefore omit these two cases from consideration.

**§4. No frequency shift present in the rotating space with self-closed time-like geodesics (Gödel's metric).** This space metric, introduced in 1949 by Kurt Gödel [18], has the form

$$ds^2 = a^2 \left[ (d\tilde{x}^0)^2 + 2e^{\tilde{x}^1} d\tilde{x}^0 d\tilde{x}^2 - (d\tilde{x}^1)^2 + \frac{e^{2\tilde{x}^1}}{2} (d\tilde{x}^2)^2 - (d\tilde{x}^3)^2 \right], \quad (4.1)$$

where  $a = \text{const} > 0$  [cm] is a constant determined as  $a^2 = \frac{c^2}{8\pi G\rho} = -\frac{1}{2\lambda}$ . Such a space is not empty, but filled with dust of density  $\rho$ , and physical vacuum ( $\lambda$ -field). Also, it rotates so that time-like geodesics are self-closed therein.

Gödel's metric was originally given in the form (4.1), through the dimensionless Cartesian coordinates  $d\tilde{x}^0 = \frac{1}{a} dx^0$ ,  $d\tilde{x}^1 = \frac{1}{a} dx^1$ ,  $d\tilde{x}^2 = \frac{1}{a} dx^2$ ,  $d\tilde{x}^3 = \frac{1}{a} dx^3$ , which emphasize the meaning of the world-constant  $a$ . We now move to the regular Cartesian coordinates  $a d\tilde{x}^0 = dx^0 = c dt$ ,  $a d\tilde{x}^1 = dx^1$ ,  $a d\tilde{x}^2 = dx^2$ ,  $a d\tilde{x}^3 = dx^3$ , which are more suitable for the calculation of the components of the fundamental metric tensor. We obtain

$$ds^2 = c^2 dt^2 + 2e^{\frac{x^1}{a}} c dt dx^2 - (dx^1)^2 + \frac{e^{\frac{2x^1}{a}}}{2} (dx^2)^2 - (dx^3)^2, \quad (4.2)$$

where, as is seen,

$$g_{00} = 1, \quad g_{02} = e^{\frac{x^1}{a}}, \quad g_{01} = g_{03} = 0. \quad (4.3)$$

Therefore the space of Gödel's metric is free of gravitation and deformation, but rotates with a three-dimensional linear velocity  $v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}$  whose only nonzero component is

$$v_2 = -c e^{\frac{x^1}{a}}, \quad (4.4)$$

which does not depend on time. In this case, the second (inertial) term of the gravitational inertial force  $F_i$  (1.5) is zero as well.

All factors which could change the frequency of a photon are absent in the space of Gödel's metric. This means that photons should not gain a frequency shift with the distance travelled therein.

**§5. Cosmological blueshift in the space of Schwarzschild's metric of a sphere of incompressible liquid.** This metric was introduced in 1916 by Karl Schwarzschild [19] with a limitation imposed on the fundamental metric tensor (he supposed that its three-dimensional components should not possess breaking). This metric in the general form, which is free of the said limitation, was obtained in 2009 by Larissa

Borissova (see formula 3.55 in [20], or (1.1) in [21]). It has the form

$$ds^2 = \frac{1}{4} \left( 3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)^2 c^2 dt^2 - \frac{dr^2}{1 - \frac{\varkappa \rho_0 r^2}{3}} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (5.1)$$

where  $\varkappa = \frac{8\pi G}{c^2}$  is Einstein's gravitational constant,  $\rho_0 = \frac{M}{V} = \frac{3M}{4\pi a^3}$  is the density of the liquid,  $a$  is the sphere's radius, and  $r$  is the radial coordinate within it. The metric is free of rotation and deformation. Only gravitation affects particles due to  $g_{00} \neq 1$ . Differentiating  $g_{00}$  of the metric (5.1), according to the definition of the gravitational inertial force  $F_i$  (1.5), we obtain the solely nonzero component of the force

$$F_1 = - \frac{c^2 \varkappa \rho_0 r}{3 \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \left( 3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)}. \quad (5.2)$$

The scalar geodesic equation for a massless particle (1.3) in this case takes the form

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_1 c^1 = 0, \quad (5.3)$$

which is  $d \ln \omega = \frac{1}{c^2} F_1 dr$ . Thus we arrive at the equation

$$d \ln \omega = \frac{\varkappa \rho_0 r}{3 \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}} \frac{dr}{\left( 3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right)} \quad (5.4)$$

which transforms into

$$d \ln \omega = - d \ln \left( 3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}} \right) \quad (5.5)$$

and solves as

$$\omega = \omega_0 \frac{3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - 1}{3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}}. \quad (5.6)$$

Herein  $\frac{\varkappa \rho_0 a^2}{3}$  is a world-constant of the space. Generally speaking, its numerical value is permitted to be within the range  $0 \leq \frac{\varkappa \rho_0 a^2}{3} \leq 1$ . This is in order to keep  $1 - \frac{\varkappa \rho_0 a^2}{3} \geq 0$  (the square root from the remain-

der should remain a real value).

To simplify and analyse the obtained solution (5.6), we use the following intermediate substitution. Because  $r_g = \frac{2GM}{c^2}$  is the gravitational radius of a sphere with the mass  $M = \rho_0 V = \frac{4}{3} \pi \rho_0 a^3$  and the radius  $a$ , we have  $r_g = \frac{\varkappa \rho_0 a^3}{3}$  and, therefore,  $\frac{\varkappa \rho_0 a^2}{3} = \frac{r_g}{a}$ . With these, the obtained solution (5.6) takes the form

$$\omega = \omega_0 \frac{3 \sqrt{1 - \frac{r_g}{a}} - 1}{3 \sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}}, \quad (5.7)$$

which is more suitable for analysis and further approximations.

Since  $r_g \ll a$ , at distances  $r$  which are very small in comparison to the radius of such a universe ( $r \ll a$ ), the obtained solution (5.6) takes the simplified form

$$\omega \simeq \omega_0 \left( 1 - \frac{\varkappa \rho_0 r^2}{12} \right). \quad (5.8)$$

The obtained solution manifests that, in a spherical universe filled with incompressible liquid, a photon should gain energy and frequency with the travelled distance. This is a blueshift effect: the more distant an object we observe in such a universe is, the more blueshifted should be the lines of its spectrum. Hence, this is a cosmological effect. We will therefore further refer to this effect as the *cosmological blueshift*.

According to our formulae (5.6) and (5.8), the cosmological blueshift increases with the *square of the distance* from the object. Let the photon have a frequency  $\omega$  being emitted by a source located at a distance  $r$  from an observer. Then, keeping in mind that  $\omega_0$  is the photon's frequency registered by the observer ( $r=0$ ), we obtain the cosmological blueshift in a spherical universe filled with incompressible liquid

$$z = \frac{\omega - \omega_0}{\omega_0} = - \frac{1 - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}}{3 \sqrt{1 - \frac{\varkappa \rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa \rho_0 r^2}{3}}}, \quad (5.9)$$

which at small distances  $r$  of the photon's travel ( $r \ll a$ ), according to our formula (5.9), takes the simplified form

$$z \simeq - \frac{\varkappa \rho_0 r^2}{12}. \quad (5.10)$$

For a photon emitted by a source, which is located at the event horizon (where  $r=a$ ), the energy and frequency gain are ultimately

high in such a universe. In this case, according to our solution (5.6), the observed frequency of the photon should be

$$\omega_{\max} = \omega_0 \frac{3 \sqrt{1 - \frac{\kappa \rho_0 a^2}{3}} - 1}{2 \sqrt{1 - \frac{\kappa \rho_0 a^2}{3}}}. \quad (5.11)$$

while the cosmological blueshift of the photon should take its ultimately high numerical value in such a space, which is

$$z_{\max} = - \frac{1 - \sqrt{1 - \frac{\kappa \rho_0 a^2}{3}}}{2 \sqrt{1 - \frac{\kappa \rho_0 a^2}{3}}}. \quad (5.12)$$

In my view, the main criterion for the applicability of a cosmological model to our Universe should be the redshift law predicted at small distances  $r \ll a$ . It is surely registered in most galaxies, which are not so much distant as the event horizon. However, the obtained solution (5.10) manifests a blueshift. This is a serious reason for us to conclude that Schwarzschild's metric of a sphere of incompressible liquid cannot be applied to our Universe as a whole.

On the other hand, the recent study by Borissova [20] shows that the Schwarzschild model is applicable to the Sun and the planets, by the assumption that these objects can be approximated as spheres of incompressible liquid.

In addition, she has obtained [20] that the space metric inside a Schwarzschild sphere of incompressible liquid in the state of gravitational collapse is equivalent to de Sitter's space metric (see de Sitter's metric below). In this case, i.e., inside a collapsed Schwarzschild liquid sphere, the gravitational force changes sign from attraction to repulsion, and, therefore, the cosmological blueshift (deduced above) changes to the corresponding cosmological redshift. This means that a collapsed Schwarzschild liquid sphere (the gravitational force inside such a sphere is a force of repulsion) can theoretically be conceivable as a model of our Universe.

**§6. Cosmological redshift and blueshift in the space of a sphere filled with physical vacuum (de Sitter's metric).** This metric, introduced in 1917 by Willem de Sitter [22, 23], has the form

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (6.1)$$

and describes a space filled with a spherically symmetric distribution of physical vacuum (determined by the  $\lambda$ -term of Einstein's equations). Such a space is free of rotation and deformation, but contains a non-Newtonian gravitational field determined by  $g_{00} = 1 - \frac{\lambda r^2}{3}$  of the metric. Differentiating the  $g_{00}$  according to the definition of the gravitational inertial force  $F_i$  (1.5), we obtain its solely nonzero component

$$F_1 = \frac{\lambda c^2}{3} \frac{r}{1 - \frac{\lambda r^2}{3}}, \quad (6.2)$$

whose magnitude increases with distance. This is a force of repulsion if  $\lambda > 0$  (physical vacuum has a negative density), and is a force of attraction if  $\lambda < 0$  (the vacuum density is positive).\*

The scalar geodesic equation for a massless particle (1.3), which in this case has the form

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2} F_1 c^1 = 0, \quad (6.3)$$

thus transforms into  $d \ln \omega = \frac{1}{c^2} F_1 dr$ , which is

$$d \ln \omega = \frac{\lambda r}{3} \frac{dr}{1 - \frac{\lambda r^2}{3}}, \quad (6.4)$$

or, in another form,

$$d \ln \omega = -\frac{1}{2} d \ln \left( 1 - \frac{\lambda r^2}{3} \right). \quad (6.5)$$

This equation solves as

$$\omega = \frac{\omega_0}{\sqrt{1 - \frac{\lambda r^2}{3}}}, \quad (6.6)$$

where  $\frac{\lambda r^2}{3}$  should be in the range  $0 \leq \frac{\lambda r^2}{3} \leq 1$ . For yet, observational astronomy provides only information about the upper boundary of the numerical value of the  $\lambda$ -term, which is as small as  $\lambda \leq 10^{-56} \text{ cm}^{-2}$ . Therefore, our obtained solution (6.6) at small distances  $r$  takes the simplified form

$$\omega \simeq \omega_0 \left( 1 + \frac{\lambda r^2}{6} \right). \quad (6.7)$$

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\*See Chapter 5 of [5], especially §5.3 and §5.5 therein. It is still unclear what sign is really attributed to the  $\lambda$ -term of Einstein's equations.



The obtained result can lead to two opposite conclusions, depending on the sign of  $\lambda$ .

Einstein's equations have the form  $R_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$ . Given a space of de Sitter's metric, the  $\lambda$ -term is connected to the density of physical vacuum  $\rho$ , the four-dimensional curvature  $K$ , and the three-dimensional observable curvature  $C$  as  $\rho = -\frac{\lambda}{\varkappa} = -\frac{3K}{\varkappa} = \frac{C}{2\varkappa}$  (see §5.3 of [5], for details).

Classically, we assume  $\lambda > 0$  so that physical vacuum has negative density and energy as any other potential field. In this case, the non-Newtonian gravitational force is a force of repulsion, the space (space-time) has a positive four-dimensional curvature  $K > 0$ , while the three-dimensional observable curvature is negative  $C < 0$ .

Having  $\lambda > 0$ , the frequency shift we have obtained in formula (6.6) and in its simplified form (6.7) implies that a photon travelling to the observer in a de Sitter universe should lose energy and frequency due to the deceleration of the photon by the non-Newtonian force of repulsion (the  $\lambda$ -field) that is present in the space. As a result, the photon's frequency should be redshifted upon arrival: the more distant an object we observe in a de Sitter universe where  $\lambda > 0$  is, the more redshifted should be the lines of its spectrum. We will therefore further refer to this effect as the *cosmological redshift*. The magnitude of the redshift, according to our solution (6.6), shall be

$$z = \frac{\omega - \omega_0}{\omega_0} = \frac{1}{\sqrt{1 - \frac{\lambda r^2}{3}}} - 1 \quad (6.8)$$

where  $\omega = \omega_{(r)}$  is the frequency of the photon being emitted by a source, which is located at a distance  $r$  from the observer, while the photon's frequency registered by the observer is  $\omega_0 = \omega_{(r=0)}$ . At small distances of the photon's travel, according to (6.7), we have the redshift

$$z \simeq \frac{\lambda r^2}{6}. \quad (6.9)$$

At the event horizon — the ultimately large distance  $r = a$ , which in a de Sitter universe is determined by the obvious condition  $\frac{\lambda r^2}{3} = \frac{\lambda a^2}{3} = 1$ , — the photon's frequency and redshift take their ultimately high numerical values. According to our solutions for the photon's frequency (6.6) and its redshift (6.8), they are

$$\omega_{\max} = \frac{\omega_0}{\sqrt{1 - \frac{\lambda a^2}{3}}} = \infty, \quad (6.10)$$

$$z_{\max} = \frac{1}{\sqrt{1 - \frac{\lambda a^2}{3}}} - 1 = \infty. \quad (6.11)$$

Contrarily, one may assume that the  $\lambda$ -field is not a potential field but a substance.

In this case, it should have positive density and energy, which means that the acting non-Newtonian gravitational force is a force of attraction, the space (space-time) has a negative four-dimensional curvature  $K < 0$ , its three-dimensional observable curvature is positive  $C > 0$ , and also  $\lambda < 0$ .

In such a de Sitter universe ( $\lambda < 0$ ), according to the solutions we have obtained, a photon travelling to the observer should gain energy and frequency with the distance travelled by it. This is due to the acceleration of the photon by the non-Newtonian force of attraction (the  $\lambda$ -field) that is present in the space. This means that the photon's frequency should be blueshifted upon arrival: the more distant an object we observe is, the more blueshifted should be the lines of its spectrum. In other words, the *cosmological blueshift* should be registered in a de Sitter universe where  $\lambda < 0$ . In this case,

$$\omega = \frac{\omega_0}{\sqrt{1 + \frac{\lambda r^2}{3}}}, \quad (6.12)$$

$$z = \frac{1}{\sqrt{1 + \frac{\lambda r^2}{3}}} - 1, \quad (6.13)$$

or, at small distances ( $r \ll a$ ),

$$\omega \simeq \omega_0 \left(1 - \frac{\lambda r^2}{6}\right), \quad (6.14)$$

$$z \simeq -\frac{\lambda r^2}{6}, \quad (6.15)$$

while the ultimate frequency and blueshift are

$$\omega_{\max} = \frac{\omega_0}{\sqrt{1 + \frac{\lambda a^2}{3}}} = \frac{\omega}{\sqrt{2}} \simeq 0.71 \omega_0, \quad (6.16)$$

$$z_{\max} = \frac{1}{\sqrt{1 + \frac{\lambda a^2}{3}}} - 1 \simeq -0.29. \quad (6.17)$$

In the present year, 2011, the highest redshifts registered by the astronomers are  $z = 10.3$  (the galaxy UDFj-39546284) and  $z = 8.55$  (the galaxy UDFy-38135539). Three dozens of other galaxies and quasars have redshifts higher than  $z = 1$ . The number of such high redshifted cosmic objects increases with each year of such observations. Also, a non-linearity of the observed redshift law was recently discovered by astronomers in the spectra of distant galaxies. We therefore should expect that our Universe is the version of de Sitter worlds that has a non-linear redshift (see above).

**§7. No frequency shift present in the space of a sphere filled with ideal liquid and physical vacuum (Einstein's metric).** This metric, introduced in 1917 by Albert Einstein [24], describes a sphere filled with homogeneous and isotropic distribution of ideal (non-viscous) liquid and physical vacuum ( $\lambda$ -field). It has the form

$$ds^2 = c^2 dt^2 - \frac{dr^2}{1 - \lambda r^2} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \quad (7.1)$$

This metric is free of gravitation ( $g_{00} = 1$ ), rotation ( $g_{0i} = 0$ ), and deformation (the three-dimensional components  $g_{ik}$  of the fundamental metric tensor do not depend on time). This means that such a space contains no one factor which could change the frequency of a photon. Hence, the photon's frequency remains unchanged with the distance travelled in the space of Einstein's metric.

**§8. Cosmological redshift and blueshift in the deforming spaces of Friedmann's metric.** The models, introduced in 1922 by Alexander Friedmann [25, 26], are free of gravitation and rotation, but are deforming, which points to the presence of the functions  $g_{ik} = g_{ik}(t)$ . In other words, the three-dimensional subspace of the space-time deforms. It may expand, compress, or oscillate. Such a space can be empty, or filled with a homogeneous and isotropic distribution of ideal (non-viscous) liquid in common with physical vacuum ( $\lambda$ -field), or filled with one of the media. In particular, it can be dust\*.

Friedmann's metric has the form

$$ds^2 = c^2 dt^2 - R^2 \left[ \frac{dr^2}{1 - \kappa r^2} - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right], \quad (8.1)$$

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\*The energy-momentum tensor of ideal liquid is the same as that of dust except that the latter is marked by the absence of the term containing pressure. In other words, dust is pressureless ideal liquid.

where  $R = R(t)$  is the curvature radius of the space, while  $\kappa = 0, \pm 1$  is the curvature factor. If  $\kappa = -1$ , the three-dimensional subspace has the hyperbolic (open) geometry. If  $\kappa = 0$ , its geometry is flat. If  $\kappa = +1$ , it has elliptic (closed) geometry. The models with  $\kappa = +1$  and  $\kappa = -1$  were considered in 1922 and 1924 by Friedmann [25, 26]. The generalized formulation of the metric containing  $\kappa = 0, \pm 1$  was first examined in 1925 by Georges Lemaître [27, 28], then in 1929 by Howard Percy Robertson [29], and in 1937 by Arthur Geoffrey Walker [30]. Therefore, Friedmann's metric in its generalized form (8.1) is also known as the Friedmann-Lemaître-Robertson-Walker metric.

Friedmann's metric is expressed here through a "homogeneous" radial coordinate  $r$ . It comes across as the regular radial coordinate divided by the curvature radius whose scales change accordingly during expansion or compression of the space. As a result, the homogeneous radial coordinate  $r$  does not change its scale with time.

The scalar geodesic equation (1.3) for a massless particle, which travels in a Friedmann universe along the radial coordinate  $x^1$ , takes the form

$$\frac{d\omega}{d\tau} + \frac{\omega}{c^2} D_{11} c^1 c^1 = 0, \quad (8.2)$$

where  $c^1$  [ $\text{sec}^{-1}$ ] is the solely nonzero component of the observable "homogeneous" velocity of the massless particle. The square of the velocity is  $h_{11} c^1 c^1 = c^2$  [ $\text{cm}^2/\text{sec}^2$ ]. The components of the chr-inv.-metric tensor  $h_{ik}$  (1.7) can be calculated according to Friedmann's metric (8.1). After some algebra, we obtain

$$h_{11} = \frac{R^2}{1 - \kappa r^2}, \quad h_{22} = R^2 r^2, \quad h_{33} = R^2 r^2 \sin^2 \theta, \quad (8.3)$$

$$h = \det \|h_{ik}\| = h_{11} h_{22} h_{33} = \frac{R^6 r^4 \sin^2 \theta}{1 - \kappa r^2}, \quad (8.4)$$

$$h^{11} = \frac{1 - \kappa r^2}{R^2}, \quad h^{22} = \frac{1}{R^2 r^2}, \quad h^{33} = \frac{1}{R^2 r^2 \sin^2 \theta}. \quad (8.5)$$

In the case of mass-bearing particles, the scalar geodesic equation being in its general form cannot be solved alone. This is because mass-bearing particles can travel at any sub-light velocity, which is unknown. We therefore are enforced to find the velocity by solving the vectorial geodesic equation. This problem was resolved in [1].

Another case — massless (light-like) particles. They travel along isotropic trajectories, which are the trajectories of light. Their velocity  $c^i = \frac{dx^i}{d\tau}$  is the observable velocity of light, where  $d\tau = \sqrt{g_{00}} dt - \frac{1}{c^2} v_i dx^i$  is

the physically observable time. The observable light velocity  $c^i$  depends on the gravitational potential  $w = c^2(1 - \sqrt{g_{00}})$  and the linear velocity  $v_i = -\frac{cg_{0i}}{\sqrt{g_{00}}}$  of the three-dimensional rotation of space.

In the case of a Friedmann universe, we have  $g_{00} = 1$  and  $g_{0i} = 0$ . Hence,  $d\tau = dt$  in this case. Thus, because  $h_{11} c^1 c^1 = c^2$ , the scalar geodesic equation of a massless particle (8.2) transforms into

$$h_{11} \frac{d\omega}{dt} + \omega D_{11} = 0, \quad (8.6)$$

thus we obtain  $h_{11} \frac{d\omega}{\omega} = -D_{11} dt$ , and, finally, the equation

$$\frac{R^2}{1 - \kappa r^2} d \ln \omega = -D_{11} dt. \quad (8.7)$$

This equation is non-solvable being considered in the general form as here. To solve this equation, we should simplify it by assuming particular forms of the space deformation (the function  $R = R(t)$  of Friedmann's metric) and the curvature factor  $\kappa$  of the space\*. Further, after the function  $R = R(t)$  is assumed, we will see that  $\kappa$  comes out from the equation. So, we need to assume only  $R = R(t)$ .

The curvature radius as a function of time,  $R = R(t)$ , can be found through the tensor of the space deformation  $D_{ik}$ , whose trace

$$D = h^{ik} D_{ik} = \frac{* \partial \ln \sqrt{h}}{\partial t} = \frac{1}{\sqrt{h}} \frac{* \partial \sqrt{h}}{\partial t} = \frac{1}{V} \frac{* \partial V}{\partial t} \quad (8.8)$$

is the speed of relative deformation (expansion or compression) of the volume [3, 4]. In an arbitrary metric space, we have

$$D = \frac{1}{V} \frac{* \partial V}{\partial t} = \gamma \frac{1}{a} \frac{* \partial a}{\partial t} = \gamma \frac{u}{a}, \quad (8.9)$$

where  $a$  is the radius of the volume ( $V \sim a^3$ ),  $u$  is the linear velocity of its deformation (positive if the space expands, and negative in the case of compression), and  $\gamma = \text{const}$  is the shape factor of the space ( $\gamma = 3$  in the homogeneous isotropic models [3, 4]).

Two main types of the space deformation, and two respective types of the function  $R = R(t)$  were introduced and then examined in [1]. They are as follows.

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\*The curvature factor  $\kappa$  is included in the spatial component  $g_{11}$  of the fundamental metric tensor of Friedmann's metric (8.1). As a result, and because the space deformation  $D_{ik}$  is determined as the time derivative of the three-dimensional components of the observable metric tensor  $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ , the curvature factor  $\kappa$  is included in the formula of the space deformation.

1. A constant-speed deforming (homotachydiastolic) universe\* deforms with a constant linear velocity  $u = \frac{\partial a}{\partial t} = \text{const}$ . Its radius undergoes linear changes with time as  $a = a_0 + ut$ . Thus

$$D = \gamma \frac{u}{a_0 + ut} \simeq \gamma \frac{u}{a_0} \left( 1 - \frac{ut}{a_0} \right), \quad (8.10)$$

where  $D = \frac{3\dot{R}}{R}$  as in any Friedmann universe ( $\gamma = 3$ ). Thus we arrive at the equation  $\frac{dR}{R} = \frac{u dt}{a_0 + ut} = \frac{d(a_0 + ut)}{a_0 + ut}$ , which is  $d \ln R = d \ln (a_0 + ut)$ . It solves as  $\ln R = \ln |a_0 + ut| + \ln B$ , i.e.,  $\frac{R}{B} = a_0 + ut$ . The integration constant is found from the condition  $R = a_0$  at the initial moment of time  $t = t_0 = 0$ . It is  $B = 1$ . Therefore,  $R = a_0 + ut$ . As a result, we obtain, that in a constant-speed deforming Friedmann universe,

$$R = a_0 + ut, \quad \dot{R} = u, \quad (8.11)$$

$$D = \frac{3\dot{R}}{R} = \frac{3u}{a_0 + ut}, \quad (8.12)$$

$$D_{11} = \frac{R\dot{R}}{1 - \kappa r^2} = \frac{(a_0 + ut)u}{1 - \kappa r^2}, \quad (8.13)$$

$$D_1^1 = \frac{\dot{R}}{R} = \frac{u}{a_0 + ut}. \quad (8.14)$$

2. In a constant-deformation (homotachydiastolic) universe†, each single volume  $V$  (including the total volume of the space), undergoes equal relative changes with time

$$D = \frac{1}{V} \frac{\partial V}{\partial t} = \gamma \frac{u}{a} = \text{const}, \quad (8.15)$$

while the linear velocity of the deformation increases with time in the case of expansion, and decreases if the space compresses. In other words, this is an accelerate expanding universe or a decelerate compressing universe, respectively.

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\*I refer to this kind of universe as *homotachydiastolic* (ομοταχυδιαστολικός). Its origin is *homotachydiastoli* — ομοταχυδιαστολή — linear expansion with a constant speed, from *όμο* which is the first part of *όμοιος* — the same, *ταχύτητα* — speed, and *διαστολή* — linear expansion (compression is the same as negative expansion).

†I refer to this kind of universe as *homotachydioncotic* (ομοταχυδιόγκωτικό). This terms originates from *homotachydioncosis* — ομοταχυδιόγκωσις — volume expansion with a constant speed, from *όμο* which is the first part of *όμοιος* (omeos) — the same, *ταχύτητα* — speed, *διόγκωσις* — volume expansion, while compression can be considered as negative expansion.

Generally speaking, a volume element, which is not affected by external factors, expands or compresses so that its volume undergoes equal relative changes with time. We therefore will further consider a *constant-deformation (homotachydioncotic)* Friedmann universe.

Because  $D = \frac{3\dot{R}}{R}$  in a Friedmann universe, we assume  $\frac{\dot{R}}{R} = A = \text{const}$  for the constant-deformation (homotachydioncotic) case. We obtain the equation  $\frac{1}{R} dR = A dt$ , which is  $d \ln R = A dt$ . As a result, in a constant-deformation Friedmann universe whose curvature radius at the present moment of time  $t = t_0$  is  $a_0$ , we obtain

$$R = a_0 e^{At}, \quad \dot{R} = a_0 A e^{At}, \quad (8.16)$$

$$D = \frac{3\dot{R}}{R} = 3A = \text{const}, \quad (8.17)$$

$$D_{11} = \frac{R\dot{R}}{1 - \kappa r^2} = \frac{a_0^2 A e^{2At}}{1 - \kappa r^2}, \quad (8.18)$$

$$D_1^1 = \frac{\dot{R}}{R} = A = \text{const}. \quad (8.19)$$

Thus, substituting  $D_{11} = \frac{R\dot{R}}{1 - \kappa r^2} = \frac{a_0^2 A e^{2At}}{1 - \kappa r^2}$  (8.18) into the scalar geodesic equation (8.7), we obtain the equation in the form

$$d \ln \omega = -A dt, \quad (8.20)$$

where  $A = \frac{\dot{R}}{R}$  is a const of the space.

As is seen, this equation is independent of the curvature factor  $\kappa$  of the particular Friedmann space under consideration. In other words, by solving this equation we will arrive at a solution which will be common for all three types of the constant-deformation (homotachydias-totic) Friedmann universe which have hyperbolic ( $\kappa = -1$ ), flat ( $\kappa = 0$ ), or elliptic ( $\kappa = +1$ ) geometry, respectively.

This equation solves, obviously, as  $\ln \omega = -At + \ln B$ , where  $B$  is an integration constant. So forth, we obtain  $\ln \frac{\omega}{B} = -At$ , then, trivially,  $\omega = B e^{-At}$ . We calculate the integration constant  $B$  from the initial condition  $\omega = \omega_0$  at the moment of time  $t = t_0 = 0$ . We have  $B = \omega_0$ . As a result, the final solution of the scalar geodesic equation (8.20) is

$$\omega = \omega_0 e^{-At}. \quad (8.21)$$

At small distances (and duration) of the photon's travel, the obtained solution takes the simplified form

$$\omega \simeq \omega_0 (1 - At). \quad (8.22)$$

The obtained solution manifests that, in a constant-deformation (homotachydiastolic) Friedmann universe which expands ( $A > 0$ ), photons should lose energy and frequency according to the travelled distance. The energy and frequency loss law is exponential (8.21) at large distances of the photon's travel, and is linear (8.22) at small distances.

Accordingly, the photon's frequency should be redshifted. The magnitude of the redshift increases with the travelled distance. This is a *cosmological redshift*, in other words.

Let a photon have an initial frequency  $\omega_0$  being emitted by a source ( $t = t_0 = 0$ ), and its frequency being registered by an observer to whom the photon has travelled during the time interval  $t$  is  $\omega$ . Then we obtain the magnitude of the cosmological redshift in an expanding constant-deformation (homotachydiastolic) Friedmann universe. It is

$$z = \frac{\omega_0 - \omega}{\omega} = e^{At} - 1, \quad (8.23)$$

which is an *exponential redshift law*. At small distances of the photon travel, it takes the linearized form

$$z \simeq At, \quad (8.24)$$

which manifests a *linear redshift law*. Expanding the world-constant  $A = \frac{\dot{R}}{R}$  and the duration of the photon's travel  $t = \frac{d}{c}$ , we have

$$z = e^{\frac{\dot{R} d}{R c}} - 1, \quad (8.25)$$

where  $d = ct$  [cm] is the distance to the source that emitted the photon. At small distances, we have, respectively, the linear approximation

$$z \simeq \frac{\dot{R} d}{R c}. \quad (8.26)$$

In the case where such a universe compresses ( $A < 0$ ), this effect changes its sign thus, becoming a *cosmological blueshift*.

Our linearized redshift formula (8.26) is the same as  $z = \frac{\dot{R} d}{R c}$  obtained by Lemaître, the "father" of the theory of an expanding universe who in 1925–1927 discovered the linear redshift law\* [28]. He followed, however, another way of deduction which limited him only to the linear formula. He did not arrive at a non-linear generalization of it. Lemaître's beliefs, therefore, remained within the range of the linear redshift law.

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\*The linear redshift law is now known as *Hubble's law* due to Edwin Hubble's publication of 1929 [32]. See more details about the dramatic history of this discovery in the newest notes [33–35] published in 2011 by the historians of science.



Suppose our world to be an expanding Friedmann universe of the constant-deformation type. Then galaxies should scatter, being carried out with the expanding space. Their spectra should therefore manifest a redshift according to the exponential redshift law (8.25) or, at small distances, according to the linear redshift law (8.26).

The world-constant  $A = \frac{\dot{R}}{R}$  can be found on the basis of astronomical observations of the objects whose redshift is within the linear range (the galaxies and quasars which are located not at cosmologically large distances). For instance, consider the brightest quasar 3C 273. Its observed redshift is  $z = 0.16$ . Such a redshift means that this object is located at a cosmologically small distance (we know distant galaxies and quasars whose redshift is much higher than  $z = 1$ ). Therefore, when calculating the redshift for this object, we use the linearized formula (8.26) of our theory. The observed luminosity distance\* to the quasar 3C 273 is  $d_L = 749 \text{ Mpc} \simeq 2.3 \times 10^{27} \text{ cm}$ . According to our formula (8.26), we obtain that the world-constant  $A = \frac{\dot{R}}{R}$  has the numerical value

$$A = \frac{\dot{R}}{R} = z \frac{c}{d_L} = 2.1 \times 10^{-18} \text{ sec}^{-1}, \quad (8.27)$$

which matches the Hubble constant, which is  $H_0 = 72 \pm 8 \text{ km/sec} \times \text{Mpc} = (2.3 \pm 0.3) \times 10^{-18} \text{ sec}^{-1}$  according to the newest data of the Hubble Space Telescope [31]. The Hubble constant was initially obtained as the coefficient of the observed linear law for scattering galaxies: this law says that galaxies and quasars scatter with the radial velocity  $u = H_0 d$  increasing with the distance  $d$  to the object as 72 km/sec per each Megaparsec.

The ultimately high redshift  $z_{\text{max}}$ , which could be registered in our Universe, is calculated by substituting the ultimately large distance into the redshift law. If following Lemaitre's theory [28],  $z_{\text{max}}$  should follow from the linear redshift law  $z = \frac{\dot{R}}{R} \frac{d}{c} = A \frac{d}{c}$ . Because  $A = \frac{\dot{R}}{R}$  is the world-constant of the Friedmann space, the ultimately large curvature radius  $R_{\text{max}}$  is determined by the ultimately high velocity of the space expansion which is the velocity of light  $R_{\text{max}} = c$ . Hence,  $R_{\text{max}} = \frac{c}{A}$ . The ultimately large distance  $d_{\text{max}}$  (the event horizon) is determined by the astronomers from the linear law for scattering galaxies  $u = H_0 d$ . This linear law is known, however, due to the observation of non-extremely distant objects. They thus interpolate the empirical linear law  $u = H_0 d$

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\*In observational astronomy, the luminosity distance  $d_L$  to a cosmic object is determined through the absolute stellar magnitude  $\mathfrak{M}$  of the object, and its apparent stellar magnitude  $\mathfrak{m}$  according to the formula  $\mathfrak{M} = \mathfrak{m} - 5(\lg d_L - 1)$ , where  $d_L$  is measured in parsecs. 1 parsec =  $3.0857 \times 10^{18} \text{ cm} \simeq 3.1 \times 10^{18} \text{ cm}$ .

upto the event horizon. Since the scattering velocity  $u$  should reach the velocity of light ( $u=c$ ) at the event horizon ( $d=d_{\max}$ ), they then obtain  $d_{\max} = \frac{c}{H_0} = (1.3 \pm 0.2) \times 10^{28}$  cm. Finally, they identify the linear coefficient  $H_0$  of the empirical law for scattering galaxies as the world-constant  $A = \frac{\dot{R}}{R}$ , which follows from the space geometry. Thus they may obtain  $d_{\max} = R_{\max}$  and, from the linear redshift law, the ultimately high redshift  $z_{\max} = H_0 \frac{d_{\max}}{c} = 1$ . How, then, to explain the very distant objects, whose redshift is much higher than  $z=1$ ?

On the other hand, it is obvious that the ultimately high redshift  $z_{\max}$ , ensuing from the space (space-time) geometry, should be a result of relativistic physics. In other words,  $z = z_{\max}$  should follow not from a straight line  $z = \frac{\dot{R}}{R} \frac{d}{c} = H_0 \frac{d}{c} = \frac{u}{c}$ , which digs in the vertical “wall”  $u=c$ , but from a non-linear relativistic function.

In this case, the Hubble constant  $H_0$  remains a linear coefficient in only the pseudo-linear beginning of the real redshift law arc, wherein the velocities of scattering  $u$  are small in comparison with the velocity of light. At velocities of scattering close to the velocity of light (close to the event horizon), the Hubble constant  $H_0$  loses the meaning of the linear coefficient and the world-constant  $A$  due to the increasing non-linearity of the real redshift law.

Such a non-linear formula has been found in the framework of our theory presented here. This is the exponential redshift law (8.25), which then gives the Lemaître linear redshift law (8.26) as an approximation at small distances.

We now use the exponential redshift law (8.25). We calculate the ultimately high redshift  $z_{\max}$ , which could be conceivable in an expanding Friedmann space of the constant-deformation type. The event horizon  $d = d_{\max}$  is determined by the world-constant  $A = \frac{\dot{R}}{R}$  of such a space. Thus, the ultimately large curvature radius is  $R_{\max} = \frac{c}{A}$ , while the distance corresponding to  $R_{\max}$  on the hypersurface is  $d_{\max} = \pi R_{\max} \frac{\pi c}{A}$ . Suppose now that a photon has arrived from a source, which is located at the event horizon. According to the obtained exponential solution (8.21), the photon’s frequency at the arrival should be

$$\omega_{\max} = e^{-\frac{\dot{R}}{R} \frac{d_{\max}}{c}} = \omega_0 e^{-\pi} \simeq 0.043 \omega_0, \quad (8.28)$$

while the exponential redshift law (8.25) gives the photon’s redshift

$$z_{\max} = e^{\frac{\dot{R}}{R} \frac{d_{\max}}{c}} - 1 = e^{\pi} - 1 = 22.14, \quad (8.29)$$

which is the ultimately high redshift in such a universe.

So, the redshift law for scattering galaxies, including its non-linear increase at “cosmologically large” distances, has been explained in the expanding constant-deformation (homotachydioncotic) space, which is an accelerate expanding Friedmann universe.

The deduced exponential law points out the ultimately high redshift  $z_{\max} = 22.14$  for the objects located at the event horizon. The highest redshifted objects, registered by the astronomers, are now the galaxies UDFj-39546284 ( $z = 10.3$ ) and UDFy-38135539 ( $z = 8.55$ ). According to the theory, they are still distantly located from the “world end”. We therefore shall expect, with years of further astronomical observation, more “high redshifted surprises” which will approach the upper limit  $z_{\max} = 22.14$  predicted by our theory.

**§9. A note on the cosmological mass-defect in a Friedmann universe.** In §9 of the previous publication [1], I suggested solving the scalar geodesic equation of mass-bearing particles in a Friedmann universe. This equation being in its general form

$$\frac{dm}{d\tau} + \frac{m}{c^2} D_{11} v^1 v^1 = 0, \quad (9.1)$$

is non-resolvable. This is because mass-bearing particles can travel at any sub-light velocity, which is therefore an unknown term of the equation\*. I then looked for the velocity by solving the vectorial geodesic equation of mass-bearing particles. As a result, I arrived at a non-resolvable integral equation. Even qualitative analysis of the integral did not give a definite conclusion.

I now understand my mistake in that way of deduction. I targeted that problem in its general form. However, now I see that the problem can easily be removed in a constant-deformation Friedmann universe, where massive bodies (mass-bearing particles) travel not arbitrarily, but are only carried out with the expanding (or compressing) Friedmann space itself. In this particular case, the linear velocity of a mass-bearing particle is the same as the speed  $\dot{R}$  at which the curvature radius  $R$  of the space changes with time,  $v = \dot{R}$ . In other words, because  $v^2 = h_{ik} v^i v^k$ , we have  $h_{ik} v^i v^k = \dot{R}^2$ . In this particular case (and with  $d\tau = dt$  according to Friedmann’s metric), the scalar geodesic equation of mass-bearing particles (9.1) takes the form

$$h_{11} \frac{dm}{dt} + \frac{m}{c^2} D_{11} \dot{R}^2 = 0, \quad (9.2)$$

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\*Massless particles travel at the velocity of light  $v^i = c^i$ , so we have not this problem when considering the geodesic equations of massless particles.

which is  $h_{11} \frac{dm}{m} = -\frac{\dot{R}^2}{c^2} D_{11} dt$ , and, finally,

$$\frac{R^2}{1 - \kappa r^2} d \ln m = -\frac{\dot{R}^2}{c^2} D_{11} dt. \quad (9.3)$$

Then, substituting  $R = a_0 e^{At}$  and  $\dot{R} = a_0 A e^{At}$  (8.16), and also  $D_{11} = \frac{R\dot{R}}{1 - \kappa r^2} = \frac{a_0^2 A e^{2At}}{1 - \kappa r^2}$  (8.18) as for a constant-deformation space, we obtain the scalar geodesic equation in the form

$$d \ln m = -\frac{a_0^2 A^3 e^{2At}}{c^2} dt, \quad (9.4)$$

or  $d \ln m = -\frac{a_0^2 A^2}{2c^2} d e^{2At}$ , where  $A = \frac{\dot{R}}{R}$  is a constant of the space.

Note that the curvature factor  $\kappa$  comes out from the obtained equation. Therefore, the further solution of the equation will be common for all three types of the constant-deformation (homotachydiastolic) Friedmann universe: the hyperbolic ( $\kappa = -1$ ), flat ( $\kappa = 0$ ), and elliptic ( $\kappa = +1$ ) space.

This equation solves, obviously, as  $\ln m = -\frac{a_0^2 A^2}{2c^2} e^{2At} + \ln B$ , where the integration constant  $B$  can be found from the condition  $m = m_0$  at the initial moment of time  $t = t_0 = 0$ . Thus, after some trivial algebra, we obtain the final solution of the scalar geodesic equation (9.4). It is the double-exponent

$$m = m_0 e^{-\frac{a_0^2 A^2}{2c^2} (e^{2At} - 1)}, \quad (9.5)$$

where  $t$  is the duration of the expansion (if  $A > 0$ ) or compression ( $A < 0$ ) of the Friedmann universe. At small distances (and durations of time), this solution takes the linearized form

$$m \simeq m_0 \left( 1 - \frac{a_0^2 A^3 t}{c^2} \right). \quad (9.6)$$

The obtained exact solution (9.5) and its linearized form (9.6) manifest the *cosmological mass-defect* in a constant-deformation (homotachydiastolic) Friedmann universe: the more distant an object we observe in an expanding Friedmann universe is, the less should be its observed mass  $m$  to its real mass  $m_0$ . Contrarily, the more distant an object we observe in a compressing Friedmann universe is, the heavier should be this object according to the observation.

Our Universe seems to be expanding. This is due to the cosmological redshift registered in the distant galaxies and quasars. Therefore, according to the cosmological mass-defect deduced here, we should expect

distantly located cosmic objects to be much heavier than we estimate on the basis of astronomical observations. The magnitude of the expected mass-defect should be, according to the obtained solutions, in the order of the redshift of the objects.

The cosmological mass-defect complies with the respective solution obtained for the frequency of a photon. Both effects are deduced in the same way, by solving the scalar geodesic equation for mass-bearing and massless particles, respectively. One effect cannot be in the absence of the other, because the geodesic equations have the same form. This is a basis of the space (space-time) geometry, in other words. Therefore, once the astronomers register the linear redshift law and its non-linearity in the very distant galaxies and quasars, they should also find the corresponding cosmological mass-defect according to the solutions outlined here. Once the cosmological mass-defect is discovered, we will be able to say, surely, that our Universe as a whole is an expanding Friedmann universe of the constant-deformation (homotachydiastolic) type.

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P.S. A thesis of this presentation has been posted on the desk of the *April Meeting 2012 of the APS*, planned for March 31 – April 03, 2012, in Atlanta, Georgia. More detailed explanation of the cosmological redshift and the cosmological mass defect, surveyed briefly in my recent papers, will be considered in my forthcoming book (under preparation).

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# Gravitational Fields Exterior to Homogeneous Spheroidal Masses

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**Abstract:** General relativistic mechanics in gravitational fields exterior to homogeneous spheroidal masses is developed using our new approach. Einstein's field equations in the gravitational field exterior to a static homogeneous prolate spheroid are derived and a solution for the first field equations constructed. Our derived field equations exterior to the mass distribution have only one unknown function determined by the mass or pressure distribution. The obtained solutions yield the unknown function as generalizations of Newton's gravitational scalar potential. Remarkably, our solution puts Einstein's geometrical theory of gravity on same footing with Newton's dynamical theory; with the dependence of the field on one and only one unknown function comparable to Newton's gravitational scalar potential. The consequences of the homogeneous spheroidal gravitational field on the motion of test particles have been theoretically investigated. The effect of the oblate nature of the Sun and planets on some gravitational phenomena has been examined. These are gravitational time dilation, gravitational length contraction and gravitational spectral shift of light. Our obtained theoretical value for the Pound-Rebka experiment on gravitational spectra shift ( $2.578 \times 10^{-15}$ ) agrees satisfactorily with the experimental value of  $2.45 \times 10^{-15}$ . Expressions for the conservation of energy and angular momentum are obtained. Planetary equations of motion and equations of motion of photons in the vicinity of spheroids are derived; having additional spheroidal terms not found in Schwarzschild's space-time.

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## Chapter 1. Introduction

### §1.1. Background of the problem

#### §1.1.1 The nature of gravitation

General Relativity is the geometrical theory of gravitation published by Albert Einstein in 1915/1916. It unifies Special Relativity and Sir Isaac Newton's law of universal gravitation with the insight that gravitation is not due to a force but rather a manifestation of curved space and time, with the curvature being produced by the mass-energy and momentum content of the space-time. General Relativity is the most widely accepted theory of gravitation.

After his theory of Special Relativity which elegantly describes mechanics in electromagnetic and empty spaces, Einstein expected gravitation to have the same nature as electromagnetism and hence fit into Special Relativity. So Einstein sought a "Maxwellian" type of laws for the gravitational field. That effort by Einstein failed [1]. Einstein concluded that gravitation is of an entirely different nature from electromagnetism which is a dynamical phenomenon. Consequently, he used geometrical quantities (tensors) for the description of gravitation instead of the dynamical quantities such as force and potential. Secondly, Einstein realized that Newton's laws of gravitation satisfied Galileo's principle of relativity according to which the laws of physics take the same form in all inertial reference frames. Consequently, Einstein also introduced his principle of General Relativity which asserts that "The laws of physics take the same form in all reference frames" Thus, Einstein constructed his theory of gravitation founded on his principle of General Relativity using tensors [1].

#### §1.1.2. The space-time of General Relativity

In Special Relativity, space-time has four dimensions ( $\mu = 0, 1, 2, 3$ ) and there always exist a global coordinate system in which the world-line element or proper time takes the form

$$c^2 d\tau^2 = \eta_{\mu\sigma} dx^\mu dx^\sigma, \quad (1.1)$$

where  $\eta_{\mu\sigma}$  is a special relativistic metric tensor given by  $\eta_{00} = 1$ ,  $\eta_{11} = \eta_{22} = \eta_{33} = -1$  ( $\eta_{\mu\sigma} = 0$ ,  $\mu \neq \sigma$ ). Such a coordinate system is said to be Cartesian. In a non-Cartesian coordinate system such as a spherical or spheroidal coordinates, the world-line element of space-time may be written as [2],

$$c^2 d\tau^2 = g_{\mu\sigma} dx^\mu dx^\sigma, \quad (1.2)$$

where  $g_{\mu\sigma}$  is the corresponding metric tensor which is generally different from the Cartesian metric tensor  $\eta_{\mu\sigma}$ . In practical calculations, the metric is most often written in coordinates in which it takes the following form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1.3)$$

According to the philosophy of General Relativity (GR), the effect of gravitation is contained in the metric tensor field  $g_{\mu\sigma}$ . Thus, in Einstein's theory of gravity, the gravitational field is promoted to a space-time metric  $g_{\mu\sigma}$ .

Schwarzschild in 1916 constructed the first exact solution of Einstein's gravitational field equations. Schwarzschild's solution is one of the physically interpretable solutions of Einstein's field equations [3]. Schwarzschild metric tensor field is that due to a static spherically symmetric body situated in empty space such as the Sun or a star. Schwarzschild metric has been the basis of theoretical investigations of gravitational phenomena in Einstein's theory of gravitation. This is in spite of the fact that the Sun and most planetary bodies in the Solar System are not perfectly spherical but oblate spheroidal in shape [4].

### §1.2. Statement of the problem

From the inception of Newton's dynamical theory of gravitation in the 17th century, the planets and Sun have been treated as perfectly spherical bodies. For example in the motion of terrestrial penduli, projectiles and satellites, the Earth is regarded as perfectly spherical in geometry. Similarly in the motions of the planets, comets and asteroids in the Solar System, the Sun is regarded as perfectly spherical in geometry and also, in Einstein's geometrical theory of gravitation (General Relativity). The motions of the planets and photons in the Solar System are treated under the assumption that the Sun is a perfect sphere. It has however, been realized experimentally that the Sun and planets in the Solar System are more precisely oblate spheroidal in geometry [4] (see Table 1 below).

Obviously, the oblate spheroidal geometries of these bodies (Table 1) has corresponding effects on their gravitational fields and hence the motions of test particles in these fields. Towards the investigation of these effects in Newton's theory of gravitation; the gravitational scalar potential due to an oblate spheroidal body and Newton's equations of motion in the gravitational field of an oblate mass have been derived [4].

The prolate spheroid is the shape of some moons in the Solar System. Examples are Mimas, Enceladus, and Tethys (moons of Saturn) and

Body	Oblateness
Sun	$9 \times 10^{-6}$
Mercury	0
Venus	0
Earth	0.0034
Mars	0.006
Jupiter	0.065
Saturn	0.108
Uranus	0.03
Neptune	0.026

Table 1: Oblateness of bodies in the Solar System.

Miranda (moon of Uranus). The prolate spheroidal geometry is also used to describe the shape of some nebulae (a nebula is a region or cloud of interstellar dust and gas appearing variously as a hazy bright or dark patch) such as the Crab Nebula [5]. Also, the existence of rotating prolate spheroidal galaxies has been known for decades, yet, a theoretical model based on Newton's or Einstein's gravitational theories remains elusive [6].

The metric tensor for a gravitational field is the fundamental starting point in the studies of gravitational fields in Einstein's geometrical theory. With the metric tensor, Einstein's field equations can be derived and solved. There is no general method yet of finding rigorous solutions of Einstein's field equations [1]. In order to study general relativistic mechanics (Einstein's theory of gravitation) in oblate spheroidal gravitational fields, Howusu and Udoh [7] sought the covariant metric tensor exterior or interior to a massive oblate spheroidal body in oblate spheroidal coordinates as;

$$g_{00} = e^{-F}, \quad (1.4)$$

$$g_{11} = -e^{-G}, \quad (1.5)$$

$$g_{22} = -e^{-H}, \quad (1.6)$$

$$g_{33} = -a^2 (1 - \eta^2) (1 + \zeta^2), \quad (1.7)$$

$$g_{\mu\nu} = 0, \quad (1.8)$$

where  $F$ ,  $G$  and  $H$  are functions of  $\eta$  and  $\zeta$  only and  $a$  is a constant parameter. With this metric, they constructed gravitational field equations exterior or interior to a massive oblate spheroidal body. The field

equations they obtained are non linear second order differential equations and have three unknown functions. The major setback of the use of this metric, equations is that the introduction of three unknown functions,  $F$ ,  $G$  and  $H$  makes the field equations obtained very complex and this compounds with the non linearity of the field equations to make them almost practically unsolvable and physically uninteresting.

Howusu [8] in an attempt to address the loop holes, difficulties and shortcomings in their previous approach, realized that a general and standard metric tensor exterior to all distributions of mass or pressure within regions of all regular geometries can be obtained by extending the Schwarzschild's metric to the particular regular geometry. The most interesting and important fact about this new method is that the generalized metric tensor obtained is not an exponential function and has only one unknown function. It is also instructive to note that the unknown function in this case can be satisfactorily approximated to the Newtonian gravitational scalar potential exterior to the astrophysical body under consideration and hence makes physical interpretations simpler. This new approach is thus computationally less cumbersome and physically more applicable in principle than the previous approach.

This work examines the effect of the oblate spheroidal nature of the Sun and planets on some gravitational phenomena using this new approach. The motion of planets and photons in the Solar System are also investigated. These effects include gravitational spectral shift of light, gravitational length contraction and gravitational time dilation. We equally construct the generalized Lagrangian for this field and use it to study orbits in homogeneous oblate spheroidal space-time [9–11]. In this research work, we also start the study of static homogeneous prolate spheroidal gravitational fields using this new approach. Einstein's gravitational field equations exterior to static homogenous prolate spheroids are derived. Solutions to the derived field equations are also constructed and the consequences of the field on the motion of test particles are also investigated [12, 13].

### §1.3. Objectives of the study

- Derivation of Einstein's gravitational field equations exterior to static homogeneous (time independent) prolate spheroidal distributions of mass as an extension of Schwarzschild's metric (that is using the new approach);
- Solutions to field equations derived and consequences to the motion of test particles;

- Derivation of the planetary equation of motion and the equation for the deflection of light in the gravitational field exterior to homogenous (time independent) spheroids (prolate and oblate);
- Investigation of the effect of the oblate spheroidal nature of the Sun and planets on some gravitational phenomena (gravitational length contraction, gravitational time dilation and gravitational spectral shift of light) using the new approach.

#### §1.4. Scope of the study

The philosophy of General Relativity describes gravitation as a geometrical phenomenon with the effect of gravitation contained in the covariant metric tensor for a gravitational field [14]. The metric tensor exterior to all possible distributions of mass within oblate spheroidal and prolate spheroidal geometries given by Howusu [8] is made explicit and used to study these gravitational fields. Our knowledge of orthogonal curvilinear coordinates, tensor analysis, Schwarzschild gravitational field and general relativistic mechanics is used to achieve the objectives.

Basically, we concentrate on gravitational sources with time independent and axially-symmetric distributions of mass within spheroids, characterized by at most two typical integrals of geodesic motion, namely, energy and angular momentum. From an astrophysical point of view, such an assumption, although not necessary, could, however, prove useful, because it is equivalent to the assumption that the gravitational source is changing slowly in time so that partial time derivatives are negligible compared to the spatial ones. We stress that the mass source considered is not the most arbitrary one from a theoretical point of view, but on the other hand, many astrophysically interesting systems are usually assumed to be time independent (or static from another point of view) and axially symmetric continuous sources [15].

#### §1.5. Significance of the study

Gravity is the least understood of all the fundamental forces in nature; but mass and space, which are governed by gravity, are the building blocks and fabric of our universe. General Relativity is the most fundamental theorem of physics about the nature of gravity. If we better understand the nature of mass and space, we may be able to do things previously undreamed of. So far studies of General Relativity have yielded atomic clocks, guidance systems for spacecrafts and the Global Positioning Systems (GPS). We cannot foresee all that can come from a better understanding of space-time and mass-energy, but a theorem

about these fundamental subjects must be thoroughly examined if we are to use it to our advantage [16]. This research work is a step in this direction.

This research work substantially extends Einstein's theory of gravitation (General Relativity) from the well known Schwarzschild space-time to the experimentally more precise oblate and prolate spheroidal space-times in the universe. Thus, the theoretical analysis of the motion of particles of non-zero rest masses, gravitational length contraction, gravitational time dilation and gravitational spectral shift is extended from the gravitational field exterior to a spherical mass to the gravitational field exterior to spheroidal masses. Our approach in this research work unlike in earlier attempts makes it possible for us to obtain physically interpretable theoretical values for the above listed gravitational phenomena in approximate gravitational fields exterior to bodies in the Solar System. Our newly obtained expression for gravitational time dilation can now be incorporated into the contribution of gravitation in the design of Global Positioning System (GPS). It is hoped that when this is done, the precision rate of GPS will be greatly improved. This work also opens the door for the theoretical investigation of the contributions of the oblateness of the Sun and planets on other gravitational phenomena such as geodetic deviation, radar sounding and anomalous orbital precession, using this new approach. An insight is also provided for the theoretical investigation of the contributions of the oblateness and prolateness of some astronomical bodies on gravitational phenomena. It is thus eminent that this work will serve as an eye opener for the verification of small departures of theory from reality in astronomy in the near future.

## **Chapter 2. Methodology**

### **§2.1. General relativistic mechanics in Schwarzschild's field**

#### **§2.1.1. Einstein's gravitational field equations**

It is well known that Einstein's gravitational field equations are tensorially given as [1]

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}, \quad (2.1)$$

where  $G_{\mu\nu}$  is the Einstein tensor constructed from the metric tensor  $g_{\mu\nu}$  of the space-time;  $c$  is the speed of light in vacuum,  $G$  is the universal gravitational constant and  $T_{\mu\nu}$  is the stress tensor which is the source of the gravitational metric field. There are actually ten independent scalar

equations in (2.1) because of the symmetry of the tensors involved in the field equations. These equations are actually second order partial differential equations and are generally non-linear.

### §2.1.2. Schwarzschild's metric

If one considers a spherical body of radius  $R_0$  and total rest mass  $M$  distributed uniformly with density  $\rho_0$ , then the general relativistic field equations in its exterior region are given tensorially as [1]

$$G_{\mu\nu} = 0. \quad (2.2)$$

Thus, Einstein's equations (2.2) give ten different differential equations with zero elements on the right-hand-side. Schwarzschild in 1916 constructed the first exact solution of Einstein's gravitational field equations. It was the metric due to a static spherically symmetric body situated in empty space such as the Sun or a star [1]. The result is as follows

$$g_{00} = 1 + \frac{2f(r)}{c^2}, \quad (2.3)$$

$$g_{11} = - \left( 1 + \frac{2f(r)}{c^2} \right)^{-1}, \quad (2.4)$$

$$g_{22} = -r^2, \quad (2.5)$$

$$g_{33} = -r^2 \sin^2 \theta, \quad (2.6)$$

where  $f(r)$  is an arbitrary function determined by the distribution. It is a function of the radial coordinate  $r$  only; since the distribution and hence its exterior gravitational field possess spherical symmetry. From the condition that these metric components should reduce to the field of a point mass located at the origin and contain Newton's equations of motion in the gravitational field of the spherical body, it follows that  $f(r)$  is the Newtonian gravitational scalar potential in the exterior region of the body [17].

### §2.1.3. Schwarzschild's singularity

The world-line element in Schwarzschild field is given by [8]

$$c^2 d\tau^2 = c^2 \left( 1 + \frac{2f(r)}{c^2} \right) dt^2 - \left( 1 + \frac{2f(r)}{c^2} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \quad (2.7)$$



In this field

$$f(r) = -\frac{GM}{r}, \quad r > R_0. \quad (2.8)$$

It has been known since 1916 that it is possible for a spherical body to have a point outside it at which the Schwarzschild metric has a singularity. This singularity is denoted by  $r_s$  and is called the Schwarzschild singularity. It is given by the condition

$$1 - \frac{2GM}{c^2 r_s} = 0,$$

thus,

$$r_s = \frac{2GM}{c^2}. \quad (2.9)$$

For the Earth,  $r_s = 0.89$  cm. This radius lies in the interior of the Earth where the metric is precisely the interior metric and hence the exterior metric is not applicable. For most physical bodies in the universe, the Schwarzschild radius is much smaller than the radius of their surface. Hence for most bodies, there does not exist a Schwarzschild singularity. It is however, speculated that there exist some bodies in the universe with the Schwarzschild radius in the exterior region. Such bodies are called black holes.

#### §2.1.4. Gravitational length contraction in Schwarzschild field

In Schwarzschild field, the space part of the metric is given by

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (2.10)$$

Thus, in the neighborhood of a massive body, two points of the same angle  $\theta$  and  $\phi$  now have a separation which is different from the corresponding separation in empty space. That is

$$ds = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr \cong \left(1 + \frac{GM}{c^2 r} + \dots\right) dr. \quad (2.11)$$

This equation implies that  $ds > dr$ . In other words  $r$  is no longer the measure of radial distances. Also, it follows that the length of physical bodies is not conserved in a gravitational field. That is, length is contracted in a gravitational field. This is the phenomenon of length contraction. It is highly speculated that not only are material objects (such as meter rules) contracted by gravitational fields but also space itself is contracted by gravitational fields [2].

### §2.1.5. Gravitational time dilation in the spherical field

Consider, a clock at rest at a fixed point in Schwarzschild gravitational field around a spherical body, then  $dr = d\theta = d\phi = 0$  and hence Schwarzschild's world line element, reduces to

$$dt = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} d\tau \cong \left(1 + \frac{GM}{c^2 r} + \dots\right) d\tau. \quad (2.12)$$

It can be deduced that  $dt > d\tau$  and therefore, the coordinate time of a clock in the gravitational field is dilated relative to the proper time.

### §2.1.6. Motion of particles of non-zero rest masses in Schwarzschild field

A test mass is one which is so small that the gravitational field produced by it is so negligible that it does not have any effect on the space metric. A test mass is a continuous body, which is approximated by its geometrical centre; it has nothing in common with a point mass whose density should obviously be infinite [1].

The general relativistic equation of motion for particles of non-zero rest masses in a gravitational field are given by

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0, \quad (2.13)$$

where  $\Gamma_{\nu\lambda}^\mu$  are the coefficients of affine connection for the gravitational field. For Schwarzschild field, the equations of motion are

$$\ddot{t} + \frac{k}{c^2 r^2 \left(1 - \frac{2k}{c^2 r}\right)} \dot{t} \dot{r} = 0, \quad (2.14)$$

$$\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0, \quad (2.15)$$

$$\ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0, \quad (2.16)$$

$$\begin{aligned} \ddot{r} + \frac{1}{2} c^2 f^1 (1+f) \dot{t}^2 - \frac{1}{2} f^1 (1+f)^{-1} \dot{r}^2 - \\ - r (1+f) \dot{\theta}^2 - r (1+f) \sin^2 \theta \dot{\phi}^2 = 0, \end{aligned} \quad (2.17)$$

where the dot denotes differentiation with respect to proper time,  $k \equiv GM$ ,  $f \equiv -\frac{2k}{c^2 r}$  and  $f^1 \equiv \frac{df}{dr}$  [1].

## §2.2 General relativistic mechanics in static homogeneous spheroidal fields

### §2.2.1. Oblate and prolate spheroidal coordinate systems

The oblate spheroidal coordinates are related to the Cartesian coordinates by

$$\left. \begin{aligned} x &= a \cosh u \cos v \cos \phi \\ y &= a \cosh u \cos v \sin \phi \\ z &= a \sinh u \sin v \end{aligned} \right\}, \quad (2.18)$$

where  $u \geq 0$ ,  $0 \leq v \leq \pi$  and  $0 \leq \phi \leq 2\pi$ .

It is convenient to use the following transformations to eliminate the hyperbolic functions and ease computation with this coordinate system;  $\xi = \sinh u$ ,  $\eta = \sin v$  and thus, the relation between Cartesian and oblate spheroidal coordinate systems can be written as [18]

$$\left. \begin{aligned} x &= a (1 - \eta^2)^{1/2} (1 + \xi^2)^{1/2} \cos \phi \\ y &= a (1 - \eta^2)^{1/2} (1 + \xi^2)^{1/2} \sin \phi \\ z &= a \eta \xi \end{aligned} \right\}, \quad (2.19)$$

where  $0 \leq \xi < \infty$ ,  $-1 \leq \eta \leq 1$ ,  $0 \leq \phi \leq 2\pi$  and  $a$  is a constant parameter.

For a prolate spheroid unlike an oblate spheroid, the polar diameter is longer than the equatorial diameter. The derivation of the prolate spheroidal coordinate system is quite similar to the above derivation of the oblate spheroidal coordinate system. The relation between the Cartesian and prolate spheroidal coordinate systems is [18]

$$\left. \begin{aligned} x &= a (1 - \eta^2)^{1/2} (1 + \xi^2)^{1/2} \cos \phi \\ y &= a (1 - \eta^2)^{1/2} (1 + \xi^2)^{1/2} \sin \phi \\ z &= a \eta \xi \end{aligned} \right\}, \quad (2.20)$$

where  $0 \leq \xi < \infty$ ,  $-1 \leq \eta \leq 1$  and  $0 \leq \phi \leq 2\pi$ .

### §2.2.2. Metric tensor exterior to an oblate spheroid and a prolate spheroid

The invariant world line element in the exterior region of a static spherical body is given generally according to [8], where  $f(r, \theta, \phi)$  is a generalized arbitrary function determined by the distribution of mass or pressure and possess all the symmetries of the mass distribution. Thus,

according to [8], the invariant world line element is

$$c^2 d\tau^2 = c^2 \left( 1 + \frac{2f(r, \theta, \phi)}{c^2} \right) dt^2 - \left( 1 + \frac{2f(r, \theta, \phi)}{c^2} \right)^{-1} dr^2 - r^2 \sin^2 \theta d\phi^2. \quad (2.21)$$

It is a well known fact of General Relativity that  $f(r, \theta, \phi)$  is approximately equal to Newton's gravitational scalar potential in the space-time exterior to the mass or pressure distributions within spherical geometry.

Now, let the spherical body be transformed, by deformation, into an oblate spheroidal body in such a way that its density  $\rho_0$  and total mass  $M$  remain the same and its surface parameter is given in oblate spheroidal coordinates as

$$\xi = \xi_0 = \text{constant}. \quad (2.22)$$

Then, the general relativistic field equations exterior to an oblate spheroidal body are mathematically equivalent to those of the spherical body. This is because they are both tensorially the same. Hence, they are only related by the transformation from spherical to oblate spheroidal coordinates. Therefore, to get the corresponding invariant world line element in the exterior region of an oblate spheroidal mass one could do the following:

- 1) Replace  $f(r, \theta, \phi)$  by the corresponding function  $f(\eta, \xi, \phi)$  exterior to oblate spheroidal bodies. Thus, a sound and astrophysically satisfactory approximate expression for the function  $f(\eta, \xi, \phi)$  is obtained by equating it to the gravitational scalar potential exterior to the distribution of mass within oblate spheroidal regions [8];
- 2) Transform coordinates from spherical to oblate spheroidal

$$(ct, r, \theta, \phi) \rightarrow (ct, \eta, \xi, \phi) \quad (2.23)$$

on the right hand side of equation (2.7). The following components of the metric tensor in the region exterior to a homogeneous oblate spheroid in oblate spheroidal coordinates are obtained

$$g_{00} = \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right), \quad (2.24)$$

$$g_{11} = -\frac{a^2}{1 + \xi^2 - \eta^2} \left[ \eta^2 \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} + \frac{\xi^2 (1 + \xi^2)}{(1 - \eta^2)} \right], \quad (2.25)$$

$$g_{12} = g_{21} = -\frac{a^2 \eta \xi}{1 + \xi^2 - \eta^2} \left[ 1 - \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} \right], \quad (2.26)$$

$$g_{22} = -\frac{a^2}{1 + \xi^2 - \eta^2} \left[ \xi^2 \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} + \frac{\eta^2 (1 - \eta^2)}{(1 + \xi^2)} \right], \quad (2.27)$$

$$g_{33} = -a^2 (1 + \xi^2) (1 - \eta^2). \quad (2.28)$$

It may be of interest to note that this metric tensor field unlike the metric tensor field used by Howusu and Uduh [7] contains only one unknown function,  $f(\eta, \xi)$  determined by the mass distribution and has no exponential components.

The covariant metric tensor obtained above for gravitational fields exterior to oblate spheroidal masses has two additional non-zero components  $g_{12}$  and  $g_{21}$  not found in Schwarzschild field and the metric used by Howusu and Uduh [7]. Thus, the extension from Schwarzschild field to homogeneous oblate spheroidal gravitational fields has produced two additional non zero tensor components and thus this metric tensor field is unique. This confirms the assertion that oblate spheroidal gravitational fields are more complex than spherical fields and hence general relativistic mechanics in this field is more involved. This partly accounts for the scanty research carried out on this gravitational field.

Similarly, it has been shown [8] that the covariant metric tensor exterior to static homogeneous prolate spheroidal distributions of mass is given as

$$g_{00} = \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right), \quad (2.29)$$

$$g_{11} = -\frac{a^2 \eta^2}{\eta^2 + \xi^2 - 1} \left[ \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} + \frac{\xi^2 (1 - \xi^2)}{\eta^2 (\eta^2 - 1)} \right], \quad (2.30)$$

$$g_{12} = g_{21} = -\frac{a^2 \eta \xi}{\eta^2 + \xi^2 - 1} \left[ -1 + \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} \right], \quad (2.31)$$

$$g_{22} = -\frac{a^2 \xi^2}{\eta^2 + \xi^2 - 1} \left[ \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} + \frac{\eta^2 (\eta^2 - 1)}{\xi^2 (1 - \xi^2)} \right], \quad (2.32)$$

$$g_{33} = -a^2 (1 - \xi^2) (\eta^2 - 1). \quad (2.33)$$

This metric tensor has the same number of non-zero components (six) as the metric exterior to an oblate spheroid. As has been noted in

the case of oblate spheroids,  $f(\eta, \xi)$  is an arbitrary function determined by the mass or pressure and hence it possesses all the symmetries of the latter, a priori. Herein,  $f(\eta, \xi)$  can be conveniently approximated to be equal to Newton's gravitational scalar potential exterior to the mass distribution.

The metric tensors by virtue of their construction satisfy the first and second postulates of General Relativity. There are invariance of the line element; and Einstein's gravitational field equations [8].

### §2.2.3. Gravitational field equations exterior to static homogeneous prolate spheroids

To obtain the contravariant metric tensor for the gravitational field exterior to a prolate spheroid,  $g^{\mu\nu}$  we use the fact that  $g^{\mu\nu}$  is the cofactor of  $g_{\mu\nu}$  in  $g$  divided by  $g$  [18]. That is

$$g^{\mu\nu} = \frac{\text{cofactor of } g_{\mu\nu} \text{ in } g}{g}, \quad (2.34)$$

where

$$g = \det \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix}. \quad (2.35)$$

The coefficients of affine connection  $\Gamma_{\mu\sigma}^{\sigma}$  for any gravitational field are defined in terms of the covariant and contravariant metric tensor of space-time as [18]

$$\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2} g^{\sigma\nu} (g_{\mu\nu, \lambda} + g_{v\lambda, \mu} - g_{\mu\lambda, v}), \quad (2.36)$$

where the comma denotes partial differentiation with respect to  $\lambda$ ,  $\mu$  and  $v$ . In this research work, we have constructed the 64 coefficients of affine connection for this gravitational field.

The curvature tensor or the Riemann-Christoffel tensor  $R_{\alpha\beta\sigma}^{\delta}$  for this field is defined in terms of the coefficients of affine connection as

$$R_{\alpha\beta\sigma}^{\delta} = \Gamma_{\alpha\sigma, \chi}^{\delta} - \Gamma_{\alpha\beta, \sigma}^{\delta} + \Gamma_{\alpha\sigma}^{\epsilon} \Gamma_{\epsilon\beta}^{\delta} - \Gamma_{\alpha\beta}^{\epsilon} \Gamma_{\epsilon\sigma}^{\delta}, \quad (2.37)$$

where the comma denotes partial differentiation with respect to  $\beta$  and  $\sigma$ . This research work has equally constructed the 256 components of this tensor for homogeneous prolate spheroidal gravitational fields. From the curvature tensor  $R_{\alpha\beta\sigma}^{\delta}$  for this gravitational field, we have defined a second rank tensor  $R_{\alpha\beta}$  (called the Ricci tensor) for the gravitational

field exterior to the prolate spheroid as

$$R_{\alpha\beta} = R_{\alpha\beta\delta}^{\delta}. \quad (2.38)$$

The 16 components of this tensor for the static homogeneous prolate spheroids have been constructed. From the Ricci tensor for our gravitational field, we deduced a scalar  $R$  defined by

$$R = R_{\alpha}^{\alpha} = g^{\alpha\beta} R_{\alpha\beta} \quad (2.39)$$

called the curvature scalar for homogeneous spheroidal fields.

It is well known that for a region exterior to any astrophysical body, the general relativistic field equations are given tensorially as

$$G_{\mu\nu} = 0, \quad (2.40)$$

where  $G_{\mu\nu}$  is the Einstein tensor, given explicitly as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad (2.41)$$

where  $R_{\mu\nu}$  is the Ricci tensor,  $R$  the curvature scalar and  $G_{\mu\nu}$  the covariant metric tensor for the field. The Einstein field equations for the gravitational field exterior to homogeneous prolate spheroids are then built up.

These are partial differential equations with only one unknown. We constructed the solution to the first field equation using our knowledge of partial differential equations and path integral methods.

#### **§2.2.4. Motion of test particles exterior to static homogeneous prolate spheroidal masses**

The general relativistic equation of motion and the coefficients of affine connection for our field are used to study the motion of particles of non-zero rest masses in this field. Einstein's geometrical equations of motion for test particles in the gravitational fields of prolate spheroidal astronomical bodies are derived. These equations of motion have only one unknown function. The solution of the first field equation is then used to study the effect of the gravitational field on the motion of test particles.

#### **§2.2.5. Planetary motion and motion of photons in spheroidal gravitational fields**

In General Relativity, the change in energy of a freely moving photon is given by the scalar equation of the isotropic geodesic equations,

which manifest on the work produced on a photon being moved along a path [19]. Here, we use the generalized Lagrangian exterior to static homogenous oblate and prolate spheroids to study orbits in this gravitational field and obtain expressions for the conservation of total energy and angular momentum in this field. The planetary equation of motion and the equation for the deflection of light (photons) in the gravitational field exterior to homogeneous oblate and prolate spheroidal bodies are derived.

### §2.2.6. Effects of oblateness of the Sun and planets on some gravitational phenomena

**Gravitational time dilation.** In this research work, we show that our theoretical extension of Schwarzschild's gravitational field to oblate spheroidal fields conform satisfactorily to the above proven experimental and astrophysical facts. We consider a clock at rest in this gravitational field such that  $d\xi = d\eta = d\phi = 0$ . The world line element for the gravitational field exterior to an oblate spheroidal mass is then used to give a new expression for time dilation. We then use our new expression to calculate the dilated coordinate time as a function of proper time along the equator and pole of various bodies in the Solar System in approximate homogeneous gravitational fields. This has not been done in previous theoretical approaches to the subject.

**Gravitational length contraction.** In this research work, the space part of the world line element in the gravitational field exterior to an oblate spheroidal mass is used with the angular coordinates kept constant. This gives us a new expression for gravitational length contraction. As an illustration of this gravitational phenomenon in oblate spheroidal gravitational fields, we consider a long stick lying "radially" along the equator in the approximate gravitational field of a static homogenous oblate spheroidal mass such as the Earth and we let the  $\xi$ -coordinates of the ends be  $\xi_1$  and  $\xi_2$ , where  $\xi_2 > \xi_1$ . With this, we find the expression for its proper length and deduce that the length is reduced in the gravitational field. This computation affirms the soundness of our extension and confirms the assertion from Schwarzschild's metric that not only is length contracted in gravitational fields but space also.

**Gravitational spectral shift of light.** We consider a beam of light (photons) moving from a source or emitter at a fixed point in the gravitational field of the oblate spheroidal body to an observer or receiver at a fixed point in the same gravitational field. Einstein's equation of motion for a photon is used to derive an expression for the shift in



frequency of a photon moving in the gravitational field of an oblate spheroidal mass. We then as an illustration of the expression obtained, consider a signal of light emitted and received along the equator of the static homogenous oblate spheroidal Earth (in the approximate gravitational field). The ratio of the shift in frequency to the frequency of the emitted light at various points in the equatorial plane and received on the equator at the surface of the static homogeneous oblate spheroidal Earth is computed using our derived equation. Also, the ratio of the shift in frequency of light to the frequency of the emitted light on the equator at the surface and received at various points along the equator of the static homogeneous oblate spheroidal Earth is also computed. It is worth noting that we deliberately used emitters and receivers at rest in this gravitational field to avoid shifts in frequency due to Doppler effect. However, in more practical cases, the gravitational spectral shift is always compounded with the special relativistic shift (Doppler shift). This yields a general expression for the shift in frequency when there is a relative motion between the emitter and receiver.

### Chapter 3. Results and Discussion

#### §3.1. General relativistic mechanics in homogeneous oblate spheroidal gravitational fields

##### §3.1.1. Motion of particles of non-zero rest masses in homogeneous oblate spheroidal space-time

The contravariant metric tensor  $g^{\mu\nu}$  for this gravitational field is obtained as

$$g^{00} = \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1}, \quad (3.1)$$

$$g^{11} = -\frac{(1 - \eta^2)(1 + \xi^2 - \eta^2) \left[ \eta^2 (1 - \eta^2) + \frac{\xi^2(1 + \xi^2)}{1 + \frac{2}{c^2} f(\eta, \xi)} \right]}{\frac{a^2}{1 + \frac{2}{c^2} f(\eta, \xi)} [\eta^2 (1 - \eta^2) + \xi^2 (1 + \xi^2)]^2}, \quad (3.2)$$

$$g^{12} = g^{21} = -\frac{\eta \xi (1 - \eta^2) (1 + \xi^2) (1 + \xi^2 - \eta^2)}{\frac{a^2}{1 + \frac{2}{c^2} f(\eta, \xi)} [\eta^2 (1 - \eta^2) + \xi^2 (1 + \xi^2)]^2} \times \left[ 1 - \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{-1} \right], \quad (3.3)$$

$$g^{22} = - \frac{(1 + \xi^2)(1 + \xi^2 - \eta^2) \left[ \xi^2 (1 + \xi^2) + \frac{\eta^2(1-\eta^2)}{1 + \frac{2}{c^2} f(\eta, \xi)} \right]}{\frac{a^2}{1 + \frac{2}{c^2} f(\eta, \xi)} [\eta^2 (1 - \eta^2) + \xi^2 (1 + \xi^2)]^2}, \quad (3.4)$$

$$g^{33} = \left[ a^2 (1 + \xi^2) (1 - \eta^2) \right]^{-1}. \quad (3.5)$$

The contravariant metric tensor has two additional non-zero components not found in Schwarzschild field. Notice that unlike in Schwarzschild field; where all the non-zero components of the contravariant tensor are simply reciprocals of the covariant metric tensor; only equations (3.1) and (3.5) are reciprocals of their respective covariant tensors. The other non-zero components have a common denominator. The coefficients of affine connection found have fourteen non zero components., dependent on a single unknown function  $f$ . Schwarzschild's connection coefficients on the other hand have ten non-zero components dependent on the gravitational scalar potential exterior to the spherically symmetric mass [20].

Using the general relativistic equation of motion for test particles and the coefficients of affine connection for the gravitational field exterior to an oblate spheroidal mass the following equations of motion are obtained. The time equation of motion is obtained as

$$\frac{d}{d\tau}(\ln \dot{t}) + \frac{d}{d\tau} \left[ \ln \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right) \right] = 0 \quad (3.6)$$

with solution as

$$\dot{t} = A \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1}. \quad (3.7)$$

As  $t \rightarrow \tau$ ,  $f(\eta, \xi) \rightarrow 0$  and the constant  $A \equiv 1$ . Thus,

$$\dot{t} = \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1}. \quad (3.8)$$

Equation (3.8) is the expression for the variation of the time on a clock moving in this gravitational field. It is of same form as that in Schwarzschild's gravitational field. Interestingly, our expression differs greatly from that obtained by [21]. In his case, he obtains  $\dot{t}$  as an exponential function dependent on his unknown function  $F(\xi)$ . Thus, our expression in its merit stands out uniquely, as an extension of the results in Schwarzschild's field. Also, it tends out most remarkably

that our unknown function can be evaluated from the gravitational field equations.

The  $\eta$  equation of motion is

$$\ddot{\eta} + \Gamma_{00}^1 c^2 \dot{t}^2 + \Gamma_{11}^1 \dot{\eta}^2 + \Gamma_{22}^1 \dot{\xi}^2 + \Gamma_{33}^1 \dot{\phi}^2 + 2\Gamma_{12}^1 \dot{\eta} \dot{\xi} = 0. \quad (3.9)$$

The  $\xi$  equation of motion is given as

$$\ddot{\xi} + \Gamma_{00}^1 c^2 \dot{t}^2 + \Gamma_{11}^1 \dot{\eta}^2 + \Gamma_{22}^1 \dot{\xi}^2 + \Gamma_{33}^1 \dot{\phi}^2 + 2\Gamma_{12}^1 \dot{\eta} \dot{\xi} = 0. \quad (3.10)$$

The azimuthal equation of motion is obtained as

$$\dot{\phi} = \frac{l}{(1 - \eta^2)(1 + \xi^2)}, \quad (3.11)$$

where  $l$  is a constant of motion. Herein  $l$  physically corresponds to the angular momentum and hence equation (3.11) is the law of conservation of angular momentum in this gravitational field. It does not depend on the gravitational potential and is of same form as that obtained in Schwarzschild's and Newton's dynamical theory of gravitation. It is worth emphasizing that although the form is the same, it stands out unique as the parameters are in oblate spheroidal coordinates.

### §3.1.2. Planetary motion and motion of photons in the equatorial plane of homogeneous oblate spheroidal gravitational fields

The Lagrangian, in the space-time exterior to an oblate spheroid can be written explicitly in oblate spheroidal coordinates as

$$L = \frac{1}{c} \left[ -g_{00} \left( \frac{dt}{d\tau} \right)^2 - g_{11} \left( \frac{d\eta}{d\tau} \right)^2 - 2g_{12} \left( \frac{d\eta}{d\tau} \right) \left( \frac{d\xi}{d\tau} \right) - g_{22} \left( \frac{d\xi}{d\tau} \right)^2 - g_{33} \left( \frac{d\phi}{d\tau} \right)^2 \right]^{1/2}. \quad (3.12)$$

For orbits in the equatorial plane of a homogeneous oblate spheroidal mass;  $\eta \equiv 0$  and using the Lagrangian it is shown (using the fact that the gravitational field is a conservative field) that the law of conservation of energy in the equatorial plane of the gravitational field exterior to an oblate spheroidal mass is

$$\left( 1 + \frac{2}{c^2} f(\xi) \right) \dot{t} = k, \quad \dot{k} = 0, \quad (3.13)$$

where  $k$  is a constant. Notice that this equation is exactly the same as the expression obtained from the time equation of motion for test part-

icles. This expression has never been obtained before. Thus, our use of the metric tensor and Lagrangian mechanics in oblate spheroidal gravitational field yields the first ever documented expression for the conservation of energy in this field [9].

Also, the law of conservation of angular momentum in the equatorial plane of the gravitational field exterior to an oblate spheroidal body is obtained as

$$(1 + \xi^2) \dot{\phi} = l, \quad \dot{l} = 0, \quad (3.14)$$

where  $l$  is a constant. It is interesting and instructive to note that this expression is equivalent to that obtained from the general relativistic azimuthal equation of motion for test particles in the gravitational field exterior to an oblate spheroidal mass. Thus our method for obtaining the laws of conservation of total energy and angular momentum in this section is mathematically more convenient and physically more interesting than the method in the previous section. Instead of going through the rigorous tensor analysis to derive the affine connections before proceeding to derive the conservation laws; we simply need to build the covariant metric tensor and use the generalized Lagrangian to deduce the conservation laws.

Using the fact that the Lagrangian  $L = \epsilon$ , with  $\epsilon = 1$  for time like orbits and  $\epsilon = 0$  for null orbits the planetary equation of motion in this gravitational field is

$$\begin{aligned} & \frac{d^2 u}{d\phi^2} - 3u(1 + u^2) \frac{du}{d\phi} + \frac{u + u^2}{2} (u^2 - u + 2) \times \\ & \times \left( 1 + \frac{2}{c^2} f(u) \right) = \left( \frac{1 + u^2}{acl} \right)^2 (a^2 c^2 u^2 - 1 - u^2) \frac{d}{du} f(u). \end{aligned} \quad (3.15)$$

It can be solved to obtain the perihelion precision of planetary orbits. This is opened up for further research.

The photon equation of motion in the vicinity of a static massive homogenous oblate spheroidal body is obtained as

$$\begin{aligned} & \frac{d^2 u}{d\phi^2} - 3u(1 + u^2) \frac{du}{d\phi} + \frac{u + u^2}{2} (u^2 - u + 2) \times \\ & \times \left( 1 + \frac{2}{c^2} f(u) \right) = \frac{u^2}{c^2} (1 + u^2)^2 \frac{d}{du} f(u). \end{aligned} \quad (3.16)$$

In the limit of special relativity, some terms in equation (3.16) vanish and the equation becomes

$$\frac{d^2 u}{d\phi^2} - 3u(1 + u^2) \frac{du}{d\phi} + \frac{u + u^2}{2} (u^2 - u + 2). \quad (3.17)$$

The solution of the special relativistic case, equation (3.17) can be used to solve the general relativistic equation, (3.16). This can be done by taking the general solution of equation (3.17) to be a perturbation of the solution of equation (3.16). The immediate consequence of this analysis is that it will produce a new expression for the total deflection of light grazing a massive oblate spheroidal body such as the Sun. This is also open for further research and astrophysical interpretations.

### §3.1.3. Effects of oblateness of the Sun and planets on some gravitational phenomena

**Gravitational scalar potential along the pole and equator of the homogeneous oblate spheroidal Sun and planets.** The computed numerical values of the constants  $\xi_0$  and  $a$  for the oblate spheroidal bodies in the Solar System are given in Table 2.

Body	Equatorial radius $x_0 \times 10^3$	Polar radius $z_0 \times 10^3$	$\xi_0$	$a$ , m
Sun	700,00	699,994	241.52	$2.89829 \times 10^6$
Mercury	2,440	2,440	—	—
Venus	6,052	6,052	—	—
Earth	6,378	6,356	12.01	$5.29226 \times 10^5$
Mars	3,396	3,376	09.17	$3.68157 \times 10^5$
Jupiter	71,490	66,843	02.64	$2.53193 \times 10^7$
Saturn	60,270	53,761	01.97	$2.72899 \times 10^7$
Uranus	25,560	24,793	03.99	$6.21378 \times 10^6$
Neptune	24,760	24,116	04.30	$5.60837 \times 10^6$

Table 2: Computed constants  $\xi_0$  and  $a$  for the Sun and planets [10].

The gravitational scalar potential exterior to a homogeneous oblate spheroid [4] is given as

$$f(\eta, \xi) = B_0 Q_0(-i\xi) + B_2 Q_2(-i\xi) P_2(\eta), \quad (3.18)$$

where  $Q_0$  and  $Q_2$  are the Legendre functions linearly independent to the Legendre polynomials  $P_0$  and  $P_1$  respectively.  $B_0$  and  $B_2$  are constants with approximate expressions as

$$B_0 \approx i \frac{4\pi G \rho_0 a^2 \xi_0^5}{3(1 + \xi_0^2)}, \quad (3.19)$$

$$B_2 \approx i \frac{4\pi G \rho_0 a^2 \xi_0^5}{3[44\xi_0^2 + (1 + 3\xi_0^2) \binom{2}{0}]}. \quad (3.20)$$

The mean density  $\rho_0$  for various bodies in the universe is taken according to the astronomical data. With the values of  $\xi_0$  and  $a$  in Table 2, we get the values for the constants  $B_0$  and  $B_2$  for the homogeneous oblate spheroidal Sun and planets as in Table 3.

Body	Mean density		
	$\rho_0, \text{kg/m}^3$	$i B_0, \text{Nm/kg}$	$i B_2, \text{Nm/kg}$
Sun	1409	$4.67961 \times 10^{13}$	$8.91380 \times 10^7$
Mercury	5400	—	—
Venus	5200	—	—
Earth	5500	$7.43766 \times 10^8$	$1.70123 \times 10^5$
Mars	3900	$1.13049 \times 10^8$	$4.40357 \times 10^5$
Jupiter	1300	$3.76352 \times 10^9$	$1.50951 \times 10^7$
Saturn	690	$8.76690 \times 10^8$	$5.70607 \times 10^6$
Uranus	1300	$8.41939 \times 10^8$	$1.61800 \times 10^6$
Neptune	1600	$1.06534 \times 10^9$	$1.78225 \times 10^6$

Table 3: Values of the constants  $B_0$  and  $B_2$  for the Sun and Planets [17].

By considering the first two terms of the series expansion of the Legendre functions, we can write

$$f(\eta, \xi) \approx \frac{B_0}{3\xi^3} (1 + 3\xi^2) i + \frac{B_2}{30\xi^3} (7 + 15\xi^2) i, \quad (3.21)$$

$$f(\eta, \xi) \approx \frac{B_0}{3\xi^3} (1 + 3\xi^2) i - \frac{B_2}{30\xi^3} (7 + 15\xi^2) i, \quad (3.22)$$

as the respective expressions for the gravitational scalar potential along the equator and pole exterior to homogeneous oblate spheroidal bodies. Now, with the computation of the constant  $\xi_0$  for the homogeneous oblate spheroidal Sun and planets, we can now evaluate the scalar potential along the equator and the pole at various points (multiples of  $\xi_0$ ) exterior to the Sun and planets. The detailed results are presented in [10]. Our computations agree satisfactorily with the experimental fact that the Gravitational Scalar Potential exterior to any regularly shaped object has maximum magnitude on the surface of the body and decreases to zero at infinity.

The consequence of the results obtained above is that the exact shape of the planets and Sun was used to obtain the gravitational scalar potential on the surface at the pole and equator. Thus, instead of using the values obtained by considering the Sun and planets as homogeneous spheres, our experimentally convenient values obtained can now be used.

The door is now open for the computation of values for various gravitational phenomena exterior to the static homogeneous oblate spheroidal Sun and planets along the equator and pole. Some of these phenomena include gravitational length contraction and time dilation.

**Gravitational time dilation in fields exterior to static oblate spheroidal distributions of mass.** Consider a clock at rest at a fixed point  $(\eta, \xi, \phi)$  in the gravitational field exterior to an oblate spheroidal mass, the world line element for this gravitational field reduces to

$$dt = \left(1 + \frac{2}{c^2} f(\eta, \xi)\right)^{1/2} d\tau. \quad (3.23)$$

Expanding the right hand side gives

$$dt = \left(1 + \frac{2}{c^2} f(\eta, \xi) + \dots\right)^{1/2} d\tau. \quad (3.24)$$

We obtain that  $dt > d\tau$  (dilation). Thus, coordinate time of a clock in this gravitational field is dilated relative to proper time.

As an illustration, consider two events at fixed points exterior to the homogenous oblate spheroidal Earth along the equator, separated in this gravitational field by coordinate time  $dt$  and proper time  $d\tau$ . Substituting the values for the gravitational scalar potential into the equation for gravitational time dilation (3.24), (approximate fields) yields the results presented in Table 4.

Thus, we conclude that clock runs more slowly at a smaller distance from the massive oblate spheroidal body. In other words, clocks will run slower at lower gravitational potentials (deeper within a gravity well). This was first confirmed experimentally in the laboratory by the Hafele-Keating experiment [22]. Today, there are numerous direct measurements of gravitational time dilation using atomic clocks [23], while ongoing validation is provided as a side-effect of the operation of Global Positioning System (GPS). One important experiment that was conducted to support Einstein's principle of time dilation was the experiment by Rossi and Hall in 1941 and repeated recently in accelerator rings. In this experiment, muons travelling with a velocity close to the velocity of light are observed to survive longer than muons that travel with velocities that are much less than that of light. Also, in 1976, the Smithsonian Astrophysical Observatory sent aloft a Scout rocket to an altitude of 10,000 km. This expedition also confirmed gravitational time dilation.

Fixed point along the Equator	Radial distance along the Equator, km	$dt$ as a factor of $d\tau$
$\xi_0$	6,378	1.306170
$2\xi_0$	12,723	1.122655
$3\xi_0$	19,075	1.076871
$4\xi_0$	25,430	1.055996
$5\xi_0$	31,784	1.044042
$6\xi_0$	38,140	1.036296
$7\xi_0$	44,495	1.030867
$8\xi_0$	50,851	1.026852
$9\xi_0$	57,207	1.023761
$10\xi_0$	63,562	1.021308

Table 4: Coordinate time at fixed points along the equator in the gravitational field exterior to the Earth as a factor of proper time [11].

**Gravitational length contraction in fields exterior to oblate spheroidal distributions of mass.** Here, the space part of the world line element in the gravitational field exterior to an oblate spheroidal mass is used with the angular coordinates kept constant. This gives us an expression for gravitational length contraction in this field as

$$ds = \left( \frac{a^2}{1 + \xi^2 - \eta} \left[ \frac{\xi^2}{1 + \frac{2}{c^2} f(\eta, \xi)} + \frac{\eta^2 (1 - \eta^2)}{(1 + \xi^2)} \right] \right)^{1/2} d\xi. \quad (3.25)$$

Along the equatorial line,  $\eta = 0$  and equation becomes

$$ds = a \xi (1 + \xi^2)^{-1/2} \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1/2} d\xi. \quad (3.26)$$

It can be shown that  $ds > d\xi$  from equation (3.26). In other words, the coordinate distance separating these two points is contracted in this gravitational field. Thus, we can write

$$d\xi = (a \xi)^{-1} (1 + \xi^2)^{1/2} \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{1/2} ds \quad (3.27)$$

as our expression for gravitational length contraction along the equator in this gravitational field.

As an illustration of this gravitational phenomenon, we can consider a long stick lying radially along the equator in the approximate gravitational field of a static homogeneous oblate spheroidal mass such as the



Earth. Let the  $\xi$ -coordinates of the ends be  $\xi_1$  and  $\xi_2$ , where  $\xi_2 > \xi_1$ . Then the formula for its proper length will be as that found in [11].

**Gravitational spectral shift in gravitational fields exterior to oblate spheroidal distributions of mass or pressure** Here, we consider a beam of light moving from a source or emitter at a fixed point in the gravitational field of the oblate spheroidal body to an observer or receiver at a fixed point in the same gravitational field. Einstein's equation of motion for a photon is used to derive an expression for the shift in frequency of a photon moving in the gravitational field of an oblate spheroidal mass as.

Now, consider a beam of light moving from a source or emitter (E) at a fixed point in the gravitational field of an oblate spheroidal body to an observer or receiver (R) at a fixed point in the field. Let the space-time coordinates of the emitter and receiver be  $t_E, \eta_E, \xi_E, \phi_E$  and  $t_R, \eta_R, \xi_R, \phi_R$  respectively. It is a well known fact that light moves along a null geodesic given by

$$d\tau = 0. \quad (3.28)$$

Thus, the world line element for a photon (light) takes the form

$$c^2 g_{00} dt^2 = g_{11} d\eta^2 + 2g_{12} d\eta d\xi + g_{22} d\xi^2 + g_{33} d\phi^2. \quad (3.29)$$

Substituting the covariant metric tensor for this gravitational field and let  $u$  be a suitable parameter that can be used to study the motion of a photon in this gravitational field then equation (3.29) can be written as

$$\frac{dt}{du} = \frac{1}{c} \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1/2} ds, \quad (3.30)$$

where  $ds$  is defined as

$$\begin{aligned} ds^2 = & -\frac{a^2}{1 + \xi^2 - \eta^2} \left[ \eta^2 \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} + \frac{\xi^2 (1 + \xi^2)}{(1 - \eta^2)} \right] \left( \frac{d\eta}{du} \right)^2 - \\ & - \frac{2a^2 \eta \xi}{1 + \xi^2 - \eta^2} \left[ 1 - \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} \right] \frac{d\eta}{du} \frac{d\xi}{du} - \\ & - a^2 (1 + \xi^2) (1 - \eta^2) \left( \frac{d\phi}{du} \right)^2 - \\ & - \frac{a^2}{1 + \xi^2 - \eta^2} \left[ \xi^2 \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1} + \frac{\eta^2 (1 + \eta^2)}{(1 - \xi^2)} \right] \left( \frac{d\xi}{du} \right)^2. \end{aligned} \quad (3.31)$$

Integrating equation (3.30) for a signal of light moving from emitter to receiver gives

$$t_{\text{R}} - t_{\text{E}} = \frac{1}{c} \int_{u_{\text{E}}}^{u_{\text{R}}} \left[ \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1/2} ds \right] du. \quad (3.32)$$

The time interval between emission and reception of all light signals is well known to be the same for all light signals in relativistic mechanics (constancy of the speed of light) and thus the integral on the right hand side is the same for all light signals. Consider two light signals designated 1 and 2 then

$$\Delta t_{\text{R}} = \Delta t_{\text{E}}. \quad (3.33)$$

Hence, coordinate time difference of two signals at the point of emission equals that at the point of reception. From our expression for gravitational time dilation in this gravitational field, we can write

$$\Delta \tau_{\text{R}} = \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{1/2} \Delta t_{\text{R}} \quad (3.34)$$

Hence

$$\frac{\Delta \tau_{\text{R}}}{\Delta \tau_{\text{E}}} = \left( \frac{1 + \frac{2}{c^2} f_{\text{R}}(\eta, \xi)}{1 + \frac{2}{c^2} f_{\text{E}}(\eta, \xi)} \right)^{1/2}. \quad (3.35)$$

Now, consider the emission of a peak or crest of light wave as one event. Let  $n$  be the number of peaks emitted in a proper time interval  $\Delta \tau_{\text{E}}$ , then, by definition, the frequency of the light relative to the emitter,  $\nu_{\text{E}}$ , is given as

$$\nu_{\text{E}} = \frac{n}{\Delta \tau_{\text{E}}}. \quad (3.36)$$

Similarly, since the number of cycles is invariant, the frequency of light relative to the receiver,  $\nu_{\text{R}}$ , is given as

$$\nu_{\text{R}} = \frac{n}{\Delta \tau_{\text{R}}}. \quad (3.37)$$

Consequently,

$$\frac{\nu_{\text{R}}}{\nu_{\text{E}}} = \frac{\Delta \tau_{\text{E}}}{\Delta \tau_{\text{R}}} = \left( \frac{1 + \frac{2}{c^2} f_{\text{E}}(\eta, \xi)}{1 + \frac{2}{c^2} f_{\text{R}}(\eta, \xi)} \right)^{1/2}. \quad (3.38)$$

The expressions on the right hand side of equation (3.38) are converging and can be expanded binomially in approximate gravitational

fields. This gives

$$z \equiv \frac{\Delta\nu}{\nu_E} \equiv \frac{\nu_R - \nu_E}{\nu_E} \approx \frac{1}{c^2} (f_E(\eta, \xi) - f_R(\eta, \xi)) \quad (3.39)$$

to the order of  $c^{-2}$ . It follows from equation (3.39) that if the source is nearer the body than the receiver then  $f_E(\eta, \xi) - f_R(\eta, \xi)$  and hence  $\Delta\nu < 0$ . This indicates that there is a reduction in the frequency of light when the source or emitter is nearer the body than the receiver. The light is said to have undergone a red shift (that is the light moves towards red in the visible spectrum). Otherwise (source further away from body than receiver), the light undergoes a blue shift.

This was experimentally confirmed in the laboratory by the Pound-Rebka experiment in 1959 [24] (they used the Mossbauer effect to measure the change in frequency in gamma rays as they travelled from the ground to the top of Jefferson Labs at Havard University). The effect of a gravitational potential difference on the apparent energy of the 14.4 keV gamma ray of  $\text{Fe}^{57}$  was found by Pound and Rebka [25] to agree within uncertainties, with Einstein's prediction based on his principle of equivalence (General Relativity). Pound and Rebka in 1964 [26] improved on their earlier results confirming Einstein's prediction to greater precision. The resonance of the 14.4 keV  $\text{Fe}^{57}$  gamma ray between Iron foils was still employed. The same height as in the earlier experiment in the Jefferson Physical Laboratory (22.5 m) was also used. This gravitational phenomenon was later confirmed by astronomical observations.

Now, suppose the Pound-Rebka experiment was performed at the surface of the Earth on the equator. Then, since the gamma ray frequency shift was observed at a height of 22.5 m above the surface, we model our theoretical computation and calculate the theoretical value for this shift.

Recall that at the surface of the Earth, on the equator, we have  $x_0 = 6378000$  m. Numerical values of  $a$  and  $\xi_0$  are defined as in Table 2. The value of  $x$  at a height of 22.5 m above the surface is trivially  $x_0 + 22.5 = 6378022.5$  m. Using the value of  $a$  for the Earth from Table 2 it is shown that  $\xi$  at the point is 12.0100447. For spectral shift of light emitted at the surface and received 22.5 m above the surface of the Earth, along the equator, equation (3.39) holds in approximate gravitational fields. In this case,  $f_E = -6.2079113 \times 10^7$  Nm/kg; this is the gravitational scalar potential on the surface of the Earth on the equator. At the reception point, we use the value of  $\xi$  and compute  $f_R = -6.207888 \times 10^7$  Nm/kg. Thus, substituting the values of  $f_E$ ,  $f_R$  and  $c$  into equation (3.39) yields the shift in frequency as

$z \simeq 2.578 \times 10^{-15}$ . This value is quite close to that obtained by Pound and Rebka ( $z \simeq 2.45 \times 10^{-15}$ ) in 1964. The closeness of our theoretically computed value for the Pound-Rebka experiment is remarkable indeed. The difference can be accounted for by the slight discrepancy between theory and experiment. Approximations made to the gravitational scalar potential are also a possible contributing factor.

We can now conveniently predict the gravitational spectral shift for Pound-Rebka experiment, if it was performed along the equator of the Sun and oblate spheroidal planets. As in the case of the Earth, it can be shown that the predicted shift in frequency is as shown in Table 5.

Body	Distance km	$\xi$	$f_R$ Nm/kg	Predicted shift
Sun	700,022.5	241.527	$-1.9373218 \times 10^{11}$	$-2.85889 \times 10^{-21}$
Mars	3418.5	9.231	$-1.2317966 \times 10^7$	$-9.24256 \times 10^{-20}$
Jupiter	71512.5	1.971	$-1.4958977 \times 10^9$	$-1.010111 \times 10^{-20}$
Saturn	60292.5	1.971	$-4.8484869 \times 10^8$	$-1.902222 \times 10^{-20}$
Uranus	25582.5	3.994	$-2.1522082 \times 10^8$	$-4.647889 \times 10^{-20}$
Neptune	24782.5	4.304	$-2.5196722 \times 10^8$	$-5.168667 \times 10^{-20}$

Table 5: Predicted Pound-Rebka shift in frequency for the Sun and other oblate spheroidal planets.

With these predictions, astrophysicists and astronomers can now attempt carrying out similar experiments on these planets. Although, the prospects of carrying out such experiments on the surface of some of the planets and Sun are less likely (due to temperatures on their surfaces and other factors); theoretical studies of this type helps us to understand the behavior of photons as they leave or approach these astrophysical bodies. This will thus aid in the development of future astronomical instruments that can be used to study these heavenly bodies.

Also, our expression for gravitational time dilation and spectral shift can be used in place of those obtained from Schwarzschild's field in the expression of relativistic effects in the GRACE satellites. The GRACE mission consists of two identical satellites orbiting the Earth at an altitude of about 500 km. Dual-frequency carrier-phase GPS receivers are flying on both satellites. They are used for precise orbit determination and to time-tag the K-band ranging systems used to measure changes in the distance between the two satellites. Kristine et al. [27] developed an expression for the relativistic effects of low Earth orbiters (the GRACE satellites). Their expression can be re-modified by considering our de-

rived expressions in this work. This will improve on the accuracy of GRACE data.

### §3.2. General relativistic mechanics in homogeneous prolate spheroidal gravitational fields

#### §3.2.1. Gravitational field equations exterior to a homogeneous prolate spheroidal mass

The generalized covariant metric tensor exterior to static homogeneous prolate spheroidal distributions of mass or pressure is given as equations (2.29) to (2.33). The contravariant metric tensor for this gravitational field  $g^{\mu\nu}$  can be obtained with the aid of the tensor equations (2.34) and (2.35). The contravariant metric tensor has two additional non-zero components not found in Schwarzschild field.

The coefficients of affine connection for the gravitational field exterior to a static homogeneous prolate spheroidal mass can be found. The curvature tensor for this gravitational field has twenty four non-zero components.

The Ricci tensor for this gravitational field can thus be composed in terms of the curvature tensor and the curvature scalar,  $R$ , can also be obtained. The general relativistic field equations for a region exterior to any astrophysical body are given as

$$R_{00} - \frac{1}{2} R g_{00} = 0, \quad (3.40)$$

$$R_{11} - \frac{1}{2} R g_{11} = 0, \quad (3.41)$$

$$R_{12} - \frac{1}{2} R g_{12} = 0, \quad (3.42)$$

$$R_{22} - \frac{1}{2} R g_{22} = 0, \quad (3.43)$$

$$R_{33} - \frac{1}{2} R g_{33} = 0. \quad (3.44)$$

The gravitational field equations derived are second order partial differential equations that can be solved and interpreted. All its mathematically possible solutions may then be distinguished by physical considerations, such as consistency with astrophysical or astronomical observations, data and facts. Hence, in principle, our arbitrary function,  $f(\eta, \xi)$ , which uniquely and completely determines the solution of Einstein's gravitational metric tensor field exterior to the static homogeneous prolate spheroidal mass or pressure distributions can be found.

It is interesting to note that the number of distinct non-zero components of the Ricci tensor is five. The number of distinct non-zero components of the Ricci tensor is always the same, no matter the nature of the mass distribution within prolate spheroidal regions. This number corresponds to the number of distinct non-zero components of the metric tensor in this field. It is also equal to the number of distinct field equations possible in the gravitational field.

Thus, generally, in prolate spheroidal fields, the rigorous field equations are nonlinear differential equations. The Schwarzschild's solution is a rigorous solution of Einstein's field equations and we have succeeded to extend his results to fields exterior to prolate spheroidal masses. Schwarzschild's solution is significant because it is the only solution of the field equations in empty space which is static, which has spherical symmetry, and which goes over into the flat metric at infinity [1]. Also, in fields exterior to static homogenous prolate spheroidal masses (with the approximate expression for our arbitrary function given as Newton's gravitational scalar potential exterior to the body), the metric reduces conveniently to the flat space metric for prolate spheroidal masses at infinity (since the gravitational potential reduces to zero at infinity).

### §3.2.2. Solutions to gravitational field equations exterior to homogeneous prolate spheroidal masses

It can be shown trivially that no two of these five Einstein field equations possess a common simultaneous solution. Consequently these equations may only be solved separately and their different solutions applied whenever and wherever necessary and useful in physical theories.

It is also obvious that in the case of the static homogenous distribution of mass within a prolate spheroidal region in this research work, all the five nontrivial Einstein field equations possess their own different solutions which may be applied whenever and wherever useful in physical theory.

In this section, we construct the solution for the first field equation, equation (3.40). Writing the various terms of the field equation (3.40) explicitly in terms of the metric tensor and simplifying by grouping yields a more explicit expression of the field equation with only terms of order  $c^{-2}$  as

$$K_1(\eta, \xi) f_{\eta\eta} + K_2(\eta, \xi) f_{\eta\xi} + K_3(\eta, \xi) f_{\xi\xi} + K_4(\eta, \xi) f_{\eta} + K_5(\eta, \xi) f_{\xi} + K_6(\eta, \xi) f = 0, \quad (3.45)$$

where the coefficients  $K_i$  ( $i = 1, \dots, 6$ ) are functions of  $\xi$  and  $\eta$  only.

Equation (3.45) is thus our simplified exterior field equation to the order of  $c^{-2}$  for homogeneous prolate spheroidal gravitational fields. We can now conveniently seek to construct the astrophysically most satisfactory solutions of equation (3.45) which are convergent in the exterior space-time:

$$\xi > \xi_0 \quad \text{and} \quad -1 \leq \eta \leq 1. \quad (3.46)$$

Let us now seek the solution  $f$ , of our field equation (3.45) in the form of a power series

$$f(\eta, \xi) = \sum_{n=1}^{\infty} R_n(\xi) \eta^n, \quad (3.47)$$

where  $R_n$  is a function to be determined for each value of  $n = 0, 1, 2, \dots$ . Substituting equation (3.47) into (3.45) and using the fact that  $\{\eta^n\}_{n=0}^{\infty}$  is a linearly independent set, we can equate the coefficients of  $\eta^n$  on both sides and hence obtain the equations satisfied by the functions  $R_n$ . From the coefficients of  $\eta^0$  we obtain the equation

$$\begin{aligned} & \xi^2 (\xi^2 - 1) R_1'(\xi) + 2 (\xi^2 - 1) (\xi^3 - \xi - 2) R_1(\xi) + (\xi^2 - 1)^2 \times \\ & \times (4\xi^3 + 2\xi^2 + \xi - 1) R_0'(\xi) + \left[ 4(2 + \xi^2) + 8\xi^8 (\xi^2 - 1) + \right. \\ & \left. + 2 (\xi^2 - 1)^2 (2\xi^4 - 2\xi^8 - 4\xi^3 - 1) \right] R_0(\xi) = 0. \end{aligned} \quad (3.48)$$

Equation (3.48) is the first recurrence differential equation for the unknown functions  $R_n$ . Similarly all the other recurrence differential equations follow. There are infinitely many of the recurrence differential equations to determine all the unknown functions.

Firstly, it is most interesting and instructive to note that according to the first recurrence differential equation (3.48), the unknown functions  $R_0$  and  $R_1$  are actually arbitrary. Therefore we have the freedom to choose them to satisfy the physical requirements or needs of any particular distribution or area of application. Thus, we realize that they can be chosen in such a way that there are generalizations of the gravitational scalar potential exterior to the mass distribution.

Secondly, we note that the first recurrence differential equation (3.48) determines the unknown function  $R_1$  in terms of  $R_0$ . Similarly, the other recurrence differential equations will determine all the other unknown functions  $R_2, \dots$ , in terms of  $R_0$ . Hence we obtain the general exterior solution of equation (3.45) in terms of  $R_0$ . This is our mathematically most simple and astrophysically most satisfactory general exterior solution of order  $c^{-2}$ .

### §3.2.3. Motion of particles of non-zero rest masses exterior to static homogeneous prolate spheroidal space-time

The time equation of motion is obtained as

$$\frac{d}{d\tau} (\ln t) + \frac{d}{d\tau} \left[ \ln \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right) \right] = 0 \quad (3.49)$$

with solution

$$t = \left( 1 + \frac{2}{c^2} f(\eta, \xi) \right)^{-1}. \quad (3.50)$$

Equation (3.50) is the expression for the variation of the time on a clock moving in this gravitational field. It is of same form as that obtained in the oblate spheroidal gravitational field and in Schwarzschild's field. The  $\eta$ -equation and  $\xi$ -equation of motion are

$$\dot{\eta} + \Gamma_{00}^1 c^2 \dot{t}^2 + \Gamma_{11}^1 \dot{\eta}^2 + \Gamma_{22}^1 \dot{\xi}^2 + \Gamma_{33}^1 \dot{\phi}^2 + 2\Gamma_{12}^1 \dot{\eta} \dot{\xi} = 0, \quad (3.51)$$

$$\ddot{\xi} + \Gamma_{00}^2 c^2 \dot{t}^2 + \Gamma_{11}^2 \dot{\eta}^2 + \Gamma_{22}^2 \dot{\xi}^2 + \Gamma_{33}^2 \dot{\phi}^2 + 2\Gamma_{12}^2 \dot{\eta} \dot{\xi} = 0. \quad (3.52)$$

For azimuthal motion,

$$\frac{d}{d\tau} (\ln \dot{\phi}) + \frac{d}{d\tau} \left[ \ln (\eta^2 - 1) (1 - \xi^2) \right] = 0, \quad (3.53)$$

with solution

$$\dot{\phi} = \frac{l}{(\eta^2 - 1)(1 - \xi^2)}, \quad (3.54)$$

where  $l$  is a constant of motion. Herein  $l$  physically corresponds to the angular momentum. This is the law of conservation of angular momentum in this gravitational field. It has the same form as that obtained in the oblate spheroidal gravitational field and does not depend on the gravitational potential. Therefore, it is of same form as that obtained in Schwarzschild's and Newton's dynamical theory of gravitation. The significance of these results is that the law of conservation of angular momentum takes the same form in the three different gravitational fields and thus the expression for this law of mechanics is invariant with respect to the three gravitational fields.

### §3.2.4. Orbits in homogeneous prolate spheroidal space-time

The Lagrangian in the space-time exterior to a prolate spheroid is used to obtain

$$\frac{d^2 u}{d\phi^2} - \frac{2u}{1+u^2} \frac{du}{d\phi} + \left( \frac{1+u^2}{acl} \right)^2 \frac{df}{du} = 0 \quad (3.55)$$



as the planetary equation of motion and

$$\frac{d^2u}{d\phi^2} - \frac{2u}{1+u^2} \frac{du}{d\phi} = 0 \quad (3.56)$$

as the photon equation of motion in the vicinity of a static massive homogenous prolate spheroidal body.

### Conclusion

The practicability of the findings in this work is an encouraging factor. More so, that in this age of computational precision, the applications of these results is another factor. This work exposes the philosophical and theoretical successes/failures of General Relativity theory to the advancement of studies in gravitation. The astrophysical applications of our extension abound as all applications of Schwarzschild's metric in studying gravitational phenomena in the Solar System can now be studied using our new approach.

With the formulation of our mathematically most simple and astrophysically most satisfactory solutions to Einstein's gravitational field equations the way is opened for the solution of the general relativistic equations of motion for all test particles in the gravitational fields of all static homogeneous distributions of mass within prolate spheroidal regions in the universe. And precisely because these equations contain the pure Newtonian as well as post-Newtonian gravitational scalar potentials all their predictions shall be most naturally comparable to the corresponding predictions from the pure Newtonian theory. This is most satisfactory indeed.

It is now obvious how our work may be emulated to

- 1) Derive a mathematically most simple structure for all the metric tensors in the space-times exterior or interior to any distribution of mass within any region having any of the geometries in nature,
- 2) Formulate all the nontrivial Einstein geometrical gravitational field equations and derive all their general solutions, and
- 3) Derive astrophysically most satisfactory unique solutions for application to the motions of all test particles and comparison with corresponding pure Newtonian results and applications. Therefore our goal in this research work has been completely achieved: to use the case of a spheroidal distribution of mass to show how the much vaunted Einstein's geometrical gravitational field equations may be solved exactly and analytically for any given distribution of mass within any region having any geometry.

On a final note, the theme studied in this research work is obviously very attractive as it is related to the expansion of our views to boundaries far away from our everyday experience, and opens beautiful horizons for possible laboratory, astrophysical and astronomical experiments. Naturally, Einstein's equations are of great importance to mankind, even if most people don't understand it clearly. By connecting the geometrical properties of space with the physical properties of matter, the equations regulate almost all of the space-time functions of our life. We are living in not just a mere three dimensional space, but in time that is manifested as the change of all physical structures (even the most stable physical structures change). The change of geometric formations changes the coordinate nets and hence, changes the geometrical structure of the space we observe. Einstein's equations rule this process. We are very optimistic that in the future, when people will fail to use oil as the source of energy, Einstein's equations will be the main engine for a theoretical physicist working on the sources of energy or related problems. People will turn their attention to more obvious and bizarre energetics than simply using oil or other fuels. As researchers in gravitational physics, we see many excellent sources of energy around us. These are the planets orbiting the Sun, rotating stars, stellar energy and many others. These sources are working from other principles than those known to modern theoretical physics. But these sources are not obvious as the self-rotating Sun (it should come to a halt after 2.5 revolutions due to internal viscosity) or the planets orbiting it (they also should experience a halt). The energy propelling these systems can be best understood from the space-time geometry and thus Einstein's theory of gravitation has a very promising future.

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# Geometric Thermodynamics of Kerr-AdS Black Hole with the Cosmological Constant as a State Variable

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**Abstract:** The thermodynamics of the Kerr-AdS black hole is reformulated within the context of the formalism of geometrothermodynamics (GTD) and the cosmological constant is considered as a thermodynamical parameter. We conclude that the mass of the black hole corresponds to the total enthalpy of this system. Choosing appropriately the metric in the manifold of equilibrium states, we study the phase transitions as a divergence of the thermodynamical curvature scalar. This approach reproduces the Hawking-Page transition and shows that considering the cosmological constant as a thermodynamical parameter does not contribute new phase transitions to the pre-existing picture.

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**§1. Introduction.** The thermodynamics of black holes has been studied extensively since the work of Hawking [1]. The notion of critical behavior for black holes has arisen in several contexts from the Hawking-Page [2] phase transition in anti-de-Sitter (AdS) background to the pioneering work by Davies [3] on the thermodynamics of Kerr-Newman black holes and the idea of the extremal limit of various black hole families regarded as genuine critical points [4–6]. Recently, some authors have considered the cosmological constant  $\Lambda$  as a dynamical variable [7, 8] and it has further been suggested that it is better to consider  $\Lambda$  as a thermodynamic variable, [9–13]. Physically,  $\Lambda$  is interpreted as a thermodynamic pressure in [14, 15], a fact that is consistent with the observation in [16–18] that its conjugate thermodynamic variable is proportional to a volume.

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The use of geometry in statistical mechanics was pioneered by Ruppeiner [19] and Weinhold [20], who suggested that the curvature of a metric defined on the space of parameters of a statistical mechanical theory could provide information about its phase structure. When this treatment is applied to the study of black hole thermodynamics, some puzzling anomalies appear. A possible solution was suggested by Quevedo's geometrothermodynamics (GTD) whose starting point [21] was the observation that standard thermodynamics is invariant with respect to Legendre transformations. The formalism of GTD indicates that phase transitions occur at those points where the thermodynamic curvature scalar is singular.

In this paper, we apply the GTD formalism to the Kerr-AdS black hole to investigate the behavior of the thermodynamical curvature. As is well known, a black hole with a positive cosmological constant has both a cosmological horizon and an event horizon. These two surfaces have, in general, different Hawking temperatures, which complicates any thermodynamical treatment. Therefore, we will focus on the case of a negative cosmological constant, though many of the conclusions are applicable to the positive  $\Lambda$  case. Furthermore, the negative  $\Lambda$  case is of interest for studies on AdS/CFT correspondence and the subsequent considerations of this work are likely to be relevant in those studies.

**§2. Geometrothermodynamics in brief.** The formulation of GTD is based on the use of contact geometry as a framework for thermodynamics. The  $(2n + 1)$ -dimensional thermodynamic phase space  $\mathcal{T}$  is coordinatized by the thermodynamic potential  $\Phi$ , the extensive variables  $E^a$ , and the intensive variables  $I^a$ , with  $a = 1, \dots, n$ . We define on  $\mathcal{T}$  a non-degenerate metric  $G = G(Z^A)$  with  $Z^A = \{\Phi, E^a, I^a\}$ , and the Gibbs 1-form  $\Theta = d\Phi - \delta_{ab} I^a dE^b$  with  $\delta_{ab} = \text{diag}(1, 1, \dots, 1)$ . If the condition  $\Theta \wedge (d\Theta)^n \neq 0$  is satisfied, the set  $(\mathcal{T}, \Theta, G)$  defines a contact Riemannian manifold. The Gibbs 1-form is invariant with respect to Legendre transformations, while the metric  $G$  is Legendre invariant if its functional dependence on  $Z^A$  does not change under a Legendre transformation. This invariance guarantees that the geometric properties of  $G$  do not depend on the thermodynamic potential used in its construction.

Now, we define the  $n$ -dimensional subspace of equilibrium thermodynamic states,  $\mathcal{E} \subset \mathcal{T}$ , by means of the smooth mapping

$$\left. \begin{aligned} \varphi : \mathcal{E} &\longrightarrow \mathcal{T} \\ (E^a) &\longmapsto (\Phi, E^a, I^a) \end{aligned} \right\}, \quad (1)$$

with  $\Phi = \Phi(E^a)$ , and the condition  $\varphi^*(\Theta) = 0$ , which gives the first law of thermodynamics

$$d\Phi = \delta_{ab} I^a dE^b, \quad (2)$$

and the conditions for thermodynamic equilibrium (the intensive thermodynamic variables are dual to the extensive ones),

$$\frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b. \quad (3)$$

The mapping  $\varphi$  defined above implies that we know the equation  $\Phi = \Phi(E^a)$  explicitly. It is known as the fundamental equation, and from it can be derived all the equations of state. The second law of thermodynamics is equivalent to the convexity condition on the thermodynamic potential,

$$\frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0. \quad (4)$$

Since the thermodynamic potential satisfies the homogeneity condition  $\Phi(\lambda E^a) = \lambda^\beta \Phi(E^a)$  for constant parameters  $\lambda$  and  $\beta$ , it satisfies Euler's identity,

$$\beta \Phi(E^a) = \delta_{ab} I^b E^a, \quad (5)$$

and using the first law of thermodynamics, this gives the Gibbs-Duhem relation,

$$(1 - \beta) \delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0. \quad (6)$$

Defining a non-degenerate metric structure  $g$  on  $\mathcal{E}$  that is compatible with a metric  $G$  on  $\mathcal{T}$ , we state that a thermodynamic system is described by the thermodynamical metric  $G$  [21] if it is invariant with respect to transformations which do not modify the contact structure of  $\mathcal{T}$ . In particular,  $G$  must be invariant with respect to Legendre transformations in order for GTD to be able to describe thermodynamic properties in terms of geometric concepts independently of the the thermodynamic potential used. A partial Legendre transformation is written as

$$Z^A \rightarrow \tilde{Z}^A = \left\{ \tilde{\Phi}, \tilde{E}^a, \tilde{I}^a \right\}, \quad (7)$$

where

$$\left. \begin{aligned} \Phi &= \tilde{\Phi} - \delta_{kl} \tilde{E}^k \tilde{I}^l \\ E^i &= -\tilde{I}^i \\ E^j &= \tilde{E}^j \\ I^i &= \tilde{E}^i \\ I^j &= \tilde{I}^j \end{aligned} \right\}, \quad (8)$$

with  $i \cup j$  any disjoint decomposition of the set of indices  $\{1, 2, \dots, n\}$  and  $k, l = 1, \dots, i$ . As is shown in [21], a Legendre invariant metric  $G$  induces a Legendre invariant metric  $g$  on  $\mathcal{E}$  defined by the pullback  $\varphi^*$  as  $g = \varphi^*(G)$ . There is a vast number of metrics on  $\mathcal{T}$  that satisfy the Legendre invariance condition. The results of Quevedo et al. [22–24] show that phase transitions occur at those points where the thermodynamic curvature is singular and that the metric structure of the phase manifold  $\mathcal{T}$  determines the type of systems that can be described by a specific thermodynamic metric. For instance, a pseudo-Euclidean structure

$$G = \Theta^2 + (\delta_{ab} E^a I^b) (\eta_{cd} dE^c dI^d) \quad (9)$$

with  $\eta_{cd} = \text{diag}(-1, 1, 1, \dots, 1)$  is Legendre invariant because of the invariance of the Gibbs 1-form and induces on  $\mathcal{E}$  the Quevedo's metric

$$g = \left( E^f \frac{\partial \Phi}{\partial E^f} \right) \left( \eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d \right), \quad (10)$$

which describes systems characterized with second-order phase transitions. On the other hand, an Euclidean structure

$$G = \Theta^2 + (\delta_{ab} E^a I^b) (\delta_{cd} dE^c dI^d) \quad (11)$$

is also a Legendre invariant and induces on  $\mathcal{E}$  the metric

$$g = \left( E^f \frac{\partial \Phi}{\partial E^f} \right) \left( \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^c dE^d \right), \quad (12)$$

which describes systems with first-order phase transitions.

**§3. The Kerr-AdS black hole.** The Einstein action with cosmological constant  $\Lambda$  is given by

$$\mathcal{A} = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda), \quad (13)$$

and the general solution representing a black hole is given by the Kerr-AdS solution

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\varphi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left( a dt - \frac{r^2 + a^2}{\Xi} d\varphi \right)^2 + \rho^2 \left( \frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right), \quad (14)$$



where

$$\Delta_r = (r^2 + a^2) \left( 1 - \frac{\Lambda r^2}{3} \right) - 2mr, \quad (15)$$

$$\Delta_\theta = 1 + \frac{\Lambda a^2}{3} \cos^2 \theta, \quad (16)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad (17)$$

and

$$\Xi = 1 + \frac{\Lambda a^2}{3}. \quad (18)$$

The physical parameters of the black hole can be obtained by means of Komar integrals using the Killing vectors  $\frac{\partial t}{\Xi}$  and  $\partial_\varphi$ . In this way, one obtains the mass of the black hole

$$M = \frac{m}{\Xi^2} \quad (19)$$

and its angular momentum

$$J = aM = a \frac{m}{\Xi^2}. \quad (20)$$

The horizons are given by the roots of

$$\Delta_r = 0. \quad (21)$$

In particular, the largest positive root located at  $r = r_+$  defines the event horizon with an area

$$A = 4\pi \frac{(r_+^2 + a^2)}{\Xi}. \quad (22)$$

The Smarr formula for the Kerr-AdS black hole gives the relation

$$M^2 = J^2 \left( \frac{\pi}{S} - \frac{\Lambda}{3} \right) + \frac{S^3}{4\pi^3} \left( \frac{\pi}{S} - \frac{\Lambda}{3} \right)^2 \quad (23)$$

that corresponds to the fundamental thermodynamical equation  $M = M(S, J, \Lambda)$  which relates the total mass  $M$  of the black hole with the extensive variables, entropy  $S = \frac{A}{4}$ , angular momentum  $J$  and cosmological constant  $\Lambda$ , and from which all the thermodynamical information can be derived.

In the geometric formulation of thermodynamics, we will choose the extensive variables as  $E^a = \{S, J, \Lambda\}$  and the corresponding intensive

variables as  $I^a = \{T, \Omega, \Psi\}$ , where  $T$  is the temperature,  $\Omega$  is the angular velocity and  $\Psi$  is the generalized variable conjugate to the state parameter  $\Lambda$ . Therefore, the coordinates that we will use in the 7-dimensional thermodynamical space  $\mathcal{T}$  are  $Z^A = \{M, S, J, \Lambda, T, \Omega, \Psi\}$ . The contact structure of  $\mathcal{T}$  is generated by the 1-form

$$\Theta = dM - TdS - \Omega dJ - \Psi d\Lambda. \quad (24)$$

To obtain the induced metric in the space of equilibrium states  $\mathcal{E}$  we will introduce the smooth mapping

$$\begin{aligned} \varphi : \{S, J, \Lambda\} &\longmapsto \\ &\longmapsto \{M(S, J, \Lambda), S, J, \Lambda, T(S, J, \Lambda), \Omega(S, J, \Lambda), \Psi(S, J, \Lambda)\} \end{aligned} \quad (25)$$

along with the condition  $\varphi^*(\Theta) = 0$ , that corresponds to the first law  $dM = TdS + \Omega dJ + \Psi d\Lambda$ . This condition also gives the relation between the different variables with the use of the fundamental relation (23). The Hawking temperature is evaluated as

$$T = \frac{\partial M}{\partial S} = \frac{S^2}{8\pi^3 M} \left( \frac{\pi}{S} - \frac{\Lambda}{3} \right) \left( \frac{\pi}{S} - \Lambda \right) - \frac{\pi J^2}{2MS^2}, \quad (26)$$

the angular velocity is

$$\Omega = \frac{\partial M}{\partial J} = \frac{J}{M} \left( \frac{\pi}{S} - \frac{\Lambda}{3} \right) \quad (27)$$

and the conjugate variable to  $\Lambda$  is

$$\Psi = \frac{\partial M}{\partial \Lambda} = -\frac{S^3}{12\pi^3 M} \left( \frac{\pi}{S} - \frac{\Lambda}{3} \right) - \frac{J^2}{6M}. \quad (28)$$

As can be seen,  $\Psi$  has dimensions of a volume. In fact, in the limit of a non-rotating black hole,  $J \rightarrow 0$ , we have  $\Psi = -\frac{4}{3}r_+^3$  (see [25]) and it can be interpreted as an effective volume excluded by the horizon, or alternatively a regularized version of the difference in the total volume of space with and without the black hole present [14–16]. Since the cosmological constant  $\Lambda$  behaves like a pressure and its conjugate variable as a volume, the term  $\Psi d\Lambda$  has the correct dimensions of energy and is the analogue of  $VdP$  in the first law. This suggests that after expanding the set of thermodynamic variables to include the cosmological constant, the mass  $M$  of the AdS black hole should be interpreted as the *enthalpy* rather than as the total energy of the spacetime.

The  $\mathcal{T}$  becomes a Riemannian manifold by defining the metric (9),

$$G = (dM - TdS - \Omega dJ - \Psi d\Lambda)^2 + (ST + \Omega J + \Psi \Lambda)(-dSdT + dJd\Omega + d\Lambda d\Psi). \quad (29)$$

The  $G$  has non-zero curvature and its determinant is  $\det \|G\| = -\frac{(ST + \Omega J + \Psi \Lambda)^6}{64}$ . Equation (10) lets us define the induced metric structure on  $\mathcal{E}$  as

$$g = (SM_S + JM_J + \Lambda M_\Lambda) \begin{pmatrix} -M_{SS} & 0 & 0 \\ 0 & M_{JJ} & M_{J\Lambda} \\ 0 & M_{J\Lambda} & M_{\Lambda\Lambda} \end{pmatrix}, \quad (30)$$

where subscripts represent partial differentiation with respect to the corresponding coordinate. Note that the determinant of this metric is

$$\det \|g\| = M_{SS} (M_{J\Lambda}^2 - M_{JJ}M_{\Lambda\Lambda}) (SM_S + JM_J + \Lambda M_\Lambda)^3. \quad (31)$$

We can also define an Euclidean metric (11) on  $\mathcal{T}$ , but there are no phase transitions associated with this metric.

**§4. Phase transitions and the curvature scalar.** Phase transitions are an interesting subject in the study of black hole thermodynamics since there is no unanimity in their definition. In ordinary thermodynamics, phase transitions are defined by looking for singular points in the behavior of thermodynamical variables. Davis [3,26] shows that the divergences in the heat capacity indicate phase transitions. For example, using equation (23) we have that the heat capacity for the Kerr-AdS black hole is

$$C = T \frac{\partial S}{\partial T} = \frac{M_S}{M_{SS}}, \quad (32)$$

$$C = \frac{S \left( \frac{\pi}{S} - \frac{\Lambda}{3} \right) \left( \frac{\pi}{S} - \Lambda \right) - \frac{4\pi^4 J^2}{S^3}}{\left( \frac{\pi}{S} - \frac{\Lambda}{3} \right) \left( \frac{\pi}{S} - 2\Lambda \right) - \frac{\pi}{S} \left( \frac{\pi}{S} - \Lambda \right) + \frac{8\pi^3}{S^2} \left( \frac{\pi J^2}{S^2} - ST^2 \right)}. \quad (33)$$

Thus, one can expect that phase transitions occur at the divergences of  $C$ , i.e. at points where  $M_{SS} = 0$ . For negative  $\Lambda$  the divergence of  $C$  corresponds to the generalization of the well-known Hawking-Page transition [2]. In GTD, the emergence of phase transitions appears to be related with the divergences of the curvature scalar  $R$  in the space of equilibrium states  $\mathcal{E}$ . To understand this relation, remember that  $R$

$$M_{SS} = \frac{144\pi^7 J^4 (9\pi - 4\Lambda S) + 24\pi^3 J^2 S^2 (3\pi - 2\Lambda S) (\Lambda S - 3\pi)^2 + S^4 (\Lambda S - 3\pi)^3 (\Lambda S + \pi)}{8\pi^{3/2} S^4 \left[ \frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}},$$

$$M_{JJ} = -\frac{2\pi^{3/2} (\Lambda S - 3\pi)^3}{\left[ \frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}},$$

$$M_{\Lambda\Lambda} = -\frac{6\pi^{9/2} J^4}{\left[ \frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}},$$

$$M_{J\Lambda} = \frac{12\pi^{9/2} J^3 (\Lambda S - 3\pi)}{S \left[ \frac{(\Lambda S - 3\pi)(S^2(\Lambda S - 3\pi) - 12\pi^3 J^2)}{S} \right]^{3/2}}.$$

always contains the determinant of the metric  $g$  in the denominator and, therefore, the zeros of  $\det \|g\|$  could lead to curvature singularities (if those zeros are not cancelled by the zeros of the numerator).

Here we have considered the metric  $g$  given in (30) and its determinant is proportional to  $M_{SS}$  as shown in equation (31), making clear the coincidence with the divergence of the heat capacity and the existence of a second-order phase transition that corresponds to the generalization of the Hawking-Page result. There is also a factor of  $(M_{J\Lambda}^2 - M_{JJ}M_{\Lambda\Lambda})$  in the determinant which codifies the information of non-constant  $\Lambda$ . Note that the interesting second derivatives of the thermodynamic potential are shown in Page 75.

As can be seen, for negative values of  $\Lambda$  the factor  $(M_{J\Lambda}^2 - M_{JJ}M_{\Lambda\Lambda})$  is always positive. Therefore, we conclude that considering  $\Lambda$  as a new thermodynamical state parameter does not produce new phase transitions in the Kerr-AdS black hole.

**§5. Conclusion.** Quevedo's geometrothermodynamics describes in an invariant manner the properties of thermodynamic systems using geometric concepts. It indicates that phase transitions would occur at those points where the thermodynamic curvature  $R$  is singular. Following Quevedo, the choice of the metric given in equation (10) apparently describes second-order phase transitions.

In this work, we have applied the GTD formalism to the Kerr-AdS black hole, considering the cosmological constant as a new thermodynamical state variable. In this approach, the total mass of the black hole is interpreted as the total enthalpy of the system. Thus, we have obtained a curvature scalar that diverges exactly at the point where the Hawking-Page phase transition occurs. Since we have employed a metric of the form given in (10) we conclude that this is a second-order phase transition. It is also important to note that the consideration of  $\Lambda$  as a thermodynamical variable does not include new phase transitions in the system.

It is clear that the phase manifold in the GTD formalism contains information about thermodynamic systems; however, it is not clear at present where the thermodynamic information is encoded. A more detailed investigation along these lines will be reported in the future.

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# Why a Background Persistent Field Must Exist in the Extended Theory of General Relativity

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**Abstract:** In the framework of an Extended General Relativity based on a semi-affine connection, we have postulated the existence of a background persistent field filling the physical vacuum and affecting the neighboring masses. In a holonomic scheme, the original Weyl formulation for generalized variational fields leads to the energy-momentum tensor of a perfect fluid in the Einstein field equation with a massive source. Since both the Weyl and EGR connections are shown to be equivalent in a particular way, the perfect fluid tensor with its pressure appears as a Riemannian transcription of the EGR massive tensor, with its surrounding active background field. This result would lend support to our assertion regarding the persistent field within the EGR formulation.

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**Introduction.** In one of our earlier publications [1], we have worked out an extended theory of general Relativity (EGR Theory) which allows for a permanent field to exist, thus filling the physical vacuum. This field appears as a continuity of the matter-pseudo-gravity field [2] which is required to fulfill the conservation law for the corresponding energy-momentum tensor in the classical GR theory. The existence of this persistent field has been first predicted in another paper based on the Lichnerowicz matching conditions applied to two spherically symmetric metrics [3]. In the foregoing, we provide a strict demonstration based on the properties of the so-called *generalized variational manifolds* [4, 5], which are a general class of *Finslerian spaces*. This theory relies on the symmetric *Weyl connection* which can be extended to the so-called *decomposable connection* or *semi-symmetric connection* [6, p. 69–75]. The Weyl manifold is then entirely defined from a) the Riemannian metric  $ds^2$  and b) a form  $dK = K_a dx^a$  which is generally non-integrable.

In what follows, we will however restrict our study to the symmetric part of the Weyl connection which readily relate to the EGR one in a very simple way.

With respect to a holonomic frame (in the sense of Cartan), the form  $dK$  becomes integrable, and we may establish a *pure* conformal metric  $(ds^2)'$  whose conformal factor is  $e^{2K}$ .

This conformal factor enables us to define 4-velocities collinear with the unit 4-vectors of the Einstein metric and allow us to write simple conformal geodesic equations for the flow lines of a specific type of fluid. This differential system is nothing else but the geodesic equation of a *perfect fluid*, where an equation of state links its proper density and the pressure prescribed on it:  $\rho = f(p)$ . So, by choosing a Weyl connection that spans the *generalized variational spaces* and making use of a holonomic frame, we are led to find the energy-momentum tensor of a neutral perfect fluid  $T_{ab}$ .

By relating the Weyl connection to the EGR connection in a very simple way, the EGR massive tensor  $(T_{ab})_{\text{EGR}}$  appears to be formally equivalent to the form of  $T_{ab}$ .



This substantiates the existence of the EGR persistent field tensor which in the Riemannian scheme is represented by a pressure term  $g_{ab} p$ , and where the dynamical mass density  $\rho$  increases to  $(\rho + p)$ , thus confirming our postulate that the EGR Theory describes trajectories of dynamical entities comprising *bare* masses of particles and their own gravity field [7].

## Chapter 1. The Extended General Relativity (EGR)

**§1.1. The Riemannian metric.** In an open neighborhood of the pseudo-Riemannian manifold  $V_4$ , the metric of signature  $(+ - - -)$  can be expressed by

$$ds^2 = g_{ab} \theta^a \theta^b, \quad (1.1)$$

where  $\theta_a$  are the Pfaffian forms in the considered region  $a = 4, 1, 2, 3$ .

The manifold considered here is always understood to be *globally hyperbolic* [8]. We also set here  $c = 1$ .

## §1.2. Brief overview on the Extended General Relativity

**§1.2.1. Basic properties.** We first briefly recall here our previous results. The non-metricity condition is ensured by the EGR covariant derivative  $D$  or  $'$ , of the metric tensor which has been found to be

$$D_a g_{bc} = \frac{1}{3} (J_c g_{ab} + J_b g_{ac} - J_a g_{bc}). \quad (1.2)$$

The vector  $J_a$  is related to a specific  $4 \times 4$  *Hermitean*  $(\gamma^5)_{\text{EGR}}$  matrix by

$$J^a = k \operatorname{tr} (\gamma^5)_{\text{EGR}}, \quad (1.3)$$

where  $k$  is a real positive constant [9].

One then considers the *semi-affine connection* (EGR connection)

$$(\Gamma_{ab}^d)_{\text{EGR}} = \{^d_{ab}\} + (\Gamma_{ab}^d)_J \quad (1.4)$$

with

$$(\Gamma_{ab}^d)_J = \frac{1}{6} (\delta_a^d J_b + \delta_b^d J_a - 3 g_{ab} J^d) \quad (1.5)$$

and the Christoffel symbols of the second kind  $\{^d_{ab}\}$ .

If  $\nabla_a$  is the Riemannian derivative operator, we have thus inferred the EGR curvature tensor:

$$(R^a_{bcd})_{\text{EGR}} = R^a_{bcd} + \nabla_d \Gamma_{bc}^a - \nabla_c \Gamma_{bd}^a + \Gamma_{bc}^f \Gamma_{fd}^a - \Gamma_{bd}^f \Gamma_{fc}^a. \quad (1.6)$$

The contracted tensor

$$\left. \begin{aligned} (R_{ab})_{\text{EGR}} &= R_{ab} - \frac{1}{2} \left( g_{ab} \nabla_e J^e + \frac{1}{3} J_a J_b \right) + \frac{1}{6} J_{ab} \\ J_{ab} &= \partial_a J_b - \partial_b J_a \end{aligned} \right\} \quad (1.7)$$

leads to the EGR *Einstein tensor*

$$(G_{ab})_{\text{EGR}} = (R_{ab})_{\text{EGR}} - \frac{1}{2} \left( g_{ab} R_{\text{EGR}} - \frac{2}{3} J_{ab} \right) \quad (1.8)$$

with the EGR *curvature scalar*

$$R_{\text{EGR}} = R - \frac{1}{3} \left( \nabla_e J^e + \frac{1}{2} J^2 \right). \quad (1.9)$$

In the *Riemannian regime*, this tensor obviously reduces to the usual Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R.$$

**§1.2.2. The EGR world-velocity.** On the EGR manifold  $\mathfrak{M}$ , the conoids, as defined in the Riemannian scheme, do not exactly coincide with the EGR representation, because the EGR line element slightly deviates from the standard Einstein geodesic invariant [10].

The EGR line element includes a small correction to the Riemann invariant  $ds^2$  which we write as

$$(ds^2)_{\text{EGR}} = ds^2 + d(ds^2), \quad (1.10)$$

where

$$d(ds^2) = (Dg_{ab}) dx^a dx^b \quad (1.11)$$

with  $Dg_{ab} = \frac{1}{3} (J_c g_{ab} + J_b g_{ac} - J_a g_{bc}) dx^c$ .

Hence, the EGR line element is simply expressed by

$$(ds^2)_{\text{EGR}} = (g_{ab} + Dg_{ab}) dx^a dx^b, \quad (1.12)$$

which naturally reduces to the Riemannian (invariant) interval  $ds^2$  when the covariant derivative of the metric tensor  $g_{ab}$  vanishes (i.e. when we have  $J_a = 0$ ).

We can thus define an EGR *4-velocity* as

$$(u^a)_{\text{EGR}} = \frac{dx^a}{(ds)_{\text{EGR}}}. \quad (1.13)$$

This vector will always be assumed to be a unit vector according to

$$g_{ab}(u^a u^b)_{\text{EGR}} = g^{ab}(u_a u_b)_{\text{EGR}} = 1. \quad (1.14)$$

**§1.3. The EGR persistent field.** The EGR field equation with a massive source is written

$$(G_{ab})_{\text{EGR}} = \varkappa [\rho_{\text{EGR}}(u_a u_b)_{\text{EGR}} + (t_{ab})_{\text{field}}], \quad (1.15)$$

where  $\varkappa$  is Einstein's constant.

The persistent field tensor is here assumed to represent a *vacuum homogeneous background energy* which is linked to its density by

$$\sqrt{-g}(t^{ab})_{\text{field}} = (\mathcal{F}^{ab})_{\text{field}}. \quad (1.16)$$

Explicitly,  $(\mathcal{F}^{ab})_{\text{field}}$  is derived from the canonical equations

$$(\mathcal{F}_b^a)_{\text{field}} = \frac{1}{2\varkappa} \left[ \mathcal{H} \delta_b^a - \partial_b (\Gamma_{df}^e)_{\text{EGR}} \frac{\partial \mathcal{H}}{\partial (\partial_a \Gamma_{df}^e)_{\text{EGR}}} \right], \quad (1.17)$$

where the invariant density is  $\mathcal{H} = (\mathcal{R}^{ab} R_{ab})_{\text{EGR}}$  built itself with the second-rank tensor density

$$(\mathcal{R}^{ab})_{\text{EGR}} = (R^{ab})_{\text{EGR}} \sqrt{-g}.$$

## Chapter 2. The Perfect Fluid Solution

**§2.1. The Weyl formulation.** The essential work of Lichnérowicz on *The Generalized Variational Spaces* begins by defining the *symmetric Weyl connection*:

$$W_{bc}^a = \{^a_{bc}\} + g^{ad} (g_{cd} F_b + g_{bd} F_c - g_{bc} F_d). \quad (2.1)$$

From the point M in the neighborhood of the space-time manifold  $V_4$ , a congruence of differentiable lines, such that for all  $m \in V_4$  there is a unique curve joining M onto  $m$ , and thus the *Weyl metric*

$$(ds^2)_W = e^{2F} ds^2 \quad (2.2)$$

can be defined, where

$$F = \int_M^m K_a dx^a. \quad (2.3)$$

The form  $dK = K_a dx^a$  is generally *non-integrable*.

**§2.2. The holonomic scheme.** In a *holonomic frame* [11, p. 45], we have  $K_a = \partial_a K$ , and the form  $dK$  is now integrable. In this case, we simply write (2.2) as the conformal metric

$$(ds^2)' = e^{2K} ds^2 \quad (2.4)$$

and the Weyl connection (2.1) reduces to the conformal connection

$$\{^a_{bc}\}' = \{^a_{bc}\} + g^{ad}(g_{cd}K_b + g_{bd}K_c - g_{bc}K_d). \quad (2.5)$$

### §2.3. The fiber bundle framework

**§2.3.1. Short overview on fiber bundles.** Given an  $n$ -dimensional manifold  $M$ , we can construct another manifold called a *fiber bundle* which is locally a direct product of  $M$  and a suitable space  $E$ , called the *total space*. For a thorough theory see for example [12, p. 50–55]. In this section, we shall only consider the *tangent bundles* category  $T_p(M)$ , ( $p \in M$ ) which is the fiber bundle over the manifold  $M$  obtained by giving the set  $\mathcal{E} = \cup T_p$ , its natural manifold structure and its natural projection onto  $M$ . A trivial example is the manifold  $M$  representing the circle  $S_1$  and the real line  $R_1$  with which can be constructed the cylinder  $C_2$ , as a product bundle over  $S_1$ .

In the following we shall consider:

- The *differentiable manifold*  $V_4$ ;
- The *bundle fiber space*  $W_{2 \times 4}$  of all vectors tangent to various points of  $V_4$ ;
- The *bundle fiber space*  $D_{8-1}$  of all directions tangent to various points of  $V_4$ .

**§2.3.2. Variational calculation.** Most of the derivations detailed in here can be found in [13, p. 72–75]. An element  $\in W_8$  is defined by the coordinates  $x^a$  of the point  $x \in V_4$ , and by the four quantities  ${}^\circ x^a$ , contravariant components of the vector in the natural basis associated at  $x$  to the  $(x^a)$ . For an element of  $D_{8-1}$ , the  ${}^\circ x^a$  will be only defined as *directional parameters* such that

$${}^\circ x^a = x^a(u).$$

The curve  $C$  is the projection on  $V_4$  of the curve  $U$  of  $D_{8-1}$ , locus of all directions tangent to  $C$  at its various points. This parametrized representation defining  $C$  is described in  $W_8$  by another curve  $L(u)$ , locus of the *derivate vectors* at  $u$  with respect to various points  $x$  of  $C$ .

In local coordinates for  $L(u)$ , we thus have:

$${}^{\circ}x^a = \frac{dx^a}{du}. \quad (2.6)$$

In these coordinates, we consider a *scalar-valued* function  $f(x^a, {}^{\circ}x^a)$  defined on  $W_8$  which is homogeneous and of first degree with respect to  ${}^{\circ}x^a$ . On  $V_4$  to the curve  $C$  joining the points  $x^0$  onto  $x^1$  there can always be associated the integral expressed in  $L(u)$  as:

$$\Phi = \int_{u^0}^{u^1} f(x^a, {}^{\circ}dx^a) = \int_{x^0}^{x^1} f(x^a, dx^a). \quad (2.7)$$

Always in local coordinates, let us now evaluate the variation of  $\Phi$  with respect to the variable points of  $C$ :

$$\delta\Phi = f_{u^1} \delta u^1 - f_{u^0} \delta u^0 - \int_{u^0}^{u^1} \delta f du. \quad (2.8)$$

Classically, inspection shows that

$$\int_{u^0}^{u^1} \delta f du = \left( \frac{\partial f}{\partial {}^{\circ}x^a} \delta x^a \right)_{u=u^0}^{u=u^1} - \int_{u^0}^{u^1} P_a \delta x^a du, \quad (2.9)$$

where the  $P_a$  are the first members of the Euler equations associated with the function  $f$ .

We infer the expression

$$\delta\Phi = [\omega(\delta)]_{x^1} - [\omega(\delta)]_{x^0} - \int_{u^0}^{u^1} P_a \delta x^a du, \quad (2.10)$$

where  $\omega(\delta)$  has the form

$$\omega(\delta) = \left( \frac{\partial f}{\partial {}^{\circ}x^a} \right) \delta x^a - \left( {}^{\circ}x^a \frac{\partial f}{\partial {}^{\circ}x^a} - f \right) \delta u \quad (2.11)$$

and due to the homogeneity of  $f$ , it reduces to

$$\omega(\delta) = \frac{\partial f}{\partial {}^{\circ}x^a} \delta x^a. \quad (2.12)$$

The  $P_a$  are the components of a covariant vector  $\mathbf{P}$  which appear as a scalar product in (2.9):

$$\delta\Phi = [\omega(\delta)]_{x^1} - [\omega(\delta)]_{x^0} - \int_{u^0}^{u^1} \langle \mathbf{P} \delta x \rangle du. \quad (2.13)$$

**§2.3.3. Specific variational derivation.** Let us apply the above results to the function

$$f = e^K \frac{ds}{du} = e^K \sqrt{g_{ab} \circ x^a \circ x^b}, \quad (2.14)$$

where  $K$  is always defined in  $V_4$ .

Between two points  $x^0$  and  $x^1$  of  $V_4$  connected by a time-like curve, we set the integral

$$s' = \int_{x^0}^{x^1} e^K ds = \int_{x^0}^{x^1} e^K \sqrt{g_{ab} \circ x^a \circ x^b}. \quad (2.15)$$

Then upon differentiation, we readily infer

$$\left. \begin{aligned} f \frac{\partial f}{\partial \circ x^a} &= e^{2K} g_{ab} \circ x^b \\ f \frac{\partial f}{\partial x^a} &= e^K \left( \partial_a e^K g_{bc} \circ x^a \circ x^c + \frac{1}{2} e^K \partial_a g_{bc} \circ x^a \circ x^c \right) \end{aligned} \right\}. \quad (2.16)$$

We now choose  $s$  as the parameter of the curve  $C$ , so the vector

$$\circ x^a = \frac{dx^a}{ds} = u^a \quad (2.17)$$

is here regarded as the unit vector tangent to  $C$ .

Equations (2.16) then reduce to the following expressions

$$\left. \begin{aligned} \frac{\partial f}{\partial \circ x^b} &= e^K u^b \\ \frac{\partial f}{\partial x^b} &= \frac{1}{2} e^K \partial_b g_{ad} u^a u^d + \partial_b e^K \\ e^K \{abd\} u^a u^d &+ \partial_b e^K \end{aligned} \right\}, \quad (2.18)$$

where  $\{abd\}$  denotes the Christoffel symbols of the first kind.

In this parametrized formulation, the components  $P_b$  of  $\mathbf{P}$  are written

$$P_b = \frac{d}{ds} \frac{\partial f}{\partial x^b} - \frac{\partial f}{\partial x^b} = \frac{d}{ds} (e^K u_b) - e^K \{abd\} u^a u^d - \partial_b e^K, \quad (2.19)$$

i.e.

$$P_b = e^K (u^a \partial_a u_b) - \{abd\} u^a u^d - \partial_a e^K (\delta_b^a - u^a u_b), \quad (2.20)$$

hence

$$P_b = e^K (u^a \nabla_a u_b) - (\partial_a K)(\delta_b^a - u^a u_b), \quad (2.21)$$

and (2.13) becomes

$$\delta s' = [\omega(\delta)]_{x^1} - [\omega(\delta)]_{x^0} - \int_{s_0}^{s_1} \langle \mathbf{P} \delta x \rangle ds, \quad (2.22)$$

where locally:

$$\omega(\delta) = e^K u_a dx^a. \quad (2.23)$$

When the curve  $C$  varies between two fixed points  $x^0$  and  $x^1$ , (2.22) obviously reduces to

$$\delta s' = - \int_{x^0}^{x^1} \langle \mathbf{P} \delta x \rangle ds. \quad (2.24)$$

In order to extremalize  $s'$ ,  $\mathbf{P}$  must be zero, and since  $e^K \neq 0$ , we have

$$(u^a \nabla_a u_b) - (\partial_a K)(\delta_b^a - u^a u_b) = 0. \quad (2.25)$$

### Chapter 3. The Geodesic Equations

**§3.1. The Riemannian situation.** In  $V_4$ , the unit vector satisfies

$$g_{ab} u^a u^b = g^{ab} u_a u_b = 1, \quad (3.1)$$

and differentiating, we thus obtain

$$u^b \nabla_a u_b = 0. \quad (3.2)$$

Let us now consider the covariant derivative

$$\nabla_a (r u^a u_b) = r (\partial_b K), \quad (3.3)$$

where  $r$  is a scalar.

If we take into account (3.1) and (3.2), the equation (3.3) is equivalent to

$$u^a \nabla_a u_b = (\partial^a K)(g_{ab} - u_a u_b) \quad (3.4)$$

after contracted multiplication by  $u^b$  and division by  $r$ .

However, (3.3) is just the conservation condition applied to a tensor  $T_{ab}$  provided we set

$$r (\partial_b K) = \nabla_a (r \delta_b^a). \quad (3.5)$$

Explicitly, the tensor  $T_{ab}$  can be written:

$$T_{ab} = r u_a u_b - p g_{ab}. \quad (3.6)$$

We note that this equation has the form of the well-known tensor describing a *perfect fluid* with proper density  $\rho$ :

$$T_{ab} = (\rho + p) u_a u_b - p g_{ab} \quad (3.7)$$

with an equation of state  $\rho = f(p)$ , and with  $r = (\rho + p)$ , where  $p$  is the scalar pressure of the fluid.

Hence,

$$K = \int_{p_0}^p \frac{dp}{\rho + p}, \quad (3.8)$$

and

$$u^a \nabla_a u_b = (\partial^a K)(g_{ab} - u_a u_b), \quad (3.9)$$

i.e.

$$u^a \nabla_a u_b = (\partial^a K) h_{ab}, \quad (3.10)$$

where

$$h_{ab} = g_{ab} - u_a u_b \quad (3.11)$$

is the well-known *projection tensor* in the adopted signature.

The 4-vector  $\partial^a K$  is regarded as the acceleration of the flow lines given by the pressure gradient orthogonal to those lines.

We can thus draw a first conclusion: equations (2.25) and (3.9) are formally identical. They represent the differential system which the flow lines must satisfy, or in other words, they represent the geodesics of the perfect fluid flow lines.

**Theorem:** *In a holonomic frame, a perfect fluid follows time-like lines extremalizing the integral*

$$s' = \int_{x^0}^{x^1} e^{2K} ds \quad (3.12)$$

for variations between two fixed points.

Therefore, these flow lines are time-like geodesics conformal to the metric  $ds^2$ :

$$(ds^2)' = e^{2K} ds^2 = e^{2K} g_{ab} dx^a dx^b \quad (3.13)$$

with the following metric tensor components:

$$(g_{ab})' = e^{2K} g_{ab}, \quad (g^{ab})' = e^{-2K} g^{ab}. \quad (3.14)$$



One can find a similar conclusion in [14] and [15].

### §3.2. The EGR geodesics for the EGR massive tensor.

**§3.2.1. A conformal like 4-velocity.** We now define the collinear vectors  $w_a$  with  $u_a$ :

$$w_a = e^K u_a, \quad w^a = e^K u^a, \quad (3.15)$$

hence

$$(w_a)' = e^K u_a, \quad (3.16)$$

$$(w^a)' = e^{-K} u^a. \quad (3.17)$$

These are regarded as 4-velocities and are *still* unit vectors in the conformal metric  $(ds^2)'$ .

As a result, an alternative way of expressing (3.10) can be easily shown to be

$$(w^a)'(\nabla_a)'(w_b)' = 0, \quad (3.18)$$

where  $(\nabla_a)'$  is the covariant derivative in  $(ds^2)'$  which is built from the conformal connection

$$\{^a_{bc}\}' = \{^a_{bc}\} + g^{ad}(g_{cd}K_b + g_{bd}K_c - g_{bc}K_d). \quad (3.19)$$

### §3.2.2. Relation between the Weyl and the EGR connections.

We recall that the EGR theory is built from two types of curvature forms in a dual basis:

The *rotation curvature 2-form* is:

$$\Omega_b^a = -\frac{1}{2} R^a{}_{bcd} \theta^c \wedge \theta^d. \quad (3.20)$$

The *segmental curvature 2-form* is:

$$\Omega = -\frac{1}{2} R^a{}_{acd} \theta^c \wedge \theta^d. \quad (3.21)$$

This last form results from the *variation* of the paralley transported vector around a closed path, a feat which necessarily induces  $D_a g_{bc} \neq 0$ .

Therefore, the global symmetric connection is easily inferred as

$$\Gamma_{bc}^a = \{^a_{bc}\} + g^{ad}(D_b g_{cd} + D_c g_{bd} - D_d g_{bc}) \quad (3.22)$$

and the Weyl connection

$$W_{bc}^a = \{^a_{bc}\} + g^{ad}(g_{cd}F_d + g_{bd}F_c - g_{bc}F_d) \quad (3.23)$$

is simply obtained by setting

$$D_a g_{bc} = g_{bc} F_a. \quad (3.24)$$

Hence, in this particular case, we can relate the EGR connection to the Weyl counterpart by

$$g_{bc} F_a = D_a g_{bc} = \frac{1}{3} (J_c g_{ab} + J_b g_{ac} - J_a g_{bc}). \quad (3.25)$$

**§3.2.3. The EGR geodesic equation.** In a strict Weyl formulation, the equation (3.4) can be derived up to

$$u^a \nabla_a u_b = F^a (g_{ab} - u_a u_b), \quad (3.26)$$

where the form  $dF = F^a dx_a$  is *not integrable*.

The  $F$  can always be chosen so that we have the correspondence

$$\left. \begin{aligned} e^F (u_a)_{\text{EGR}} &\longrightarrow (w_a)' = e^K u_a \\ e^{-F} (u^a)_{\text{EGR}} &\longrightarrow (w^a)' = e^{-K} u^a \end{aligned} \right\}. \quad (3.27)$$

Therefore, the EGR geodesic equation for the neutral matter is analogously expressed by

$$(u^a)_{\text{EGR}} D_a (u_b)_{\text{EGR}} = F^a [g_{ab} - (u_a u_b)_{\text{EGR}}]. \quad (3.28)$$

This equation is obeyed by the flow lines of the dynamical mass-gravity field whose energy-momentum tensor is given by

$$(T_{ab})_{\text{EGR}} = [\rho_{\text{EGR}} (u_a u_b)_{\text{EGR}} + (t_{ab})_{\text{field}}], \quad (3.29)$$

which is to be compared with the tensor

$$T_{ab} = (\rho + p) u_a u_b - p g_{ab}. \quad (3.30)$$

**Consequences and conclusions.** The main goal of our successive studies is to provide a physical justification of the theory that predicts an underlying medium which contains and unifies the particle and anti-particle states, see [16–20].

To sustain this argument, we have asserted that the EGR Theory must exhibit a background persistent field filling the standard physical vacuum.

In addition, when a neutral massive source is present, the persistent field tensor should supersede the matter *gravity pseudo-tensor* necessarily required by the conservation condition which is imposed by the conserved *Riemannian* Einstein tensor.

Therefore, the EGR theory allows for describing the geodesic motion of a dynamical entity that includes the *bare* mass slightly increased by its own surrounding gravity field.

In this paper, we have strictly shown that this is indeed the case, if one places himself in the frame of the generalized variational spaces which are easily related to the EGR manifold.

Using then a holonomic frame of reference, we eventually find that the inferred *Riemannian* perfect fluid tensor matches the model of the EGR energy-momentum tensor of neutral homogeneous matter. In the Riemannian scheme, the role of the persistent field tensor is taken up by the fluid pressure, and the increased bare mass density is here modified by the fluid pressure through an equation of state.

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# Traversable Space-Time Wormholes Sustained by the Negative Energy Electromagnetic Field

Patrick Marquet\*

**Abstract:** We consider the Thorne-Morris static space-time wormhole, sustained by the so-called exotic matter which may produce a huge space-time distortion to achieve hyper-fast interstellar travel. Modifying this metric, we suggest such a particular type of matter by means of the negative electromagnetic energy density. This possibility relies on Maxwell's equations, which are applied to time-varying electromagnetic fields, and synchronously time-varying electromagnetic 4-current densities. By choosing the proper phase displacements, the time component of the electromagnetic stress-energy tensor displays a negative energy density in part produced by the interacting electromagnetic potential superimposed onto the current density. The positive energy part of this tensor does not make a contribution, since it is confined at the outer vicinity of the wormhole.

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**Introduction.** During the last decades, many published papers were devoted to space-time shortcuts among whom the *space-time wormhole model* introduced by Thorne and Morris [1] can be selected. This concept is very similar to the *Einstein-Rosen bridge* [2, p.198], but instead, the wormhole connects two distinct Universes, referred to as the *lower* and *upper* worlds. To be physically sustained, it is well-known that this model requires a negative energy density which implies the existence of *exotic matter* (as coined by Kip Thorne) and which classically violates all energy conditions [3]. Apart from the *averaged null energy condition* (ANEC), and the *averaged weak energy condition* (AWEC), we will be also interested in a global energy condition [4] which is referred to as the *volume integral quantifier* and which is linked to the Visser-Kardadhich (VKD) wormholes [5]. In this approach, the total (exotic) energy is considered by performing a specific integration with respect to the matter proper volume element, and the amount of the (global) energy condition violations is measured when the integral becomes negative. This class of energy violations provides useful information and in particular, it determines the optimum choice of the *thickness* of the exotic matter layer threading the throat of the wormhole.

In Chapter 1, we first review the standard Lorentzian wormhole model, and we modify the metric in order to include a particular breakdown in the shape function. This breakdown accounts for two regions: the *inner* throat itself and an *outer* surrounding compact shell that is asymptotically fading away in order to merge with the quasi-Minkowskian Universe. Splitting the shape function into an *inner* region and an *outer* close shell does not affect the general wormhole physics. In Chapter 2, we investigate the possibility of using a variable electromagnetic field which interacts with a time-varying electric current, so that the resulting energy-momentum tensor splits up into a positive part and a negative part. This splitting is then required to correspond to the wormhole shape function breakdown, so that the positive electromagnetic free field contribution can be generated in the shell, i.e. *outside* the throat, while keeping the negative energy part *inside* to provide the necessary exotic matter, without violating the energy conditions.

## Chapter 1. The Static Lorentzian Wormhole

**§1.1. Definitions: the basic metric.** In the signature +2, let us consider a generic static space-time

$$ds^2 = g_{ab} dx^a dx^b = -e^{2\Phi(r)} dt^2 + g_{\mu\nu} dx^\mu dx^\nu,$$

where we set  $G = c = 1$ . Latin indices  $(a, b)$  run from 0 to 3; Greek indices  $(\mu, \nu, \dots)$  run from 1 to 3.

We then recall here the general static spherically symmetric *wormhole solution*

$$ds = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (1.1)$$

where  $\Phi(r)$  is related to the gravitational redshift and it is thus the so-called *redshift function*, while  $b(r)$  is denoted the *shape function* since it determines the shape of the wormhole.

An alternative way of expressing (1.1) is

$$ds = -e^{2\Phi(r)} dt^2 + dl^2 + r^2(l) (d\theta^2 + \sin^2\theta d\phi^2), \quad (1.2)$$

where we have set the *proper radial distance*

$$l(r) = \pm \int_{r_0}^r \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}}, \quad (1.3)$$

which is required to be finite everywhere.

Herein  $l(r)$  decreases from  $l = +\infty$  in the *upper world*, to  $l = 0$  at the throat, and then from 0 to  $-\infty$  in the *lower world*.

A first *traversability condition* required for the wormhole is to be horizon-free, i.e.

$$g_{tt} = -e^{2\Phi(r)} \neq 0,$$

so that  $\Phi(r)$  must also be finite everywhere in the throat.

This is the standard definition.

**§1.2. Definitions: the modified metric.** We now bring some slight improvement and we re-define the function  $b(r)$  as follows

$$b(r) = 1 - \tanh(b_{\text{worm}} + b_{\text{out}}), \quad (1.4)$$

$b_{\text{worm}}$  and  $b_{\text{out}}$  are here two disjoint *smooth* functions of  $r$  that respectively correspond to the wormhole throat, and the *outside* Universe. *Inside* the throat  $b_{\text{out}} = 0$ , and

$$b(r) = 1 - \tanh b_{\text{worm}} \quad (1.5)$$

remains the true characteristic of the wormhole solution.

Outside the wormhole,  $b_{\text{out}} \gg b_{\text{worm}}$ , so we have

$$\tanh(b_{\text{worm}} + b_{\text{out}}) \rightarrow 1, \quad b(r) \rightarrow 0. \quad (1.6)$$

Therefore, by definition  $\Phi(r) \rightarrow 0$ , and the metric (1.1) reduces to the usual spherically symmetric solution of the Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (1.7)$$

Introducing such an intrinsic breakdown for  $b(r)$  enables us to maintain the entire mathematical construction of the wormhole through the new metric:

$$ds^2 = -e^{2\Phi(r)} + \frac{dr^2}{1 - \frac{1 - \tanh(b_{\text{worm}} + b_{\text{out}})}{r}} + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (1.8)$$

Through this modification, we see that the standard wormhole is now surrounded by a *shell*, representing a transient region where the shape function starts to decrease asymptotically from  $b(r)$  to 0.

The trick is here obvious:

This shell does not belong to the inner throat, but is still part of the wormhole geometry together with its physical properties.

We will explain this particular feature for achieving our initial goal in the final course of our theory.

### §1.3. The common concept

**§1.3.1. The geometric description.** The space-time wormhole classically depicted in the Thorne-Morris model is formed with a *static layer* of a particular matter type threading the throat, which was coined by the authors as *exotic matter* (see formal definition below).

Our viewpoint is here different: we consider a dynamical object that actually *produces* the required exotic matter to create the wormhole in which it passes through. Hence it *creates* this space-time distortion, as long as needed for its travel duration.

In the first stage, due to the spherically symmetric nature of the concept and without loss of generality, we restrict the study to the equatorial plane  $\theta = \frac{\pi}{2}$ , and the interval at  $t = \text{const}$ , so the basic metric reads

$$ds^2 = \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\phi^2 \quad (1.9)$$

still bearing in mind  $b(r) = 1 - \tanh(b_{\text{worm}} + b_{\text{out}})$ .



The coordinate  $r$  decreases from  $+\infty$  to a minimum value  $r_0$  corresponding to the location (radius) of the wormhole throat, where  $b(r_0) = r_0$ , and then it increases from  $r_0$  to  $+\infty$ . The object we have in mind can be best conceived here, as a vertical cylinder centered about the axis  $z$  with a radius  $r_0$  that reaches the inner *layer* of the *exotic matter* it carries along.

The reduced metric (1.9) can be embedded into a 3-dimensional Euclidean space, it is written in cylindrical coordinates  $r, \phi, z$  as

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2. \quad (1.10)$$

The embedded surface has equation  $z = z(r)$ , and the metric of this surface is written

$$ds^2 = \left[ 1 + \left( \frac{dz}{dr} \right)^2 \right] dr^2 + r^2 d\phi^2, \quad (1.11)$$

from these latter two equations, we infer the *slope*

$$\frac{dz}{dr} = \pm \frac{1}{\sqrt{\frac{r}{b(r)} - 1}}. \quad (1.12)$$

In our picture, the inner layer of the *exotic matter* has the minimum radius  $r = b(r) = r_0$ , at which the embedded surface is vertical (the slope  $\frac{dz}{dr} \rightarrow \infty$ ). See Fig. 1.

Fig. 1: Cross-section of the created throat.

Far from the *exotic matter layer*,  $r \rightarrow \infty$ ,  $\frac{dz}{dr} \rightarrow 0$ , the space is asymptotically flat in accordance with (1.6).

According to (1.3), an object tunnelling through the wormhole has a velocity  $v(r)$  passing the throat at  $r$  which is measured by a set of

static observers located at this point:

$$v = \frac{dl}{e^{2\Phi(t)} dt} = \pm \frac{dr}{\sqrt{1 - \frac{b}{r}} e^{2\Phi(t)} dt}. \quad (1.13)$$

Once the travel through the created wormhole is completed, the throat should *flare out* at the end of this creation. Calling  $r(z)$ , the *embedding function*, its inverse must satisfy the important equation *at or near* the distance  $r_0$  to the exotic matter inner layer:

$$\frac{d^2 r}{dz^2} = \frac{b - b' r}{2b^2} > 0, \quad (1.14)$$

where the prime denotes the derivative with respect to the radial coordinate  $r$ , and within the distance  $r_0$ , inspection shows that the form function  $b$  should satisfy to  $b'(r_0) < 1$ .

The relation (1.14) is known as the *flaring out condition*.

**§1.3.2. Acceleration gained by an object while traversing the throat.** In the present analysis, we will adopt a set of orthonormal basis vectors which we regard as the proper reference frame of a collection of observers remaining at rest in the coordinate system ( $t$ , and fixed  $r, \theta, \phi$ ).

In our case, the orthonormal basis vectors  $\hat{e}_a$  are expressed by

$$\left. \begin{aligned} \hat{e}_t &= e^{-\Phi} e_t \\ \hat{e}_r &= \sqrt{1 - \frac{b}{r}} e_r \\ \hat{e}_\theta &= \frac{e^\theta}{r} \\ \hat{e}_\phi &= \frac{e^\phi}{r \sin \theta} \end{aligned} \right\}. \quad (1.15)$$

With this particular choice, the metric components reduce to the Minkowskian system

$$\hat{e}_a \hat{e}_b = \hat{g}_{ab} = \eta_{ab} = \{-1, 0, 0, 0\}. \quad (1.16)$$

In a general basis, the four-velocity for a static observer is

$$u^a = \frac{dx^a}{d\tau} = (u^t, 0, 0, 0) = [e^{-\Phi(r)}, 0, 0, 0]. \quad (1.17)$$

From the decomposition of the covariant derivative of  $u^a$ , we can extract its four-acceleration

$$a^a = u^a_{;b} u^b, \quad (1.18)$$

which reduces, by (1.11), to the following components (denoting the proper time by  $\tau$ )

$$\left. \begin{aligned} a^t &= 0 \\ a^r &= \{^r_{tt}\} \left(\frac{dt}{d\tau}\right)^2 = \Phi' \left(1 - \frac{b}{r}\right) \end{aligned} \right\}, \quad (1.19)$$

where  $a_r$  is the non-null radial component acceleration required for the observer to follow a geodesic inside the throat (free-fall condition). Herein  $\{^a_{bc}\}$  are the Christoffel symbols of the second kind.

We now revert to our orthonormal basis  $\hat{e}_a$ , and we express the object's proper reference frame in terms of the Special Relativity transformation factor  $\gamma = \frac{1}{\sqrt{1-v^2}}$ , as

$$\left. \begin{aligned} (\hat{e}_0)_{\text{SR}} &= \gamma \hat{e}_t \pm \gamma v \hat{e}_r \\ (\hat{e}_1)_{\text{SR}} &= \gamma \hat{e}_r \pm \gamma v \hat{e}_t \\ (\hat{e}_2)_{\text{SR}} &= e^\theta \\ (\hat{e}_3)_{\text{SR}} &= e^\phi \end{aligned} \right\}. \quad (1.20)$$

Referred to this basis, the object's four-acceleration in its proper reference frame is

$$(\hat{a}^a)_{\text{SR}} = (\hat{u}^a_{;b} \hat{u}^b)_{\text{SR}}.$$

In the  $(t, r, \theta, \phi)$  coordinate frame, the object moves radially and its acceleration is specialized to  $t$ , i.e.  $a_t = u_{t;r} u^r - \{^a_{tb}\} u^a u^b$ .

Setting  $\mathbf{a} = |\mathbf{a}| \hat{e}_\mu$ , we note that  $a_t = \mathbf{a} e_t = (a \hat{e}_\mu) e_t = -\gamma v e^\Phi |\mathbf{a}|$ , and we finally arrive at

$$|\mathbf{a}| = \left[ \sqrt{1 - \frac{b}{r}} e^{-\Phi} (\gamma e^\Phi)' \right]. \quad (1.21)$$

If we now imagine that the object contains some humanoid crew, the acceleration felt by the occupants should obviously not exceed an *earth-like* gravity.

Therefore the expanded acceleration  $(\hat{a}^a)_{\text{SR}}$  should satisfy the magnitude constraint ( $\oplus$  means the Earth)

$$\mathbf{a} \leq \mathbf{g}_\oplus. \quad (1.22)$$

**§1.3.3. The exotic matter.** Consider the Einstein tensor expressed with our orthonormal basis  $\hat{e}_a$ :

$$\hat{G}_{ab} = \hat{R}_{ab} - \frac{1}{2} \hat{g}_{ab} R.$$

In this situation, the components are greatly simplified as

$$\left. \begin{aligned} \hat{G}_{tt} &= \frac{b'}{r^2} \\ \hat{G}_{rr} &= -\frac{b}{r^3} + 2 \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} \\ \hat{G}_{\theta\theta} &= \left(1 - \frac{b}{r}\right) \left[ \Phi'' + (\Phi')^2 - \frac{(b'r-b)\Phi'}{2r(r-b)} - \frac{b'r-b}{2r^2(r-b)} + \frac{\Phi'}{r} \right] \\ \hat{G}_{\phi\phi} &= \hat{G}_{\theta\theta} \end{aligned} \right\}. \quad (1.23)$$

If we stick to Birkhoff's theorem [6, p.157], which states that the only vacuum solutions with (static) spherical symmetry is the *Schwarzschild solution*, we are led to introduce a stress-energy tensor  $T_{ab}$ . Then, the field equations with a source  $\hat{G}_{ab} = 8\pi \hat{T}_{ab}$  induce here the sole non-zero diagonal components of the energy-momentum tensor which classically receive the following (but somewhat arbitrary) physical meanings

$$\hat{T}_{tt} = \rho(r), \quad (1.24)$$

$$\hat{T}_{rr} = -T_{\text{tens}}(r), \quad (1.25)$$

$$\hat{T}_{\theta\theta} = \hat{T}_{\phi\phi} = p(r), \quad (1.26)$$

where  $\rho(r)$  is the mass density of the layer,  $T_{\text{tens}}(r)$  is the radial tension (or transverse pressure) which is opposed to the radial pressure  $p(r)$  ascribed to the mass density  $\rho$  of the special layer and which is necessary to sustain the throat. Based on the evident proportionality with the Einstein tensor  $\hat{G}_{ab}$ , the components of the energy-momentum tensor are

$$\rho(r) = \frac{1}{8\pi} \frac{b'}{r^2}, \quad (1.27)$$

$$T_{\text{tens}}(r) = \frac{1}{8\pi} \left[ \frac{b}{r^3} - 2 \left(1 - \frac{b}{r}\right) \frac{\Phi'}{r} \right], \quad (1.28)$$

$$p(r) = \frac{1}{8\pi} \left(1 - \frac{b}{r}\right) \left[ \Phi'' + (\Phi')^2 - \frac{(b'r-b)\Phi'}{2r(r-b)} - \frac{b'r-b}{2r^2(r-b)} + \frac{\Phi'}{r} \right]. \quad (1.29)$$

From the center of the object to the inner layer of exotic matter, those components reduce to

$$\rho(r_0) = \frac{1}{8\pi} \frac{b'(r_0)}{r_0^2}, \quad (1.30)$$

$$T_{\text{tens}}(r_0) = \frac{1}{8\pi r_0^2}, \quad (1.31)$$

$$p(r_0) = \frac{1}{8\pi} \frac{1 - b'(r_0)}{2r_0^2} [1 + r_0 \Phi'(r_0)], \quad (1.32)$$

where  $b(r_0) = 1 - \tanh(b_{\text{worm}})$ .

In the classical wormhole theory, it is customary to introduce the dimensionless function

$$\varsigma = \frac{T_{\text{tens}} - \rho}{|\rho|}, \quad (1.33)$$

which is also known as the *exoticity function*.

Using equations (1.27) and (1.28), we find

$$\varsigma = \frac{\frac{b}{r} - b' - 2r \left(1 - \frac{b}{r}\right) \Phi'}{|b'|}. \quad (1.34)$$

Taking into account the flaring out condition (1.14), the equation (1.33) takes the form

$$\varsigma = \frac{2b^2}{r|b|} \frac{d^2 r}{dz^2} - \frac{2r \left(1 - \frac{b}{r}\right) \Phi'}{|b'|}, \quad (1.35)$$

as  $\rho$  is finite and so is  $b'$ , while given the fact that  $\left(1 - \frac{b}{r}\right) \Phi' \rightarrow 0$  at the throat, we obtain the fundamental relation

$$\varsigma(r_0) = \frac{(T_0)_{\text{tens}} - \rho_0}{|\rho_0|} > 0. \quad (1.36)$$

The restriction

$$(T_0)_{\text{tens}} > \rho_0 \quad (1.37)$$

tells us that the radial tension that is required to sustain the throat, must exceed the layer's mass density.

This is a manifest violation of the *weak energy conditions* (WEC) which states that for any timelike vector  $u^a$ , we must have

$$T_{ab} u^a u^b \geq 0. \quad (1.38)$$

Indeed, consider a radially moving observer inside the throat: for a sufficiently fast velocity, in his basis (1.20), this observer measures an

energy density given by the time component of the stress-energy tensor

$$\hat{T}_{00} = \gamma^2 \hat{T}_{tt} + 2\gamma^2 v^2 \hat{T}_{tr} + \gamma^2 v^2 \hat{T}_{rr} = \gamma^2(\rho_0 - T_{\text{tens}}) + T_{\text{tens}}, \quad (1.39)$$

which is seen *negative*, thus violating the energy conditions (1.38).

If the observer is static, we have the strict condition

$$\rho_0 < 0. \quad (1.40)$$

#### §1.3.4. The totally exotic matter.

We now define the *averaged null energy condition* (ANEC) which is satisfied along a null curve as

$$\int T_{ab} k^a k^b d\lambda \geq 0, \quad (1.41)$$

where  $k^a$  is a null vector and  $\lambda$  is a generic affine parameter.

We then consider an extended type of energy condition involving the *volume integral quantifier*, which is expressed by the two inequalities

$$\int T_{ab} u^a u^b dV \geq 0, \quad \int T_{ab} k^a k^b dV \geq 0, \quad (1.42)$$

where the integral is performed with respect to the proper volume element  $dV$  of the exotic matter. With the null vector  $\hat{k}^a = (1, 1, 0, 0)$ , the expression  $\hat{T}_{ab} \hat{k}^a \hat{k}^b$  is given by

$$\rho - (T_0)_{\text{tens}} = \frac{1}{8\pi} \left(1 - \frac{b}{r}\right) \left[ \ln \left( \frac{e^{2\Phi}}{1 - \frac{b}{r}} \right) \right]'. \quad (1.43)$$

Performing an integration by parts

$$\begin{aligned} I_V &= \int [\rho - (T_0)_{\text{tens}}] dV = \\ &= -\frac{1}{8\pi} \left[ (r-b) \ln \left( \frac{e^{2\Phi}}{1 - \frac{b}{r}} \right) \right]_{r_0}^{\infty} - \int_{r_0}^{\infty} (1-b') \left[ \ln \left( \frac{e^{2\Phi}}{1 - \frac{b}{r}} \right) \right] dr. \end{aligned}$$

At the throat  $r_0 = b = [1 - \tanh(b_{\text{worm}})]$  and far from it, the space is asymptotically Euclidean i.e.  $\Phi = 0$ , so the first part of the right-hand side vanishes, and the (global) energy violation condition is represented by the volume integral

$$I_V = \int [\rho - (T_0)_{\text{tens}}] dV = -\frac{1}{8\pi} \int_{r_0}^{\infty} (1-b') \ln \left( \frac{e^{2\Phi}}{1 - \frac{b}{r}} \right) dr. \quad (1.44)$$

If we now want to evaluate the radial *thickness* of the negative energy layer denoted by

$$\Delta = r_E - r_0, \quad (1.45)$$

we just *slice out* a portion of the volume integral (1.44) as

$$I_V = \int [\rho - (T_0)_{\text{tens}}] dV = -\frac{1}{8\pi} \int_{r_0}^{r_E} (1 - b') \ln \left( \frac{e^{2\Phi}}{1 - \frac{b}{r}} \right) dr. \quad (1.46)$$

In the equatorial plane representation  $\theta = \frac{\pi}{2}$ , we will use this volume portion to insert an electromagnetic fluid circulation self-provided by an object (space ship) that creates the wormhole for a limited duration necessary to jump between the upper and the lower worlds.

REMARK: At first glance, one might be tempted to assume an arbitrary small quantities of ANEC-violating matter when  $r_E \rightarrow r_0$ , however further analysis would show that the smaller the amount of exotic matter, the longer the traversable time as measured by external clocks.

Indeed, setting the proper distance  $l = -l_1$  in the lower world, and  $l = +l_2$  in the upper world, (assuming  $\gamma \approx 1$ ), we let  $v = \frac{dl}{d\tau}$ , so that  $d\tau = \frac{dl}{v}$ , and

$$\Delta t = \int_{t_2}^{t_1} dt = \int_{-l_2}^{+l_1} e^{-\Phi(r)} \frac{dl}{v} = \int_{r_2}^{r_1} \frac{e^{-\Phi(r)}}{v} \left( 1 - \frac{b}{r} \right) dr. \quad (1.47)$$

## Chapter 2. Achieving the Production of Exotic-Like Matter

### §2.1. The electromagnetic field contribution

§2.1.1. **The physical stress-energy tensor.** Due to the radially symmetric model, Birkhoff's theorem still apply to our modified metric (1.6). However, this theorem does not forbid another type of energy-momentum tensor as a source of the Einstein equations:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}.$$

We will then postulate that these equations will possess another type of energy source. Let us first consider the general electrical four-current density

$$j^a = \mu u^a, \quad (2.1)$$

where  $\mu$  is a *time-varying charge density*, coupled to an electromagnetic field characterized by a four-potential  $A^a$ .

The resulting energy-momentum tensor for the interacting system is expressed in a general basis as

$$(T^{ab})_{\text{elec}} = \frac{1}{4\pi} \left( \frac{1}{4} g^{ab} F_{cd} F^{cd} + F^{ae} F_{e \cdot b} \right) + g^{ab} j_e A^e - j^a A^b, \quad (2.2)$$

from which we extract the energy density as

$$(T^{00})_{\text{elec}} = \frac{1}{4\pi} \left( \frac{1}{4} F_{cd} F^{cd} + F^{0e} F_{e \cdot 0} \right) + j_e A^e - j^0 A^0. \quad (2.3)$$

In our specially chosen orthonormal basis (1.15), the following relations hold

$$(\hat{T}^{00})_{\text{elec}} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \mathbf{j} \mathbf{A}, \quad (2.4)$$

where  $\mathbf{E}$  and  $\mathbf{B}$  are respectively the electric and magnetic field strengths derived from the Maxwell tensor  $F_{cd} = \partial_c F_d - \partial_d F_c$ .

We suppose that the field potential  $A_a(\varphi, \mathbf{A})$  is given in the Lorentz gauge, and we set for the three-potential  $\mathbf{A} = A_\beta$  and for the three-current  $\mathbf{j} = j_\beta$ .

Now, the key feature of our theory consists of implementing the following decomposition equivalence:

$$(\hat{T}_{00})_{\text{elec}} = [(\hat{T}^{00})_{\text{elec}}]_{\text{out}} + [(\hat{T}^{00})_{\text{elec}}]_{\text{worm}} = \frac{(\mathbf{E}^2 + \mathbf{B}^2)}{8\pi} + \mathbf{j} \mathbf{A}. \quad (2.5)$$

In this situation, the *positive* free radiative energy density  $\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}$  is *de facto* generated from an electromagnetic field which is located *outside* the wormhole external-layer thickness (see form. 1.45):

$$b_{\text{out}} > r_{\text{E}}. \quad (2.6)$$

This region is obviously the shell precisely defined by  $b_{\text{out}}$  in the modified metric (1.8).

The finite volume of exotic matter computed as per (1.46) in the equatorial plane representation  $\theta = \frac{\pi}{2}$  should here contain a *round-shaped circuit* (e.g. *superconductive medium*) wherein the time-varying current  $\mathbf{j}$  is circulating. In this case, the interacting term  $\mathbf{j} \mathbf{A}$  is exhibiting its energy density *inside* this volume where the three-current density  $\mathbf{j}$  must take the *angular* form

$$j_\phi = \mu r \left( \frac{d\phi}{dt} \right) \quad (2.7)$$

with the mean radius  $r = \frac{r_{\text{E}} + r_0}{2}$ .



**§2.1.2. Negative energy density of the interacting term.** As described in Maxwell's equations,  $\partial_a F^{ab} = 4\pi j^b$ , the time component of  $j^b$  is just  $\mu$  and the interacting term  $j\mathbf{A}$  can be decomposed as

$$\left. \begin{aligned} [(\hat{T}^{00})_{\text{elec}}]_{\text{out}} &= \frac{\mathbf{E} \nabla \varphi}{4\pi} + \mu \varphi \\ [(\hat{T}^{00})_{\text{elec}}]_{\text{worm}} &= \frac{(-\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t}) \nabla \varphi}{4\pi} + \mu \varphi \end{aligned} \right\}, \quad (2.8)$$

since  $E = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ .

In (2.8) the first term in the brackets is always negative. As to the last term, it is made negative when the time-varying scalar charge density  $\mu$  and the scalar potential  $\varphi$  are  $180^\circ$  out of phase (method reached by the use of *phasors*).

Eventually, in our coordinate basis we need only consider now the equivalence

$$-\left(\nabla \varphi + \frac{\partial \mathbf{A}}{\partial t}\right) + 4\pi \mu \varphi = \frac{b'(r_0)}{2r_0^2} \quad (2.9)$$

always with  $b(r_0) = 1 - \tanh(b_{\text{worm}})$  and with  $\mu \varphi < 0$ .

**§2.2. The exotic matter.** Reverting now to the energy density expression (1.39) expressed in the basis (1.20)

$$\hat{T}_{00} = \gamma^2 \hat{T}_{tt} + 2\gamma^2 v^2 \hat{T}_{tr} + \gamma^2 v^2 \hat{T}_{rr} = \gamma^2 (\rho_0 - T_{\text{tens}}) + T_{\text{tens}},$$

we remember that for a collection of *static observers* the energy density

$$\hat{T}_{00} = \rho_0 < 0$$

is seen negative to match the exoticity condition (1.36).

If we then set for the negative matter

$$[(\hat{T}^{00})_{\text{worm}}]_{\text{elec}} \equiv \rho_0 < 0, \quad (2.10)$$

we do have an adequate density mass equivalent.

The substitution (2.10) should not conflict with the other diagonal components of the stress tensor which now correspond to

$$[(\hat{T}_{rr})_{\text{worm}}]_{\text{elec}} = -(T_{\text{tens}})_{\text{worm}}(r_0), \quad (2.11)$$

$$[(\hat{T}_{\theta\theta})_{\text{worm}}]_{\text{elec}} = [(\hat{T}_{\phi\phi})_{\text{worm}}]_{\text{elec}} = p(r_0). \quad (2.12)$$

Note that the mass of the charge has been here discarded in the Einstein field equations since we here assume that the electromagnetic

effects greatly prevail over masses which can thus be neglected. For example, if we use the fundamental leptonic charge which is the electron  $e$ , its rest mass is  $m_e = 9.1091 \times 10^{-31}$  kg. In this case, for a given three-volume  $V$ , we consider  $e$  as a point-wise charge, and the charge density  $\mu$  is then given by

$$\mu = \sum_i e_i \delta(x - x^i),$$

where  $\delta(x - x^i)$  is the known Dirac function.

All the above reasoning naturally holds for the generalized metric (1.8) when we drop out the constant equatorial plane restriction  $\theta = \frac{\pi}{2}$ .

**Concluding remarks and outlook.** We have just here briefly sketched the basic principle of a theory using an electromagnetic field suitably interacting with a time-varying current in order to produce negative energy needed to sustain the space-time wormhole co-generation. Our approach heavily relies on the equivalences (2.10), (2.11) and (2.12) which certainly deserve further scrutiny.

Far reaching traversability and stability conditions are beyond the scope of this paper, as well as additional improved models tending to reduce exotic matter regions. Numerical estimates for the electromagnetic field magnitudes can also be predicted in a separate paper.

However, the story of the space-time wormhole theory does not end here. As soon as 1988, in their famous article [7], Morris, Thorne and Yurtsever have suggested that the traversable wormhole (if feasible) could be used as a time machine. Briefly speaking, they consider two nearby whormoles' mouths labeled 1 and 2. At  $t = \tau = 0$ , the mouths 1 and 2 are at rest. The mouth 2 is next given an acceleration to reach a near-light velocity, then it reverses its motion to return to its initial (spatial) location. From an exterior observer who measures both mouths, the (proper) time attached to the wormhole 2 is dilated with respect to wormhole 1's time, which has thus *aged* with respect to the second one, when it has come back to its position. As a result, at any later time, if one is tunnelling through the mouth 1, one emerges from the mouth 2 as though one has travelled backward in time. Of course, the reachable past can only *start* from the *date* of creation of the time machine. We cannot however exclude that possible advanced civilizations have already long engineered such a concept so that they are able to travel in a far past perhaps anterior to our human existence.

So much for the principle. There are however a wide range of further complex constraints which are to be overcome.

Starting from *chronal* domains separated from an *achronal* domain by a *future chronology horizon* (special type of the *Cauchy horizon*), Hawking [8] asserted that tunnelling high frequency electromagnetic wave packets would *pile up* against the separation line and finally drive the energy density on the boundary of the hypothetical time machine to infinity, thus destroying this machine at the instant it was created or at least preventing anyone outside of it from entering through it. This is known as the *chronology protection conjecture*: the so-called *closed time curves* (CTCs) at the *chronology horizon* are thus deemed a physical impossibility. This condition is supposed to be required in order to avoid any time paradox, a cliché which has since been strictly ruled out by several physicists (see for instance Klinkhammer, 1992 private communication).

Quite recently, several authors [9, 10] have thus challenged these statements, and the physicist Li-Xin Li [11] has even *rejected* the Hawking conjecture. Without going into sound technicalities, it suffices to know that the vacuum metric fluctuations (close to the Planck length scale) produce unwanted effects on the *defocusing* exerted by the wormhole on the amplitude of any classical high-frequency waves propagating along a null geodesic (light) following the inner *walls* of the wormhole axis, thus eventually causing it to collapse.

Analyzing the total cross sections for various particles' pair collisions or formations (see, for example, form. 94.6 in [12], and related formal derivation), Li has strictly demonstrated that by inserting an opaque absorption material with a definite transmission coefficient including any of these cross sections into the inner wall of the wormhole, the metric fluctuations tend to zero, leading to a stable Lorentzian wormhole.

Nevertheless, and although some real positive progresses are increasingly emerging, we clearly see that a deeper amount of research work remains to be carried out, before feasible solutions can eventually be found.

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# On the Komar Energy and the Generalized Smarr Formula for a Charged Black Hole Inspired by Noncommutative Geometry

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**Abstract:** We calculate the Komar energy  $E$  for a charged black hole inspired by noncommutative geometry and identify the total mass ( $M_0$ ) by considering the asymptotic limit. We also found the generalized Smarr formula, which shows a deformation from the well known relation  $M_0 - Q_0^2/r_+ = 2ST$  depending on the noncommutative scale length  $\ell$ .

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**§1. Introduction.** There is a deep connection between gravity and thermodynamics that has been known for a long time, from the works of Bekenstein and Hawking [1–3] to the recent research of Padmanabhan [4, 5]. In a thermodynamical system like Schwarzschild black hole, the entropy  $S$ , the Hawking temperature  $T$  and energy  $E$  are related by the first law of thermodynamics

$$dE = T dS, \quad (1)$$

where  $E$  is identified with the Komar energy [6, 7] and specifically for a Schwarzschild black hole it equals the total mass of the black hole,  $M$ . There is also an integral version of this equation

$$E = M = 2TS. \quad (2)$$

known as the Smarr formula [8] and it can be verified by using the expressions for temperature and entropy,

$$T = \frac{1}{8\pi M}, \quad (3)$$

$$S = \frac{A}{4} = 4\pi M^2. \quad (4)$$

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Eq. (2) has been obtained in different ways [5, 9] and the Komar energy is identified with the conserved charge associated with the Killing vector defined at the event horizon (see for example [10]). Recently, some generalized expressions for Smarr formula in different spacetimes have been studied [9–11] and in particular, the Kerr-Newman black hole with electric charge  $Q$  and angular momentum  $J$  satisfies the Smarr relation [12]

$$M = 2TS + \Phi_H Q + 2\Omega_H J, \quad (5)$$

where  $\Phi_H$  and  $\Omega_H$  are the electric potential and angular velocity at the horizon, respectively.

As a continuation of the research in black holes inspired by noncommutative geometry started in [13], in this paper we investigate the specific case of a 4-dimensional spherically symmetric charged black hole studied in [14–21]. This solution is obtained by introducing the noncommutativity effect through a coherent state formalism [22–24], which implies the replacement of the point distributions by smeared structures throughout a region of linear size  $\ell$ . We perform the analysis by obtaining the Komar energy by direct integration and found the generalized Smarr formula, which shows a deformation from the usual relation depending on the noncommutative parameter  $\ell$ .

## §2. Komar energy of the charged noncommutative black hole.

Many formulations of noncommutative field theory are based on the Weyl-Wigner-Moyal  $*$ -product [25–27] that lead to some important problems such as Lorentz invariance breaking, loss of unitarity or UV divergences of the quantum field theory. However, Smilagic and Spallucci [14–18, 20] explained recently a model of noncommutativity that can be free from the problems mentioned above. They assume that a point-like mass  $M$  and charge  $Q$ , instead of being quite localized at a point, must be described by a smeared structure throughout a region of linear size  $\ell$ . The metric for this distribution is given by [21]

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (6)$$

where

$$f(r) = 1 - \frac{2M(r)}{r} + \frac{Q^2(r)}{r^2}, \quad (7)$$

$$Q(r) = \frac{Q_0}{\sqrt{\pi}} \sqrt{\gamma^2 \left( \frac{1}{2}, \frac{r^2}{4\ell^2} \right) - \frac{r}{\sqrt{2}\ell} \gamma \left( \frac{1}{2}, \frac{r^2}{2\ell^2} \right) + \frac{\sqrt{2}r}{\ell} \gamma \left( \frac{3}{2}, \frac{r^2}{4\ell^2} \right)}, \quad (8)$$

$$M(r) = \frac{2M_0}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\ell^2}\right), \quad (9)$$

and

$$\gamma\left(\frac{a}{b}, x\right) = \int_0^x du u^{\frac{a}{b}-1} e^{-u} \quad (10)$$

is the lower incomplete gamma function. Considering a spatial 2-sphere  $V$  with boundary  $\partial V$ , the Komar integral for the energy is

$$E(V) = \frac{a}{16\pi} \oint_{\partial V} \nabla^\mu \xi^\nu d\Sigma_{\mu\nu}, \quad (11)$$

where the killing vector is  $\xi = \frac{\partial}{\partial t}$ ,  $d\Sigma_{\mu\nu}$  is the surface element at the boundary and the value of constant  $a$  will be found by comparison with the noncommutative Schwarzschild case. This is

$$E = \frac{2a}{16\pi} \oint_{\partial V} \nabla^\mu \xi^t d\Sigma_{\mu t}, \quad (12)$$

where the factor 2 appears because of the symmetry of the integrand. The covariant derivative involved is

$$\nabla_\mu \xi^t = \partial_\mu \xi^t + \Gamma_{\mu\sigma}^t \xi^\sigma = \Gamma_{\mu t}^t, \quad (13)$$

and for the noncommutative charged solution the nonvanishing connections are

$$\Gamma_{rt}^t = \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2} \frac{dQ^2}{dr} - Q^2}{r(r^2 - 2Mr + Q^2)}, \quad (14)$$

$$\Gamma_{tt}^t = \Gamma_{\theta t}^t = \Gamma_{\varphi t}^t = 0, \quad (15)$$

giving

$$E = \frac{a}{8\pi} \oint_{\partial V} \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2} \frac{dQ^2}{dr} - Q^2}{r^3} d\Sigma_{rt}. \quad (16)$$

The surface element corresponds to

$$d\Sigma_{rt} = -d\Sigma_{tr} = -r^2 \sin^2 \theta d\theta d\varphi \quad (17)$$

and therefore

$$E = -\frac{a}{8\pi} \frac{-\frac{dM}{dr} r^2 + rM + \frac{r}{2} \frac{dQ^2}{dr} - Q^2}{r} \oint_{\partial V} \sin^2 \theta d\theta d\varphi, \quad (18)$$

$$E = \frac{a}{2} \left[ \frac{dM}{dr} r - M - \frac{1}{2} \frac{dQ^2}{dr} + \frac{Q^2}{r} \right]. \quad (19)$$

By comparison with the Komar energy of the Schwarzschild black hole, we shall identify  $a = -2$ . Hence, the energy of the noncommutative charged black hole is finally given by

$$E = M - \frac{dM}{dr} r - \frac{Q^2}{r} + Q \frac{dQ}{dr}. \quad (20)$$

The horizons of the metric (6) can be found by setting  $f(r_{\pm}) = 0$ , i.e.

$$r_{\pm}^2 - 2r_{\pm}M(r_{\pm}) + Q^2(r_{\pm}) = 0, \quad (21)$$

which can be solved as

$$r_{\pm} = M(r_{\pm}) \pm \sqrt{M^2(r_{\pm}) - Q^2(r_{\pm})}. \quad (22)$$

The Hawking temperature is defined in terms of the surface gravity at the event horizon by

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \partial_r f(r)|_{r=r_+}, \quad (23)$$

which gives in this case

$$T = \frac{1}{2\pi r_+^2} \left[ M(r_+) - \frac{Q^2(r_+)}{r_+} - r_+ \frac{dM}{dr} \Big|_{r=r_+} + Q(r_+) \frac{dQ}{dr} \Big|_{r=r_+} \right]. \quad (24)$$

The entropy in terms of the area of the horizon is given by the well known relation

$$S = \frac{A}{4} = \pi r_+^2 \quad (25)$$

and therefore, the Komar energy (20) at the event horizon becomes

$$E = 2\pi r_+^2 T = 2ST. \quad (26)$$

Using the Reissner-Nordström values  $r_{\pm} = M_0 \pm \sqrt{M_0^2 - Q_0^2}$  as a first approximation of the horizons (22) and putting them into the incomplete gamma functions of Eqs. (8) and (9) one obtains

$$r_{\pm} = M_{\pm} \pm \sqrt{M_{\pm}^2 - Q_{\pm}^2} \quad (27)$$

where we have defined  $M_{\pm}$  and  $Q_{\pm}$  in Page 112.

For a large value of its argument (i.e. large masses), function  $\varepsilon$  tends to unity while the exponential term goes to zero, giving the classical Reissner-Nordström horizons  $r_{\pm} \rightarrow r_{RN\pm} = M_0 \pm \sqrt{M_0^2 - Q_0^2}$ .



$$M_{\pm} = M_0 \left[ \varepsilon \left( \frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{2\ell} \right) - \frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{\sqrt{\pi} \ell} \exp \left( -\frac{\left( M_0 \pm \sqrt{M_0^2 - Q_0^2} \right)^2}{4\ell^2} \right) \right],$$

$$Q_{\pm} = Q_0 \sqrt{\varepsilon^2 \left( \frac{M_0 \pm \sqrt{M_0^2 - Q_0^2}}{2\ell} \right) - \frac{\left( M_0 \pm \sqrt{M_0^2 - Q_0^2} \right)^2}{\sqrt{2\pi} \ell^2} \exp \left( -\frac{\left( M_0 \pm \sqrt{M_0^2 - Q_0^2} \right)^2}{4\ell^2} \right)},$$

and  $\varepsilon(x)$  is the Gauss error function,

$$\varepsilon(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

Using the same first approximation for the event horizon  $r_+$  in the Hawking temperature (23) one obtains [29]

$$T \approx \frac{1}{4\pi} \frac{r_+ - r_-}{r_+^2}. \quad (28)$$

This approximation permits us to write the Komar energy at the horizon, using Eqs. (26), (28) and (27), as

$$E = 2\pi r_+^2 T = \frac{r_+ - r_-}{2}, \quad (29)$$

$$E = \frac{1}{2} \left[ M_+ + M_- + \sqrt{M_+^2 - Q_+^2} - \sqrt{M_-^2 - Q_-^2} \right]. \quad (30)$$

By considering the behavior of the functions  $M_{\pm}$  and  $Q_{\pm}$ , it is easy to see that the limit of large masses of (30), as well as taking the limit  $\ell \rightarrow 0$ , recover the Reissner-Nordström energy while for  $Q_0 = 0$ , it gives the result of Banerjee and Gangopadhyay [28] for the noncommutative Schwarzschild black hole with the usual  $E = M_0$ . These results let us identify the quantity  $M_0$  as the total mass of the black hole and  $Q_0$  as its total electric charge.

With a similar procedure, the entropy can be approximated by

$$S = \pi r_+^2 \approx \pi \left( M_+ + \sqrt{M_+^2 - Q_+^2} \right)^2, \quad (31)$$

which give in the limit of large masses, or in the limit  $\ell \rightarrow 0$ , the usual result for the Reissner-Nordström black hole,

$$S \rightarrow S_{RN} = \pi \left( M_0 + \sqrt{M_0^2 - Q_0^2} \right)^2. \quad (32)$$

Using Eqs. (8) and (9) and the property of the gamma function

$$\frac{\partial}{\partial u} \gamma \left( \frac{a}{b}, u \right) = e^{-u} u^{-1 + \frac{a}{b}} \quad (33)$$

to perform the derivatives, the Komar energy (20) for this spacetime yields

$$\begin{aligned} E = & M(r) - \frac{Q^2(r)}{r} - \frac{M_0}{2\sqrt{\pi}} \frac{r^3}{\ell^3} e^{-\frac{r^2}{4\ell^2}} + \\ & + \frac{Q_0^2}{2\pi} \left[ \frac{2}{\ell} e^{-\frac{r^2}{4\ell^2}} \gamma \left( \frac{1}{2}, \frac{r^2}{4\ell^2} \right) - \frac{1}{\sqrt{2}\ell} \gamma \left( \frac{1}{2}, \frac{r^2}{2\ell^2} \right) \right] + \\ & + \frac{\sqrt{2}}{\ell} \gamma \left( \frac{3}{2}, \frac{r^2}{4\ell^2} \right) - \frac{r}{\ell^2} e^{-\frac{r^2}{2\ell^2}} + \frac{\sqrt{2}}{4} \frac{r^3}{\ell^4} e^{-\frac{r^2}{4\ell^2}} \Big]. \quad (34) \end{aligned}$$

Using the long distance approximations for the gamma functions

$$\gamma\left(\frac{3}{2}, \frac{r^2}{4\ell^2}\right) \simeq \frac{\sqrt{\pi}}{2} - \frac{r}{2\ell} e^{-r^2/4\ell^2}, \quad (35)$$

$$\gamma\left(\frac{1}{2}, \frac{r^2}{2\ell^2}\right) \simeq \sqrt{\pi} - \sqrt{2}\ell \frac{e^{-r^2/2\ell^2}}{r}, \quad (36)$$

$$\gamma\left(\frac{1}{2}, \frac{r^2}{4\ell^2}\right) \simeq \sqrt{\pi} - 2\ell \frac{e^{-r^2/4\ell^2}}{r}, \quad (37)$$

we obtain finally

$$\begin{aligned} M_0 - \frac{Q_0^2}{r_+} &= 2TS + \frac{M_0}{\sqrt{\pi}} \frac{r_+}{\ell} e^{-\frac{r_+^2}{4\ell^2}} \left(1 + \frac{r_+^2}{2\ell^2}\right) + \\ &+ \frac{Q_0^2}{\pi r_+} \left[ e^{-\frac{r_+^2}{2\ell^2}} \left(\frac{5}{2} + \frac{r_+^2}{2\ell^2} + \frac{4\ell^2}{r_+^2}\right) - \right. \\ &\left. - e^{-\frac{r_+^2}{4\ell^2}} \left(4\sqrt{\pi} \frac{\ell}{r_+} + \sqrt{\pi} \frac{r_+}{\ell} + \frac{\sqrt{2}}{4} \frac{r_+^2}{\ell^2} + \frac{\sqrt{2}}{8} \frac{r_+^4}{\ell^4}\right) \right]. \quad (38) \end{aligned}$$

Since  $M_0$  and  $Q_0$  have been identified as the mass and charge of the black hole, Eq. (38) corresponds to the generalization of the *Smarr formula for the noncommutative charged black hole*. Note that this relation deviates from the usual one (5) by the two last terms in the right hand side, but it is clear that in the limit  $\ell \rightarrow 0$  these terms disappear. In the case  $Q_0 = 0$  we recover the relation for the noncommutative Schwarzschild black hole presented in [28, 30, 31].

**§3. Conclusion.** We have computed the Komar energy for a charged black hole inspired in noncommutative geometry and its asymptotic limit that let us identify the constant  $M_0$  as its total mass and  $Q_0$  as its electric charge. With these results, we obtained the noncommutative version of the Smarr formula (38) which show a deformation from the usual relation and the new terms depend on the noncommutative parameter  $\ell$ .

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# Lichnérowicz's Theory of Spinors in General Relativity: the Zelmanov Approach

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**Abstract:** In this paper, we apply Abraham Zelmanov's theory of chronometric invariants to the spinor formalism, based on Lichnérowicz's initial spinor formalism extended to the General Theory of Relativity. From the classical theory, we make use of the Dirac current which is shown to be a real four-vector, and by an appropriate choice of the compatible gamma matrices, this current retains all the properties of a space-time vector. Its components are uniquely expressed in terms of spinor components, and we eventually obtain the desired physically observable spinors in the sense of Zelmanov, next to the scalar, vector, and tensor quantities.

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**Introduction.** Preliminary conventions:

- $\hbar = 1$  and  $c \neq 1$ ;
- Four-dimensional general basis  $e_\mu$  and numbering of the four  $4 \times 4$  Dirac gamma matrices with Greek indices:  $\mu, \nu = 0, 1, 2, 3$ ;
- Three-dimensional general basis  $e_\alpha$  with Greek indices:  $\alpha, \beta = 1, 2, 3$ ;
- Four-dimensional coordinates with Latin indices:  $a, b, c, \dots, f = 0, 1, 2, 3$ ;
- Three-dimensional coordinates with Latin indices:  $i, j, k, l, m, \dots = 1, 2, 3$ ;
- Spinor indices with capital Latin indices:  $A, B = 1, 2, 3, 4$ .

We first briefly recall the essence of Zelmanov's theory: the dynamic fundamental observer can be coupled with his physical referential system whose general space-time possesses a gravitational field, generally subject to rotation and deformation. Physical-mathematical quantities (scalars, vectors, tensors) as measured in the observer's accompanying frame of reference, are called *physically observable quantities* if and only if they result *uniquely* from the *chronometric projection* of the generally covariant four-dimensional quantities onto the time line and the spatial section in such a way that the new semi-three-dimensional quantities depend everywhere on the monad vector (world-velocity): those are known as *chronometrically invariant quantities*.

If the spatial sections are everywhere orthogonal to the time line, the enveloping space is said to be *holonomic*. In general, the real space-time (e.g. of the Metagalaxy) is *non-homogeneous* and *non-isotropic*, i.e. it is *non-holonomic*.

Besides vectors and tensors, we intend here to derive a law setting Zelmanov's physically observable properties for another type of quantities, namely *spinors*. To achieve this, we will first follow the Lichnérowicz analysis which formally defines spinors within General Relativity, and which leads to the well-known Dirac equation for the electron in a pseudo-Riemannian space-time. We will then infer the *Dirac current* from the *spinor Lagrangian*, which is shown to retain all properties of a real four-vector.

With a special choice of the gamma matrices compatible with the regular spinor theory [1], we define the Dirac current as a space-time vector whose components are exclusively expressed in terms of spinors.

These spinors, as they are physically observed, are thus modified through the *chronometric properties* of the general space, i.e. the *linear*

velocity of space rotation  $v_i$ , and the gravitational force  $F_i$ , as well as the gravitational potential  $w$ .

Three-dimensional *non-holonomy* and *deformation* of space, which are respectively represented by the antisymmetric and symmetric chronometrically invariant tensors  $A_{ik}$ , and  $D_{ik}$ , appear in the Christoffel symbols' components formulated uniquely in terms of physically observable quantities [2], which are to be part of the Riemann spinor connection.

Accordingly, we can construct a *Dirac-Zelmanov equation* for the electron interacting with a four-potential  $A^\mu$ , whose chronometrically invariant projections (physically observable components) only apply here to  $A_0$ .

It is essential to note that the inferred Dirac-Zelmanov equation does also comply with the positron equation, as easily shown below.

## §1. The Riemannian spinor field

**§1.1. General Riemannian space-time.** Let  $V_4$  be a  $C^\infty$ -differentiable Riemann four-manifold which admits a structural group, namely the *homogeneous* (or *full*) *Lorentz group* denoted by  $\mathbf{L}(4)$ .

The metric is locally written on an open neighbourhood of the manifold  $V_4$  as

$$ds^2 = g_{ab} dx^a dx^b, \quad (1)$$

which is equivalent to writing

$$ds^2 = \eta_{\mu\nu} \theta^\mu \theta^\nu, \quad (2)$$

where  $\eta_{\mu\nu}$  is the *Minkowski metric tensor*:  $(1, -1, -1, -1)$ .

The  $\theta^\mu$  are the four *Pfaffian forms* (see a formal definition of these in Appendix) which are related to the coordinate bases by

$$\theta^\mu = \mathbf{a}_a^\mu dx^a, \quad (3)$$

where the  $\mathbf{a}_a^\mu$  form the *tetrad part* that carries the curved space-time properties.

We systematically refer  $V_4$  to orthonormal bases, which are the elements of a fiber space  $E(V_4)$ , whose structural group is the full Lorentz group  $\mathbf{L}(4)$ .

Let now  $\mathbf{F}$  be a matrix  $\mathbf{F} \in \mathbf{L}(4)$ . The *global orientation* of the manifold  $V_4$  is a *pseudoscalar* denoted by  $\mathbf{o}$ , whose square is 1, and which is defined, with respect to the system of frames  $y$  of  $E(V_4)$ , as the component  $\mathbf{o}_y = \pm 1$ , such that if  $y = y' \mathbf{F}$ , we get  $\mathbf{o}_y = \mathbf{o}_{y'} \mathbf{o}_{\mathbf{F}}$ .



We next define the antisymmetric tensor  $\varepsilon_{b_1 \dots b_0}^{a_1 \dots a_0}$  as follows: its components are +1 if the upper indices' series is an even permutation of the series of lower indices all assumed distinct, and -1 for an odd permutation, and 0 otherwise. The covariant derivative of this tensor is zero. To simplify the notation, we may just write  $\varepsilon_{1 \dots 0}^{a_1 \dots a_0} = \varepsilon^{a_1 \dots a_0}$ ,  $\varepsilon_{a_1 \dots a_0}^{1 \dots 0} = \varepsilon_{a_1 \dots a_0}$ .

If  $V_4$  is oriented, we set the orientable volume element tensor  $\eta$  (whose covariant derivative is also zero), as

$$\eta_{abcd} = \sqrt{-g} \varepsilon_{abcd}, \quad \eta^{abcd} = \frac{1}{\sqrt{-g}} \varepsilon^{abcd}.$$

In a positively-oriented basis, the Levi-Civita tensor  $\varepsilon$  would have components defined by  $\varepsilon_{0123} = 1$ . This amounts to a choice of orientation of the four-dimensional manifold in where the orthonormal basis  $e_\mu$  represents a Lorentz frame with  $e_0$  pointing toward the future and  $e_\delta$  being a *right-handed triad*.

As a result, for two global orientations  $\mathbf{o}$  and  $\mathbf{o}'$  on  $V_4$ , we have at most two total orientations  $\mathbf{o}$  and  $-\mathbf{o}$ , which define the *orientable* volume element  $\eta$  as

$$\mathbf{o} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3.$$

Now, on  $V_4$ , we define the *temporal orientation* or *time orientation*  $\mathbf{t}$  with respect to the set  $y \in E(V_4)$ , by one component  $\mathbf{t}_y = \pm 1$ , such that if  $y = y' \mathbf{F}$ , we have

$$\mathbf{t}_y = \mathbf{t}_{y'} \mathbf{t}_{\mathbf{F}}$$

where  $V_4$  admits *at most* two time orientations  $\mathbf{t}$  and  $-\mathbf{t}$ .

Any vector  $e_0$  ( $e_0^2 = 1$ ) at  $x$  is oriented towards the *future* (respectively the *past*), when the components of  $\mathbf{t}$  with respect to the orthonormal frames  $(e_0, e_\delta)$  is 1 (respectively -1).

**§1.2. The gamma matrices.** In what follows,  $\Lambda^*$  is the complex conjugate of an arbitrary matrix  $\Lambda$ ,  $\Lambda^T$  is the transpose of  $\Lambda$ , while  $\tilde{\Lambda}$  is the classical adjoint of  $\Lambda$ .

Let  $\mathfrak{R}$  be the real numbers set on which the vector space is defined. This vector space is spanned by the 16 matrices

$$\mathbf{I}, \quad \gamma_\mu, \quad \gamma_\mu \gamma_\nu, \quad \gamma_\mu \gamma_\nu \gamma_\alpha, \quad \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta, \quad (4)$$

where  $\mathbf{I}$  is the *unit matrix*, and the four *gamma matrices* are denoted by

$$\gamma^\mu \equiv \gamma_B^{\mu A}. \quad (5)$$

Note that all indices are distinct here.

With the tensor  $\eta_{\mu\nu}$ , we write the fundamental relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\eta_{\mu\nu} \mathbf{I}, \quad (6)$$

which is verified by the gamma matrices, with the following elements

$$\left. \begin{aligned} \gamma^0 = \gamma_0 &= i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ -\gamma^1 = \gamma_1 &= i \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ -\gamma^2 = \gamma_2 &= i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ -\gamma^3 = \gamma_3 &= i \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} \end{aligned} \right\}. \quad (7)$$

This system of matrices is also called a *standard* representation, as opposed, for example, to the Majorana representation [3, p.108], or to the usual spinorial representation.

We note that

$$\tilde{\gamma}_\mu = -\eta^{\mu\mu} \gamma_\mu \quad (8)$$

and also

$$\det(\gamma_\mu) = 1, \quad \text{tr}(\gamma_\mu) = 0, \quad \text{tr}(\gamma_\mu \gamma_\nu) = 0, \quad \text{tr}(\gamma_\mu \gamma_\nu \gamma_\sigma) = 0$$

for  $\mu \neq \nu \neq \sigma$ , etc.

## §2. The spinor concept

**§2.1. The isomorphism  $\mathbf{p}$ .** We shortly recall the definition of the spin group  $\text{Spin}(4)$  which is said to be the *covering* of  $\mathbf{L}(4)$ . The projection  $\mathbf{p}$  of  $\text{Spin}(4)$  onto  $\mathbf{L}(4)$  is such that if

$$\mathbf{A} = (A_\nu^{\mu'}) = \mathbf{p}\Lambda, \quad \mathbf{A} \in \mathbf{L}(4), \quad \Lambda \in \text{Spin}(4)$$

we must have

$$\Lambda \gamma_\mu \Lambda^{-1} = A_\mu^{\nu'} \gamma_{\nu'},$$

where  $\mathbf{p}$  is regarded as the *isomorphism* of the spin group  $\mathbf{S}(4)$  on the Lorentz group  $\mathbf{L}(4)$ .

In view of expressing the spinors as physically observable quantities, one could start with the regular Riemannian line-element

$$ds^2 = g_{ab} dx^a dx^b,$$

which becomes in Zelmanov's theory

$$ds = c^2 d\tau^2 - d\sigma^2,$$

where

$$d\tau = \sqrt{g_{00}} dt + \frac{1}{c} \frac{g_{0i} dx^i}{\sqrt{g_{00}}},$$

$$d\sigma^2 = h_{ik} dx^i dx^k$$

and

$$h_{ik} = -g_{ik} + \frac{g_{0i} g_{0k}}{g_{00}}.$$

(See [4], formula 1.29, in accordance with formula 84.6 of §84 [5].)

Hence considering  $h_\mu^i h_k^\mu = \delta_k^i$ , one could then start by writing the physically observable Dirac matrices as simply verifying the relation

$$\gamma_i \gamma_k + \gamma_k \gamma_i = -2\delta_{ik} \mathbf{I}$$

but the matrix  $\gamma_0(x)$  is omitted and the isomorphism  $\mathbf{p}$  can no longer apply.

Therefore, writing the *Pfaffian* interval as  $ds^2 = \eta_{\mu\nu} \theta^\mu \theta^\nu$  allows us to preserve this isomorphism, which is the fundamental feature of any relativistic spinor theory. Like in the classical treatment, we thus maintain the relation (6), so that the gamma matrices are kept non-local. Proceeding with the Lichnérowicz formalism, we shall see that there is another way to obtain the spinors as unique observable quantities within Zelmanov's theory.

**§2.2. Spinor definitions.** From  $E(V_4)$ , we define a principal *fiber space* denoted by  $S(V_4)$  whose each point  $z$  represents a general *spinor frame*: at this stage, it is essential to understand that  $\text{Spin}(4)$  is here its structural group.  $V_4$  is therefore a vector space of  $4 \times 4$  matrices with complex elements, which is acted upon by the  $\text{Spin}(4)$  group [6].

Let us denote by  $\pi$  the *canonical projection* of  $S(V_4)$  onto  $V_4$ , and  $\mathbf{p}$  the projection of  $S(V_4)$  onto  $E(V_4)$ , so that a tensor of  $V_4$  is referred to its frame by  $y = \mathbf{p}z$ .

The *contravariant 1-spinor*  $\psi$  at  $x \in V_4$  is defined as a mapping  $z \rightarrow \psi(z)$  of  $\pi^{-1}(x)$  onto  $V_4$ .

The *covariant 1-spinor*  $\phi$  at  $x$  is a mapping  $z \rightarrow \phi(z)$  of  $\pi^{-1}(x)$  onto the space  $V_4$ , *dual* to  $V_4$ .

The contravariant 1-spinor  $\psi$  forms a vector space  $Sx$  on the complex numbers, whereas the covariant 1-spinor forms the vector space  $S'x$  *dual* to  $Sx$ . The contravariant 1-spinor  $\psi$  has also its covariant 1-spinor counterpart expressed by

$$\bar{\psi} = \mathbf{t}\tilde{\psi}\beta, \quad (9)$$

which is classically known as the *Dirac conjugate*, also expressed by

$$\bar{\psi} = \psi^*\gamma^0 \quad (10)$$

with  $\mathbf{t} = +1$ . Herein,  $\beta$  is a matrix defined below in (12).

Conversely, any covariant 1-spinor  $\phi$  is now the image of the contravariant 1-spinor  $\mathbf{t}\beta\phi$ .

**§2.3. The charge conjugation and the adjoint operation.** An antilinear mapping  $\mathbf{C}$  of  $Sx$  onto itself exists. It maps a 1-spinor  $\psi$  to another 1-spinor such as

$$\mathbf{C} : \psi \longrightarrow \psi\mathbf{C} = \psi^*.$$

We readily see that  $\mathbf{C}^2 = \text{Identity}(\psi \rightarrow \psi)$ , while  $\mathbf{C}$  is known as the *charge conjugate operation*.

In particular, the charge conjugate of the covariant 1-spinor  $\phi$  is

$$\mathbf{C}\phi = \psi^*$$

hence

$$(\mathbf{C}\phi, \psi) = (\phi, \mathbf{C}\psi)^*.$$

The relation (6) results from the identity  $(u^\mu\gamma_\mu)^2 = -(u^\mu u^\nu \eta_{\mu\nu})\mathbf{I}$ , where  $u^\mu \in \mathcal{C}$  (complex numbers), thus from (8) we find

$$\gamma_0\gamma_\mu\gamma_0^{-1} = -\tilde{\gamma}_\mu. \quad (11)$$

Introducing now the real matrix  $\beta = i\gamma_0$  which verifies  $\beta^2 = \mathbf{I}$ , the important relation can be derived from (10)

$$\beta\gamma_\mu\beta^{-1} = -\tilde{\gamma}_\mu. \quad (12)$$

By defining an *antilinear mapping*  $\mathbf{A}$  of  $Sx$  onto  $S'x$  as

$$\mathbf{A} : \psi \longrightarrow \bar{\psi} = \mathbf{t}\tilde{\psi}\beta \quad (13)$$

we have the *Dirac adjoint operation*  $\mathbf{A}$ .

We now consider a contravariant 1-spinor  $\psi$  which satisfies  $\mathbf{C}\psi = \psi^*$ . Thus

$$\mathbf{A}\mathbf{C}\psi = \mathbf{t}\psi^T\beta.$$

On the other hand,  $\mathbf{A}\psi = \mathbf{t}\tilde{\psi}\beta$ , from which we infer

$$\mathbf{C}\mathbf{A}\psi = \mathbf{t}\psi^T,$$

i.e.  $\mathbf{C}\mathbf{A}\psi = -\mathbf{t}\psi^T\beta$ . Therefore

$$\mathbf{A}\mathbf{C}\psi = -\mathbf{C}\mathbf{A}\psi.$$

The Dirac adjoint operator and the charge conjugation *anticommute* on the 1-spinors [7].

### §3. The Riemannian Dirac equation

**§3.1. The spinor connection.** In order to write the Schrödinger equation under a relativistic form, Dirac introduced a four-component wave function  $\psi_A$  (see [8] and [9, p. 252]) expressed with the gamma matrices. In the classical theory, the expression  $\gamma_B^{\mu A}\partial_\mu$  is known as the *Dirac operator*, and it is customary to omit the *spinor indices*  $A, B$  by simply writing  $\gamma_{\mu B}^A$  so as to get  $\gamma^\mu\partial_\mu$ .

In a Riemannian situation, the derivative  $\partial_\mu$  becomes  $D_\mu$  with a Riemannian spinor connection defined as follows:

$$\mathbf{N} = -\frac{1}{4}\Gamma^{\mu\nu}\gamma_\mu\gamma_\nu = -\frac{1}{4}\Gamma_\nu^\mu\gamma_\mu\gamma^\nu. \quad (14)$$

Within a neighbourhood  $\mathfrak{U}$  of  $E(V_4)$ , we define a connection 1-form  $\mathbf{\Gamma}$  that is represented by either of the two matrices  $\Gamma_\nu^\mu$  or  $\Gamma^{\mu\nu}$ , and whose elements are linear forms.

The matrix  $\mathbf{N}$  defines the *spinor connection* corresponding to  $\mathbf{\Gamma}$ . The elements of  $\mathbf{N}$  are given by the local 1-forms

$$N_B^A = -\frac{1}{4}\Gamma_\nu^\mu\gamma_\nu^A\gamma_B^{\nu C}.$$

By means of the Riemannian connection  $\Gamma_{\nu\alpha}^\mu$  with respect to the frames in  $\mathfrak{U}$ , the corresponding coefficients of  $\mathbf{N}$  are written

$$N_{B\alpha}^A = -\frac{1}{4}\Gamma_{\nu\alpha}^\mu\gamma_{\mu C}^A\gamma_B^{\nu C}. \quad (15)$$

**§3.2. The Riemannian Dirac operators and subsequent Dirac equation.** Some inspection shows that the absolute differential of the gamma matrices is given by

$$D\gamma^\mu = d\gamma^\mu + \Gamma_\nu^\mu \gamma^\nu + (\mathbf{N}\gamma^\mu - \gamma^\mu \mathbf{N}) \quad (16)$$

and this differential is shown to be zero. With (15) it can also be inferred that the covariant derivative of a spinor  $\psi^A$  is

$$D_\mu \psi^A = \partial_\mu \psi^A + N_{B\mu}^A \psi^B \quad (17)$$

and for the covariant 1-spinor  $\phi$

$$D_\nu \phi_A = \partial_\nu \phi_A - N_{A\nu}^B \phi_B.$$

Introducing now the *Riemannian Dirac operators*  $W$  and  $\bar{W}$  as

$$W\psi = \gamma^\mu D_\mu \psi, \quad \bar{W}\phi = -D_\mu \phi \gamma^\mu \quad (18)$$

for a massive spin- $\frac{1}{2}$ -field, the *Riemannian Dirac equations* [10] are written as

$$(W - m_0 c)\psi = 0, \quad (19)$$

$$(\bar{W} - m_0 c)\phi = 0, \quad (20)$$

where the rest mass  $m_0$  is usually attributed to the associated particle.

Classically, the Dirac massive equation is always written with the contravariant 1-spinor  $\psi$ , satisfying the *free field equation* (19) (no external interacting field).

In accordance with our previous results, the Dirac adjoint  $\bar{\psi}$  thus satisfies

$$(\bar{W} - m_0 c)\bar{\psi} = 0. \quad (21)$$

**§3.3. The Dirac current vector density.** In order to obtain the physically observable spinor quantities we are aiming at, we first define a space-time vector which is entirely expressed through the spinor formulation. For this purpose, we will rely on the Dirac current vector which is formally inferred from the Dirac Lagrangian of a massive fermion field. Such a Lagrangian is shown to be [11]

$$\mathcal{L}_D = \frac{1}{2} \left[ \bar{\psi} \gamma^\mu D_\mu \psi - (D_\mu \bar{\psi}) \gamma^\mu \psi \right] - m_0 c \bar{\psi} \psi. \quad (22)$$

An alternative formula is given by

$$\mathcal{L}_D = \bar{\psi} (\gamma^\mu - m_0 c) \psi.$$

Since these forms differ only by a divergence which vanishes at infinity, they generate the same action and correspond to the same physics.

Following Noether's theorem, we now apply the invariance rule to  $\mathcal{L}_D$  (22) upon the *global* transformations (where  $U$  is a positive scalar).

$$\psi \longrightarrow e^{iU} \psi, \quad \bar{\psi} \longrightarrow \bar{\psi} e^{-iU}$$

for linear transformations of  $\psi$ , the respective Lagrangian variation is

$$\delta \mathcal{L}_D = i \bar{\psi} \gamma^\mu \psi D_\mu \delta U = D_\mu (i \bar{\psi} \gamma^\mu \psi \delta U) - D_\mu (i \bar{\psi} \gamma^\mu \psi) \delta U$$

from which we expect to infer a current density  $(j^\mu)_D$  through a classical action variation

$$\delta \mathbf{S}_D = - \int D_\mu (j^\mu)_D \delta U \boldsymbol{\eta}, \quad (23)$$

where we have set

$$(j^\mu)_D = i \bar{\psi} \gamma^\mu \psi. \quad (24)$$

If  $\psi$  is a solution of the Dirac field equations (19),  $\delta \mathbf{S}_D$  vanishes for any  $\delta U$ , so

$$D_\mu (j^\mu)_D = 0. \quad (25)$$

Thus we have defined the *conserved Dirac current vector density*  $(j^\mu)_D$  which is a *real vector*. To prove this, we write  $(j^\mu)_D$  with the aid of (12)

$$(j^\mu)_D = i \mathbf{t} \tilde{\psi} \beta \gamma^\mu \psi.$$

Applying the usual adjoint operation  $(j^\mu)_D^* = -i \mathbf{t} \tilde{\psi} \tilde{\gamma}^\mu \beta \psi$  and taking into account  $\tilde{\gamma}^\mu \beta = -\beta \gamma^\mu$ , we eventually find

$$(j^\mu)_D^* = i \mathbf{t} \tilde{\psi} \gamma^\mu \psi = i \bar{\psi} \gamma^\mu \psi = (j^\mu)_D \quad (26)$$

which concludes the demonstration.

#### §4. The Dirac-Zelmanov equation

**§4.1. The unique physically observable spinor.** We shall now suggest a way to express the 1-contravariant spinor through the characteristics of Zelmanov's *formalism of General Relativity and Riemannian geometry* (i.e. the theory of chronometric invariants).

Consider the monad world-vector

$$b^\mu = \frac{dx^\mu}{ds}. \quad (27)$$

According to Zelmanov's theorem, for any vector  $Q^\mu$ , two quantities are physically observable:

$$b^\mu Q_\mu = \frac{Q_0}{\sqrt{g_{00}}} \quad (28)$$

and

$$h^i_\mu Q^\mu = Q^i. \quad (29)$$

At first glance, all one would have to do, is simply replace  $Q^\mu$  with  $j^\mu$ . However, in this case, we readily see that  $Q_0$  (or  $j_0$ ) must differ from  $Q^0$  (or  $j^0$ ) which seems to contradict the matrix definition for  $\gamma^0$  (7). Therefore, we will tackle the problem in a different manner.

With the aid of the standard gamma matrices (7), the components of  $j^\mu$  are easily derived:

$$\left. \begin{aligned} j^0 &= \mathbf{t} (\psi^1 \psi^{1*} + \psi^2 \psi^{2*} + \psi^3 \psi^{3*} + \psi^4 \psi^{4*}) \\ j^1 &= -\mathbf{t} (\psi^2 \psi^{1*} + \psi^1 \psi^{2*} - \psi^4 \psi^{3*} - \psi^3 \psi^{4*}) \\ j^2 &= -i\mathbf{t} (\psi^2 \psi^{1*} - \psi^1 \psi^{2*} - \psi^4 \psi^{3*} + \psi^3 \psi^{4*}) \\ j^3 &= -\mathbf{t} (\psi^1 \psi^{1*} - \psi^2 \psi^{2*} - \psi^3 \psi^{3*} + \psi^4 \psi^{4*}) \end{aligned} \right\}. \quad (30)$$

The vector  $j^\mu$  retains all the space-time properties as  $j^0$  is shown to lie within the (future) half-light cone for  $\mathbf{t} = +1$ . Analogously, the vector  $\mathbf{j}$  can be isotropic in which case  $j^\mu j_\mu$  must be zero.

Like in the Riemannian picture, we make explicit  $j^\mu j_\mu$  as follows:

$$j^\mu j_\mu = (j^0)^2 - (j^i)^2 = (j^0)^2 - j^i j_i. \quad (31)$$

Then, we remark that (31) is formally similar to Zelmanov's expression for an arbitrary vector  $A^\mu$

$$A^\mu A_\mu = a^2 - a^i a_i = a^2 - h_{ik} a^i a^k \quad (32)$$

setting  $j^\mu = A^\mu$ , we then have

$$a = \frac{j_0}{\sqrt{g_{00}}}, \quad (33)$$

$$a^i = j^i, \quad (34)$$

with

$$j^0 = \frac{a + \frac{v_i a^i}{c}}{1 - \frac{w}{c^2}}, \quad j_i = -a_i - \frac{v_i a}{c}, \quad (35)$$

where  $v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}$  is the linear velocity of space rotation, while  $w = c^2(1 - \sqrt{g_{00}})$  is the gravitational potential.

Thus, we see that the only combination of the observable spinor components are (with  $\mathbf{t} = +1$ , see above)

$$\mathbf{t} (\psi^1 \psi^{1*} + \psi^2 \psi^{2*} + \psi^3 \psi^{3*} + \psi^4 \psi^{4*}) = \frac{a + \frac{v_i a^i}{c}}{1 - \frac{w}{c^2}}. \quad (36)$$



We have now reached our goal by linking the first spinor combination of (30) with relations (34) and (36), i.e. we have found the unique physically observable spinors. This mathematical approach also enables us to note that  $j_0$  and  $j^0$  have distinct expressions within the framework of Zelmanov's theory.

**§4.2. The Zelmanov spinor connection.** Consider now the Riemannian connection coefficients  $\Gamma_{\nu\alpha}^\mu$  which are just the Christoffel symbols (i.e. Levi-Civita connection) with respect to any coordinate basis.

We define the *Zelmanov spinor connection* as

$$(N_{B\alpha}^A)_{\text{Zel}} = -\frac{1}{4} (\Gamma_{\nu\alpha}^\mu)_{\text{Zel}} \gamma_{\mu C}^A \gamma_B^{\nu C}. \quad (37)$$

The components of the  $(\Gamma_{\nu\alpha}^\mu)_{\text{Zel}}$  have been deduced by Zelmanov as the unique physically observable quantities

$$\begin{aligned} \Gamma_{00}^0 &= -\frac{1}{c^3} \left[ \frac{1}{1 - \frac{w}{c^2}} \frac{\partial w}{\partial t} + \left(1 - \frac{w}{c^2}\right) v_k F^k \right], \\ \Gamma_{00}^k &= -\frac{1}{c^2} \left(1 - \frac{w}{c^2}\right)^2 F^k, \\ \Gamma_{0i}^0 &= \frac{1}{c^2} \left[ -\frac{1}{1 - \frac{w}{c^2}} \frac{\partial w}{\partial x^i} + v_k \left( D_i^k + A_i^{\cdot k} + \frac{1}{c^2} v_i F^k \right) \right], \\ \Gamma_{0i}^k &= \frac{1}{c} \left(1 - \frac{w}{c^2}\right) \left( D_i^k + A_i^{\cdot k} + \frac{1}{c^2} v_i F^k \right), \\ \Gamma_{ij}^0 &= -\frac{1}{c \left(1 - \frac{w}{c^2}\right)} \left\{ -D_{ij} + \frac{1}{c^2} v_n \times \right. \\ &\quad \times \left[ v_j (D_i^n + A_i^{\cdot n}) + v_i (D_j^n + A_j^{\cdot n}) + \frac{1}{c^2} v_i v_j F^n \right] + \\ &\quad \left. + \frac{1}{2} \left( \frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} \right) - \frac{1}{2c^2} (F_i v_j + F_j v_i) - \Delta_{ij}^n v_n \right\}, \\ \Gamma_{ij}^k &= \Delta_{ij}^k - \frac{1}{c^2} \left[ v_i (D_j^k + A_j^{\cdot k}) + v_j (D_i^k + A_i^{\cdot k}) + \frac{1}{c^2} v_i v_j F^k \right]. \end{aligned}$$

where

$$\Delta_{jk}^i = \frac{1}{2} h^{im} \left( \frac{* \partial h_{jm}}{\partial x^k} + \frac{* \partial h_{km}}{\partial x^j} - \frac{* \partial h_{jk}}{\partial x^m} \right)$$

are the chronometrically invariant Christoffel symbols, for which the fundamental differential operator is

$$\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}} \frac{\partial}{\partial x^0}.$$

Associated with the global non-holonomic vector  $v_i$ , Zelmanov's angular momentum tensor  $A_{ik}$  — ultimately characterizing the space as non-holonomic and non-isotropic — is given by

$$A_{ik} = \frac{1}{2} \left( \frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} (F_i v_k - F_k v_i).$$

Here

$$D_{ik} = \frac{1}{2\sqrt{g_{00}}} \frac{\partial h_{ik}}{\partial t}$$

is the space deformation tensor and

$$F_i = \frac{1}{1 - \frac{w}{c^2}} \left( \frac{\partial w}{\partial x^i} - \frac{\partial v_i}{\partial t} \right)$$

is the gravitational inertial force vector.

**§4.3. The massive Dirac field interacting with an electromagnetic field.** Let us consider here the Lagrangian for a charged Dirac massive field coupled with a potential  $A_\mu$

$$\mathcal{L}(\psi, A_\mu) = \mathcal{L}(\psi) + \mathcal{L}(A_\mu) - e\psi A_\mu (j^\mu)_D$$

the coupling constant  $e$  is taken as a negative charge (i.e. the electron).

Taking into account the expression of the Dirac current density

$$(j^\mu)_D = i(\bar{\psi}\gamma^\mu\psi) \quad (38)$$

we shall evaluate the variation of the Lagrangian  $\mathcal{L}(\psi, A_\mu)$ .

After a simple but lengthy calculation, we obtain (omitting  $D$  in  $j$ )

$$\begin{aligned} \delta\mathcal{L}(\psi, A_\mu) &= [\delta\bar{\psi}(\gamma^\mu(D_\mu - ieA_\mu)\bar{\psi} - m_0c\psi)] + \\ &+ [(-D_\mu + ieA_\mu)\bar{\psi}\gamma^\mu - m_0c\psi]\delta\psi + \\ &+ [-D^\nu(D_\nu A_\mu - D_\mu A_\nu) - ej_\mu]\delta A^\mu + \text{divergence term.} \end{aligned} \quad (39)$$

Extremalizing the action defined from  $\mathcal{L}(\psi, A_\mu)$ , we get two (massive) field equations

$$\gamma^\mu(D_\mu - ieA_\mu)\psi = m_0c\psi, \quad (40)$$

$$-(D_\mu + ieA_\mu)\bar{\psi}\gamma^\mu = m_0c\bar{\psi}, \quad (41)$$

and

$$\delta d\mathbf{A} = e\mathbf{j}. \quad (42)$$

Under the conjugate operation, the following transformation

$$\psi \longrightarrow \psi_{(c)}$$

takes a place, and the first equation (40) becomes

$$\gamma^\mu (D_\mu + ieA_\mu)\psi_{(c)} = m_0 c \psi_{(c)} \quad (43)$$

and it is interpreted as the *positron equation* which accounts for the *anti-electron* or *positron* with a *positive* charge, when  $\psi_{(c)}$  is substituted into  $\psi$ . Thus, in equation (43), the rest mass  $m_0$  represents the positron.

Within the Zelmanov picture, the Dirac equation (40) will be uniquely written as

$$\gamma^\mu [(D_\mu)_{\text{Zel}} - ie(A_\mu)_{\text{Zel}}]\psi' = m_0 c \psi', \quad (44)$$

where  $\psi'$  is the modified spinor according to (36) and  $(D_\mu)_{\text{Zel}}$  is the spinor derivative constructed from the components of (37).

As regards the four-potential  $(A_\mu)_{\text{Zel}}$ , we have the following components

$$A_0 = b^\mu A_\mu \sqrt{g_{00}}. \quad (45)$$

This scalar potential is a chronometrically invariant quantity, with the associated vector components:

$$A_i = -h_{ik}A^k - \frac{A_0 v_i}{c\sqrt{g_{00}}}. \quad (46)$$

**Concluding remarks.** There is no ambiguity neither any loss of generality regarding the special choice of gamma matrices (7), as long as they verify the fundamental relation (6). Therefore, assuming that the inferred Dirac current is a space-time vector is here legitimate all the more as this vector can be isotropic.

Based on this weak constraint, it has thus been possible to express the contravariant 1-spinor  $\psi$  (or its combination) in terms of the gravitational potential  $w$  and the linear velocity of space rotation  $v_i$ .

The Dirac-Zelmanov equation for a massive field completes the scope of the theory by implicitly displaying the *non-holonomy* and *non-isotropy tensor*  $A_{ik}$  and the *tensor of deformation of space*  $D_{ik}$  through the Zelmanov spinor connection.

In equation (44), the electron and its rest mass  $m_0$  are constant and are independent of the observer, so even when the contravariant 1-spinor

$\psi$  is modified when viewed in the observer's physical frame, the positron equation (43) should also be true in Zelmanov's theory.

But above all, the most salient feature of the present theory certainly lies in that the Zelmanov theory entirely *confirms* the way Lichnérowicz approached the spinor analysis in General Relativity.

It is indeed remarkable to note that the observable spinor formulation requires the fundamental relation (6) to be maintained.

Any other current attempts to write down (6) as

$$\gamma_a(x)\gamma_b(x) + \gamma_b(x)\gamma_a(x) = -2g_{ab}\mathbf{I},$$

where  $\gamma_a(x) = a_a^\mu(x)\gamma_\mu$  are the mere local generalization of the gamma matrices (see for instance [11, p. 25] or [12, p. 515]), can be definitely ruled out.

It also means that the adopted metric form  $ds^2 = \eta_{\mu\nu}\theta^\mu\theta^\nu$  clearly appears to be the right choice for describing the spinor in General Relativity.

This is certainly the *essential* result of our short study, as it dramatically shows that the chronometric (physically observable) properties of Zelmanov are equivalent to a pure mathematical analysis (Lichnérowicz) in perfect harmony with quantum theory (Dirac) and resulting experimental data (i.e. existence of the positron).

**Appendix.** In mathematics, a vector cannot in general be considered as an arrow connecting two points on the manifold M [14]. A tangent vector  $\mathbf{V}$  along a curve  $\gamma(t)$  at  $p$ , is considered as an operator (linear functional) which assigns to each differential function  $f$  on M, a real number  $\mathbf{V}(f)$ .

This operator satisfies the axioms:

Axiom I:  $\mathbf{V}(f + h) = \mathbf{V}(f) + \mathbf{V}(h),$

Axiom II:  $\mathbf{V}(fh) = h\mathbf{V}(f) + f\mathbf{V}(h),$

Axiom III:  $\mathbf{V}(cf) = c\mathbf{V}(f),$  where  $c = \text{constant}.$

One shows that such a tangent vector can be expressed by

$$\mathbf{V} = V^a \frac{\partial}{\partial x^a},$$

where the real coefficients  $V^a$  are the components of  $\mathbf{V}$  at  $p$ , with respect to the *local* coordinates  $(x^1, \dots, x^4)$  in the neighbourhood of  $p$ . Accordingly, the directional derivatives along the coordinates lines at  $p$  form a basis of the four-dimensional vector space whose elements are the tangent vectors at  $p$ , i.e. the tangent space  $\mathbf{T}_p$ .

The basis  $(\frac{\partial}{\partial x^a})$  is called a *coordinate basis*. On the contrary, a general basis  $e_\nu$  is formed by four linearly independent vectors  $e_\nu$ ; any vector  $\mathbf{V} \in T_p$  is a *linear combination* of these basis vectors:

$$\mathbf{V} = V^\alpha e_\alpha.$$

By definition, a *1-form (Pfaffian form)*  $\zeta$  maps a vector  $\mathbf{V}$  into a real number, with the *contraction* denoted by the symbol  $\langle \zeta, \mathbf{V} \rangle$ , and this mapping is *linear*.

The four linearly independent 1-forms  $\theta^\mu$  which are uniquely determined by

$$\langle \theta^\mu, e_\nu \rangle = \delta_\nu^\mu$$

form a general basis of the dual space  $T_p^*$  to the tangent space  $T_p$ . This basis is said to be dual to the basis  $e$  of  $T_p$ . The bases  $e_\nu, \theta^\mu$  are linear combinations of coordinates bases:

$$e_\nu = b_\nu^a \left( \frac{\partial}{\partial x^a} \right), \quad \theta^\mu = a_a^\mu dx^a.$$

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# Earth Flyby Anomalies Explained by a Time-Retarded Causal Version of Newtonian Gravitational Theory

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**Abstract:** Classical Newtonian gravitational theory does not satisfy the causality principle because it is based on instantaneous action-at-a-distance. A causal version of Newtonian theory for a large rotating sphere is derived herein by time-retarding the distance between interior circulating point-mass sources and an exterior field-point. The resulting causal theory explains exactly the Earth flyby anomalies reported by NASA in 2008. It also explains exactly an anomalous decrease in the Moon's orbital speed. No other known theory has been shown to explain both the flyby anomalies and the lunar orbit anomaly.

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**Introduction.** It has long been known that electromagnetic fields propagate at or near the vacuum speed of light. The actual speed of light depends on whether the field is propagating in a vacuum or in a material medium. In either case, to calculate the electromagnetic fields of a moving point-charge, the concept of “time retardation” must be used [1]. The causality principle indicates that the “effect” of a “causal” physical field requires a certain amount of time to propagate from a point-source to a distant field-point. Classical Newtonian theory is “acausal” because the Newtonian gravitational field is based on instantaneous action-at-a-distance [2].

Gravitational fields are believed to propagate in empty space with exactly the same speed as the vacuum speed of light [3]. In 1898 the speed of the Sun’s gravitational field was found by a high school math teacher, P. Gerber, by calculating what it would need to be to cause the (in 1898) “anomalous” advance of the perihelion of Mercury [4]. Gerber’s value,  $3.05500 \times 10^8$  m/s, is about 2% greater than the vacuum speed of light. In 2002 a group of radio astronomers measured the speed of Jupiter’s gravitational field by detecting the rate of change in the gravitational bending of radio waves from a distant quasar as the giant planet crossed the line-of-sight [5]. They concluded that the speed of Jupiter’s gravitational field is  $1.06 \pm 0.21$  times the vacuum speed of light. These results suggest that the speed of propagation of the gravitational field near a massive central object may not be exactly the same as the vacuum speed of light.

The first terrestrial measurement that proved a connection between gravity and light, the gravitational red-shift, was carried out by R. V. Pound and G. A. Rebka in 1959 [6]. In 1972 J. C. Hafele and R. E. Keating reported the results of their experiments which detected the relativistic time dilation and the gravitational red-shift for precision clocks flown around the world using commercial jet flights [7]. This experiment showed conclusively that clocks at a deeper gravitational potential run slower and that moving clocks run slower. It also showed that the Sagnac effect [8], which originally was for electromagnetic fields, also applies to gravitational fields. To correct for these relativistic effects, the precision clocks used in the GPS system are adjusted before they are launched into space [9].

In 2008 Anderson *et al.* [10] reported that anomalous orbital-energy changes have been observed during six spacecraft flybys of the Earth. The reported speed-changes range from +13.28 mm/s for the NEAR flyby to -4.6 mm/s for the Galileo-II flyby. Anderson *et al.* state in their abstract:



“These anomalous energy changes are consistent with an empirical prediction formula which is proportional to the total orbital energy per unit mass and which involves the incoming and outgoing geocentric latitudes of the asymptotic spacecraft velocity vectors.”

Let the calculated speed-change be designated by  $\delta v_{\text{emp}}$ . The empirical prediction formula found by Anderson *et al.* can be expressed as follows

$$\begin{aligned}\delta v_{\text{emp}} &= \frac{2v_{\text{eq}}}{c} v_{\text{in}} (\cos \lambda_{\text{in}} - \cos \lambda_{\text{out}}) = \\ &= -\frac{2v_{\text{eq}}}{c} v_{\text{in}} \int_{t_{\text{in}}}^{t_{\text{out}}} \sin(\lambda(t)) \frac{d\lambda}{dt} dt,\end{aligned}\quad (1.1)$$

where  $v_{\text{eq}}$  is the Earth’s equatorial rotational surface speed,  $c$  is the vacuum speed of light,  $v_{\text{in}}$  is the initial asymptotic inbound speed,  $\lambda_{\text{in}}$  is the asymptotic inbound geocentric latitude, and  $\lambda_{\text{out}}$  is the asymptotic outbound geocentric latitude. If  $t$  is the observed coordinate time for the spacecraft in its trajectory, then  $\lambda_{\text{in}} = \lambda(t_{\text{in}})$  and  $\lambda_{\text{out}} = \lambda(t_{\text{out}})$ . If  $d\lambda/dt = 0$ , then  $\delta v_{\text{emp}} = 0$ . An order of magnitude estimate for the maximum possible value for  $\delta v_{\text{emp}}$  is  $2(5 \times 10^2 / 3 \times 10^8) v_{\text{in}} \sim 30 \text{ mm/s}$ .

The following is a direct quote from the conclusions of Anderson *et al.* (the ODP means the Orbit Determination Program):

“Lämmerzahl *et al.* [11] studied and dismissed a number of possible explanations for the Earth flyby anomalies, including Earth atmosphere, ocean tides, solid Earth tides, spacecraft charging, magnetic moments, Earth albedo, solar wind, coupling of Earth’s spin with rotation of the radio wave, Earth gravity, and relativistic effects predicted by Einstein’s theory. All of these potential sources of systematic error, and more, are modeled in the ODP. None can account for the observed anomalies.”

The article by Lämmerzahl *et al.* [11], which is entitled “Is the physics within the Solar system really understood?”, was published in 2006.

A direct quote from the abstract for a more recent article, one published in 2009 by M. M. Nieto and J. D. Anderson, follows [12]:

“In a reference frame fixed to the solar system’s center of mass, a satellite’s energy will change as it is deflected by a planet. But a number of satellites flying by Earth have also experienced energy changes in the Earth-centered frame — and that’s a mystery.”

Nieto and Anderson then conclude their article with the following comments:

“Several physicists have proposed explanations of the Earth flyby anomalies. The least revolutionary invokes dark matter bound to Earth. Others include modifications of special relativity, of general relativity, or of the notion of inertia; a light speed anomaly; or anisotropy in the gravitational field — all of those, of course, deny concepts that have been well tested. And none of them have made comprehensive, precise predictions of Earth flyby effects. For now the anomalous energy changes observed in Earth flybys remain a puzzle. Are they the result of imperfect understandings of conventional physics and experimental systems, or are they the harbingers of exciting new physics?”

When the article by Nieto and Anderson was published, “conventional” or “mainstream” physics had not resolved the mystery of the Earth flyby anomalies. It appears that a new and possibly unconventional theory is needed.

The empirical prediction formula (1.1) found by Anderson *et al.* is not based on any mainstream theory of physics (it was simply “picked out of the air”). However, the empirical prediction formula is remarkably simple and gives calculated speed-change values that are surprisingly close to the observed speed-change values. The empirical prediction formula gives three clues for that which must be satisfied by any theory that is developed to explain the flyby anomaly:

- 1) the theory must produce a formula for the speed-change that is proportional to the ratio  $v_{\text{eq}}/c$ ,
- 2) the anomalous force acting on the spacecraft must change the  $\lambda$ -component of the spacecraft’s velocity, and
- 3) it must be proportional to  $v_{\text{in}}$ .

The objective of *this article* is threefold:

- 1) derive a new causal version of classical acausal Newtonian theory,
- 2) show that this new version is able to produce exact agreement with all six of the anomalous speed-changes reported by Anderson *et al.*, and
- 3) show that it is also able to explain exactly the “lunar orbit anomaly”, an anomalous change in the Moon’s orbital speed which will be described below.

This new version for Newtonian theory uses only mainstream physics:

- 1) classical Newtonian theory, and
- 2) the causality principle which requires time-retardation of the gravitational force.

It also satisfies the three requirements of the empirical prediction formula.

This article proposes a simple correction that converts Newton's acausal theory into a causal theory. The resulting causal theory has a new, previously overlooked, time-retarded transverse component, designated  $\mathbf{g}_{\text{trt}}$ , which depends on  $1/c_g$ , where  $c_g$  is the speed of gravity, which approximately equals the speed of light. The new total gravitational field for a large spinning sphere,  $\mathbf{g}$ , has two components, the standard well-known classical acausal radial component,  $\mathbf{g}_r$ , and a new relatively small previously undetected time-retarded transverse vortex component,  $\mathbf{g}_{\text{trt}}$ . The total vector field  $\mathbf{g} = \mathbf{g}_r + \mathbf{g}_{\text{trt}}$ . The zero-divergence vortex transverse vector field  $\mathbf{g}_{\text{trt}}$  is orthogonal to the irrotational radial vector field  $\mathbf{g}_r$ .

This new vector field is consistent with Helmholtz's theorem, which states that any physical vector field can be expressed as the sum of the gradient of a zero-rotational scalar potential and the curl of a zero-divergence vector potential [13, p. 52]. This means that  $\mathbf{g}_r$  can be derived in the standard way from the gradient of a scalar potential, and  $\mathbf{g}_{\text{trt}}$  can be derived from the curl of a vector potential, but  $\mathbf{g}_{\text{trt}}$  cannot be derived from the gradient of a scalar potential.

The time retarded gravitational fields for a moving point-mass can be derived by using the slow-speed weak-field approximation for Einstein's general relativity theory. Let  $\varphi$  be the time-retarded scalar potential, let  $\mathbf{e}$  be the time-retarded "gravitoelectric" acceleration field, let  $\mathbf{a}$  be the time-retarded vector potential, and let  $\mathbf{h}$  be the time-retarded "gravitomagnetic" induction field. It is shown in §2 that the formulas for  $\varphi$ ,  $\mathbf{e}$ ,  $\mathbf{a}$ , and  $\mathbf{h}$ , have been derived by W. Rindler in his popular textbook, *Essential Relativity* [14]. They are as follows

$$\left. \begin{aligned} \varphi &= G \iiint \left[ \frac{\rho}{r''} \right] dV, & \mathbf{a} &= \frac{G}{c} \iiint \left[ \frac{\rho \mathbf{u}}{r''} \right] dV \\ \mathbf{e} &= -\nabla \varphi, & \mathbf{h} &= \nabla \times 4\mathbf{a} \end{aligned} \right\}, \quad (1.2)$$

where  $\rho$  is the mass-density of the central object,  $\mathbf{u}$  is the inertial velocity of a source-point-mass which is held solidly in the central rotating object by nongravitational forces (inertial velocity means the velocity in an inertial frame),  $\mathbf{r}''$  is the vector distance from a source-point-mass to the field-point, the square brackets denote that the enclosed value is to be retarded by the light travel time from the source-point to the field-point, and  $dV$  is an element of volume of the central body.

Let the origin for an inertial (nonrotating and nonaccelerating)

frame-of-reference coincide with the center-of-mass of a contiguous central object. Let  $\mathbf{r}'$  be the radial vector from the origin to a source-point-mass in the central object, and let  $\mathbf{r}$  be the radial vector from the origin to an external field-point, so that the vector distance from the source-point to the field-point  $\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$ . The triple integrals in (1.2) indicate that the retarded integrands  $[\rho/r'']$  and  $[\rho \mathbf{u}/r'']$  are to be integrated over the volume of the central object at the retarded time.

Let  $m$  be the mass of a test-mass that occupies the field-point at  $\mathbf{r}$ , and let  $\mathbf{v}$  be the inertial velocity of the test mass. The analogous Lorentz force law, i.e., the formula for the time-retarded gravitational force  $\mathbf{F}$  acting on  $m$  at  $\mathbf{r}$ , is [14]

$$\mathbf{F} = -m \left( \mathbf{e} + \frac{1}{c} (\mathbf{v} \times \mathbf{h}) \right) = -m \nabla \left( G \iiint \left[ \frac{\rho}{r''} \right] dV \right) - m \left( \mathbf{v} \times \left( \nabla \times \left( \frac{4G}{c^2} \iiint \left[ \frac{\rho \mathbf{u}}{r''} \right] dV \right) \right) \right). \quad (1.3)$$

This shows that Rindler's time-retarded version for the slow-speed weak-field approximation gives a complete stand-alone time-retarded solution. The time-retarded fields were derived from general relativity theory, but there is at this point no further need for reference to the concepts and techniques of general relativity theory. Needed concepts and techniques are those of classical Newtonian theory.

Furthermore, Rindler's version satisfies the causality principle because the fields are time-retarded. It is valid as a first order approximation only if

$$v^2 \ll c^2, \quad u^2 \ll c^2, \quad \frac{GM}{r} = |\varphi| \ll c^2, \quad (1.4)$$

where  $M$  is the total mass of the central object.

Notice in (1.3) that the acceleration caused by the gravitoelectric field  $\mathbf{e}$  is independent of  $c$ , but the acceleration caused by the gravitomagnetic induction field  $\mathbf{h}$  is reduced by the factor  $1/c^2$ . The numerical value for  $c$  is on the order of  $3 \times 10^8$  m/s. If the magnitude for  $\mathbf{e}$  is on the order of  $10 \text{ m/s}^2$  (the Earth's field at the surface), and the magnitudes for  $\mathbf{u}$  and  $\mathbf{v}$  are on the order of  $10^4$  m/s, the relative magnitude for the acceleration caused by  $\mathbf{h}$  would be on the order of  $10 \times 4 (10^4 / 3 \times 10^8)^2 \text{ m/s}^2 \sim 10^{-8} \text{ m/s}^2$ . This estimate shows that, for most slow-speed weak-field practical applications in the real world, the acceleration caused by  $\mathbf{h}$  is totally negligible compared to the acceleration caused by  $\mathbf{e}$ .

The empirical prediction formula (1.1) indicates that the flyby speed-

change is reduced by  $1/c$ , not by  $1/c^2$ , which rules out the gravitomagnetic induction field  $\mathbf{h}$  as a possible cause for the flyby anomalies. The acceleration of the gravitomagnetic field is simply too small to explain the flyby anomalies.

Consequently, if the gravitomagnetic component is ignored for being negligible, the practicable version for Rindler's Lorentz force law (1.3) becomes the same as a time-retarded version for Newton's well-known inverse-square law

$$\mathbf{F} = -Gm \nabla \iiint \left[ \frac{\rho}{r''} \right] dV, \quad (1.5)$$

where  $\mathbf{F}$  is the time-retarded gravitational force acting on  $m$ .

Let  $d^3\mathbf{F}$  be the time-retarded elemental force of an elemental point-mass source  $dm'$  at  $\mathbf{r}'$ . With this notation, the time-retarded version for Newton's inverse-square law becomes

$$d^3\mathbf{F} = -Gm \frac{dm'}{r''^2} \frac{\mathbf{r}''}{r''}, \quad (1.6)$$

where  $\mathbf{r}''/r''$  is a unit vector directed towards increasing  $\mathbf{r}''$ . The notation  $d^3\mathbf{F}$  indicates that the differential element of force must be integrated over 3-dimensional space to get the total force.

By definition, the gravitational field of a point-source  $dm'$  at  $\mathbf{r}'$  is the gravitational force of the source that acts on a test-mass of mass  $m$  at  $\mathbf{r}$  per unit mass of the test-mass.

The traditional symbol for the Newtonian gravitational vector field is  $\mathbf{g}$ . Therefore, the formula for the time-retarded elemental gravitational field  $d^3\mathbf{g}$  of an elemental point-mass-source  $dm'$  at  $\mathbf{r}'$  for a field-point at  $\mathbf{r}$  occupied by a test-mass of mass  $m$  becomes

$$d^3\mathbf{g} = \frac{d^3\mathbf{F}}{m} = -G \frac{dm'}{r''^2} \frac{\mathbf{r}''}{r''}. \quad (1.7)$$

The negative sign indicates that the gravitational force is attractive.

Let  $\rho(\mathbf{r}')$  be the mass-density of the central object at  $\mathbf{r}'$ . Then

$$dm' = \rho(\mathbf{r}') dV. \quad (1.8)$$

The resulting formula for the elemental gravitational field  $d^3\mathbf{g}$ , which consists of the radial component  $d^3\mathbf{g}_r$  and the transverse component  $d^3\mathbf{g}_{\text{trt}}$ , becomes  $d^3\mathbf{g} = d^3\mathbf{g}_r + d^3\mathbf{g}_{\text{trt}}$ . The formula for each component becomes

$$d^3\mathbf{g}_r = -G \frac{dm'}{r''^2} \left( \frac{\mathbf{r}''}{r''} \right)_r, \quad d^3\mathbf{g}_{\text{trt}} = -G \frac{dm'}{r''^2} \left( \frac{\mathbf{r}''}{r''} \right)_{\text{trt}}, \quad (1.9)$$

where  $(\mathbf{r}''/r'')_r$  is the radial component of the unit vector and  $(\mathbf{r}''/r'')_{\text{trt}}$  is the transverse component of the unit vector. The total field is obtained by a triple integration over the volume of the central object at the retarded time.

The triple integral is rather easy to solve by numerical integration (such as by using the integration algorithm provided in Mathcad15) if the central object can be approximated by a large spinning isotropic sphere. To get a good first approximation for this article, the Earth is simulated by a large spinning isotropic sphere. It is shown in §3 that the triple integration for  $g_{\text{trt}}$  leads to the necessary factor  $1/c_g$ , where  $c_g$  is the speed of propagation of the Earth's gravitational field.

It is also shown in the forthcoming §3 that the formula for the magnitude of  $\mathbf{g}_{\text{trt}}$  is

$$g_{\text{trt}}(\theta) = -G \frac{I_E}{r_E^4} \frac{v_{\text{eq}}}{c_g} \frac{\Omega_\phi(\theta) - \Omega_E}{\Omega_E} \cos^2(\lambda(\theta)) PS(r(\theta)), \quad (1.10)$$

where  $G$  is the gravity constant,  $I_E$  is the Earth's spherical moment of inertia,  $r_E$  is the Earth's spherical radius,  $\Omega_E$  is the Earth's spin angular speed,  $v_{\text{eq}}$  is the Earth's equatorial surface speed,  $c_g$  is the speed of propagation of the Earth's gravitational field,  $\theta$  is the spacecraft's parametric polar coordinate angle in the plane of the orbit or trajectory ( $\Omega_\theta = d\theta/dt$  is the spacecraft's orbital angular speed),  $\Omega_\phi$  is the azimuthal  $\phi$ -component of  $\Omega_\theta$ ,  $\lambda$  is the spacecraft's geocentric latitude,  $r$  is the spacecraft's geocentric radial distance, and  $PS(r)$  is an inverse-cube power series representation for the triple integral over the Earth's volume. If the magnitude is negative, i.e., if  $\Omega_\phi > \Omega_E$  (prograde), the vector field component  $\mathbf{g}_{\text{trt}}$  is directed towards the east. If  $\Omega_\phi < 0$  (retrograde), it is directed towards the west.

The magnitude for  $\mathbf{g}_{\text{trt}}$  satisfies the first requirement of the empirical prediction formula. It is proportional to  $v_{\text{eq}}/c_g \cong v_{\text{eq}}/c$ . But the empirical prediction formula also requires that the speed-change must be in the  $\lambda$ -component of the spacecraft's velocity,  $\mathbf{v}_\lambda$ . The magnitude for the  $\lambda$ -component is defined by

$$v_\lambda = r_\lambda \frac{d\lambda}{dt} = r_\lambda \frac{d\lambda}{d\theta} \frac{d\theta}{dt} = r_\lambda \Omega_\theta \frac{d\lambda}{d\theta}, \quad (1.11)$$

where  $r_\lambda$  is the  $\lambda$ -component of  $r$ . The velocity component,  $\mathbf{v}_\lambda$ , is orthogonal to  $\mathbf{g}_{\text{trt}}$ . Consequently,  $\mathbf{g}_{\text{trt}}$  cannot directly change the magnitude of  $\mathbf{v}_\lambda$  (it only changes the direction).

However, a hypothesized "induction-like" field, designated  $\mathbf{F}_\lambda$ , can be directed perpendicularly to  $\mathbf{g}_{\text{trt}}$  in the  $\mathbf{v}_\lambda$ -direction. Assume that the

$\phi$ -component of the curl of  $\mathbf{F}_\lambda$  equals  $-k dg_{\text{trt}}/dt$ , where  $k$  is a constant\*. This induction-like field can cause a small change in the spacecraft's speed. The reciprocal of the constant  $k$ ,  $v_k = 1/k$ , called herein the "induction speed", becomes an adjustable parameter for each case. The average for all cases gives an overall constant for the causal version of Newton's theory.

The formula for the magnitude of  $\mathbf{F}_\lambda$  is shown in §3 to be

$$F_\lambda(\theta) = \frac{v_{\text{eq}}}{v_k} \frac{r_E}{r(\theta)} \int_0^\theta \frac{r(\theta)}{r_E} \frac{\Omega_\theta(\theta)}{\Omega_E} \frac{1}{r_E} \frac{dr}{d\theta} \frac{dg_{\text{trt}}}{d\theta} d\theta. \quad (1.12)$$

The acceleration caused by  $\mathbf{F}_\lambda$  satisfies the second requirement of the empirical prediction formula, the one that requires that the anomalous force must change the  $\lambda$ -component of the spacecraft's velocity. It needs to be emphasized at this point that the acceleration field  $\mathbf{F}_\lambda$  is a hypothesis, and is subject to proof or disproof by the facts-of-observation. This hypothesis is needed to satisfy the requirement that the speed-change must be in the  $\lambda$ -component of the spacecraft's velocity.

The anomalous time rate of change in the spacecraft's kinetic energy is given by the dot product,  $\mathbf{v} \cdot \mathbf{F}_\lambda$ . It is shown in §3 that the calculated asymptotic speed-change,  $\delta v_{\text{trt}}$ , is given by

$$\delta v_{\text{trt}} = \delta v_{\text{in}} + \delta v_{\text{out}}, \quad (1.13)$$

where

$$\delta v_{\text{in}} = \delta v(\theta_{\text{min}}), \quad \delta v_{\text{out}} = \delta v(\theta_{\text{max}}), \quad (1.14)$$

and

$$\delta v(\theta) = \frac{v_{\text{in}}}{2} \int_0^\theta \frac{r_\lambda(\theta) F_\lambda(\theta)}{v_{\text{in}}^2} \frac{d\lambda}{d\theta} d\theta. \quad (1.15)$$

The angles  $\theta_{\text{min}}$  and  $\theta_{\text{max}}$  are the minimum and maximum values for  $\theta$ . The initial speed  $v_{\text{in}} = v(\theta_{\text{min}})$ . The speed-change  $\delta v(\theta)$  is proportional to  $v_{\text{in}}$ , which satisfies the third requirement of the empirical prediction formula.

It is shown in §4 that this "neoclassical" causal version for acausal Newtonian theory explains exactly the flyby anomalies. Table 1 lists, for each of the six Earth flybys reported by Anderson *et al.*, the observed speed change from Appendix A,  $\delta v_{\text{obs}}$ , the calculated speed change from (1.13),  $\delta v_{\text{trt}}$ , the ratio that was used for the speed of gravity,  $c_g/c$ , the value for the induction speed ratio that gives exact agreement with the observed speed-change,  $v_k/v_{\text{eq}}$ , and the value for the eccentricity of the

\*For Maxwell's theory, the numerical value for the constant  $k = 1/c$  [1].

trajectory,  $\varepsilon$ .

Notice in Table 1 that the required values for  $v_k$  cluster between  $6v_{\text{eq}}$  and  $17v_{\text{eq}}$ . Also notice that the two high-precision flybys, NEAR and Rosetta, put very stringent limits on the speed of gravity,  $c_g$ . If the “true” value for  $v_k$  had been known with high precision, the two high-precision flybys would have provided first-ever measured values for the speed of propagation of the Earth’s gravitational field.

In 1995, F.R. Stephenson and L.V. Morrison published a study of records of eclipses from 700 BC to 1990 AD [15]. They conclude (LOD means length-of-solar-day, ms cy<sup>-1</sup> means milliseconds per century): 1) the LOD has been increasing on average during the past 2700 years at the rate of  $+1.70 \pm 0.05$  ms cy<sup>-1</sup>, i.e.  $(+17.0 \pm 0.5) \times 10^{-6}$  s per year, 2) tidal braking causes an increase in the LOD of  $+2.3 \pm 0.1$  ms cy<sup>-1</sup>, i.e.  $(+23 \pm 1) \times 10^{-6}$  s per year, and 3) there is a non-tidal decrease in the LOD, numerically  $-0.6 \pm 0.1$  ms cy<sup>-1</sup>, i.e.  $(-6 \pm 1) \times 10^{-6}$  s per year.

Stephenson and Morrison state that the non-tidal decrease in the LOD probably is caused by “post-glacial rebound”. Post-glacial rebound decreases the Earth’s moment of inertia, which increases the Earth’s spin angular speed, and thereby decreases the LOD. But post-glacial rebound cannot change the Moon’s orbital angular momentum.

According to Stephenson and Morrison, tidal braking causes an increase in the LOD of  $(23 \pm 1) \times 10^{-6}$  seconds per year, which causes a decrease in the Earth’s spin angular momentum, and by conservation of angular momentum causes an increase in the Moon’s orbital angular momentum. It is shown in §5 that tidal braking alone would cause an increase in the Moon’s orbital speed of  $(19 \pm 1) \times 10^{-9}$  m/s per year, which corresponds to an increase in the radius of the Moon’s orbit of  $(14 \pm 1)$  mm per year.

But lunar-laser-ranging experiments have shown that the radius of the Moon’s orbit is actually increasing at the rate of  $(38 \pm 1)$  mm per year [16]. This rate for increase in the radius corresponds to an increase in the orbital speed of  $(52 \pm 2) \times 10^{-9}$  m/s per year. Clearly there is an unexplained or anomalous difference in the change in the radius of the orbit of  $(-24 \pm 2)$  mm per year ( $38 - 14 = 24$ ), and a corresponding anomalous difference in the change in the orbital speed of  $(-33 \pm 3) \times 10^{-9}$  m/s per year ( $52 - 19 = 33$ ). This “lunar orbit anomaly” cannot be caused by post-glacial rebound, but it can be caused by the proposed neoclassical causal version of Newton’s theory.

It is shown in §5 that the proposed neoclassical causal theory produces a change in the Moon’s orbital speed of  $(-33 \pm 3) \times 10^{-9}$  m/s per year if the value for the induction speed  $v_k = (8 \pm 1)v_{\text{eq}}$ . The eccen-



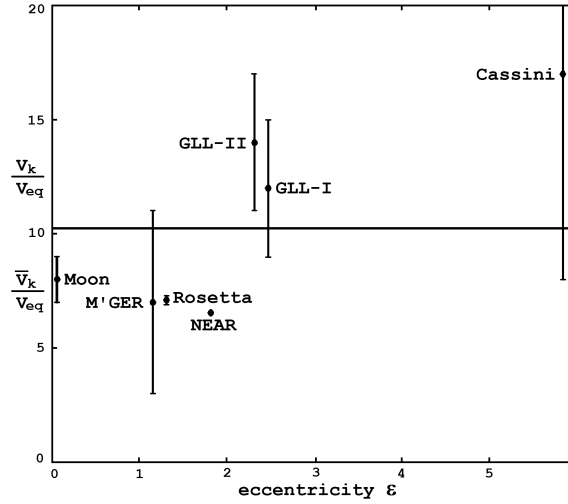


Fig. 1: Required induction speed ratio (designated by  $\bullet$ ),  $v_k/v_{eq} \pm a$  rough estimate for the uncertainty, versus eccentricity  $\epsilon$ . The average value for all seven ratios,  $\bar{v}_k = 10.2 v_{eq}$ , is shown by the horizontal line.

tricity for the Moon's orbit,  $\epsilon = 0.0554$ , indicates that it revolves in a nearly circular closed orbit. Consequently, a new closed orbit case can be added to the open orbit flybys listed in Table 1. A graph of the required induction speed ratio,  $v_k/v_{eq}$ , versus eccentricity  $\epsilon$ , Fig. 1, shows that the required value for the induction speed for the Moon is consistent with the required values for the induction speed for the six flyby anomalies.

The average  $\pm$  standard deviation for the seven induction speed ratios in Fig. 1 is

$$\bar{v}_k = (10.2 \pm 3.8) v_{eq} = 4.8 \pm 1.8 \text{ km/s.} \quad (1.16)$$

It will be interesting to compare this average value with parameter values for other theories which explain the flyby anomalies.

The neoclassical causal theory can be further tested by doing high precision Doppler-shift research studies of the orbital motions of spacecrafts in highly eccentric and inclined near-Earth orbits. The predicted annual speed-change  $\delta v_{yr}$  (prograde) and  $\delta v_{yr}$  (retrograde) for orbits with eccentricity  $\epsilon = 0.5$ , inclination  $\alpha_{eq} = 45^\circ$ , and geocentric latitude at perigee  $\lambda_p = 45^\circ$ , with the induction speed set equal to its maximum probable value  $v_k = 14 v_{eq}$ , and with the radial distance at perigee  $r_p$  ranging from  $2r_E$  to  $8r_E$ , are calculated in §6 and listed in Table 2.

Flyby	NEAR*	GLL-I	Rosetta*	M'GER	Cassini	GLL-II
$\delta v_{\text{obs}}$ (mm/s)	+13.46 $\pm 0.01$	+3.92 $\pm 0.30$	+1.80 $\pm 0.03$	+0.02 $\pm 0.01$	-2 $\pm 1$	-4.6 $\pm 1$
$\delta v_{\text{trt}}$ (mm/s)	+13.46 $\pm 0.01$	+3.92 $\pm 0.30$	+1.80 $\pm 0.03$	+0.02 $\pm 0.01$	-2 $\pm 1$	-4.6 $\pm 1$
$c_g/c$	1.000 $\pm 0.001$	1	1.00 $\pm 0.02$	1	1	1
$v_k/v_{\text{eq}}$	6.530 $\pm 0.005$	12 $\pm 3$	7.1 $\pm 0.2$	7 $\pm 4$	17 $\pm 9$	14 $\pm 3$
$\varepsilon$	1.8142	2.4731	1.3122	1.3596	5.8456	2.3186

Table 1: Listing of the observed speed-change,  $\delta v_{\text{obs}}$ , the calculated speed-change from (1.13),  $\delta v_{\text{trt}}$ , the ratio used for the speed of gravity,  $c_g/c$ , the required value for the induction speed ratio,  $v_k/v_{\text{eq}}$ , and the eccentricity  $\varepsilon$ , for each of the six Earth flybys reported by Anderson *et al.* [10]. Listed uncertainties are rough estimates based on the uncertainty estimates of Anderson *et al.* The induction speed,  $v_k$ , was adjusted to make the calculated speed change,  $\delta v_{\text{trt}}$ , be identically equal to the observed speed change,  $\delta v_{\text{obs}}$ . The two high-precision flybys are marked by an asterisk.

$r_p/r_E$	2	3	4	5	6	7	8
$P$	11.2	20.7	31.8	44.4	58.4	73.6	89.9
$\delta v_{\text{yr}}$	+315	+29.5	+3.93	+0.173	-0.422	-0.422	-0.362
$\delta v r_{\text{yr}}$	-517	-76.8	-21.0	-7.97	-3.69	-1.95	-1.14

Table 2: Predicted period,  $P$  in hours, the speed-change for prograde orbits,  $\delta v_{\text{yr}}$  in mm/s per year, and the speed-change for retrograde orbits,  $\delta v r_{\text{yr}}$  in mm/s per year, for a near-Earth orbiting spacecraft with orbital parameters  $\varepsilon = 0.5$ ,  $\alpha_{\text{eq}} = 45^\circ$ , and  $\lambda_p = 45^\circ$ , with  $v_k = 14 v_{\text{eq}}$ , and for  $r_p$  ranging from  $2r_E$  to  $8r_E$ .

If the proposed neoclassical causal theory is to be viable, it cannot conflict with Einstein's general relativity theory. The only possible conflict is with the excess advance in the perihelion of the planet Mercury, +43 arc seconds per century, which is explained exactly by general relativity theory. The predicted rate for change in the angle for perihelion for the neoclassical causal theory is shown in §7 to be less than 0.04 arc seconds per century, which is very much less than the relativistic advance and is undetectable. Therefore, there is no conflict with general relativity theory. Furthermore, the neoclassical causal theory does not require any change of any kind for general relativity theory. In fact, it is derived from general relativity theory.

There are at least two other published theories that try to provide an explanation for the Earth flyby anomalies. These are: 1) the 3-space flow theory of R. T. Cahill [17] and 2) the exponential radial field theory authored by H. J. Busack [18].

In [17] Cahill reviews numerous Michelson interferometer and one-way light-speed experiments which clearly show an anisotropy in the velocity of light. His calculated flyby speed-changes depend on the direction and magnitude for 3-space inflow at the spacecraft on the date and time of the flyby. Cahill found that the average speed for 3-space inflow is  $12 \pm 5$  km/s. Cahill's average,  $12 - 5 = 7$  km/s, essentially equals the average value for  $v_k$ , see (1.16),  $4.8 + 1.8 = 6.6$  km/s.

In [18] Busack applies a small exponential correction for the Earth's radial gravitational field. If  $f(\mathbf{r}, \mathbf{v})$  is Busack's correction, the inverse-square law becomes

$$\mathbf{g}_r(\mathbf{r}, \mathbf{v}) = -\frac{GM_E}{r^2} \frac{\mathbf{r}}{r} (1 + f(\mathbf{r}, \mathbf{v})),$$

where  $f(\mathbf{r}, \mathbf{v})$  is expressed as

$$f(\mathbf{r}, \mathbf{v}) = A \exp\left(-\frac{r - r_E}{B - C(\mathbf{r} \cdot \mathbf{v})/(\mathbf{r} \cdot \mathbf{v}_{\text{Sun}})}\right).$$

The velocity  $\mathbf{v}$  is the velocity of the field-point in the "gravitational rest frame in the cosmic microwave background", and  $\mathbf{v}_{\text{Sun}}$  is the Sun's velocity in the gravitational rest frame. Numerical values for the adjustable constants are approximately  $A = 2.2 \times 10^{-4}$ ,  $B = 2.9 \times 10^5$  m, and  $C = 2.3 \times 10^5$  m. Busack found that these values produce rather good agreement with the observed values for the flyby speed-changes.

The maximum possible value,  $f(\mathbf{r}, \mathbf{v}) = A$ , occurs where  $r = r_E$ . At this point,  $g_r = (GM_E/r_E^2)(1 + A) \sim 10(1 + 2 \times 10^{-4}) \text{ m/s}^2$ . Compare this estimate with an estimate for the peak value for the neoclassical causal

transverse field for the NEAR flyby, which is  $g = g_r \sqrt{1 + g_{\text{rt}}^2/g_r^2} \sim 10(1 + 8 \times 10^{-12}) \text{ m/s}^2$ .

Both of these alternative theories require a preferred frame-of-reference. Neither has been tested for the lunar orbit anomaly, and neither satisfies the causality principle because neither depends on the speed of gravity.

In conclusion, the proposed neoclassical causal version for acausal Newtonian theory has passed seven tests: 1) explanation of the six flyby anomalies, and 2) explanation of the lunar orbit anomaly. It will be very difficult if not impossible for any other rational theory to be causal and pass all seven of these tests.

**§2. Slow-speed weak-field approximation for general relativity theory.** The following comment from F. Rohrlich's article tells us about one problem that Sir Isaac Newton could not solve [2]:

“Historians tell us that Newton was quite unhappy over the fact that his law of gravitation implies an action-at-a-distance interaction over very large distances such as that between the sun and the earth. But he was unable to resolve this problem.”

The great author of Newtonian theory stood on the shoulders of giants, but he was not able to see Maxwell's theory or the slow-speed weak-field approximation for Einstein's theory.

The time-retarded version for the slow-speed weak-field approximation for general relativity theory provides a valid first-order approximation for the gravitational field of a moving point mass and a moving field-point. This approximation applies for “slowly” moving particles in “weak” gravitational fields. The word “slowly” means that  $|\mathbf{u}| \ll c$ , where  $|\mathbf{u}|$  is the maximum magnitude for the source-particle velocity, that  $|\mathbf{v}| \ll c$ , where  $|\mathbf{v}|$  is the maximum magnitude for the field-point test-particle velocity, and the word “weak” means  $|\varphi| \ll c^2$ , where  $|\varphi| = GM/r$  is the maximum absolute magnitude for the scalar gravitational potential.

The chapter entitled *The Linear Approximation to GR* in W. Rindler's popular textbook starts on page 188 [14]. The following is a direct quote from pages 190 and 191:

“In the general case, Equations (8.180) can be integrated by standard methods. For example, the first yields as the physically relevant solution,

$$\gamma_{\mu\nu} = -\frac{4G}{c^4} \iiint \frac{[T_{\mu\nu}] dV}{r}, \quad (8.184)$$

where  $[ ]$  denotes the value “retarded” by the light travel time to the origin of  $r$ .

As an example, consider a system of sources in stationary motion (e.g., a rotating mass shell). All  $\gamma$ 's will then be time-independent. If we neglect stresses and products of source velocities (which is not really quite legitimate<sup>14</sup>), the energy tensor (8.128) becomes

$$T_{\mu\nu} = \begin{pmatrix} \mathbf{0}_3 & -c^2\mathbf{v} \\ -c^2\mathbf{v} & c^4\rho \end{pmatrix} \quad (8.185)$$

where  $\mathbf{0}_3$  stands for the  $3 \times 3$  zero matrix, and so, from (8.184),

$$\gamma_{ij} = 0, \quad i, j = 1, 2, 3. \quad (8.186)$$

For slowly moving test particles,  $ds = cdt$ . If we denote differentiation with respect to  $t$  by dots, the first three geodesic equations of motion become [cf. (8.15)]

$$\ddot{x}^i = -\Gamma_{\mu\nu}^i \dot{x}^\mu \dot{x}^\nu \quad (8.187)$$

$$= -\left( \gamma^i_{\mu,\nu} - \frac{1}{2} \gamma_{\mu\nu,}^i - \frac{1}{4} \eta^i_{\mu} \gamma_{,\nu} - \frac{1}{4} \eta^i_{\nu} \gamma_{,\mu} + \frac{1}{4} \eta_{\mu\nu} \gamma_{,}^i \right) \dot{x}^\mu \dot{x}^\nu, \quad (8.188)$$

where we have substituted into (8.187) from (8.168) and (8.172) and used  $\gamma = \eta^{\mu\nu} \gamma_{\mu\nu} = -h$ . Moreover,  $\gamma = c^2 \gamma_{44}$ . Now if we let  $\dot{x}^\mu = (u^i, 1)$  and neglect products of the  $u$ 's, Equation (8.188) reduces to

$$\ddot{x}^i = -\gamma^i_{4,j} u^j + \gamma_{j4,}^i u^j + \frac{1}{4} \gamma_{44,}^i.$$

This can be written vectorially in the form

$$\ddot{\mathbf{r}} = \mathbf{grad} \varphi - \frac{1}{c} (\mathbf{u} \times \mathbf{curl} \mathbf{a}) = - \left[ \mathbf{e} + \frac{1}{c} (\mathbf{u} \times \mathbf{h}) \right], \quad (8.189)$$

where [cf. (8.184), (8.185)]

$$\begin{aligned} \varphi &= -\frac{1}{4} \gamma_{44} = G \iiint \frac{[\rho] dV}{r}, \\ \mathbf{a} &= -\frac{c}{4} \gamma_{4,}^i = \frac{1}{c} G \iiint \frac{[\rho \mathbf{u}] dV}{r}, \end{aligned} \quad (8.190)$$

and

$$\mathbf{e} = -\mathbf{grad} \varphi, \quad \mathbf{h} = \mathbf{curl} 4\mathbf{a}. \quad (8.191)$$

The formal similarity with Maxwell's theory is striking. The only differences are: the minus sign in (8.189) (because the force is attractive); the factor  $G$  in (8.190) (due to the choice of units); and the novel factor 4 in (8.191) (ii)."

We need to change from Rindler's symbols to the symbols being used in this article. For the distance from the source-point to the field-point:  $r \rightarrow r''$ . For the integrands:  $[\rho] \rightarrow [\rho/r'']$ , and  $[\rho\mathbf{u}] \rightarrow [\rho\mathbf{u}/r'']$ . For the gradient and the curl:  $\mathbf{grad} \rightarrow \nabla$ ,  $\mathbf{curl} \rightarrow \nabla \times$ .

The converted formulas for  $\varphi$  and  $\mathbf{e}$  give the time-retarded scalar gravitational potential and the time-retarded gravitoelectric field

$$\varphi = G \iiint \left[ \frac{\rho}{r''} \right] dV, \quad \mathbf{e} = -\nabla \varphi. \quad (2.1)$$

The converted formulas for  $\mathbf{a}$  and  $\mathbf{h}$  give the time-retarded vector gravitational potential and the time-retarded gravitomagnetic field

$$\mathbf{a} = \frac{G}{c} \iiint \left[ \frac{\rho\mathbf{u}}{r''} \right] dV, \quad \mathbf{h} = -\nabla \times 4\mathbf{a}. \quad (2.2)$$

Let  $m$  be the mass of a test-mass that occupies the field point. Then the time-retarded gravitational force  $\mathbf{F}$  that acts on the test-mass  $m$  becomes

$$\mathbf{F} = -m \left( \mathbf{e} + \frac{1}{c} (\mathbf{u} \times \mathbf{h}) \right). \quad (2.3)$$

**§3. Derivation of the formulas for the speed-change caused by the neoclassical causal version of Newton's theory.** Let the Earth be simulated by a large rotating isotropic sphere of radius  $r_E$ , mass  $M_E$ , angular speed  $\Omega_E$ , moment of inertia  $I_E$ , and radial mass-density distribution  $\rho(r')$ . The radial mass-density distribution and values for the Earth's parameters are shown in Appendix B.

Consider a spacecraft in an open or closed orbit around this sphere. Let  $(X, Y, Z)$  be the rectangular coordinate axes for an inertial frame-of-reference, let the sphere's center coincide with the origin, and let the  $(X, Y)$  plane coincide with the equatorial plane, so that the  $Z$ -axis coincides with the sphere's rotational axis. Let  $\mathbf{r}''$  be the vector distance from  $\mathbf{r}'$  to  $\mathbf{r}$ ,  $\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$ .

Let the elemental gravitational field of an interior circulating point-mass  $dm'$  at  $\mathbf{r}'$  be designated by  $d^3\mathbf{g}$ . As depicted in Fig. 2,  $d^3\mathbf{g}$  has two components, a radial component designated by  $d^3\mathbf{g}_r$ , and a transverse component designated by  $d^3\mathbf{g}_{\text{ttt}}$ . Therefore,  $d^3\mathbf{g} = d^3\mathbf{g}_r + d^3\mathbf{g}_{\text{ttt}}$ .



Obviously,  $t(t')$  is here a function of  $t'$ , and *vice versa*,  $t'(t)$  is a function of  $t$ .

By definition, the derivative  $dt/dt'$  is the Jacobian for the transformation from  $t$  to  $t'$ . Let  $f(t, t')$  be an implicit function of  $t$  and  $t'$ , and let  $F(t)$  be the function that results from integration of  $f(t, t')$  over  $t$

$$\begin{aligned} F(t) &= \int_0^t f(t, t') dt = \int_{t'(0)}^{t'(t)} f(t, t') \frac{dt}{dt'} dt' = \\ &= \int_{t'(0)}^{t'(t)} f(t, t') (\text{Jacobian}) dt'. \end{aligned} \quad (3.3)$$

The formulas for the components  $d^3g_r$  and  $d^3g_{trt}$  can be found by substituting into the formulas of (1.9)

$$\left. \begin{aligned} d^3g_r &= \left( -G \frac{dm'}{r''^2} \right) (RC)(\text{Jacobian}) \\ d^3g_{trt} &= \left( -G \frac{dm'}{r''^2} \right) (TC_Z)(\text{Jacobian}) \end{aligned} \right\}. \quad (3.4)$$

Because the speed of gravity  $c_g \cong c$ , it has a very large numerical value, and the derivative of  $r''$  with respect to  $t'$  approximately equals the derivative of  $r''$  with respect to  $t$

$$\begin{aligned} \frac{dt}{dt'} &= 1 + \frac{1}{c_g} \frac{dr''}{dt'} = 1 + \frac{1}{c_g} \frac{dr''}{dt} \frac{dt}{dt'} = \\ &= 1 + \frac{1}{c_g} \frac{dr''}{dt} \left( 1 + \frac{1}{c_g} \frac{dr''}{dt'} \right) \cong 1 + \frac{1}{c_g} \frac{dr''}{dt}. \end{aligned} \quad (3.5)$$

The formulas for the geocentric radial distance to the field-point and its derivative are

$$\left. \begin{aligned} r(\theta) &= \frac{r_p(1+\varepsilon)}{1+\varepsilon \cos \theta} \\ \frac{dr}{d\theta} &= \frac{r(\theta)^2}{r_p} \frac{\varepsilon}{1+\varepsilon} \sin \theta \end{aligned} \right\}, \quad (3.6)$$

where  $\theta$  is the parametric polar coordinate angle for the spacecraft,  $r_p$  is the geocentric radial distance at perigee, and  $\varepsilon$  is the eccentricity of the orbit or trajectory.

Let the rectangular coordinates for  $\mathbf{r}$  be  $r_X, r_Y, r_Z$  and those for  $\mathbf{r}'$  be  $r'_X, r'_Y, r'_Z$ . Let the spherical coordinates for  $\mathbf{r}$  be  $r, \lambda, \phi$  and those



for  $\mathbf{r}'$  be  $r', \lambda', \phi'$ . Then

$$\left. \begin{aligned} r_X &= r \cos \lambda \cos \phi, & r'_X &= r' \cos \lambda' \cos \phi' \\ r_Y &= r \cos \lambda \sin \phi, & r'_Y &= r' \cos \lambda' \sin \phi' \\ r_Z &= r \sin \lambda, & r'_Z &= r' \sin \lambda' \end{aligned} \right\}. \quad (3.7)$$

The formula for  $dm'$  is

$$dm' = \rho(r') r'^2 \cos \lambda' dr' d\lambda' d\phi'. \quad (3.8)$$

The square of the magnitude for  $\mathbf{r}''$  is

$$\begin{aligned} r''^2 &= (r_X - r'_X)^2 + (r_Y - r'_Y)^2 + (r_Z - r'_Z)^2 = \\ &= (r \cos \lambda \cos \phi - r' \cos \lambda' \cos \phi')^2 + \\ &+ (r \cos \lambda \sin \phi - r' \cos \lambda' \sin \phi')^2 + \\ &+ (r \sin \lambda - r' \sin \lambda')^2. \end{aligned}$$

Expanding the square and using the trig identity for  $\cos(\phi - \phi')$  gives

$$r''^2 = r^2 (1 + x), \quad (3.9)$$

where  $x$  is defined by

$$x \equiv \frac{r'^2}{r^2} - 2 \frac{r'}{r} (\cos \lambda \cos \lambda' \cos(\phi - \phi') + \sin \lambda \sin \lambda'). \quad (3.10)$$

The derivative  $dr''/dt'$ , which depends on the derivatives  $(dr/dt', d\lambda/dt', d\phi/dt')$  and  $(dr'/dt', d\lambda'/dt', d\phi'/dt')$ , will be needed to find the Jacobian

$$\left. \begin{aligned} \frac{dr}{dt'} &\cong \frac{dr}{dt} = v_r = \Omega_\theta \frac{dr}{d\theta}, & \frac{dr'}{dt'} &= 0 \\ \frac{d\lambda}{dt'} &\cong \frac{d\lambda}{dt} = \frac{v_\lambda}{r_\lambda} = \Omega_\lambda, & \frac{d\lambda'}{dt'} &= 0 \\ \frac{d\phi}{dt'} &\cong \frac{d\phi}{dt} = \frac{v_\phi}{r_\phi} = \Omega_\phi, & \frac{d\phi'}{dt'} &\cong \frac{d\phi'}{dt} = \Omega_E \end{aligned} \right\}. \quad (3.11)$$

We actually need the derivatives that contribute to the transverse motion of the projection of the orbit onto the equatorial plane. All but  $\Omega_\phi$  and  $\Omega_E$  can be for now disregarded, because the other derivatives produce terms that will not survive the triple integration.

If the integrand is an odd function, e.g.,  $f(\phi') \sin \phi'$ , where  $f(\phi')$  is any even function,  $f(-\phi') = f(\phi')$ , the integral  $\int f(\phi') \sin \phi' d\phi'$  over  $\phi'$  from  $-\pi$  to  $+\pi$  vanishes, i.e., reduces to zero.

Consequently the derivative for the square of  $r''$  reduces to

$$\begin{aligned}\frac{dr''^2}{dt'} &= 2r'' \frac{dr''}{dt'} = \\ &= 2r'r \left( \frac{d\phi}{dt'} - \frac{d\phi'}{dt'} \right) \cos \lambda \cos \lambda' \sin(\phi - \phi').\end{aligned}\quad (3.12)$$

The derivatives  $d\phi/dt' \cong d\phi/dt = \Omega_\phi$  and  $d\phi'/dt' \cong d\phi'/dt = \Omega_E$ . The formula for the Jacobian reduces to

$$\begin{aligned}\text{Jacobian} &= 1 + \frac{1}{c_g} \frac{dr''}{dt'} = \\ &= 1 + \frac{r}{c_g} \frac{r'}{r''} (\Omega_\phi - \Omega_E) \cos \lambda \cos \lambda' \sin(\phi - \phi').\end{aligned}\quad (3.13)$$

The formulas for  $RC$  and  $TC_Z$ , (2.1), reduce to

$$\begin{aligned}RC &= \frac{r}{r''} - \frac{1}{r''r} (r_X r'_X + r_Y r'_Y + r_Z r'_Z) = \\ &= \frac{r}{r''} - \frac{r'}{r''} (\cos \lambda \cos \lambda' \cos(\phi - \phi') + \sin \lambda \sin \lambda'),\end{aligned}\quad (3.14)$$

and also

$$TC_Z = \frac{1}{r''r} (r'_X r_Y - r_X r'_Y) = \frac{r'}{r''} \cos \lambda \cos \lambda' \sin(\phi - \phi').\quad (3.15)$$

Firstly consider the radial component  $g_r$ . Substituting (3.8), (3.13), and (3.14) into (3.4) gives

$$\begin{aligned}d^3 g_r &= \left( -G \frac{dm'}{r''^2} \right) (RC) (\text{Jacobian}) = \\ &= \left( -G \frac{\rho(r')}{r''^2} \cos \lambda' dr' d\lambda' d\phi' \right) \times \\ &\times \frac{r}{r''} \left( 1 - \frac{r'}{r} \sin \lambda \sin \lambda' - \frac{r'}{r} \cos \lambda \cos \lambda' \cos(\phi - \phi') \right) \times \\ &\times \left( 1 + \frac{r}{c_g} \frac{r'}{r''} (\Omega_\phi - \Omega_E) \cos \lambda \cos \lambda' \sin(\phi - \phi') \right).\end{aligned}\quad (3.16)$$

This formula contains both time-retarded or “causal” terms and non-time-retarded or “acausal” terms. The causal terms, those which contain the factor  $1/c_g$ , contain either  $\sin(\phi - \phi')$  or  $\sin(\phi - \phi') \cos(\phi - \phi')$ . These terms will vanish upon integration over  $\phi'$  from  $-\pi$  to  $+\pi$ , which means that all the effects of time-retardation cancel out for the radial

component. The only terms that survive the integration are the acausal terms. In other words, the radial component can be regarded as being acausal. It can be found by using the standard methods. Gauss' theorem gives the standard well-known inverse square law [13, p. 37]

$$\mathbf{g}_r = -G \frac{M_E \mathbf{r}}{r^2 r}. \quad (3.17)$$

The radial component, until recently the only known component, has been studied by many researchers for more than 300 years. It is well-known that  $\mathbf{g}_r$  obeys the conservation laws for orbital energy and orbital angular momentum. The orbital energy is conserved for an isotropic solid central sphere (with no tidal bulges) because the central force is conservative, i.e., there is no mechanism (e.g., friction) by which orbital kinetic and potential energy can be dissipated into another form of energy. The orbital angular momentum is conserved because the central radial force cannot exert a torque on the orbiting body.

Now consider the transverse component. Substituting (3.8), (3.13), and (3.15) into (3.4) gives

$$\begin{aligned} d^3 g_{\text{trt}} \left( -G \frac{dm'}{r'^2} \right) (TC_Z) (\text{Jacobian}) &= \\ &= \left( -G \frac{\rho(r')}{r'^2} \cos \lambda' dr' d\lambda' d\phi' \right) \times \\ &\times \left( \frac{r'}{r''} \cos \lambda \cos \lambda' \sin(\phi - \phi') \right) \times \\ &\times \left( 1 + \frac{r}{c_g} \frac{r'}{r''} (\Omega_\phi - \Omega_E) \cos \lambda \cos \lambda' \sin(\phi - \phi') \right). \quad (3.18) \end{aligned}$$

This formula contains one causal term and one acausal term. The acausal term contains  $\sin(\phi - \phi')$ , which vanishes upon integration over  $\phi'$  from  $-\pi$  to  $+\pi$ . The surviving term, which contains  $\sin^2(\phi - \phi')$ , does not vanish upon the integration. Consequently, the time-retarded transverse field is causal, and it can be found by using (Jacobian-1).

Substituting (Jacobian-1) for (Jacobian) in (3.18) gives

$$\begin{aligned} d^3 g_{\text{trt}} &= \left( -G \frac{\rho(r')}{r'^2} \cos \lambda' dr' d\lambda' d\phi' \right) \times \\ &\times \left( \frac{r'}{r''} \cos \lambda \cos \lambda' \sin(\phi - \phi') \right) \times \\ &\times \left( \frac{r}{c_g} \frac{r'}{r''} (\Omega_\phi - \Omega_E) \cos \lambda \cos \lambda' \sin(\phi - \phi') \right). \quad (3.19) \end{aligned}$$

Rearranging and combining factors gives

$$d^3 g_{\text{trt}} = -A \left( \frac{\Omega_\phi - \Omega_E}{\Omega_E} \right) \cos^2(\lambda) IG \frac{dr'}{r_E} d\lambda' d\phi', \quad (3.20)$$

where the definitions for the equatorial surface speed  $v_{\text{eq}}$ , and for the coefficient  $A$ , are

$$v_{\text{eq}} \equiv r_E \Omega_E, \quad A \equiv G \bar{\rho} r_E \frac{v_{\text{eq}}}{c_g}, \quad (3.21)$$

and the integrand for the triple integration is

$$IG \equiv \left( \frac{r_E}{r} \right)^3 \left( \frac{\rho(r')}{\bar{\rho}} \frac{r'^4}{r_E^4} \right) (\cos^3 \lambda') \left( \frac{\sin^2(\phi - \phi')}{(1+x)^2} \right), \quad (3.22)$$

where  $\bar{\rho}$  is the mean value for  $\rho(r')$ . The formula for  $\rho(r')$  and value for  $\bar{\rho}$  can be found in Appendix B.

The solution for  $g_{\text{trt}}$  becomes

$$g_{\text{trt}} = -A \left( \frac{\Omega_\phi - \Omega_E}{\Omega_E} \right) \cos^2(\lambda) TI, \quad (3.23)$$

where the triple integral  $TI$  is defined by

$$TI \equiv \int_0^{r_E} \frac{dr'}{r_E} \int_{-\pi/2}^{\pi/2} d\lambda' \int_{-\pi}^{\pi} IG d\phi'. \quad (3.24)$$

Most of the integrals in this article are solved by using the numerical integration algorithm in Mathcad15. It can be shown that the solution for the triple integral  $TI$  is independent of  $\lambda$  and  $\phi$ , which means that it can be solved with  $\lambda=0$  and  $\phi=0$ . But solving a triple integral by numerical integration takes a lot of computer time, particularly if  $r$  is near the singularity at  $r=r_E$ , which must be avoided.

A suitable power series approximation for the triple integral is needed. Let  $PS'(r)$  be a four-term power series, defined as follows

$$PS'(r) \equiv \frac{I_E}{\bar{\rho} r_E^5} \left( \frac{r_E}{r} \right)^3 \left( C_0 + C_2 \left( \frac{r_E}{r} \right)^2 + C_4 \left( \frac{r_E}{r} \right)^4 + C_6 \left( \frac{r_E}{r} \right)^6 \right). \quad (3.25)$$

Let  $PS(r)$  be the same power series without the unitless coefficient, which for the Earth has the value 1.3856 (see Appendix B)

$$PS(r) \equiv \left( \frac{r_E}{r} \right)^3 \left( C_0 + C_2 \left( \frac{r_E}{r} \right)^2 + C_4 \left( \frac{r_E}{r} \right)^4 + C_6 \left( \frac{r_E}{r} \right)^6 \right). \quad (3.26)$$

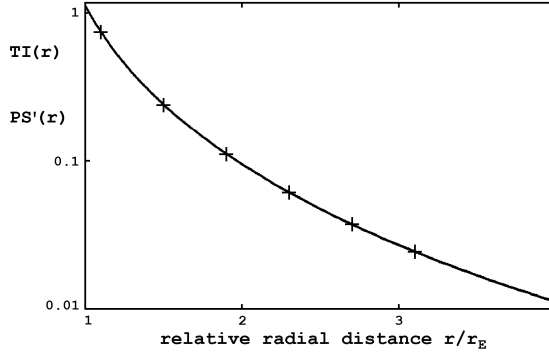


Fig. 3: Semilog graph of the triple integral  $TI(r)$  of (3.24) designated by +, and the power series  $PS'(r)$  of (3.25) designated by the solid curve, versus  $r/r_E$ , using the coefficients of (3.27). The maximum relative difference  $(TI - PS')/TI$  is less than  $2 \times 10^{-4}$ . Notice that there is no singularity in the power series and it can be extrapolated all the way down to the surface where  $r = r_E$ .

By using the least-squares fitting routine in Mathcad15, the following values for the coefficients were found to give an excellent fit of  $PS'(r)$  to the volume integral  $TI(r)$

$$\left. \begin{aligned} C_0 = 0.50889, & \quad C_2 = 0.13931 \\ C_4 = 0.01013, & \quad C_6 = 0.14671 \end{aligned} \right\}. \quad (3.27)$$

The quality of the fit using these coefficients is shown in Fig. 3. The maximum relative difference at the values for  $r$  shown by + in Fig. 3 is less than  $2 \times 10^{-4}$ .

The solution for  $g_{trt}(\theta)$  can now be rewritten with  $PS(r)$  as follows

$$g_{trt}(\theta) = -G \frac{I_E v_{eq}}{r_E^4 c_g} \left( \frac{\Omega_\phi(\theta) - \Omega_E}{\Omega_E} \right) \cos^2(\lambda(\theta)) PS(r(\theta)). \quad (3.28)$$

The radial component  $g_r$  satisfies the conservation laws, but the relatively small transverse component  $g_{trt}$  does not satisfy the conservation laws. Because the strength of  $g_{trt} \sim 10^{-6} g_r$ , a good first approximation is obtained by applying the conservation laws.

Let  $L$  be the magnitude for the spacecraft's orbital angular momentum. Then

$$\begin{aligned} \text{constant} = L &= m v_p r_p = m (r_p \Omega_p) r_p = \\ &= m r_p^2 \Omega_p = m r(\theta)^2 \Omega_\theta(\theta). \end{aligned}$$

Here  $r_p$  is the value for  $r$  at perigee,  $v_p$  is the orbital speed at perigee, and  $\Omega_p$  is the orbital angular speed at perigee. Therefore, by conservation of orbital angular momentum, the formula for the spacecraft's orbital angular speed becomes

$$\Omega_\theta(\theta) \equiv \frac{d\theta}{dt} = \frac{r_p v_p}{r(\theta)^2} = \frac{r_p^2}{r(\theta)^2} \Omega_p. \quad (3.29)$$

Let  $E$  be the spacecraft's orbital kinetic energy plus the orbital potential energy. Then

$$\begin{aligned} \text{constant} = E &= \frac{1}{2} m v(\theta)^2 - \frac{GM_E m}{r(\theta)} = \\ &= \frac{1}{2} m v_\infty^2 + \frac{1}{2} m v_p^2 - \frac{GM_E m}{r_p}. \end{aligned}$$

Here  $v_\infty$  is the spacecraft's speed as  $r \rightarrow \infty$ . Therefore, by conservation of energy, the formula for the orbital speed becomes

$$v(\theta) = \sqrt{v_\infty^2 + v_p^2 + 2 \frac{GM_E}{r(\theta)} - 2 \frac{GM_E}{r_p}}. \quad (3.30)$$

Let  $(x, y, z)$  be the rectangular coordinates for an inertial frame with the origin at the center of the sphere and with the  $(x, y)$  plane coinciding with the plane of the orbit. Let  $\theta_p$  be the angle which rotates the  $(x, y)$  plane so that perigee occurs at  $\theta = \theta_p$ . The formulas for  $r_x$  and  $r_y$  are

$$\left. \begin{aligned} r_x(\theta) &= r(\theta) \cos(\theta - \theta_p) \\ r_y(\theta) &= r(\theta) \sin(\theta - \theta_p) \end{aligned} \right\}. \quad (3.31)$$

The formulas for  $v_x$  and  $v_y$  are

$$\left. \begin{aligned} v_x &= \frac{dr_x}{dt} = v_r \cos(\theta - \theta_p) - r \Omega_\theta \sin(\theta - \theta_p) \\ v_y &= \frac{dr_y}{dt} = v_r \sin(\theta - \theta_p) + r \Omega_\theta \cos(\theta - \theta_p) \end{aligned} \right\}. \quad (3.32)$$

The radial component  $v_r$  is given by (3.6) and (3.11)

$$\begin{aligned} v_r &= \Omega_\theta \frac{dr}{d\theta} = \Omega_\theta \frac{r(\theta)^2}{r_p} \frac{\varepsilon}{1 + \varepsilon} \sin \theta, \\ r(\theta) &= \frac{r_p(1 + \varepsilon)}{1 + \varepsilon \cos \theta}. \end{aligned}$$

Let  $\alpha_{\text{eq}}$  be the inclination of the orbital plane to the equatorial plane, and let  $\lambda_{\text{p}}$  be the geocentric latitude for perigee. If  $\theta_{\text{p}} = 0$ , let the  $x$ -axis of the orbital frame coincide with the  $X$ -axis of the equatorial plane. Then the formulas for the transformation from the  $(x, y, z)$  orbital frame to the  $(X, Y, Z)$  equatorial frame are

$$\left. \begin{aligned} r_X &= r(\theta) \cos(\theta - \theta_{\text{p}}) \\ r_Y &= r(\theta) \cos \alpha_{\text{eq}} \sin(\theta - \theta_{\text{p}}) \\ r_Z &= -r(\theta) \sin \alpha_{\text{eq}} \sin(\theta - \theta_{\text{p}}) \end{aligned} \right\}. \quad (3.33)$$

The formulas for the velocity components in the  $(X, Y, Z)$  frame are

$$\left. \begin{aligned} v_X &= \frac{dr_X}{dt} = v_r \cos(\theta - \theta_{\text{p}}) - r \Omega_{\theta} \sin(\theta - \theta_{\text{p}}) \\ v_Y &= \frac{dr_Y}{dt} = v_r \cos \alpha_{\text{eq}} \sin(\theta - \theta_{\text{p}}) + r \Omega_{\theta} \cos \alpha_{\text{eq}} \cos(\theta - \theta_{\text{p}}) \\ v_Z &= \frac{dr_Z}{dt} = -v_r \sin \alpha_{\text{eq}} \sin(\theta - \theta_{\text{p}}) - r \Omega_{\theta} \sin \alpha_{\text{eq}} \cos(\theta - \theta_{\text{p}}) \end{aligned} \right\}. \quad (3.34)$$

Let  $r_{\phi}(\theta)$  be the geocentric radial distance to the projection of the field-point onto the  $(X, Y)$  equatorial plane, and let  $r_{\lambda}(\theta)$  be the geocentric radial distance to the projection of the field point onto a vertical  $(X, Z)$  plane. Then

$$r_{\phi}(\theta) = \sqrt{r_X(\theta)^2 + r_Y(\theta)^2}, \quad (3.35)$$

and

$$r_{\lambda}(\theta) = \sqrt{r_X(\theta)^2 + r_Z(\theta)^2}. \quad (3.36)$$

Let  $v_{\phi}$  be the speed of the projection of the field-point onto the  $(X, Y)$  equatorial plane, and let  $v_{\lambda}$  be the speed of the projection of the field point onto a vertical  $(X, Z)$  plane. Then

$$v_{\phi}(\theta) = \sqrt{v_X(\theta)^2 + v_Y(\theta)^2}, \quad (3.37)$$

and

$$v_{\lambda}(\theta) = \sqrt{v_X(\theta)^2 + v_Z(\theta)^2}. \quad (3.38)$$

The formula for the tangent of the azimuthal angle  $\phi$  is

$$\tan \phi(\theta) = \frac{r_Y(\theta)}{\sqrt{r_X(\theta)^2 + r_Z(\theta)^2}}.$$

Solving for  $\phi$  gives

$$\phi(\theta) = \tan^{-1} \left( \frac{r_Y(\theta)}{\sqrt{r_X(\theta)^2 + r_Z(\theta)^2}} \right). \quad (3.39)$$

The formula for the tangent of the geocentric latitude is

$$\tan \lambda(\theta) = \frac{r_Z(\theta)}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}}.$$

Solving for  $\lambda$  gives

$$\lambda(\theta) = \tan^{-1} \left( \frac{r_Z(\theta)}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}} \right). \quad (3.40)$$

The formula for  $\Omega_\phi$  becomes

$$\Omega_\phi(\theta) = \Omega_\theta(\theta) \frac{d\phi}{d\theta} = \pm \frac{v_\phi(\theta)}{r_\phi(\theta)}. \quad (3.41)$$

Use the + sign if  $\alpha_{\text{eq}}$  takes numerical values in the range  $0^\circ < \alpha_{\text{eq}} < 90^\circ$ , and the - sign if  $90^\circ < \alpha_{\text{eq}} < 180^\circ$ .

The value for the angle  $\theta_p$  depends on the latitude for perigee  $\lambda_p$ , which ranges from  $-90^\circ$  to  $+90^\circ$ , and  $\alpha_{\text{eq}}$ , which ranges from  $0^\circ$  to  $+180^\circ$ . If  $\alpha_{\text{eq}} = 0^\circ$  or  $180^\circ$ , then  $\theta_p = 0^\circ$ . If  $\alpha_{\text{eq}} = 90^\circ$ , then  $\theta_p = \lambda_p$ . If  $0 < \alpha_{\text{eq}} < \pi$  radians and  $\alpha_{\text{eq}} \neq \frac{\pi}{2}$  and  $\sin \lambda_p \leq \sin \alpha_{\text{eq}}$ , the formula for  $\theta_p$  (the angle is taken here in radians) is

$$\theta_p = \sin^{-1} \left( \frac{\sin \lambda_p}{\sin \alpha_{\text{eq}}} \right). \quad (3.42)$$

If  $\sin \lambda_p > \sin \alpha_{\text{eq}}$ , the inverse sine function is shifted from the primary branch and the value for  $\theta_p$  is greater than  $90^\circ$ . Of the six flybys reported by Anderson *et al.* [10], only the MESSENGER flyby has a value for  $\theta_p$  that is greater than  $90^\circ$ ; (from Appendix A)  $\alpha_{\text{eq}} = 133.1^\circ$ ,  $\lambda_p = 46.95^\circ$ , which gives  $\theta_p = 90.0467^\circ$ .

We need a method to determine the numerical values for the minimum and maximum permissible values for  $\theta$ , designated  $\theta_{\text{min}}$  and  $\theta_{\text{max}}$ . One method is to solve  $r(\theta)$  (3.6) for the value for  $\theta$  which causes the denominator to be zero. Let  $\theta_\infty$  be that value

$$\theta_\infty = \cos^{-1} \left( \frac{-1}{\varepsilon} \right). \quad (3.43)$$



Consequently, to avoid integration over the singularity in  $r(\theta)$ , the value for  $\theta_{\min}$  must be greater than  $-\theta_\infty$  and the value for  $\theta_{\max}$  must be less than  $+\theta_\infty$ .

Let  $t_{\text{in}}(\theta_{\text{in}})$  be the (negative) time for the start of inbound data accumulation before  $t = 0$  at  $\theta = 0$  at perigee, and let  $t_{\text{out}}(\theta_{\text{out}})$  be the (positive) time for the end of outbound data accumulation after  $t = 0$  at  $\theta = 0$  at perigee.

The formulas for calculating  $t_{\text{in}}$  and  $t_{\text{out}}$  are

$$\begin{aligned} t_{\text{in}} &= \int_{t(0)}^{t(\theta_{\text{in}})} dt = \int_0^{\theta_{\text{in}}} \frac{dt}{d\theta} d\theta = \int_0^{\theta_{\text{in}}} \frac{1}{\Omega_\theta(\theta)} d\theta = \\ &= \int_0^{\theta_{\text{in}}} \frac{r(\theta)^2}{v_p r_p} d\theta \longrightarrow -\infty \text{ if } \theta_{\text{in}} = -\theta_\infty, \end{aligned}$$

and

$$\begin{aligned} t_{\text{out}} &= \int_{t(0)}^{t(\theta_{\text{out}})} dt = \int_0^{\theta_{\text{out}}} \frac{dt}{d\theta} d\theta = \int_0^{\theta_{\text{out}}} \frac{1}{\Omega_\theta(\theta)} d\theta = \\ &= \int_0^{\theta_{\text{out}}} \frac{r(\theta)^2}{v_p r_p} d\theta \longrightarrow +\infty \text{ if } \theta_{\text{out}} = +\theta_\infty. \end{aligned} \quad (3.44)$$

Numerical values for  $t_{\text{in}}$  and  $t_{\text{out}}$  were included in the report of Anderson *et al.* [10] only for the NEAR flyby (see Appendix A).

Let  $a$  and  $b$  be the semimajor and semiminor axes for an elliptical (closed) orbit ( $0 \leq \varepsilon < 1$ ). Kepler's laws give the orbital angular speed in terms of  $a$ ,  $b$ , and the period  $P$  [19]

$$\left. \begin{aligned} a &= \frac{1}{2}(r_a + r_p) && \text{(semimajor axis)} \\ b &= a\sqrt{1 - \varepsilon^2} && \text{(semiminor axis)} \\ P &= \frac{a^{3/2}}{\sqrt{GM_E}} && \text{(Kepler's 3rd law)} \\ \Omega_\theta(\theta) &= \frac{2\pi}{P} \frac{ab}{r(\theta)^2} && \text{(Kepler's 2nd law)} \end{aligned} \right\}, \quad (3.45)$$

where  $r_a$  is the geocentric radial distance at apogee.

The equivalent circular orbit for an elliptical orbit will be needed. Let  $r_{\text{co}}$ ,  $v_{\text{co}}$  and  $\Omega_{\text{co}}$  be the radius, orbital speed, and orbital angular speed for an equivalent circular orbit which has the period  $P$ , respectively. The formulas for  $r_{\text{co}}$ ,  $v_{\text{co}}$ , and  $\Omega_{\text{co}}$  can be found by rearranging

Kepler's 3rd law

$$\left. \begin{aligned} P &= \frac{a}{\sqrt{GM_E}} \sqrt{a} = \frac{a}{\sqrt{GM_E/a}} = \frac{r_{\text{co}}}{v_{\text{co}}} = \frac{1}{\Omega_{\text{co}}} \\ v_{\text{co}} &= \sqrt{\frac{GM_E}{a}} = r_{\text{co}} \Omega_{\text{co}} \\ r_{\text{co}} &= a, \quad \Omega_{\text{co}} = \frac{v_{\text{co}}}{a} \end{aligned} \right\}. \quad (3.46)$$

The formulas for  $\lambda(\theta)$  and  $\Omega_\phi(\theta)$  are greatly simplified for elliptical orbits if  $\alpha_{\text{eq}} = \lambda_p$ . In this case, continue to use (3.6) for  $r(\theta)$ , and use (3.45) for  $\Omega_\theta(\theta)$ . Then use the following formulas for  $\lambda(\theta)$ ,  $\Omega_\phi(\theta)$ , and  $r_\lambda(\theta)$

$$\left. \begin{aligned} \lambda(\theta) &= \tan^{-1}(\tan \alpha_{\text{eq}} \cos \theta) \\ \Omega_\phi(\theta) &= \Omega_\theta(\theta) \cos \alpha_{\text{eq}} \\ r_\lambda(\theta) &= r(\theta) \cos \theta \end{aligned} \right\}. \quad (3.47)$$

For closed orbits, the value for  $\theta_{\text{min}} = -\pi$  radians and the value for  $\theta_{\text{max}} = +\pi$  radians.

Let  $\mathbf{F}_\lambda$  be an induction-like field, and let the  $\phi$ -component of the curl of  $\mathbf{F}_\lambda$  equal  $-k dg_{\text{trt}}/dt$ , where  $k$  is a constant. The formula for the curl operator in spherical coordinates can be found in J. D. Jackson's textbook [1]

$$\nabla \times \mathbf{F}_\lambda = \mathbf{e}_\phi \frac{1}{r} \frac{\partial}{\partial r} (r F_\lambda) = -k \frac{dg_{\text{trt}}}{dt} = \mathbf{e}_\phi k \frac{dg_{\text{trt}}}{dt}, \quad (3.48)$$

where  $\mathbf{e}_\phi$  is a unit vector directed towards the east. Solving for  $\partial(r F_\lambda)/\partial r$  and integrating both sides from  $t(0)$  to  $t(\theta)$  gives

$$\begin{aligned} \int_{t(0)}^{t(\theta)} \frac{\partial}{\partial r} (r F_\lambda) dt &= \int_0^\theta \frac{\partial}{\partial r} (r F_\lambda) \frac{dt}{d\theta} d\theta = \int_0^\theta \frac{d}{d\theta} (r F_\lambda) \frac{d\theta}{dr} \frac{dt}{d\theta} d\theta = \\ &= k \int_{t(0)}^{t(\theta)} r \frac{dg_{\text{trt}}}{dt} dt = k \int_0^\theta r \frac{dg_{\text{trt}}}{d\theta} d\theta. \end{aligned} \quad (3.49)$$

Therefore,

$$\int_0^\theta \left( \frac{d}{d\theta} (r F_\lambda) \frac{d\theta}{dr} \frac{dt}{d\theta} - k r \frac{dg_{\text{trt}}}{d\theta} \right) d\theta = 0. \quad (3.50)$$

This equation is satisfied for all values of  $\theta$ , if and only if,

$$\frac{d}{d\theta} (r F_\lambda) = k r \frac{dr}{d\theta} \frac{d\theta}{dt} \frac{dg_{\text{trt}}}{d\theta}. \quad (3.51)$$

Integrating both sides from 0 to  $\theta$  gives

$$F_\lambda(\theta) = \frac{k}{r(\theta)} \int_0^\theta r(\theta) \Omega_\theta(\theta) \frac{dr}{d\theta} \frac{dg_{\text{trt}}}{d\theta} d\theta. \quad (3.52)$$

Units for  $F_\lambda$  are the units for acceleration,  $\text{m/s}^2$ . The constant  $k$  has units of  $(\text{m/s})^{-1}$ . Let  $v_k$  be the reciprocal of  $k$ ,  $v_k \equiv 1/k$ , which will be called the “induction speed”. Regard  $v_k$  as an adjustable parameter for each case, and regard the average for all cases as a fixed parameter for the neoclassical causal theory. The formula for  $F_\lambda$  can be rewritten in terms of  $v_k$  and  $v_{\text{eq}}$

$$F_\lambda(\theta) = \frac{v_{\text{eq}}}{v_k} \frac{r_E}{r(\theta)} \int_0^\theta \frac{r(\theta)}{r_E} \frac{\Omega_\theta(\theta)}{\Omega_E} \frac{1}{r_E} \frac{dr}{d\theta} \frac{dg_{\text{trt}}}{d\theta} d\theta. \quad (3.53)$$

Let  $\delta v/v_{\text{in}} \ll 1$  be the relative change in the magnitude of  $\mathbf{v}$  due to  $\mathbf{F}_\lambda$ , where  $v_{\text{in}}$  is the initial speed. The dot product  $\mathbf{v} \cdot \mathbf{F}_\lambda$  gives the time rate at which the orbital energy is changed. Therefore,

$$\begin{aligned} \left(1 + \frac{\delta v}{v_{\text{in}}}\right)^2 &\cong 1 + 2 \frac{\delta v}{v_{\text{in}}} = \\ &= 1 + \frac{1}{v_{\text{in}}^2} \int_{t(0)}^{t(\theta)} F_\lambda v_\lambda dt = \\ &= 1 + \frac{1}{v_{\text{in}}^2} \int_0^\theta r_\lambda F_\lambda \frac{d\lambda}{d\theta} d\theta. \end{aligned} \quad (3.54)$$

Therefore,

$$\delta v(\theta) = \frac{v_{\text{in}}}{2} \int_0^\theta \frac{r_\lambda(\theta) F_\lambda(\theta)}{v_{\text{in}}^2} \frac{d\lambda}{d\theta} d\theta. \quad (3.55)$$

As previously defined,  $\theta_{\text{min}}$  and  $\theta_{\text{max}}$  are the minimum and maximum values for  $\theta$ . Let  $\delta v_{\text{trt}}$  be the speed-change for a flyby or for one revolution. Then

$$\delta v_{\text{in}} = \delta v(\theta_{\text{min}}), \quad \delta v_{\text{out}} = \delta v(\theta_{\text{max}}), \quad \delta v_{\text{trt}} = \delta v_{\text{in}} + \delta v_{\text{out}}. \quad (3.56)$$

These formulas will be used in §4 to calculate the speed-changes listed in Table 1.

**§4. Calculated speed-changes for six Earth flybys caused by the neoclassical causal version of Newton’s theory.** The trajectory parameters given by Anderson *et al.* [10] are listed in Appendix A.

The parameters for the NEAR spacecraft flyby will be used for the following example calculation. The same method will be applied to derive the time-retarded speed-change for each of the remaining five flybys.

Numerical values for the Earth's parameters and the Earth's radial mass-density distribution are given in Appendix B.

As previously defined,  $r(\theta)$  is the geocentric radial distance to the spacecraft in the plane of the trajectory. The formulas for  $r(\theta)$  and its derivative (3.6) are

$$\left. \begin{aligned} r(\theta) &= \frac{r_p(1+\varepsilon)}{1+\varepsilon\cos\theta} \\ \frac{dr}{d\theta} &= \frac{r(\theta)^2}{r_p} \frac{\varepsilon}{1+\varepsilon} \sin\theta \end{aligned} \right\}, \quad (4.1)$$

where  $\theta$  is the parametric polar coordinate angle,  $\varepsilon$  is the eccentricity for the hyperbolic trajectory, and  $r_p$  is the geocentric radial distance to the spacecraft at perigee (at  $\theta=0$ ).

The following method will be used to find  $\varepsilon$ . The asymptotic angle  $\alpha_{\text{asm}}$ , Fig. 4, depends on the deflection angle,

$$\alpha_{\text{asm}} = \frac{1}{2}(180^\circ - DA) = \frac{1}{2}(180^\circ - 66.9^\circ) = 56.55^\circ. \quad (4.2)$$

The radial distance at perigee,  $r_p$ , depends on the altitude at perigee  $h_p$ , through  $r_E$ , as follows

$$r_p = r_E + h_p = r_E + 539 \text{ km} = 1.0846 r_E. \quad (4.3)$$

The impact parameter  $FP$  is given by conservation of angular momentum,

$$FP v_\infty = r_p v_p, \quad (4.4)$$

where  $v_\infty$  and  $v_p$  are values listed in Appendix A and  $r_p$  is given by (4.3). Given numerical values for the NEAR flyby are

$$v_\infty = 6.851 \text{ km/s}, \quad v_p = 12.739 \text{ km/s}.$$

The numerical value for  $FP$  becomes

$$FP = r_p \frac{v_p}{v_\infty} = 2.0167 r_E. \quad (4.5)$$

The ratio  $FP/OF = \sin \alpha_{\text{asm}}$ . Therefore,

$$OF = \frac{FP}{\sin \alpha_{\text{asm}}} = 2.4171 r_E. \quad (4.6)$$

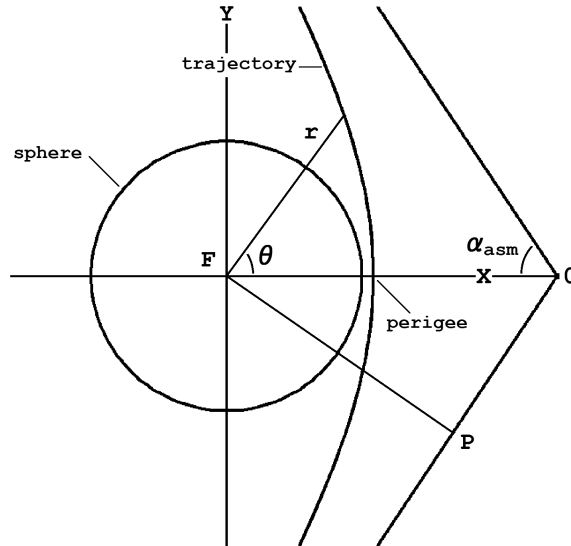


Fig. 4: Hyperbolic trajectory for the NEAR spacecraft flyby in the  $(x, y)$  trajectory plane for a central sphere of radius  $r_E$  (using (3.31) with  $\theta_p = 0$ ). The geocentric radial distance to the spacecraft is  $r(\theta)$  at the parametric angle  $\theta$ . The least geocentric distance  $r_p$  is at perigee. The asymptote angle  $\alpha_{asm}$  is defined by the deflection angle. The center of the sphere is at the focus  $F$ . The impact parameter is the distance  $FP$ . Another trajectory parameter is the distance  $OF$ .

The parameter  $a$  is the distance  $OF - r_p$ ,

$$a = OF - r_p = 1.3325 r_E. \quad (4.7)$$

The parameter  $b$  depends on the asymptotic angle  $\alpha_{asm}$ ,

$$b = a \tan \alpha_{asm} = 2.0170 r_E. \quad (4.8)$$

The eccentricity  $\varepsilon$  depends on  $a$  and  $b$ ,

$$\varepsilon = \frac{\sqrt{a^2 + b^2}}{a} = 1.8142. \quad (4.9)$$

This gives the numerical value for  $\varepsilon$  to be used in  $r(\theta)$  (4.1), which is the geocentric radial distance to the spacecraft in the plane of the trajectory.

The value for  $\theta_\infty$ , (3.43), for the NEAR flyby is

$$\theta_\infty = \cos^{-1} \left( \frac{-1}{\varepsilon} \right) = 123.45^\circ. \quad (4.10)$$

Notice that  $180^\circ - \alpha_{\text{asm}}$  also equals  $\theta_\infty$ .

For the NEAR spacecraft flyby,  $\alpha_{\text{eq}} = 108.0^\circ$  and  $\lambda_p = 33.0^\circ$ , which from (3.42) gives

$$\theta_p = \sin^{-1} \left( \frac{\sin \lambda_p}{\sin \alpha_{\text{eq}}} \right) = 34.9364^\circ. \quad (4.11)$$

Numerical values for  $r_p$ ,  $\alpha_{\text{asm}}$ ,  $\varepsilon$ ,  $\theta_\infty$ , and  $\theta_p$  for each of the six flybys reported by Anderson *et al.* are listed in Table 3.

The formula for  $\lambda(\theta)$  is given by (3.40),

$$\lambda(\theta) = \tan^{-1} \left( \frac{r_Z(\theta)}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}} \right), \quad (4.12)$$

where  $r_X$ ,  $r_Y$ , and  $r_Z$  are given by (3.33),

$$\left. \begin{aligned} r_X(\theta) &= r(\theta) \cos(\theta - \theta_p) \\ r_Y(\theta) &= r(\theta) \cos \alpha_{\text{eq}} \sin(\theta - \theta_p) \\ r_Z(\theta) &= -r(\theta) \sin \alpha_{\text{eq}} \sin(\theta - \theta_p) \end{aligned} \right\}. \quad (4.13)$$

Table 4 compares the listed inbound and outbound asymptotic latitudes from Appendix A,  $\lambda_{\text{in}}$  and  $\lambda_{\text{out}}$ , with the calculated asymptotic latitudes  $\lambda(-0.9999\theta_\infty)$  and  $\lambda(+0.9999\theta_\infty)$  using (4.12). This table shows that some of the listed latitudes are inconsistent with the maximum and minimum permissible calculated values.

The starting and ending times for the NEAR flyby are known from Appendix A to be  $-88.4$  hours and  $+95.6$  hours. The calculated values for  $t_{\text{in}}$  and  $t_{\text{out}}$  are given by (3.44)

$$\left. \begin{aligned} &\text{if } \theta_{\text{in}} = -0.9973\theta_\infty = -123.1192^\circ \\ t_{\text{in}} &= \int_0^{\theta_{\text{in}}} \frac{r(\theta)^2}{v_p r_p} d\theta = -87.75 \text{ hours} \\ &\text{if } \theta_{\text{out}} = +0.9975\theta_\infty = +123.1437^\circ \\ t_{\text{out}} &= \int_0^{\theta_{\text{out}}} \frac{r(\theta)^2}{v_p r_p} d\theta = +94.88 \text{ hours} \end{aligned} \right\}. \quad (4.14)$$

This shows that known values for  $t_{\text{in}}$  and  $t_{\text{out}}$  can be used to calculate precise values for  $\theta_{\text{in}}$  and  $\theta_{\text{out}}$ . Values for  $t_{\text{in}}$  and  $t_{\text{out}}$  for the other flybys were not listed, so another method is used herein to get estimated values

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
$r_p/r_E$	1.0846	1.1507	1.3070	1.3684	1.1844	1.0476
$\alpha_{\text{asm}}$	$56.55^\circ$	$66.15^\circ$	$40.35^\circ$	$42.65^\circ$	$80.15^\circ$	$64.45^\circ$
$\varepsilon$	1.8142	2.4731	1.3122	1.3596	5.8456	2.3186
$\theta_\infty$	$123.45^\circ$	$113.85^\circ$	$139.65^\circ$	$137.35^\circ$	$99.85^\circ$	$115.55^\circ$
$\theta_p$	$+34.9364^\circ$	$+44.8989^\circ$	$+36.9067^\circ$	$+90.0468^\circ$	$-68.3765^\circ$	$-57.4444^\circ$

Table 3: Trajectory parameter values for each of the six Earth flybys reported by Anderson *et al.* [10]. The ratio  $r_p/r_E$  is the geocentric radial distance at perigee relative to the Earth's radius,  $\alpha_{\text{asm}}$  is the angle for the asymptotes (see Fig. 4),  $\varepsilon$  is the eccentricity for the trajectory,  $\theta_\infty$  is the value for  $\theta$  which makes  $r(\theta_\infty)$  go to infinity, and  $\theta_p$  is the value for  $\theta$  which rotates the  $(x, y)$  orbital plane so that the latitude for perigee equals the value for  $\lambda_p$  listed in Appendix A.

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
$\lambda_{\text{in}}$	$+20.76^\circ?$	$+12.52^\circ$	$+2.81^\circ?$	$-31.44^\circ$	$+12.92^\circ$	$+34.26^\circ?$
$\lambda(-0.9999\theta_\infty)$	$+20.52^\circ$	$+12.63^\circ$	$+1.99^\circ$	$-32.50^\circ$	$+12.94^\circ$	$+34.08^\circ$
$\lambda_{\text{out}}$	$-71.96^\circ?$	$-34.15^\circ$	$-34.29^\circ?$	$-31.92^\circ$	$-4.99^\circ$	$-4.87^\circ?$
$\lambda(+0.9999\theta_\infty)$	$-71.94^\circ$	$-34.26^\circ$	$-34.12^\circ$	$-32.45^\circ$	$-5.02^\circ$	$-4.62^\circ$

Table 4: Comparison of the listed asymptotic inbound and outbound geocentric latitudes,  $\lambda_{\text{in}}$  and  $\lambda_{\text{out}}$  (from Appendix A) with the calculated latitudes,  $\lambda(-0.9999\theta_\infty)$  and  $\lambda(+0.9999\theta_\infty)$  by using (4.12). Cases where the listed latitude is incompatible with the calculated latitude are marked with “?”.

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
$\theta_{\text{in}}/\theta_{\infty}$	-0.9973	-0.998	-0.983	-0.993	-0.999	-0.998
$\theta_{\text{out}}/\theta_{\infty}$	+0.9975	+0.998	+0.993	+0.994	+0.999	+0.998
$t_{\text{in}}$ (hours)	-88	-87	-88	-85	-89	-81
$t_{\text{out}}$ (hours)	+95	+87	+88	+100	+89	+81
$r_{\text{in}}/r_{\text{E}}$	346	444	206	207	808	412
$r_{\text{out}}/r_{\text{E}}$	374	444	206	241	808	412
$v_{\text{in}}/v_{\infty}$	+1.416	+1.415	-1.426	+1.427	+1.414	-1.415

Table 5: The above values for  $\theta_{\text{in}}$  and  $\theta_{\text{out}}$  for the NEAR flyby are based on listed values for  $t_{\text{in}} = 88.4$  hours and  $t_{\text{out}} = 95.6$  hours. The values for  $\theta_{\text{in}}$  and  $\theta_{\text{out}}$  for the other flybys are rough estimates by using plausible values for  $t_{\text{in}}$  and  $t_{\text{out}}$ . The last three rows list the corresponding ratios for  $r_{\text{in}}/r_{\text{E}}$ ,  $r_{\text{out}}/r_{\text{E}}$ , and  $v_{\text{in}}/v_{\infty}$ . The curious required sign reversal for  $v_{\text{in}}$  for the Rosetta and GLL-II flybys may be a manifestation of the covariance and contravariance of vectors [20].

for  $\theta_{\text{in}}$  and  $\theta_{\text{out}}$ . Table 5 lists the NEAR values and the estimated values for  $\theta_{\text{in}}$  and  $\theta_{\text{out}}$ , corresponding values for  $t_{\text{in}}$  and  $t_{\text{out}}$ , and corresponding values for  $r_{\text{in}}/r_{\text{E}}$ ,  $r_{\text{out}}/r_{\text{E}}$ , and  $v_{\text{in}}/v_{\infty}$ , where  $r_{\text{in}} = r(\theta_{\text{min}})$ ,  $r_{\text{out}} = r(\theta_{\text{out}})$ , and  $v_{\text{in}} = v(\theta_{\text{min}})$  by using (3.30). The required sign change for  $v_{\text{in}}$  for the Rosetta and GLL-II flybys may be a manifestation of the covariance and contravariance of vectors [20].

The formula for  $\Omega_{\theta}$  is given by (3.29)

$$\Omega_{\theta}(\theta) = \frac{r_{\text{P}} v_{\text{P}}}{r(\theta)^2}. \quad (4.15)$$

The formula for  $\Omega_{\phi}(\theta)$  is given by (3.41)

$$\Omega_{\phi}(\theta) = \pm \frac{v_{\phi}(\theta)}{r_{\phi}(\theta)} = \pm \frac{\sqrt{v_X(\theta)^2 + v_Y(\theta)^2}}{\sqrt{r_X(\theta)^2 + r_Y(\theta)^2}}, \quad (4.16)$$

where the  $X$  and  $Y$  components of  $r$  are given by (4.13). The  $X$ ,  $Y$ , and  $Z$  components of  $v$ , given by (3.34), are

$$\left. \begin{aligned} v_X(\theta) &= v_r \cos(\theta - \theta_{\text{P}}) - r(\theta) \Omega_{\theta}(\theta) \sin(\theta - \theta_{\text{P}}) \\ v_Y(\theta) &= v_r \cos \alpha_{\text{eq}} \sin(\theta - \theta_{\text{P}}) + r(\theta) \Omega_{\theta}(\theta) \cos \alpha_{\text{eq}} \cos(\theta - \theta_{\text{P}}) \\ v_Z(\theta) &= -v_r \sin \alpha_{\text{eq}} \sin(\theta - \theta_{\text{P}}) - r(\theta) \Omega_{\theta}(\theta) \sin \alpha_{\text{eq}} \cos(\theta - \theta_{\text{P}}) \end{aligned} \right\}. \quad (4.17)$$



The formula for  $v_r$ , given by (3.6) and (3.11), is

$$v_r(\theta) = \Omega_\theta(\theta) \frac{dr}{d\theta} = \Omega_\theta(\theta) \frac{r(\theta)^2}{r_p} \frac{\varepsilon}{1 + \varepsilon} \sin \theta. \quad (4.18)$$

A graph of  $\Omega_\phi$  relative to  $\Omega_E$ , Fig. 5, shows that this component of the angular speed is negative (retrograde) with a minimum value of about 50 times the Earth's angular speed.

The formula for the time-retarded transverse gravitational field is given by (3.28)

$$g_{\text{trt}}(\theta) = -G \frac{I_E v_{\text{eq}}}{r_E^4 c_g} \left( \frac{\Omega_\phi(\theta) - \Omega_E}{\Omega_E} \right) \cos^2(\lambda(\theta)) PS(r(\theta)). \quad (4.19)$$

Numerical values for  $G$ ,  $I_E$ ,  $r_E$ ,  $\Omega_E$ , and  $v_{\text{eq}}$  are listed in Appendix B. To start, let's assume that  $c_g = 1.000 c$  for the NEAR flyby. The formula for  $\Omega_\phi$  is given by (4.16). The formula for  $\lambda$  is given by (4.12). The formula for  $PS(r)$  is given by (3.26)

$$PS(r) \equiv \left( \frac{r_E}{r} \right)^3 \left( C_0 + C_2 \left( \frac{r_E}{r} \right)^2 + C_4 \left( \frac{r_E}{r} \right)^4 + C_6 \left( \frac{r_E}{r} \right)^6 \right). \quad (4.20)$$

Numerical values for the coefficients are given by (3.27)

$$\left. \begin{array}{ll} C_0 = 0.50889, & C_2 = 0.13931 \\ C_4 = 0.01013, & C_6 = 0.14671 \end{array} \right\}. \quad (4.21)$$

The formula for  $r(\theta)$  is given by (4.1).

A graph of  $g_{\text{trt}}(\theta)$  versus  $\theta$  with  $c_g = 1.000 c$ , Fig. 6, shows that the transverse field rises from zero to a sharp peak near  $\theta = 0^\circ$ , then decreases to zero. An expanded view near the peak, Fig. 7, shows a significant difference in the peak values for  $c_g = 1.000 c$  and  $c_g = 1.060 c$ .

The formula for  $F_\lambda(\theta)$  given by (3.53) is

$$F_\lambda(\theta) = \frac{v_{\text{eq}} r_E}{v_k r(\theta)} \int_0^\theta \frac{r(\theta)}{r_E} \frac{\Omega_\theta(\theta)}{\Omega_E} \frac{1}{r_E} \frac{dr}{d\theta} \frac{dg_{\text{trt}}}{d\theta} d\theta, \quad (4.22)$$

where  $v_k$  is the induction speed (an adjustable parameter),  $r(\theta)$  and  $dr/d\theta$  are given by (4.1),  $\Omega_\theta(\theta)$  is given by (4.15), and  $dg_{\text{trt}}/d\theta$  is found by using the numerical differentiation algorithm in Mathcad15 for the derivative of the time-retarded transverse field  $g_{\text{trt}}$  given by (4.19).

A couple of trial calculations indicated that for the NEAR flyby  $v_k = 6.530 v_{\text{eq}}$  gives a speed-change that agrees exactly with the observed

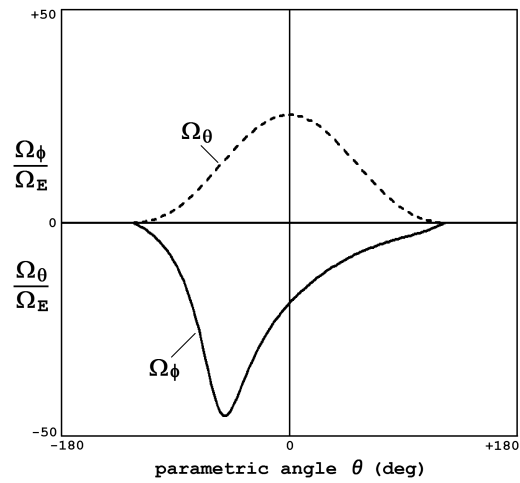


Fig. 5: Graph of the ratio  $\Omega_\phi/\Omega_E$  versus  $\theta$  for the NEAR flyby. The relative angular speed is negative (retrograde) because the inclination  $\alpha_{\text{eq}} = 108.0^\circ > 90^\circ$ . The ratio  $\Omega_\theta/\Omega_E$  is shown for reference.

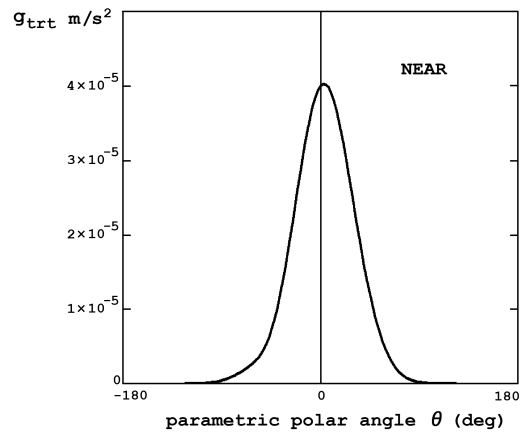


Fig. 6: Time-retarded transverse gravitational field  $g_{\text{trt}}(\theta)$  versus  $\theta$  for the NEAR flyby using  $c_g = 1.000 c$ .

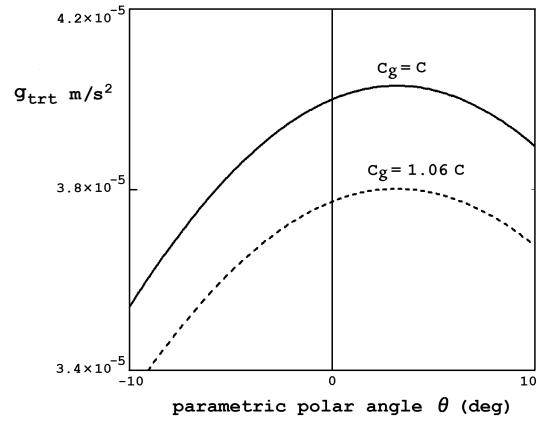


Fig. 7: Expanded view near the peak for  $g_{\text{trt}}(\theta)$  versus  $\theta$  for the NEAR flyby using  $c_g = 1.000 c$  and  $c_g = 1.060 c$ . There is a 6% difference in the peak values.

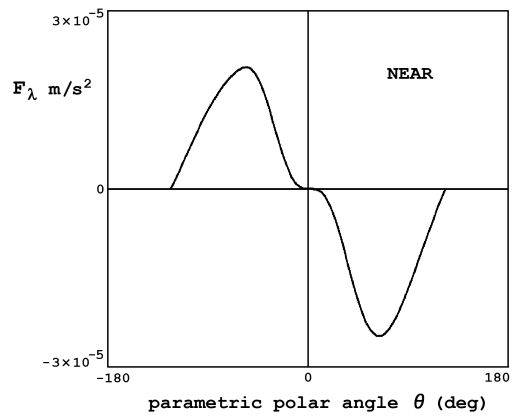


Fig. 8: Graph of the induction field  $F_\lambda$  versus  $\theta$  for the NEAR flyby with  $v_k/v_{\text{eq}} = 6.530$ . During the inbound there is a positive peak and during the outbound there is a slightly stronger negative peak.

speed-change. A graph of  $F_\lambda$  versus  $\theta$ , Fig. 8, shows a positive peak during the inbound and a slightly stronger negative peak during the outbound.

The calculated speed-change is given by (3.55)

$$\delta v(\theta) = \frac{v_{\text{in}}}{2} \int_0^\theta \frac{r_\lambda(\theta) F_\lambda(\theta)}{v_{\text{in}}^2} \frac{d\lambda}{d\theta} d\theta, \quad (4.23)$$

where  $v_{\text{in}} = v(\theta_{\text{min}}) = 1.416 v_\infty$  (from Table 5),  $r_\lambda(\theta)$  is given by (3.36),  $F_\lambda(\theta)$  is given by (4.22), and  $d\lambda/d\theta$  is found by using the numerical differentiation algorithm in Mathcad15 for the derivative of the latitude  $\lambda(\theta)$  given by (4.12).

For the NEAR flyby,  $\theta_{\text{min}} = -123.1192^\circ$  and  $\theta_{\text{max}} = +123.1437^\circ$ , given by (4.14). Then by (3.56),

$$\left. \begin{aligned} \delta v_{\text{in}} &= \delta v(\theta_{\text{min}}) = -15.9577 \text{ mm/s} \\ \delta v_{\text{out}} &= \delta v(\theta_{\text{max}}) = +29.4184 \text{ mm/s} \\ \delta v_{\text{trt}} &= \delta v_{\text{in}} + \delta v_{\text{out}} = +13.4607 \text{ mm/sec} \end{aligned} \right\}. \quad (4.24)$$

The observed speed-change for the NEAR flyby (Appendix A) is

$$\delta v_{\text{obs}} = (+13.46 \pm 0.01) \text{ mm/s}. \quad (4.25)$$

The calculated value  $\delta v_{\text{trt}}$  equals exactly the observed value  $\delta v_{\text{obs}}$  if

$$\delta v_{\text{k}} = (6.530 \pm 0.005) v_{\text{eq}}, \quad \text{with } c_{\text{g}} = 1.000 c. \quad (4.26)$$

Repeating the calculation with  $c_{\text{g}} = 1.060 c$  requires a slightly smaller value for  $v_{\text{k}}$  to make  $\delta v_{\text{trt}} = \delta v_{\text{obs}}$ ,

$$v_{\text{k}} = (6.160 \pm 0.005) v_{\text{eq}}, \quad \text{if } c_{\text{g}} = (1.060 \pm 0.001) c. \quad (4.27)$$

If the “true” value for  $v_{\text{k}}$  were known with a precision of 1 part in a thousand, 0.1%, this calculation for the NEAR flyby speed-change would provide a first-ever measured value for the Earth’s speed of gravity!

Results for all six flybys using the parameter values of Table 3 and Table 5 are listed in Table 1. Table 1 lists the observed anomalous speed change,  $\delta v_{\text{obs}}$ , with the reported uncertainty, the calculated time-retarded speed change,  $\delta v_{\text{trt}}$ , with the corresponding uncertainty, the ratio gravity-speed/light-speed that was used in the calculation,  $c_{\text{g}}/c$ , the required relative induction speed,  $v_{\text{k}}/v_{\text{eq}}$ , with the corresponding uncertainty, and the calculated value for the eccentricity for the trajectory,  $\varepsilon$ .

**§5. Anomalous decrease in the Moon's orbital speed caused by the neoclassical causal version of Newton's theory.** Numerical values for the Earth's parameters,  $M_E$ ,  $r_E$ ,  $\Omega_E$ ,  $I_E$ , and  $v_{eq}$ , are listed in the Appendix B. Needed numerical values for the Moon,  $M_M$ ,  $r_p$ ,  $r_a$ ,  $\varepsilon$ , and  $\alpha_{eq}$ , are also listed in the Appendix B.

Let  $r(\theta)$  be the radial distance from the center of the Earth to the center of the Moon

$$\left. \begin{aligned} r(\theta) &= \frac{r_p(1+\varepsilon)}{1+\varepsilon\cos\theta} \\ \frac{dr}{d\theta} &= \frac{r(\theta)^2}{r_p} \frac{\varepsilon}{1+\varepsilon} \sin\theta \end{aligned} \right\}. \quad (5.1)$$

Let  $rM(\theta)$  be the radial distance from the origin of a barycentric frame to the Moon, and let  $rM_a$  and  $rM_p$  be the value for  $rM$  at apogee and at perigee. Let  $a_M$  and  $b_M$  be the semimajor and semiminor axes for the Moon's elliptical orbit

$$\left. \begin{aligned} rM(\theta) &= \frac{M_E}{M_E + M_M} r(\theta) = 0.9879 r(\theta) \\ \frac{drM}{d\theta} &= \frac{M_E}{M_E + M_M} \frac{dr}{d\theta} \\ rM_a &= \frac{M_E}{M_E + M_M} r_a = 62.905 r_E \\ rM_p &= \frac{M_E}{M_E + M_M} r_p = 56.301 r_E \\ a_M &= \frac{1}{2} (rM_a + rM_p) = 59.603 r_E \\ b_M &= a_M \sqrt{1 - \varepsilon^2} = 59.511 r_E \end{aligned} \right\}. \quad (5.2)$$

By Kepler's 3rd law, (3.45), the calculated lunar period  $PM$  is

$$PM = \frac{2\pi a_M^{3/2}}{\sqrt{G(M_E + M_M)}} = 26.78 \text{ days}. \quad (5.3)$$

By Kepler's 2nd law, (3.45), the orbital angular speed is

$$\Omega M(\theta) = \frac{2\pi}{PM} \frac{a_M b_M}{rM(\theta)^2}. \quad (5.4)$$

Let the Moon's orbital speed at perigee be  $vM_p$  and at apogee be

$vM_a$ . Numerical values are

$$\left. \begin{aligned} vM_p &= rM_p\Omega M(0) = 10.90 \times 10^2 \text{ m/s} \\ vM_a &= rM_a\Omega M(-\pi) = 9.755 \times 10^2 \text{ m/s} \end{aligned} \right\}. \quad (5.5)$$

Let  $v_{co}$  and  $\Omega_{co}$  be the orbital speed and orbital angular speed for an equivalent circular orbit which has the radius  $a_M$  and the period  $PM$  (3.46). We then have, respectively,

$$\left. \begin{aligned} v_{co} &= \sqrt{\frac{G(M_E + M_M)}{a_M}} = 1.031 \times 10^3 \text{ m/s} \\ \Omega_{co} &= \frac{v_{co}}{a_M} = 2.715 \times 10^{-6} \text{ rad/s} \end{aligned} \right\}. \quad (5.6)$$

Let  $\delta v_{co} \ll v_{co}$  be a small change in the orbital speed, let  $\delta a_M \ll a_M$  be the corresponding change in the radius of the orbit, and let  $\delta \Omega_{co} \ll \Omega_{co}$  be the corresponding change in the angular speed. Then

$$\left. \begin{aligned} v_{co}^2 &= \frac{\text{constant}}{a_M} \\ \left(1 + \frac{\delta v_{co}}{v_{co}}\right)^2 &\cong 1 + 2 \frac{\delta v_{co}}{v_{co}} = \frac{1}{1 + \delta a_M/a_M} \cong 1 - \frac{\delta a_M}{a_M} \\ 2 \frac{\delta v_{co}}{v_{co}} &\cong \frac{\delta a_M}{a_M} \\ \frac{\delta \Omega_{co}}{\Omega_{co}} &\cong \frac{\delta v_{co}}{v_{co}} - \frac{\delta a_M}{a_M} = -\frac{\delta v_{co}}{v_{co}} \end{aligned} \right\}. \quad (5.7)$$

According to Stephenson and Morrison, tidal braking increases the  $LOD$  by  $23 \times 10^{-6}$  seconds per year [15]. Let  $\delta LOD \ll LOD$  be this change in the  $LOD$

$$\left. \begin{aligned} LOD &= 60 \times 60 \times 24 = 86400 \text{ s} \\ \delta LOD &= 23 \times 10^{-6} \text{ s per year} \end{aligned} \right\}. \quad (5.8)$$

The Earth's sidereal rotational period in seconds is

$$\frac{2\pi}{\Omega_E} = 86164.1 \text{ s}. \quad (5.9)$$

Therefore,

$$LOD = \frac{2\pi}{\Omega_E} 1.002738 = 86400 \text{ s}. \quad (5.10)$$

Let  $\delta\Omega_E$  be a small change in  $\Omega_E$ . Then

$$\left. \begin{aligned} 1 + \frac{\delta LOD}{LOD} &= \frac{1}{1 + \delta\Omega_E/\Omega_E} \cong 1 - \frac{\delta\Omega_E}{\Omega_E} \\ \delta\Omega_E &= -\Omega_E \frac{\delta LOD}{LOD} = -1.941 \times 10^{-14} \text{ rad/s per year} \end{aligned} \right\}. \quad (5.11)$$

Let  $S_E$  be the magnitude for the Earth's spin angular momentum, and let  $\delta S_E$  be a small change in  $S_E$ . Assume there is no change in  $I_E$ . Then we have

$$\left. \begin{aligned} S_E &= I_E \Omega_E = 5.851 \times 10^{33} \text{ kg} \times \text{m}^2/\text{s} \\ \delta S_E &= S_E \frac{\delta\Omega_E}{\Omega_E} = -1.558 \times 10^{24} \text{ kg} \times \text{m}^2/\text{s per year} \end{aligned} \right\}. \quad (5.12)$$

Let  $L_M$  be the magnitude for the Moon's orbital angular momentum, and let  $\delta L_M$  be a small change in  $L_M$ . By conservation of the Earth's spin angular momentum and the Moon's orbital angular momentum,

$$\left. \begin{aligned} L_M &= M_M v_{co} a_M = 2.877 \times 10^{34} \text{ kg} \times \text{m}^2/\text{s} \\ \frac{\delta L_M}{L_M} &= \frac{\delta v_{co}}{v_{co}} + \frac{\delta a_M}{a_M} = 3 \frac{\delta v_{co}}{v_{co}} \\ L_M + S_M &= \text{constant} \\ \delta L_M &= -\delta S_E = +1.558 \times 10^{24} \text{ kg} \times \text{m}^2/\text{s per year} \end{aligned} \right\}. \quad (5.13)$$

The resulting change in the orbital speed is

$$\delta v_{co} = \frac{v_{co}}{3} \frac{\delta L_M}{L_M} = +18.6 \times 10^{-9} \text{ m/s per year}. \quad (5.14)$$

Equation (5.5) gives

$$\delta a_M = 2a_M \frac{\delta v_{co}}{v_{co}} = +13.7 \times 10^{-3} \text{ m per year}. \quad (5.15)$$

This shows that tidal braking alone causes an increase in the radius of 14 mm per year, and a corresponding increase in the orbital speed of  $19 \times 10^{-9}$  m/s per year.

But lunar-laser-ranging experiments have shown that the radius is actually increasing by [16]

$$\delta a_M = +38 \text{ mm per year}. \quad (5.16)$$

The corresponding increase in the orbital speed (5.7) takes the following numerical value

$$\delta v_{\text{co}} = \frac{v_{\text{co}}}{2} \frac{\delta a_{\text{M}}}{a_{\text{M}}} = +51.6 \times 10^{-9} \text{ m/s per year.} \quad (5.17)$$

There is an obvious difference! An unexplained hidden action is causing the rate for change in the radius to decrease from 38 to 14 mm per year ( $-24$  mm per year). The corresponding rate for change in the orbital speed is decreased from  $52 \times 10^{-9}$  to  $19 \times 10^{-9}$  m/s per year ( $-33 \times 10^{-9}$  m/s per year). This unexplained difference is the “lunar orbit anomaly”.

The lunar orbit anomaly can be explained exactly by the neoclassical causal theory. Let  $\alpha_{\text{eq}}$  be the average inclination of the Moon’s orbital plane (Appendix B)

$$\alpha_{\text{eq}} = 23^\circ \pm 5^\circ. \quad (5.18)$$

Let the latitude for perigee,  $\lambda_{\text{p}}$ , equal  $\alpha_{\text{eq}}$ . Then by (3.47),

$$\lambda M(\theta) = \tan^{-1}(\tan \alpha_{\text{eq}} \cos \theta), \quad (5.19)$$

and the  $\phi$ -component of  $\Omega M$  becomes

$$\Omega_{\phi}(\theta) = \Omega M(\theta) \cos \alpha_{\text{eq}}. \quad (5.20)$$

By (3.47), the formula for the  $\lambda$ -component of  $rM$  becomes

$$r_{\lambda}(\theta) = r M(\theta) \cos \theta. \quad (5.21)$$

The formula for the Earth’s transverse field at the Moon with  $c_{\text{g}} = c$  (3.28) is

$$g_{\text{trt}}(\theta) = -G \frac{I_{\text{E}}}{r_{\text{E}}^4} \frac{v_{\text{eq}}}{c} \left( \frac{\Omega_{\phi}(\theta) - \Omega_{\text{E}}}{\Omega_{\text{E}}} \right) \cos^2(\lambda M(\theta)) PS(r(\theta)). \quad (5.22)$$

The formula for  $PS(r)$  is given by (3.26).

A trial run gave the following value for the induction speed which gives the observed speed-change

$$v_{\text{k}} = 7.94 v_{\text{eq}}. \quad (5.23)$$

The formula for the transverse induction-like field (3.53) becomes

$$F_{\lambda}(\theta) = \frac{v_{\text{eq}}}{v_{\text{k}}} \frac{r_{\text{E}}}{r(\theta)} \int_0^{\theta} \frac{r(\theta)}{r_{\text{E}}} \frac{\Omega M(\theta)}{\Omega_{\text{E}}} \frac{1}{r_{\text{E}}} \frac{dr}{d\theta} \frac{dg_{\text{trt}}}{d\theta} d\theta. \quad (5.24)$$



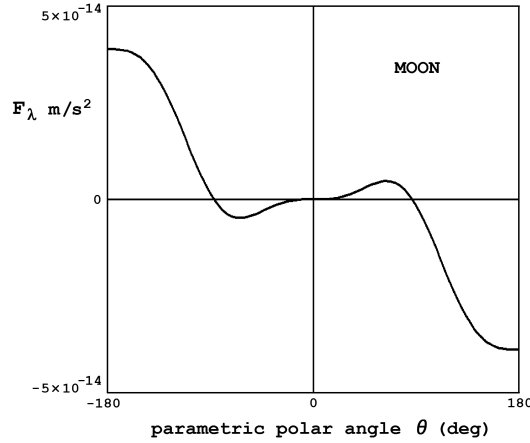


Fig. 9: Induction-like field  $F_\lambda$  versus the parametric angle  $\theta$  for the Moon with  $v_k = 7.94 v_{eq}$ .

The derivative  $dg_{trt}/d\theta$  is found by using the differentiation algorithm in Mathcad15. A graph of  $F_\lambda$  versus  $\theta$  for the Moon, Fig. 9, shows that there is asymmetry about  $\theta = 0$ .

The formula for the speed change, (3.55), becomes

$$\delta v(\theta) = \frac{v M_a}{2} \int_0^\theta \frac{r_\lambda(\theta) F_\lambda(\theta)}{v M_a^2} \frac{d\lambda}{d\theta} d\theta. \tag{5.25}$$

By numerical differentiation and integration,

$$\left. \begin{aligned} \delta v_{in} &= -\delta v(-\pi) = -1.21 \times 10^{-9} \text{ m/s} \\ \delta v_{out} &= +\delta v(+\pi) = -1.21 \times 10^{-9} \text{ m/s} \\ \delta v_{trt} &= \delta v_{in} + \delta v_{out} = -2.42 \times 10^{-9} \text{ m/s per revolution} \end{aligned} \right\}. \tag{5.26}$$

Let  $N_{rev}$  be the number of lunar revolutions per year, let  $yr$  be the number of seconds in a year, and let  $\delta v M$  be the accumulated orbital speed-change per year. Then

$$\left. \begin{aligned} N_{rev} &= yr/PM = 13.64 \\ \delta v M &= N_{rev} \delta v_{trt} = -33.0 \times 10^{-9} \text{ m/s per year} \end{aligned} \right\}. \tag{5.27}$$

This shows that, with  $v_k = 7.94 v_{eq}$ , the calculated value for the Moon's orbital speed-change is  $-33 \times 10^{-9}$  m/s per year, which explains exactly

the lunar orbit anomaly. The final value for  $v_k$  with uncertainty reduces to

$$\frac{v_k}{v_{eq}} = 8 \pm 1. \quad (5.28)$$

**§6. Predicted annual speed-change for spacecrafts in highly eccentric and inclined near-Earth orbits.** The speed-change caused by the causal version of Newton's theory depends on the speed of propagation of the gravitational field,  $c_g$ , the properties of the central sphere;  $M_E$ ,  $r_E$ ,  $\Omega_E$ ,  $I_E$ , and  $v_{eq}$ , the orbital properties of the spacecraft;  $r_p$ ,  $\varepsilon$ ,  $\alpha_{eq}$ , and  $\lambda_p$ , and the induction speed,  $v_k$ . If  $\varepsilon=0$ , the speed-change  $\delta v_{trt}=0$ , regardless of the value for  $\alpha_{eq}$ . If  $\alpha_{eq}=0$ , the speed-change  $\delta v_{trt}=0$ , regardless of the value for  $\varepsilon$ . Even if both  $\varepsilon$  and  $\alpha_{eq}$  are not zero, the speed-change is still zero if perigee is over the equator or one of the poles. The maximum speed-change occurs for spacecrafts with highly eccentric and inclined near-Earth orbits, such as with the inclination  $\alpha_{eq}=45^\circ$  and the latitude at perigee  $\lambda_p=45^\circ$ .

Suppose the orbital properties for a spacecraft are  $\varepsilon=0.5$ ,  $\alpha_{eq}=45^\circ$ , and  $\lambda_p=45^\circ$ . Let  $r_p$  range from  $2r_E$  to  $8r_E$ . Shown below are the numerical values for  $r_p=2r_E$ . The period is given by Kepler's 3rd law (3.45)

$$P = \frac{2\pi a^{3/2}}{\sqrt{GM_E}} = 11.2 \text{ hours}, \quad (6.1)$$

where

$$\left. \begin{aligned} r_a &= r_p \frac{1+\varepsilon}{1-\varepsilon} = 6r_E \\ a &= \frac{1}{2}(r_a + r_p) = 4r_E \\ b &= a\sqrt{1-\varepsilon^2} = 3.464r_E \end{aligned} \right\}. \quad (6.2)$$

The formula for  $\Omega_\theta$  is given by (3.45)

$$\Omega_\theta(\theta) = \frac{2\pi}{P} \frac{ab}{r(\theta)^2}, \quad (6.3)$$

where

$$\left. \begin{aligned} r(\theta) &= \frac{r_p(1+\varepsilon)}{1+\varepsilon\cos\theta} \\ \frac{dr}{d\theta} &= \frac{r(\theta)^2}{r_p} \frac{\varepsilon}{1+\varepsilon} \sin\theta \end{aligned} \right\}. \quad (6.4)$$

Let  $\Omega_a$  be the spacecraft's orbital angular speed at apogee and let

$v_a$  be the orbital speed at apogee. Then

$$\left. \begin{aligned} \Omega_a = \Omega_\theta(-\pi) &= 0.819 \Omega_E \\ v_a = \Omega_a r_a &= 4.92 v_{eq} \end{aligned} \right\}. \quad (6.5)$$

If the latitude for perigee  $\lambda_p = \alpha_{eq}$ , and if the value for  $\theta$  at perigee is zero, then by (3.47)

$$\lambda(\theta) = \tan^{-1}(\tan \alpha_{eq} \cos \theta), \quad (6.6)$$

and the  $\phi$ -component of  $\Omega_\theta$  becomes

$$\left. \begin{aligned} \Omega_\phi(\theta) &= \Omega_\theta(\theta) \cos \alpha_{eq} && \text{for prograde orbits} \\ \Omega r \phi(\theta) &= -\Omega_\phi(\theta) && \text{for retrograde orbits} \end{aligned} \right\}. \quad (6.7)$$

The projection of “prograde” orbits onto the equatorial plane revolves in the same direction as the Earth’s spin, and the projection of “retrograde” orbits onto the equatorial plane revolves opposite to the direction of the Earth’s spin.

The formula for the  $\lambda$ -component of  $r$  becomes

$$r_\lambda(\theta) = r(\theta) \cos \theta. \quad (6.8)$$

The formula for the Earth’s time-retarded transverse field with  $c_g = c$ , (3.28), is

$$g_{trt}(\theta) = -G \frac{I_E v_{eq}}{r_E^4 c} \left( \frac{\Omega_\phi(\theta) - \Omega_E}{\Omega_E} \right) \cos^2(\lambda(\theta)) PS(r(\theta)), \quad (6.9)$$

where  $PS(r)$  is given by (3.26).

The “true” value for  $v_k$  probably lies between  $10v_{eq}$  and  $14v_{eq}$  (1.16). To minimize the predicted speed-change, choose its maximum probable value,  $v_k = 14v_{eq}$ . The formula for the induction-like field is given by (3.53)

$$F_\lambda(\theta) = \frac{v_{eq}}{v_k} \frac{r_E}{r(\theta)} \int_0^\theta \frac{r(\theta)}{r_E} \frac{\Omega_\theta(\theta)}{\Omega_E} \frac{1}{r_E} \frac{dr}{d\theta} \frac{dg_{trt}}{d\theta} d\theta. \quad (6.10)$$

A graph of  $F_\lambda$  for  $r_p = 2r_E$  with  $v_k = 14v_{eq}$  is shown in Fig. 10.

The speed-change for one period is given by (3.56)

$$\left. \begin{aligned} \delta v(\theta) &= \frac{v_a}{2} \int_0^\theta \frac{r_\lambda(\theta) F_\lambda(\theta)}{v_a^2} \\ \delta v_{trt} &= \delta v_{in} + \delta v_{out} = -\delta v(-\pi) + \delta v(+\pi) = \\ &= 28.3 \text{ mm/s per revolution} \end{aligned} \right\}. \quad (6.11)$$

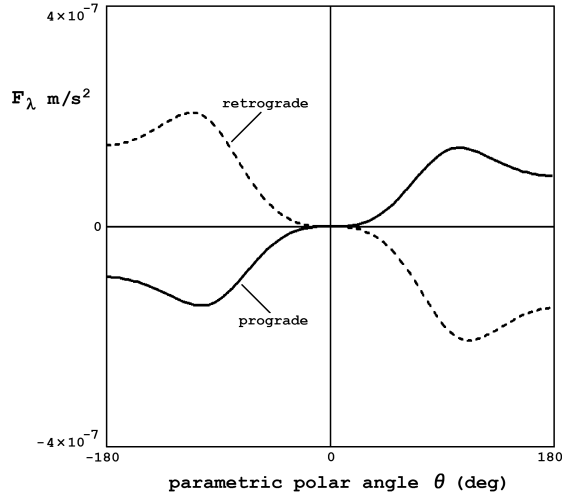


Fig. 10: Induction-like field  $F_\lambda$  versus the parametric angle  $\theta$  for a spacecraft in a near-Earth orbit with  $\varepsilon = 0.5$ ,  $\alpha_{\text{eq}} = 45^\circ$ ,  $\lambda_p = 45^\circ$ , and  $r_p = 2r_E$  with  $v_k = 14v_{\text{eq}}$ . The solid curve is for a prograde orbit and the dashed curve is for a retrograde orbit.

Let  $N_{\text{rev}}$  be the number of revolutions in one year, let  $\delta v_{\text{yr}}$  be the total speed-change accumulated during one year, and let  $yr$  be the number of seconds in a year ( $P$  is the period in seconds)

$$\left. \begin{aligned} N_{\text{rev}} &= yr/P = 780 \text{ revolutions per year} \\ \delta v_{\text{yr}} &= N_{\text{rev}} \delta v_{\text{trt}} = 315 \text{ mm/s per year} \end{aligned} \right\}. \quad (6.12)$$

The resulting calculated periods and speed-changes for  $r_p$  ranging from  $2r_E$  to  $8r_E$  are listed in Table 2.

**§7. Is there a conflict between the neoclassical causal theory and general relativity theory?** The only possible case where there could be a conflict is the excess for the advance in the perihelion of Mercury [21]. This section shows that the Sun's time-retarded transverse gravitational field causes a change in the angle for Mercury's perihelion of less than 0.04 arc seconds per century, which is negligibly less than the relativistic advance of 43 arc seconds per century and therefore is undetectable.

Let  $M_S$ ,  $r_S$ ,  $\Omega_S$ ,  $I_S$ , and  $v_{\text{eq}}$  be the Sun's mass, radius, spin angular speed, moment of inertia, and equatorial surface speed. Numerical

values for the Sun are listed in Appendix B. A 4-term power series approximation for the triple integral over the Sun's volume also can be found in Appendix B.

Let  $r_a$  and  $r_p$  be Mercury's heliocentric radial distance at aphelion and at perihelion, let  $\varepsilon$  be the eccentricity, let  $P_M$  be the observed sidereal orbital period, let  $\alpha_{\text{eq}}$  be the inclination to the Sun's equatorial plane, and let  $\lambda_p$  be the heliocentric latitude at perihelion. Numerical values from Appendix B are

$$\left. \begin{aligned} r_a &= 69816900 \times 10^3 \text{ m} \\ r_p &= 46001200 \times 10^3 \text{ m} \\ \varepsilon &= 0.205630 \\ P_M &= 87.969 \text{ days} = 7.6005 \times 10^6 \text{ s} \\ \alpha_{\text{eq}} &= 3.38^\circ \\ \lambda_p &= 3.38^\circ \end{aligned} \right\}. \quad (7.1)$$

The semimajor and semiminor axes are

$$\left. \begin{aligned} a_M &= \frac{1}{2}(r_a + r_p) = 5.791 \times 10^{10} \text{ m} \\ b_M &= a_M \sqrt{1 - \varepsilon^2} = 5.667 \times 10^{10} \text{ m} \end{aligned} \right\}. \quad (7.2)$$

The calculated period given by Kepler's 3rd law is

$$P = \frac{2\pi a_M^{3/2}}{\sqrt{GM_S}} = 7.5998 \times 10^6 \text{ s} = 1.01 P_M. \quad (7.3)$$

The formula for  $\Omega_\theta$ , given by (2.28), is

$$\Omega_\theta(\theta) = \frac{2\pi}{P} \frac{a_M b_M}{r(\theta)^2}, \quad (7.4)$$

where

$$\left. \begin{aligned} r(\theta) &= \frac{r_p(1 + \varepsilon)}{1 + \varepsilon \cos \theta} \\ \frac{dr}{d\theta} &= \frac{r(\theta)^2}{r_p} \frac{\varepsilon}{1 + \varepsilon} \sin \theta \end{aligned} \right\}. \quad (7.5)$$

Let  $\Omega_a$  be Mercury's orbital angular speed at aphelion and let  $v_a$  be the orbital speed at aphelion. Then

$$\left. \begin{aligned} \Omega_a &= \Omega_\theta(-\pi) = 5.559 \times 10^{-7} \text{ rad/s} \\ v_a &= \Omega_a r_a = 3.881 \times 10^4 \text{ m/s} \end{aligned} \right\}. \quad (7.6)$$

Mercury's heliocentric latitude is

$$\lambda(\theta) = \tan^{-1}(\tan \alpha_{\text{eq}} \cos \theta), \quad (7.7)$$

and the  $\phi$ -component of  $\Omega_\theta$  is

$$\Omega_\phi(\theta) = \Omega_\theta(\theta) \cos \alpha_{\text{eq}}. \quad (7.8)$$

The  $\lambda$ -component of  $r$  is

$$r_\lambda(\theta) = r(\theta) \cos \theta. \quad (7.9)$$

The formula for the Sun's time-retarded transverse field with  $c_g = c$  substituted is

$$g_{\text{trt}}(\theta) = -G \frac{I_S v_{\text{eq}}}{r_S^4 c} \left( \frac{\Omega_\phi(\theta) - \Omega_S}{\Omega_S} \right) \cos^2(\lambda(\theta)) PS(r(\theta)). \quad (7.10)$$

The average value for  $v_k$  is  $10v_{\text{eq}}$ . The formula for the induction-like field becomes

$$F_\lambda(\theta) = \frac{v_{\text{eq}} r_S}{v_k r(\theta)} \int_0^\theta \frac{r(\theta) \Omega_\theta(\theta)}{r_S \Omega_S} \frac{1}{r_S} \frac{dr}{d\theta} \frac{dg_{\text{trt}}}{d\theta} d\theta. \quad (7.11)$$

By numerical differentiation and integration, the speed-change becomes

$$\delta v_{\text{trt}} = -4.71 \times 10^{-7} \text{ m/s per revolution.} \quad (7.12)$$

Numerical values for the orbital speed  $v_{\text{co}}$  and the angular speed  $\Omega_{\text{co}}$  for an equivalent circular orbit for Mercury, by (3.46), are

$$\left. \begin{aligned} v_{\text{co}} &= \sqrt{\frac{GM_S}{a_M}} = 4.788 \times 10^4 \text{ m/s} \\ \Omega_{\text{co}} &= \frac{v_{\text{co}}}{a_M} = 8.268 \times 10^{-7} \text{ rad/s} \end{aligned} \right\}. \quad (7.13)$$

Let  $\theta_p$  be the value for  $\theta$  at perihelion, let  $\delta\theta_p$  be the change in  $\theta_p$  per revolution, and set  $\delta v_{\text{co}} = \delta v_{\text{trt}}$ . Then

$$\left. \begin{aligned} \frac{\delta\theta_p}{2\pi} &= \frac{\delta\Omega_{\text{co}}}{\Omega_{\text{co}}} = 3 \frac{\delta v_{\text{co}}}{v_{\text{co}}} = 3 \frac{\delta v_{\text{trt}}}{v_{\text{co}}} \\ \delta\theta_p &= 6\pi \frac{\delta v_{\text{trt}}}{v_{\text{co}}} = -1.86 \times 10^{-10} \text{ rad per revolution} \end{aligned} \right\}. \quad (7.14)$$

Let  $N_{\text{rev}}$  be the number of Mercury's revolutions in one year, let  $\Delta\theta_p$  be the accumulated angular change of Mercury during one year, and let

$yr$  be the number of seconds in a year

$$\left. \begin{aligned} N_{\text{rev}} &= \frac{yr}{P_M} = 4.125 \\ \Delta\theta_p &= N_{\text{rev}}\delta\theta_p = -7.71 \times 10^{-9} \text{ rad per year} \\ \Delta\theta_p \times \frac{180}{\pi} \times 60 \times 60 \times 100 &= -0.016 \text{ arc sec per century} \end{aligned} \right\} . \quad (7.15)$$

Thus we find that the absolute magnitude for the change in the angle for perihelion is less than 0.04 arc seconds per century, which is totally negligible compared with the relativistic change of 43 arc seconds per century.

**§8. Conclusions and recommendations.** There is here within conclusive evidence that the proposed neoclassical causal version of Newton's theory agrees with the facts-of-observation to the extent that such facts are currently available. The proposed causal version is a natural rational extension of Newton's acausal theory. It applies only for slow-speeds and weak-fields, i.e., for  $v^2 \ll c^2$  and  $GM/r \ll c^2$ . Effects of time retardation appear at the relatively large first-order  $v/c_g$  level, but they are normally very small and are previously undetected because they decrease inversely with the cube of the bypass distance. If the bypass is very close, however, time retardation effects can be relatively large. It is recommended that future research projects utilize various available methods to detect new first-order effects of the causality principle.

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#### Appendix A. Parameter values for six Earth flybys

Table A1 lists needed parameter values which can be found in the report by Anderson *et al.* [10]. The symbols are changed to be those used for this article.

The start time for the incoming and the end time for the outgoing data intervals for the NEAR flyby are stated in the caption for Fig. 3 of the report

Flyby	NEAR	GLL-I	Rosetta	M'GER	Cassini	GLL-II
$h_p$ (km)	539	960	1956	2347	1175	303
$\lambda_p$ (deg)	+33.0	+25.2	+20.20	+46.95	-23.5	-33.8
$v_p$ (km/s)	12.739	13.740	10.517	10.389	19.026	14.080
$v_\infty$ (km/s)	6.851	8.949	3.863	4.056	16.010	8.877
$DA$ (deg)	66.9	47.7	99.3	94.7	19.7	51.1
$\alpha_{eq}$ (deg)	+108.0	+142.9	+144.9	+133.1	+25.4	+138.7
$\lambda_{in}$ (deg)	+20.76	+12.52	+2.81	-31.44	+12.92	+34.26
$\lambda_{out}$ (deg)	-71.96	-34.15	-34.29	-31.92	-4.99	-4.87
$\delta v_{obs}$ (mm/s)	+13.46 $\pm 0.01$	+3.92 $\pm 0.3$	+1.80 $\pm 0.03$	+0.02 $\pm 0.01$	-2 $\pm 1$	-4.6 $\pm 1$
$\delta v_{emp}$ (mm/s)	+13.28	+4.12	+2.07	+0.06	-1.07	-4.67

Table A1: Earth flyby parameter values for the NEAR, Galileo-I, Rosetta, MESSENGER (M'GER), Cassini, and Galileo-II spacecraft flybys. The altitude at perigee  $h_p$  is referenced to the Earth geoid,  $\lambda_p$  is the geocentric latitude at perigee,  $v_p$  is the magnitude of the spacecraft's inertial velocity at perigee,  $v_\infty$  is the magnitude for the osculating hyperbolic excess velocity,  $DA$  is the deflection angle between the incoming and outgoing asymptotic velocity vectors,  $\alpha_{eq}$  is the inclination of the orbital plane to the Earth's equatorial plane,  $\lambda_{in}$  and  $\lambda_{out}$  are the geocentric latitudes for the incoming and outgoing osculating asymptotic velocity vectors, and  $\delta v_{obs}$  is the measured change in the spacecraft's orbital speed with an estimated realistic uncertainty for the measured value. The last row gives the calculated speed-change values from the empirical prediction formula  $\delta v_{emp}$ .

by Anderson *et al.*

$$t_{in} = -88.4 \text{ hours}, \quad t_{out} = +95.6 \text{ hours}. \quad (\text{A.1})$$

The data time intervals for the other flybys were not given.

Anderson *et al.* report the asymptotic flyby "declinations" instead of the asymptotic geocentric latitudes. From Fig. 1 of their report, there is no doubt that the inbound asymptotic latitude  $\lambda_{in}$  is positive for the NEAR flyby (+ for northern latitudes) and the outbound asymptotic latitude  $\lambda_{out}$  is negative (- for southern latitudes). This recognition for the correct signs is applied in Table A1.

Notice in Table A1 that both of the flybys which have negative speed-changes (the flybys in the case of Cassini and GLL-II) have negative values for the latitude at perigee  $\lambda_p$ .



### Appendix B. Various numerical values and radial mass-density distributions.

Various numerical values are needed to evaluate the formulas for the transverse gravitational field. The following values were found in [22, 23]:

$G = 6.6732 \times 10^{-11} \text{ m}^3/\text{kg}\times\text{s}^2$	Gravity constant,
$c = 2.997925 \times 10^8 \text{ m/s}$	Vacuum speed of light,
$\Omega_E = 7.292115 \times 10^{-5} \text{ rad/s}$	Earth's sidereal angular speed,
$M_E = 5.9761 \times 10^{24} \text{ kg}$	Earth's total mass,
$r_E = 6,371,034 \text{ m}$	Earth's equivalent spherical radius,
$v_E = r_E \Omega_E = 464.58 \text{ m/s}$	Earth's equatorial surface speed,
$V_E = 1.08322 \times 10^{21} \text{ m}^3$	Earth's volume,
$\bar{\rho}_E = 5.517 \times 10^3 \text{ kg/m}^3$	Earth's mean mass-density,
$I_E = 8.0238 \times 10^{37} \text{ kg/m}^2$	Earth's spherical moment of inertia,
$I_E/\bar{\rho} r_E^5 = 1.3856$	Unitless ratio for the moment of inertia.

The Earth's interior consists of four major regions: inner core, outer core, mantle, and crust [22]. The formula for the radial mass-density distribution, derived from seismic data, is

$$\rho(r') = \text{if} \left( r' < r_{ic}, \rho_{ic}, \text{if} \left( r' < r_{oc}, \rho_{oc}(r'), \text{if} \left( r' < r_{man}, \rho_{man}(r'), \rho_{cst}(r') \right) \right) \right),$$

where (radii are in meters and densities are in  $\text{kg/m}^3$ )

$$\begin{aligned} r_{ic} &= 1230 \times 10^3, \\ \rho_{ic} &= 13 \times 10^3, \\ r_{oc} &= 3486 \times 10^3, \\ \rho_{oc}(r') &= 12 \times 10^3 + 2.0 \times 10^3 \left( \frac{r_{ic} - r'}{r_{oc} - r_{ic}} \right) - 0.6 \times 10^3 \left( \frac{r_{ic} - r'}{r_{oc} - r_{ic}} \right)^2, \\ r_{man} &= 6321 \times 10^3, \\ \rho_{man}(r') &= 5.75 \times 10^3 + 0.4 \times 10^3 \left( \frac{r_{oc} - r'}{r_{man} - r_{oc}} \right) - 2.05 \times 10^3 \left( \frac{r_{oc} - r'}{r_{man} - r_{oc}} \right)^2, \\ r_{cst} &= r_E = 6378 \times 10^3, \\ \rho_{cst}(r') &= 3.3 \times 10^3 + 0.6 \times 10^3 \left( \frac{r_{man} - r'}{r_{cst} - r_{man}} \right) - 0.5 \times 10^3 \left( \frac{r_{man} - r'}{r_{cst} - r_{man}} \right)^2. \end{aligned}$$

The following numerical values for the Moon are taken from [24]:

$M_M = 7.3477 \times 10^{22} \text{ kg}$	Moon's mass,
$r_p = 363,104 \times 10^3 \text{ m}$	Moon's radial distance at perigee,
$r_a = 405,696 \times 10^3 \text{ m}$	Moon's radial distance at apogee,

$\varepsilon = 0.0554$	Eccentricity,
$P_{\text{sid}} = 27.321582$ days	Moon's sidereal orbital period,
$P_{\text{sol}} = 29.530589$ days	Moon's solar orbital period,
$\alpha_{\text{eq}} = 23.4^\circ \pm 5.2^\circ$	Moon's inclination (average $\pm$ variation).

The following numerical values for the Sun are taken from [25]:

$M_{\text{S}} = 1.9891 \times 10^{30}$ kg	Sun's mass,
$r_{\text{S}} = 6.955 \times 10^8$ m	Sun's equatorial radius,
$\bar{\rho} = 1.408 \times 10^3$ kg/m <sup>3</sup>	Sun's mean mass-density,
$\rho_{\text{ctr}} = 1.622 \times 10^5$ kg/m <sup>3</sup>	Mass-density at the center,
$\rho_{\text{photo}} = 2 \times 10^{-4}$ kg/m <sup>3</sup>	Mass-density at the photosphere,
$P_{\text{S}} = 25.05$ days = $2.158 \times 10^4$ s	Equatorial rotational period,
$\Omega_{\text{S}} = 2\pi/P_{\text{S}} = 2.911 \times 10^{-4}$ rad/s	Equatorial angular speed,
$v_{\text{eq}} = r_{\text{S}} \Omega_{\text{S}} = 2.025 \times 10^3$ m/s	Equatorial rotational surface speed,
$I_{\text{S}} = 3.367 \times 10^{46}$ kg $\times$ m <sup>2</sup>	Sun's moment of inertia,
$I_{\text{S}}/\bar{\rho}r_{\text{S}}^5 = 0.147$	Unitless ratio for the moment of inertia,
$\alpha_{\text{S}} = 7.25^\circ$	Obliquity to the ecliptic.

An exponential function provides a reasonably valid approximation for the Sun's radial mass-density distribution

$$\rho(r') = \text{if} \left( r' \leq r_{\text{S}}, \rho_{\text{ctr}} \exp \left( - \left( \frac{r'}{r_{\text{core}}} \right)^2 \right), 0 \right),$$

where the numerical value for  $r_{\text{core}}$  is

$$r_{\text{core}} = 0.18707 r_{\text{S}}.$$

The following four-term power series

$$PS(r) = \left( \frac{r_{\text{S}}}{r} \right)^3 \left( C_0 + C_2 \left( \frac{r_{\text{S}}}{r} \right)^2 + C_4 \left( \frac{r_{\text{S}}}{r} \right)^4 + C_6 \left( \frac{r_{\text{S}}}{r} \right)^6 \right),$$

provides an excellent fit to the triple integral over the Sun's volume with the following values for the coefficients

$$\begin{aligned} C_0 &= 0.500000, & C_2 &= 0.017498, \\ C_4 &= 0.001376, & C_6 &= 0.000173. \end{aligned}$$

The following numerical values for the planet Mercury are taken from [21]:

$r_{\text{p}} = 46,001,200 \times 10^3$ m	Radial distance at perihelion,
$r_{\text{a}} = 69,816,900 \times 10^3$ m	Radial distance at aphelion,
$\varepsilon = 0.205630$	Eccentricity,

$P_M = 87.969$ days	Mercury's sidereal orbital period,
$\alpha_{\text{eq}} = 3.38^\circ$	Inclination to the Sun's equator,
$\lambda_p = 3.38^\circ$	Heliocentric latitude at perihelion.

*Submitted on November 27, 2012*

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# Possibility of Experimental Study of the Properties of Time

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**Foreword of the Submitter:** This article, an English translation from N. A. Kozyrev's book: *Causal or Unsymmetrical Mechanics in Linear Approximation*, Pulkovo Observatory, 1958 (in Russian), based on work done more than 50 years ago, is unique in the archives because it contains both theoretical and experimental results that bear on the causality principle. It is submitted for publication here to show some of the results that are similar to results from the previous article by J. C. Hafele: *Earth Flyby Anomalies Explained by a Time-Retarded Causal Version of Newtonian Gravitational Theory* (in this issue, pages 134–187). Kozyrev's experimental results are mostly based on measurements of the motions of various arrangements of an aircraft navigation gyroscope suspended from a torsion pendulum. Here are some reliable results from Kozyrev's article that are similar to results from Hafele's article: 1) on page 202, Kozyrev states that the effect on the motion of the pendulum: "is caused by the rotation of the Earth", 2) on page 203 he states: "the ratio of the horizontal force of the weight ... is  $3.5 \times 10^{-5}$ ", 3) on page 207 he states: "it follows that  $\Delta Q_N/Q = 2.8 \times 10^{-5}$  at  $\varphi = 48^\circ$ ", 4) on page 209 he states: "For Pulkovo,  $\Delta Q_Z/Q = 2.8 \times 10^{-5}$ ", and 5) on page 212 he states: "To this deviation there corresponds the relative displacement  $\Delta l/l = 0.85 \times 10^{-5}$ ". All of these ratios essentially equal the ratio of the time-retarded transverse component to the radial component of the Earth's gravitational field. The submitter does not endorse Kozyrev's metaphysical speculations.

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**Part I. Theoretical concepts**

Time is the most important and most enigmatic property of nature. The concept of time surpasses our imagination. The recondite attempts to understand the nature of time by the philosophers of antiquity, the scholars in the Middle Ages, and the modern scientists, possessing a knowledge of sciences and the experience of their history, have proven fruitless. Probably this occurs because time involves the most profound and completely unknown properties of the world which can scarcely be envisaged by the bravest flight of human fancy. Past these properties of the world there passes the triumphal procession of modern science and technical progress. In reality, the exact sciences negate the existence in time of any other qualities other than the simplest quality of “duration” or time intervals, the measurement of which is realized in hours. This quality of time is similar to the spatial interval. The theory of relativity by Einstein made this analogy more profound, considering time intervals and space as components of a 4-dimensional interval of a Minkowski universe. Only the pseudo-Euclidian nature of the geometry of the Minkowski universe differentiates the time interval from the space interval. Under such a conception, time is scalar and quite passive. It only supplements the spatial arena, against which the events of the universe are played out. Owing to the scalarity of time, in the equations of theoretical mechanics the future is not separated from the past; hence, the causes are not separated from the results. In the result, classical mechanics brings to the universe a strictly deterministic, but deprived, causality. At the same time, causality comprises the most important quality of the real world.

The concept of causality is the basis of natural science. The natural scientist is convinced that the question “why” is a legitimate one, that a question can be found for it. However, the content of the exact sciences is much more impoverished. In the precise sciences, the legitimate question is only “how?”: i. e., in what manner a given chain of occurrences takes place. Therefore, the precise sciences are descriptive. The description is made in a 4-dimensional world, which signifies the possibility of predicting events. This possibility of prediction is the key to the power of the precise sciences. The fascination of this power is so great that it often compels one to forget the basic, incomplete nature of their basis. It is therefore probable that the philosophical concept of Mach, derived strictly logically from the basis of the exact sciences, attracted great attention, in spite of its nonconformity to our knowledge concerning the universe and daily experience.

The natural desire arises to introduce into the exact science the principles of natural science. In other words, the tendency is to attempt to introduce into theoretical mechanics the principle of causality and directivity of time. Such a mechanics can be called “causal” or “asymmetrical” mechanics. In such mechanics, there should be realizable experience, indicating where the cause is and where the result is. It can be demonstrated that in statistical mechanics there is a directivity of time and that it satisfies our desires. In reality, statistical mechanics constructs a certain bridge between natural and theoretical mechanics. In the statistical grouping, an asymmetrical state in time can develop, owing to unlikely initial conditions caused by the direct intervention of a proponent of the system, the effect of which is causal. If, subsequently, the system will be isolated, in conformity with the second law of thermodynamics, its entropy will increase, and the directivity of time will be associated with this trend in the variation of entropy. As a result, the system will lead to the most likely condition; it will prove to be in equilibrium, but then the fluctuations in the entropy of various signs will be encountered with equal frequency. Therefore, even in the statistical mechanics of an isolated system, under the most probable condition, the directivity of time will not exist. It is quite natural that in statistical mechanics, based on the conventional mechanics of a point, the directivity of time does not appear as a quality of time itself but originates only as a property of the state of the system. If the directivity of time and other possible qualities are objective, they should enter the system of elementary mechanics of isolated processes. However, the statistical generalization of such mechanics can lead to a conclusion concerning the unattainability of equilibrium conditions. In reality, the directivity of time signifies a pattern continuously existing in time, which, acting upon the material system, can cause it to transfer to an equilibrium state. Under such a consideration, the events should occur not only in time, as in a certain arena, but also with the aid of time. Time becomes an active participant in the universe, eliminating the possibility of thermal death. Then, we can understand harmony of life and death, which we perceive as the essence of our world. Already, owing to these possibilities alone, one should carefully examine the question as to the manner in which the concept of the directivity of time or its pattern can be introduced into the mechanics of elementary processes.

We shall represent mechanics in the simplest form, as the classical mechanics of a point or a system of material points. Desiring to introduce thus into mechanics the principle of causality of natural science, we immediately encounter the difficulty that the idea of causality has not

been completely formulated in natural science. In the constant quests for causes, the naturalist is guided rather by his own intuition than by fixed procedures. We can state only that causality is linked in the closest way with the properties of time, specifically with the difference in the future and the past. Therefore, we will be guided by the following hypothesis:

- I. Time possesses a quality, creating a difference in causes from effects, which can be evoked by directivity or pattern. This property determines the difference in the past from the future.

The requirement for this hypothesis is indicated by the difficulties associated with the development of the Leibnitz idea concerning the definition of the directivity of time through the causal relationships. The profound studies by H. Reichenbach [1] and G. Withrow [2] indicate that one can never advance this idea strictly, without tautology. Causality provides us with a concept of the existence of the directivity of time and concerning certain properties of this directivity; at the same time, it does not constitute the essence of this phenomenon, but only its result.

Let us now attempt, utilizing the simplest properties of causality, to provide a quantitative expression of hypothesis I. Proceeding from those circumstances in which: (1) cause is always outside of the body in which the result is realized and (2) the result sets in after the cause; we can formulate the next two axioms:

- II. Causes and results are always separated by space. Therefore, between them there exists an arbitrarily small, but not equaling zero, spatial difference  $\delta x$ .
- III Causes and results are separated in time. Therefore, between their appearance there exists an arbitrarily small, but not equaling zero, time difference  $\delta t$  of a fixed sign.

Axiom II forms the basis of classical Newtonian mechanics. It is contained in a third law, according to which a variation in a quantity of motion cannot occur under the effect of external forces. In other words, in a body there cannot develop an external force without the participation of another body. Hence, based on the impenetrability of matter,  $\delta x \neq 0$ . However, on the basis of the complete reversibility of time, axiom III is lacking in the Newtonian mechanics:  $\delta t = 0$ .

In atomic mechanics, just the opposite takes place. In it, the principle of impenetrability loses its value and, based on the possibility of the superposition of fields, it is obviously assumed that  $\delta x = 0$ . However, in atomic mechanics there is a temporal irreversibility, which did not exist in Newtonian mechanics; the influence upon the system of a



macroscopic body, according to theorists, introduces a difference between the future and the past, because the future proves predictable, while the past is not. Therefore, in the temporal environs of the experiment,  $\delta t \neq 0$ , although it can be arbitrarily small. In this manner, classical mechanics and atomic mechanics enter into our axiomatics as two extreme systems. This circumstance becomes especially clear if we introduce the relationship:

$$\frac{\delta x}{\delta t} = c_2. \quad (1)$$

In a real world,  $c_2$  most likely constitutes a finite value. However, in classical mechanics,  $\delta x \neq 0$ ,  $\delta t = 0$ , and hence  $c_2 = \infty$ . In atomic mechanics,  $\delta x = 0$ ,  $\delta t \neq 0$ , and therefore  $c_2 = 0$ .

Let us now discuss the concept of the symbols  $\delta x$  and  $\delta t$  introduced by us. In a long chain of causal-resultant transformations, we are considering only that elementary chain wherein the cause produces the result. According to the usual physical viewpoints, this chain comprises a spatial time point, not subject to further analysis. However, on the bases of our axioms of causality, this elementary causal-resultant chain should have a structure caused by the impossibility of spatial-time superimposition of causes and effects. The condition of non-superimposition in the case of the critical approach is stipulated by the symbols  $\delta x$  and  $\delta t$ . Hence, these symbols signify the limit of the infinitely-small values under the condition that they never revert to 0. These symbols determine the point distances or dimensions of an “empty” point, situated between the material points, with which the causes and effects are linked. However, in the calculation of the intervals of the entire causal-resultant chain, they should be considered equal to 0 with any degree of accuracy. However, if they have infinitely low values of one order, their ratio  $c_2$  can be a finite value and can express a qualitatively physical property of the causal-resultant relationship. This physical property is included in the pattern of time, formulated qualitatively by hypothesis I.

In reality, according to definition (I), the value  $c_2$  has the dimensionality of velocity and yields a value to the rate of the transition of the cause to the effect. This transition is accomplished through the “empty” point, where there are no material bodies and there is only space and time. Hence, the value  $c_2$  can be associated only with the properties of time and space, not with the properties of bodies. Therefore,  $c_2$  should be a universal constant, typifying the pattern of time in our world. The conversion of the cause to an effect requires the overcoming of the “empty” point in space. This point is an abyss, through which the transition can be realized only with the aid of the time pattern.

From this, there follows directly the active participation of time in the processes of the material systems.

In Equation (1), the symbol  $\delta t$  has a definite meaning. It can be established by the standard condition: the future minus the past comprises a positive value. However, the sign of the value for  $\delta x$  is quite arbitrary, since space is isotropic and in it there is no principal direction. At the same time, the sign of  $c_2$  should be definite, because logically we should have a possibility of conceiving of the world with an opposite time pattern: i. e., of another sign. A difficulty arises, which at first glance seems insurmountable, disrupting the entire structure formulated until now. However, owing to just this difficulty, it becomes possible to make an unequivocal conclusion:  $c_2$  is not a scalar value but a pseudo-scalar value: i. e., a scalar changing sign in the case of the mirror image or inversion of the coordinate system. In order to be convinced of this, let us rewrite Equation (1) in a vector form, having signified by  $i$  the unit vector of the direction of the causal-resultant relationship:

$$c_2(i\delta t) = \delta x. \quad (1a)$$

If  $c_2$  is pseudo-scalar,  $i\delta t$  should be a critical of a pseudo-vector collinear with the critical vector  $\delta x$ . The pseudo-vector nature of  $i\delta t$  signifies that in the plane ( $yz$ ) of a perpendicular to the  $x$ -axis there occurs a certain turning, the sign of which can be determined by the sign of  $\delta t$ . This means that with the aid of  $\delta t$ , we can orient the plane perpendicular to the  $x$ -axis: i. e., we can allocate the arrangement of the  $y$  and  $z$  axes. Let us now alter in Equation (1) the sign of  $\delta x$ , retaining the sign of  $\delta t$  and signifying the retention of the orientation of the plane ( $yz$ ). Then the constant  $c_2$  changes its sign, as it should, since our operation is tantamount to a mirror image. However, if we change the sign not only of  $\delta x$  but also of  $\delta t$ , the constant  $c_2$  based on Equation (1) does not change sign. This should be the case, because in the given instance we effected only a turning of the coordinate system. Finally, changing the sign of  $\delta t$  only, we once again obtain a mirror (specular) image of the coordinate system under which the sign of the pseudo-scalar should change. This proof of the pseudo-scalar property of the time pattern can be explained by the following simple discussion. The time pattern should be determined in relation to a certain invariant. Such an invariant, independent of the properties of matter, can be only space. The absolute value of the time pattern is obtained when the absolute difference in the future and the past will be linked with the absolute difference between right and left, although these concepts per se are quite tentative. Therefore, the time pattern also should be

established by a value having the sense of a linear velocity of turning (rotation). From this it follows that  $c_2$  cannot equal the speed of light,  $c_1$ , comprising the conventional scalar.

From the pseudo-scalar properties of the time pattern, there immediately follows the basic theorem of causal mechanics: **a world with an opposite time pattern is equivalent to our world, reflected in a mirror.**

In a world reflected in a mirror, causality is completely retained. Therefore, in a world with an opposite time pattern the events should develop just as regularly as in our world. It is erroneous to think that, having run a movie film of our world in a reverse direction, we would obtain a pattern of the world of an opposite time direction. We can in no way formally change the sign in the time intervals. This leads to a disruption of causality: i. e., to an absurdity, to a world which cannot exist. In a variation of the directivity of time, the influences that the time pattern exerts upon the material system should appear as modified. Therefore, the world reflected in a mirror should differ in its physical properties from our world. Up until modern times, this identity was assumed in atomic mechanics and was said to be the law of the preservation of parity. However, studies by Lie and Young of the nuclear processes during weak interactions led to experiments that demonstrated the erroneous position of this law. This result is quite natural under the actual existence of time directivity, which is confirmed by direct experiments to be described later. At the same time, one can never make the opposite conclusion. Numerous investigations of the observed phenomena of the nonpreservation of parity have demonstrated the possibility of other interpretations. It is necessary to conclude that further experiments in the field of nuclear physics will narrow the scope of possible interpretations to such an extent that the existence of time directivity in the elementary processes will become quite obvious.

The difference in the world from the mirror image is graphically indicated especially by biology. The morphology of animals and plants provides many examples of asymmetry, distinguishing right from left, independently of which hemisphere of the Earth the organism is living in. Asymmetry of organisms is manifested not only in their morphology. The chemical asymmetry of protoplasm discovered by Louis Pasteur demonstrates that the asymmetry constitutes a basic property of life. The persistent asymmetry of organisms being transmitted to their descendants cannot be random. This asymmetry cannot only be a passive result of the laws of nature, reflecting the time directivity. Most likely, under a definite asymmetry, corresponding to the given time pattern,

an organism acquires an additional viability: i. e., it can use it for the reinforcement of life processes. Then, on the basis of our fundamental theorem, we can conclude that in a world with an opposite time pattern, the heart in the vertebrates would be located on the right, the shells of mollusks would be mainly turned leftward, and in protoplasm there would be observed an opposite qualitative inequality of the right and left molecules. It is possible that specially formulated biological experiments will be able to prove directly that life actually uses the time pattern as an additional source of energy.

Let us now comment on yet another important circumstance, connected with the determination of the time pattern by Equation (1). Each causal-resultant relationship has a certain spatial direction, the base vector of which is signified by  $i$ . Therefore, in an actual causal relationship, the pseudo-scalar  $ic_2$  will be oriented by the time pattern. Let us prove that at one point the values for — the cause — and at another point — the result — should be in opposite directions. In reality, the result in the future will be situated in relation to the cause, while the cause in the past will be situated in relation to the result. This means that at the points cause and effect,  $\delta t$  should have opposite signs, meaning that there should also be an opposite orientation of the plane perpendicular to  $i$ . Then, at a definite  $i$ -value we have a change in the type of the coordinate system, and the expression  $ic_2$  will have a different sign. However, if during the transition from the cause to the effect we have a change in the sign of  $i$ , the sign of  $c_2$  will remain unchanged; and, hence,  $ic_2$  will change sign in this case also. This means that the time pattern is characterized by the values  $\pm ic_2$  and constitutes a physical process, the model of which can be the relative rotation of a certain ideal top (gyroscope).

By an ideal gyroscope, we connote a body, the entire mass of which is located at a certain single distance from the axis. This top can have an effect on another body through a material axis of rotation and material relationships with this axis, the masses of which can be disregarded. Therefore, the mechanical property of an ideal gyroscope will be equivalent to the properties of a material point having the mass of the gyroscope and its rotation. Let us assume that the point with which the top interacts is situated along the direction of its axis. Let us signify by  $j$  the base vector of this direction and consider it to be a standard vector. We can tentatively, independent of the type of the coordinate system, place it in another point: for example, in the direction from which the rotation of the top appears to be originating — in this case, in a clockwise direction. The rotation of the top can be described by

the approximate pseudo-scalar  $ju$ , where  $u$  equals the linear velocity of rotation. With such a description and the direction selected by us,  $u$  should be a pseudo-scalar, positive in the left-hand system of coordinates. Let us now consider the motion of a point upon which the gyroscope axis is acting from the position of the point of its rim. Since the distance of this point from the plane of the rim is arbitrarily small, its velocity, computed from the position of the rim in respect to the radius and the period, will be the same value for  $u$ . We can draw on a sheet of paper the motion of the points of the rim relative to the center and to the motion of the center from the position of the rim points. The motion is obtained in one direction if we examine the paper from the same side: e. g., from above. However, the infinitely small emergence of a stationary point from the plane of the rim compels us to examine the rotation from another position: i. e., to examine the paper from beneath. We obtain a rotation in the opposite direction, as a result of which we should compare with the gyroscope the approximate pseudo-scalar: i. e.,  $ju$ . This signifies that the time pattern being determined by the values  $\pm ic_2$  actually has an affinity with the relative rotation, which is determined by the values  $\pm ju$  of the same type. Understandably, this formal analogy does not fully explain the essence of a time pattern. However, it opens up the remarkable possibility of an experimental study of the properties of time.

In reality, if into the causal relationship there enters a rotating body, we can expect that in a system with rotation, the time pattern will change instead of  $\pm ic_2$ : it becomes equal to  $\pm(ic_2 + ju)$ . Let us now attempt to explain which variations from this can occur in a mechanical system. For this, it is necessary to refine the concept of cause and effect in mechanics.

The forces are the causes altering the mutual arrangement of bodies and their quantity of motion. The change in the arrangement of bodies can lead to the appearance of new forces, and according to the d'Alembert principle, the variation of a quantity of motion for unit time, taken with an opposite sign, can be regarded as the force of inertia. Therefore, in mechanics the forces are comprised of the causes and all possible effects. However, in the movement of a body (1) under the effect of force  $F$ , the force of inertia,  $-dp/dt$ , does not constitute a result. Both of these forces originate at one point. According to axiom II, owing to this there cannot be a causal-resultant relationship between them, and they are identical concepts. Therefore, as Kirchhoff operated in his mechanics, the force of inertia can serve as a determination of the force  $F$ . The force  $F$ , applied to point (1), can evoke an effect only in

another point (2). Let us call this force of the result the effect  $\Phi_0$  of the first point upon the second:

$$\Phi_0 = F - \frac{dp_1}{dt} = \frac{dp_2}{dt}. \quad (2)$$

For the first point, however, it comprises the lost d'Alembert force:

$$\frac{dp_1}{dt} = F - \frac{dp_2}{dt}.$$

In conformity with these expressions, we can consider that for one time,  $dt$ , point (1) loses the pulse  $dp_2$  which is transmitted to point (2). In the case for which there is a causal relationship between point (1) and (2),  $\delta t \neq 0$ , and between them there exists the approximate difference  $\delta p_2 \neq 0$ . When the cause is situated at point (1), the transition of  $dp_2$  from point (1) to point (2) corresponds to an increase in the time. Therefore:

$$\frac{\delta p_2}{\delta t} = \frac{dp_2}{dt} = \Phi_0. \quad (3)$$

Let us signify by  $i$  the unit vector of effect  $\Phi_0$ . Then, according to (3):

$$\Phi_0 = i |\Phi_0| = i \frac{\delta p_2}{\delta t} = i \left| \frac{\delta p_2}{\delta x} \right| \left| \frac{\delta x}{\delta t} \right|.$$

According to (1), the value  $|\delta x|/\delta t$  can be replaced by  $c_2$  if we tentatively utilize that system of coordinates in which  $c_2$  is positive.

Under this condition:

$$\Phi_0 = i c_2 \left| \frac{\delta p_2}{\delta x} \right|. \quad (4)$$

The factor  $i c_2$  comprises a value independent of a time pattern: i. e., a force invariant. In reality, during any pattern of time not only the spatial intervals but also the time intervals should be measured by the unchanging scales (weights). Therefore, the velocity and, consequently, also the pulses should not depend on the pattern (course) of time. As was demonstrated above, in case of the existence of a time pattern  $i c_2$  in point (2), there must be in point (1) the time pattern  $-i c_2$ . This means that during the effect upon point (2), there must be a counter effect or a reaction force  $R_0$  in point (1):

$$R_0 = -i c_2 \left| \frac{\delta p_2}{\delta x} \right|. \quad (5)$$

Thus, the third Newtonian law proves to be the direct result of the properties of causality and pattern of time. The effect and the counter

effect comprise two facets of the identical phenomenon, and between them a time discontinuity cannot exist. In this manner, the law of the conservation of a pulse is one of the most fundamental laws of nature.

Let us now assume that the time pattern has varied and, instead of  $\pm ic_2$  it has become equal to  $\pm(ic_2 + ju)$ . Then, based on Eqs. (4) and (5), the following transformation of forces should occur:

$$\Phi = (ic_2 + ju) \left| \frac{\delta p_2}{\delta x} \right|, \quad R = -(ic_2 + ju) \left| \frac{\delta p_2}{\delta x} \right|.$$

The additional forces are obtained:

$$\begin{aligned} \Delta\Phi &= \Phi - \Phi_0 = +j \frac{u}{c_2} |\Phi_0|, \\ \Delta R &= R - R_0 = -j \frac{u}{c_2} |\Phi_0|. \end{aligned} \tag{6}$$

Thus, in the causal relationship with a spinning top (gyroscope), we can expect the appearance of additional forces (6) acting along the axis of rotation of the top. The proper experiments described in detail in the following section indicate that, in reality, during the rotation, forces develop acting upon the axis and depending on the time direction. The measured value of the additional forces permits us to determine, based on (6), the value of  $c_2$  of the time pattern not only in magnitude but also in sign: i. e., to indicate the type of the coordinate system in which  $c_2$  is positive. **It turns out that the time pattern of our world is positive in a laevorotary system of coordinates.** From this, we are afforded the possibility of an objective determination of left and right; the left-hand system of coordinates is said to be that system in which the time progress is positive, while the right-hand system is one in which it is negative. In this manner, the time progress linking all of the bodies in the world, even during their complete isolation, plays the role of that material bridge concerning the need, of which Gauss [3] has already spoken, for the coordination of the concepts of left and right.

The appearance of additional forces can perhaps be graphically represented in the following manner: Time enters a system through the cause to the effect. The rotation alters the possibility of this inflow, and, as a result, the time pattern can create additional stresses in the system. These variations produce the time pattern. From this it follows that **time has energy**. Since the additional forces are directed oppositely, the pulse of the system does not vary. This signifies that time does not have a pulse, although it possesses energy.

In Newtonian mechanics,  $c_2 = \infty$ . The additional forces according to

(6) disappear, as should occur in this mechanics. This is natural because the infinite pattern of time can in no way be altered. Therefore, time proves to be an imparted fate and invincible force. However, **the actual time has a finite pattern and can be effective**, and this signifies that **the principle of time can be reversible**. How, in reality, these effects can be accomplished should be demonstrated by experiments studying the properties of time.

In atomic mechanics,  $c_2 = 0$ . Equations (6), obtained by a certain refinement of the principles of Newtonian mechanics, are approximate and do not give the critical transition at  $c_2 = 0$ . They only indicate that the additional effects not envisaged by Newtonian mechanics will play the predominant part. The causality becomes completely intertwined (confused), and the occurrences of nature will remain to be explained statistically.

The Newtonian mechanics corresponds to a world with infinitely stable causal relationships, while atomic mechanics represents another critical state of a world with infinitely weak causal relationships. Equations (6) indicate that the mechanics corresponding to the principles of causality of natural science should be developed from the aspect of Newtonian mechanics, and not from the viewpoint of atomic mechanics. In this connection, there can appear features typical for atomic mechanics. For instance, we can expect the appearance of quantum effects in macroscopic mechanics.

The theoretical concepts expounded here are basically necessary only in order to know how to undertake the experiments on the study of the properties of time. Time represents an entire world of enigmatic phenomena, and they can in no way be pursued by logical deliberations. The properties of time must be gradually explained by physical experiment.

For the formulation of experiments, it is important to have a foreknowledge of the value of the expected effects, which depend upon the value  $c_2 = 0$ . We can attempt to estimate the numerical value of  $c_2 = 0$ , by using the atomic mechanics and proceeding from dimensionality concepts. The single universal constant which can have the meaning of a pseudo-scalar is the Planck constant,  $h$ . In reality, this constant has the dimensionality of a moment of a quantity of motion and determines the spin of elementary particles. Now, utilizing the Planck constant in any scalar universal constant, it is necessary to obtain a value having the dimensionality of velocity. It is easy to establish that the expression

$$c_2 = \frac{\alpha e^2}{h} = \alpha \times 350 \text{ km/sec} \quad (7)$$



comprises a unique combination of this type. Here  $e$  equals the charge of an elementary particle and  $\alpha$  equals a certain dimensionless factor. Then, based on (6), at  $u = 100$  m/sec, the additional forces will be of the order of  $10^{-4}$  or  $10^{-5}$  (at a considerable  $\alpha$ -value) from the applied forces. At such a value for  $c_2$ , the forces of the time pattern can easily be revealed in the simplest experiments not requiring high accuracy of measurements.

## **Part II. Experiments on studying the properties of time, and basic findings**

The experimental verification of the above-developed theoretical concepts was started as early as the winter of 1951–1952. From that time, these studies have been carried on continuously over the course of a number of years with the active participation by graduate student V. G. Labeysh. At the present time, they are underway at the laboratory of the Pulkovo Observatory with engineer V. V. Nasonov. The work performed by Nasonov imparted a high degree of reliability to the experiments. During the time of these investigations, we accumulated numerous and diversified data, permitting us to form a number of conclusions concerning the properties of time. We did not succeed in interpreting all of the material, and not all of the material has a uniform degree of reliability. Here we will discuss only those data which were subjected to a recurrent checking and which, from our viewpoint, are completely reliable. We will also strive to form conclusions from these data.

The theoretical concepts indicate that the tests on the study of causal relationships and the pattern of time need to be conducted with rotating bodies: namely, gyroscopes. The first tests were made in order to verify that the law of the conservation of a pulse is always fulfilled, and independently of the condition of rotation of bodies. These tests were conducted on lever-type weights (scales). At a deceleration of the gyroscope, rotating by inertia, its moment of rotation should be imparted to the weights (scales), causing an inevitable torsion of the suspensions. In order to avert the suspension difficulties associate with this, the rotation of the gyroscope should be held constant. Therefore, we utilized gyroscopes from aviation automation, the velocity of which was controlled by a variable 3-phase current with a frequency of the order of 500 cps. The gyroscope's rotor turned with this same frequency. It appeared possible, without decreasing significantly the suspension precision, to supply current to the gyroscope suspended on weights (scales) with the

aid of three very thin uninsulated conductors. During the suspension the gyroscope was installed in a hermetically sealed box, which excluded completely the effect of air currents. The accuracy of this suspension was of the order of 0.1–0.2 mg. With a vertical arrangement of the axis and various rotation velocities, the readings of the weights (scales) remained unchanged. For example, proceeding from the data for one of the gyroscopes (average diameter  $D$  of rotor equals 4.2 cm: rotor weight  $Q$  equals 250 gm), we can conclude that with a linear rotational velocity,  $u = 70$  m/sec, the effective force upon the weights (scales) will remain unchanged, with a precision higher than up to the sixth place. In these experiments, we also introduced the following interesting theoretical complication: The box with the gyroscope was suspended from an iron plate, which attracted the electromagnets fastened together with a certain mass. This entire system was suspended on weights (scales) by means of an elastic band. The current was supplied to the electromagnets with the aid of two very thin conductors. The system for breaking the current was accomplished separately from the weights (scales). At the breaking of the circuit, the box with the gyroscope fell to a clipper fastened to the electromagnets. The amplitude of these drops and the subsequent rise could reach 2 mm. The test was conducted for various directions of suspension and rotation masses of the gyroscope, at different amplitudes, and at an oscillation frequency ranging from units to hundreds of cps. For a rotating gyroscope, just as for a stationary one, the readings of the weights (scales) remained unchanged. We can consider that the experiments described substantiate fairly well the theoretical conclusion concerning the conservation of a pulse in causal mechanics.

In spite of their theoretical interest, the previous experiments did not yield any new effects capable of confirming the role of causality in mechanics. However, in their fulfillment it was noted that in the transmission of the vibrations from the gyroscope to the support of the weights (scales), variations in the readings of the weights (scales) can appear, depending on the velocity and direction of rotation of the gyroscopes. When the vibrations of the weights (scales) themselves begin, the box with the gyroscope discontinues being strictly a closed system. However, the weights (scales) can go out of equilibrium if the additional effect of the gyroscope developing from rotation proves to be transferred from the shaft of the gyroscopes to the weights' [scales'] support. From these observations, we developed a series of tests with these gyroscopes.

In the first type, the vibrations were due to the energy of the rotor

and its pounding in the bearings, depending on the clearance in them. It is understandable that the vibrations interfere with accurate suspension. Therefore, it was necessary to abandon the precision weights (scales) of the analytical type and convert to engineering weights (scales), in which the ribs of the prisms contact small areas having the form of caps. Nevertheless, in this connection we managed to maintain an accuracy of the order of 1 mg in the differential measurements. The support areas in the form of caps are also convenient by virtue of the fact that with them we can conduct the suspension of gyroscopes rotating by inertia.

A gyroscope suspended on a rigid support can transmit through a yoke its vibrations to the support of the weights (scales). With a certain type of vibration, which was chosen completely by feel, there occurred a considerable decrease in the effect of the gyroscope upon the weights (scales) during its rotation in a counterclockwise direction, if we examined it from above. During the rotation in a clockwise direction, under the same conditions, the readings of the weights (scales) remained practically unchanged. Measurements conducted with gyroscopes of varying weight and rotor radius, at various angular velocities, indicated that **reduction of the weight, in conformity with (6), is actually proportional to the weight and to the linear rate of rotation.** For example, at a rotation of the gyroscope ( $D = 4.6$  cm,  $Q = 90$  gm,  $u = 25$  m/sec), we obtained the weight difference of  $-8$  mg. With rotation in a clockwise direction, it always turned out that (the weight difference)  $= 0$ . However, with a horizontal arrangement of the axis, in azimuth, we found the average value  $= -4$  mg. From this, we can conclude that any vibrating body under the conditions of this experiment should indicate a reduction in weight. Further studies demonstrated that **this effect is caused by the rotation of the Earth**, which will be discussed in detail later.

Presently, the only fact of importance to us is that during the vibration there develops a new zero reading relative to which **with a rotation in a counterclockwise direction, we obtain a weight reduction, while during a rotation in a clockwise direction we obtain a completely uniform increase in weight** ( $+4$  mg). In this manner, (6) is given a complete, experimental confirmation.

It follows from the adduced data that  $c_2 = 550$  km/sec. According to this condition, the vector  $j$  is oriented in that direction in which the rotation appears to be originating in a clockwise direction. This means that during the rotation of the gyroscope in a clockwise direction it is directed downward. With such a rotation, the gyroscope becomes lighter, meaning that its additional effect upon the support of the weights is di-

rected downward: i. e., in respect to the base vector  $j$ . This will obtain in the case in which  $u$  and  $c_2$  have the same signs. Under our condition relative to the direction of the base vector  $j$ , the pseudo-scalar  $u$  is positive in a left-hand system of coordinates. Consequently, a time pattern of our world is also positive in a left-hand system. Therefore, subsequently we will always utilize a left-hand system of coordinates. The aggregation of the tests conducted then permitted us to refine the value of  $c_2$ :

$$c_2 = +700 \pm 50 \text{ km/sec in a left-hand system.} \quad (8)$$

This value always makes probable the relationship of the time pattern with other universal constants based on (7) at  $\alpha = 2$ . Then, the dimensionless constant of the thin Sommerfeld structure becomes simply a ratio of the two velocities  $c_2/c_1$ , each of which occur in nature.

The tests conducted on weight (scales) with vibrations of a gyroscope also yield a new basic result. It appears that the additional force of effect and counter effect can be situated at different points in the system: i. e., on the support of the weights (scales) and on the gyroscope. We derive a pair of forces rotating the balance arm of the weights (scales). Hence, **time possesses not only energy but also a rotation moment which it can transmit to a system.**

A basic checking of the results obtained with the weights (scales) yields a pendulum in which the body constitutes a vibrating gyroscope with a horizontal axis suspended on a long fine thread. As in the tests conducted with the weights (scales), during the rotation of a gyroscope under quiescent conditions nothing took place and this filament (thread) did not deflect from the perpendicular. However, at a certain stage of the vibrations in the gyroscope the filament deflected from the perpendicular, always at the same amount (with a given  $u$ -value) and in the direction from which the gyroscope's rotation occurred in a counter-clockwise direction. With a filament length  $l = 2$  m and  $u = 25$  m/sec, the deflection amounted to 0.07 mm, which yields, for the ratio of the horizontal force of the weight, the value  $3.5 \times 10^{-5}$ , sufficiently close to the results of this suspension.

A significant disadvantage of the tests described is the impossibility of a simple control over the conditions of vibration. Therefore, it is desirable to proceed to tests in which the vibrations are developed not by the rotor but by the stationary parts of the system.

In the weights, the support of the balance arm was gripped by a special clamp, which was connected by a flexible cable with a long metal plate. One end of this plate rested on a ball-bearing, fitted eccentrically

to the shaft of an electric motor, and was connected by a rubber clamp with the bearing. The other end of the plate was fastened by a horizontal shaft. Changing the speed of the electric motor and the position of the cable on the plate, we were able to obtain harmonic oscillations from the balance arm support of the weights (scales) at any frequency and amplitude. The guiding devices for raising the balance arm support during a stopping of the weights eliminated the possibility of horizontal swaying. For the suspension of the gyroscope, it was necessary to find the optimal conditions under which the vibration was transmitted to the rotor and, at the same time, maintain that one end of the balance arm remain quasi-free relative to the other end, to which the balancing load was rigidly suspended. Under such conditions, the balance arm can vibrate freely, rotating around its end, fastened by a weight to a rigid suspension. Oscillations of this type could be obtained by suspending the gyroscope on a steel wire 0.15 mm in diameter and with a length of the order of 1–1.5 m. With this arrangement, we observed the variation in the weight of the gyroscope during its rotation around the vertical axis. It was remarkable that, in comparison with the previous tests, the effect proved to be of the opposite sign. During the turning of the gyroscope counterclockwise, we found, not a lightening, but a considerable weight increase. This means that in this case there operates on the gyroscope an additional force, oriented in a direction from which the rotation appears to be originating in a clockwise direction. This result signifies that the causality in the system and the time pattern introduced a vibration and that the source of the vibration established the position of the cause. In these tests, a source of vibration is the non-rotating part of the system, while in the initial model of the tests, a rotor constituted a source. Transposing in places the cause and the effect, we alter in respect to them the direction of rotation: i. e., the sense of base vector  $j$ . From this, based on (6), there originates the change in the sign of the additional forces. In conventional mechanics all of the forces do not depend entirely on what comprises the source of the vibration, but also on what is the effect. However, in causal mechanics, observing the direction of the additional forces, we can immediately state where the cause of the vibrations is located. This means that in reality it is possible to have a mechanical experiment distinguishing the cause from the effects.

The tests with the pendulum provided the same result. A gyroscope suspended on a fine wire, during the vibration of a point of this suspension, deflected in a direction from which its rotation transpired in a clockwise direction. The vibration of the suspension was accomplished

with the aid of an electromagnetic device. To the iron plate of a relay installed horizontally, we soldered a flexible metal rod, on which the pendulum wire was fastened. Owing to the rod, the oscillations became more harmonic. The position of the relay was regulated in such a way that there would not be any horizontal displacements of the suspension point. For monitoring the control, we connected a direct current, with which the electromagnet attracted the plate and raised the suspension point. The position of the filament (thread) was observed with a laboratory tube having a scale with divisions of 0.14 mm for the object under observation. Estimating by eye the fractions of this wide division, we could, during repeated measurements, obtain a result with an accuracy up to 0.01 mm. At a pendulum length  $l = 3.30$  m and a rotation velocity  $u = 40$  m/sec, the deflection of the gyroscope  $\Delta l$  was obtained as equaling 0.12 mm. In order to obtain a value of the additional force  $\Delta Q$  in relation to the weight of the rotor ( $Q = 250$  gm), it is necessary to introduce a correction for the weight of the gyroscope mounting  $a = 150$  gm: i. e., to multiply  $\Delta l/l$  by  $(Q + a)/Q$ . From this, we derive just that value of  $c_2$  which is represented above (8). In these tests it turned out that to obtain the effect of deflection of the filament, the end of the gyroscope shaft, from which the rotation appears to be originating in a clockwise direction, must be raised somewhat. Hence, in this direction there should exist a certain projection of force, raising the gyroscope during the vibrations. In reality, the effect of the deflection turns out to be even less when we have accomplished a parametric resonance of the thread with oscillations, the plane of which passed through the gyroscope axis. Evidently, the existence of forces acting in the direction  $ju$  intensifies the similarity of  $ju$  with the time pattern and facilitates the transformation  $\pm c_2$  by  $\pm (ic_2 + ju)$ . It is also necessary to comment that the gyroscope axis needs to be located in the plane of the first vertical. With a perpendicular arrangement of the axis — i. e., in the plane of the meridian — a certain additional displacement develops. Obviously, this displacement is created by the force evoked by the Earth's rotation, which we mentioned in describing the first experiments of the vibrations on weights. Let us now return to an explanation of these forces.

Let us signify by  $u$  the linear velocity of the rotation of a point situated on the Earth's surface. This point is situated in gravitational interaction with all other points of the Earth's volume. Their effect is equivalent to the effect of the entire mass of the Earth at a certain average velocity  $\bar{u}$ , the value of which is located between zero and  $u$  at the equator. Therefore, in the presence of a causal relationship there

can originate additional forces, directed along the axis of the Earth, and similar forces acting upon the gyroscope during its rotation with the velocity  $(u - \bar{u})$  relative to the mounting. If the causal occurrences of the cosmic life of the Earth are associated with the outer layers, these forces should act upon the surface in the direction from which the rotation appears to be originating counterclockwise: i. e., toward the north. Thus, in this case on the Earth's surface there should operate the forces of the time pattern:

$$\Delta Q = \frac{-j(u - \bar{u})}{c_2} |Q|, \quad (9)$$

where  $j$  is the Earth's orthonormal vector directed at south, while  $Q$  is the force of weight. In the interior of the Earth, forces act in the opposite direction, and according to the law of conservation of momentum, the Earth's center does not become displaced. In the polar regions  $u < \bar{u}$ , and therefore in both hemispheres  $\Delta Q$  will be directed southward. Hence, in each hemisphere there is found a typical parallel where  $\Delta Q = 0$ . Under the effect of such forces, the Earth will acquire the shape of a cardioid, extending to the south. One of the parameters characterizing a cardioid is the coefficient of asymmetry  $\eta$ :

$$\eta = \frac{b_s - b_N}{2a}, \quad (10)$$

where  $a$  equals the major semi-axis and  $b_s$  and  $b_N$  are the distances of the poles to the equatorial plane.

On Jupiter and Saturn the equatorial velocity  $u$  is around 10 km/sec. Therefore, on planets with a rapid rotation the factor can be very high and reach nonconformity with expressions (8) and (9) by several units of the third place. Careful measurements of photographs of Jupiter made by the author and D. O. Mokhnach [4] showed that on Jupiter the southern hemisphere is more extended and  $\eta = + 3 \times 10^{-3} \pm 0.6 \times 10^{-3}$ . A similar result, only with less accuracy, was also obtained for Saturn:  $\eta = 7 \times 10^{-3} \pm 3 \times 10^{-3}$ .

The measurements of the force of gravity of the surface of the Earth and the motion of artificial Earth satellites indicate that there exists a certain difference of accelerations of gravity in the northern and southern hemispheres:  $\Delta g = b_N - b_s > 0$ ,  $\Delta g/g = 3 \times 10^{-5}$ . For a homogenous planet this should also be the case for an extended southern hemisphere, because the points of this hemisphere are located farther from the center of gravity. The factor  $\eta$  should be of the order of  $\Delta g/g$ . It is necessary to stress that the conclusion is in direct contradiction with the adopted

interpretation of the above-presented data concerning the acceleration of gravity. The gist of this difference consists in the fact that without allowance for the forces of the time pattern, the increase in gravity in the northern hemisphere can be explained only by the presence there of denser rocks. In this case, the leveled surface of the same value should regress farther. Identifying the level surface with the surface of the Earth, it will remain to be inferred that the northern hemisphere is more extended. However, the sign of  $\eta$  obtained directly for Jupiter and Saturn provide evidence against this interpretation, containing in itself a further contradictory assumption concerning the disequilibrium distribution of the rocks within the Earth.

The sign obtained for the asymmetry of the shapes of planets leads to the paradoxical conclusion to the effect that the cause of the physical occurrences within the celestial bodies is situated in the peripheral layers. However, such a result is possible if, e. g., the energetics of a planet are determined by its compression. In his studies on the internal structure of a star [5], the author concluded that the power of stars is very similar to the power of cooling and compressing bodies. The inadequacy of the knowledge of the essence of the causal relationships prevents us from delving into this question. At the same time, we are compelled to insist on the conclusions which were obtained from a comparison of the asymmetry of the planets with the forces acting upon the gyroscope.

The direction of the perpendicular on the Earth's surface is determined by the combined effect of the forces of gravity, of centrifugal forces, and of the forces of the time pattern  $\Delta Q$  operating toward the north in our latitudes. In the case of a free fall, the effect on the mounting is absent ( $\Delta Q = 0$ ) and therefore  $\Delta Q = 0$ . As a result, the freely falling body should deflect from the perpendicular to the south by the value  $\Delta l_s$ :

$$\Delta l_s = -\frac{\Delta Q_N}{Q}, \quad (11)$$

where  $l$  equals the height of the body's fall and  $\Delta Q_N$  equals the horizontal component of the forces of the time pattern in the moderate latitudes. A century or two ago this problem of the deflection of falling bodies toward the south attracted considerable attention. Already the first experiments conducted by Hook in January of 1680 at the behest of Newton for the verification of the deflection of falling bodies eastward led Hook to the conviction that a falling body deflects not only eastward but also southward. These experiments were repeated many times and often led to the same result. The best determinations were made by engineer Reich in the mine shafts of Freiburg [6]. At  $l=158\text{m}$ , he ob-



tained  $\Delta l_s = 4.4 \text{ mm}$  and  $\Delta l_{\text{east}} = 28.4 \text{ mm}$ . These deflections agree well with the theory. Based on (11) from these determinations, it follows that

$$\frac{\Delta Q_N}{Q} = 2.8 \times 10^{-5} \quad \text{at } \varphi = 48^\circ, \quad (12)$$

which agrees well with our approximate concepts concerning the asymmetry of the Earth's shape. The experiments on the deflection of falling bodies from a perpendicular are very complex and laborious. The interest in these tests disappeared completely after Hagen in the Vatican [7] with the aid of an Atwood machine obtained a deflection eastward in excellent agreement with the theory, but he did not derive any deflection southward. On the Atwood machine, owing to the tension of the filament, the eastward deflection decreases by only one half. However, the southward deflection during the acceleration equals  $1/25$  (as was the case for Hagen) and, according to Eqs. (9) and (11), it should decrease by 25 times. Therefore, the Hagen experiments do not refute to any extent the effect of the southward deflection.

Let us now return to the occurrences developing during the vibration of a heavy body on the surface of the Earth. The causal-resultant relationship within the Earth creates on the surface, in addition to the standard time pattern  $\pm ic_2$ , the time pattern  $\pm [ic_2 - j(u - \bar{u})]$ . Therefore, on the surface of the Earth, on a body with which a cause is connected, there should act the additional force  $\Delta Q$ , directed northward along the axis of the Earth and being determined by (9). In the actual place where the effect is located, there should operate a force of opposite sign: i. e., southward. This means that during vibrations a heavy body should become lighter. In the opposite case, where the source of vibration is connected with the mounting, the body should become heavier. In a pendulum, during a vibration of the suspension point, there should occur a deflection toward the south. These phenomena have opened up the remarkable possibility not only of measuring the distribution of the forces of the time pattern of the surface of the Earth but also of studying the causal relationships and the properties of time by the simplest mode, for the conventional bodies, without difficult experiments with gyroscopes.

The tests on the study of additional forces caused by the Earth's rotation have the further advantage that the vibration of the point of the mounting cannot reach the body itself. The damping of the vibrations is necessary in order to express better the difference in the positions of cause and effect. Therefore, it is sufficient to suspend a body on weights on a short rubber band, assuring an undisturbed mode of operation of

the weights during the vibrations. In a pendulum, one should use a fine capron thread. In the remaining objects the tests were conducted in the same way as with the gyroscopes.

In the weights, during vibrations of the mounting of the balance arm, an increase actually occurs in the weights of a load suspended on an elastic. From the results of many experiments it was proved that the increase in the weight — i. e., the vertical component of the additional force  $\Delta Q_z$  — is proportional to the weight of the body  $Q$ . For Pulkovo,  $\Delta Q_z/Q = 2.8 \times 10^{-5}$ . The horizontal components,  $\Delta Q_s$ , were determined from the deflection of pendulums of various length (from 2 to 11 meters) during the vibration of a suspension point. During such vibrations the pendulums, in conformity with the increased load of the weights, deflected southward. For example, at  $l = 3.2$  m, we obtained  $\Delta l = 0.052$  mm. From this,  $\Delta Q_s/Q = \Delta l_z/l = 1.6 \times 10^{-5}$ , which corresponds fully to the Reich value [6] found for the lower latitude. If the force  $\Delta Q$  is directed along the Earth's axis, there should be fulfilled the condition:  $\Delta Q_s/Q_z = \tan \varphi$ , where  $\varphi$  equals the latitude of the site of the observations. From the data presented, it follows that  $\tan \varphi = 1.75$ , which completely conforms with the latitude at Pulkovo.

Similar tests were made for a higher latitude in the city of Kirovsk, and here also a good agreement with the latitude was obtained. For the weights and the pendulums, the amplitudes of the vibrations of the mounting point were of the order of tenths of a millimeter, while the frequency changed within the limits of tens of cycles per second.

The measurements conducted at various latitudes of the Northern Hemisphere demonstrated that, in reality, **there exists a parallel where the forces of time are lacking**:  $\Delta Q = 0$  at  $\varphi = 73^\circ 05'$ . Extrapolating the data from these measurements, we can obtain for the pole the estimation  $\Delta Q/Q = 6.5 \times 10^{-5}$ . Having taken the value  $c_2$  found from the tests conducted with a gyroscope (8), let us find from this for the pole:  $\bar{u} \simeq 45$  m/sec. At the equator the velocity of the Earth's rotation is 10 times higher. Therefore, the indicated  $u$ -value can prove to be less than that expected. However, it is necessary to have it in mind that presently we do not have the knowledge of the rules of combining the time pattern which are necessary for the strict calculation for the  $\bar{u}$ . Taking into account the vast distance in the kinematics of the rotations of a laboratory gyroscope and of the Earth, we can consider the results obtained for both cases as being in very good agreement.

On the weights (scales), we conducted a verification of the predicted variation in the sign, when the load itself becomes a source of vibration. For this, under the mounting area of the balance arm we introduce a

rubber lining, and in place of the load on the elastic, we rigidly suspend an electric motor with a flywheel which raises and lowers a certain load. In the case of such vibrations, the entire linkage of the balance arm of the weights remained as before. At the same time, we did not obtain an increase in the weight, but a lightening of the system suspended to the fluctuating end of the balance arm. This result excludes completely the possibility of the classic explanation of the observed effects and markedly indicates the role of causality.

In the experiments with vibrations on weights (scales) the variation in the weight of a body,  $\Delta Q_z$ , occurs in jumps, starting from a certain vibration energy. With a further increase in the frequency of the vibrations, the variation in the weight remains initially unchanged, then increases by a jump in the same value. In this manner, it turns out that in addition to the basic separating stage  $\Delta Q_z$ , that good harmonic state of the oscillations, we can observe a series of quantized values:  $\frac{1}{2}\Delta Q$ ,  $\Delta Q$ ,  $2\Delta Q$ ,  $3\Delta Q$ ,  $\dots$ , corresponding to the continuous variation in the frequency of vibrations. From the observations, it follows that the energy of the vibrations of the beginning of each stage evidently forms such a series. In other words, to obtain multiple values, the frequencies of the vibrations must be  $\sqrt{2}$ ,  $\sqrt{3}$ , etc. The impression gained is that weights in the excited stage behave like weights without vibrations: the addition of the same energy of vibrations leads to the appearance of the stage  $\Delta Q_z$ . However, we have not yet managed to find a true explanation of this phenomenon.

The appearance of the half quantum number remains quite incomprehensible. These quantum effects also occurred in the tests conducted with pendulums. Subsequently, it turned out that the quantum state of the effects is obtained in almost all of the tests. It should be noted that with the weights, we observed yet another interesting effect, for which there is no clear explanation. The energy of the vibrations, necessary for the excitation of a stage, depends upon the estimate of the balance arm of the weights (scales). The energy is minimal when the load on the elastic is situated to the south of the weights' (scales') supports, and maximal when it is located to the north. The tests conducted with vibrations have the disadvantage that the vibrations always affect, to some extent, the accuracy of the measuring system. At the same time, in our tests vibrations were necessary in order to establish the position of the causes and effects. Therefore, it is extremely desirable to find another method of doing this.

For example, we can pass a direct electric current through a long metal wire, to which the body of the pendulum is hung. The current

can be introduced through a point of the suspension and passed through a very fine wire at the body of the pendulum without interfering with its oscillations. The Lorentz forces, the interaction of current, and the magnetic field of the Earth operated in the first vertical and cannot cause a meridional displacement of interest to us. These experiments were crowned with success. Thus, in starting from 15 V and a current of 0.03 amps, there appeared a jump-like deflection toward the south by an amount of 0.024 mm, which was maintained during a further increase of the voltage up to 30 V. To this deviation there corresponds the relative displacement  $\Delta l/l = 0.85 \times 10^{-5}$ , which is almost exactly half of the stage observed during the vibrations. In the case of a plus voltage at the point of the suspension, we obtained a similar deflection northward. In this manner, knowing nothing of the nature of the electrical current, we could already conclude, from only a few of these tests, that the cause of the current is the displacement of the negative charges.

It turns out that in the pendulum, the position of the cause and effect can be established even more simply by heating or cooling the point of the suspension. For this, the pendulum must be suspended on a metal wire which conducts heat well. The point of the suspension was heated by an electrical coil. During a heating of this coil until it glowed, the pendulum deflected southward by half of the stage, as during the tests conducted with the electrical current. With a cooling of the suspension point with dry ice, we obtained a northward deflection. A southward deflection can also be obtained by cooling the body of the pendulum, such as placing it in a vessel containing dry ice. In these experiments, only under quite favorable circumstances did we succeed in obtaining the full effect of the deflection. It is obvious that the vibrations have a certain basic advantage. It is likely that not only dissipation of the mechanical energy is significant during the vibrations; it is probable that the forces of the vibrations directed along  $ju$  cause the appearance of additional forces.

In the study of the horizontal forces, the success in the heat experiments permitted us to proceed from long pendulums to a much more precise and simpler device: namely, the torsion balance. We applied torsion balances of optimal sensitivity, for which the expected deflection was 5–20 degrees. We utilized a balance arm of apothecary weights (scales), to the upper handle of which we soldered a special clamp, to which was attached a fine tungsten wire with a diameter of 35 microns and a length of around 10 cm. The higher end of the wire was fastened by the same clamp to a stationary support. To avoid the accumulation of electrical charges and their electrostatic effect, the weights (scales)

were reliably grounded through the support. From one end of the balance arm we suspended a metal rod along with a small glass vessel, into which it entered. At the other end was installed a balancing load of the order of 20 grams. The scale, divided into degrees, permitted us to determine the turning angle of the balance arm. The vessel into which the metal rod entered was filled with snow or water with ice. Thereby, there developed a flow of heat along the balance arm to the rod, and the weights (scales), mounted beforehand in the first vertical, were turned by this end toward the south. The horizontal force  $\Delta Q_s$  was computed from the deflection angle  $a$  with the aid of the formula:

$$a = \frac{T^2 - T_0^2}{4\pi^2} \frac{g}{2l} \left( \frac{\Delta Q}{Q} \right), \quad (13)$$

where  $T$  equals the period of the oscillation of the torsion balances;  $T_0$  equals the period of oscillations of one balance arm, without loads;  $g$  equals the acceleration of gravity; and  $2l$  equals the length of the balance arm: i.e., between the suspended weights. In this equation the angle  $a$  is expressed in radians. For example, in the weights with  $l = 9.0$  cm,  $T = 132$  seconds, and  $T_0 = 75$  seconds, we observed a southward deflection by an angle of  $17.5^\circ$ . Thence, based on (13), it follows that  $\Delta Q_s/Q = 1.8 \times 10^{-5}$  is in good agreement with the previously derived value of the horizontal forces. Half and multiple displacements were also observed in these experiments conducted with the torsion balances. Another variation of the experiment was the heating, by a small alcohol lamp, of a rod suspended together with a vessel containing ice. The same kind of alcohol lamp was placed at the other end of the balance arm with a compensating weight, but in such a way that it could not heat the balance arm. During the burning of both alcohol lamps, the weights did not deviate from equilibrium. In these experiments we invariably obtained the opposite effect: i.e., a turning to the north of the end of the balance arm with the rod.

It is necessary to mention one important conclusion which follows from the combination of the occurrences which have been observed. In the case of the effect on the mounting, this might not influence a heavy body; and at the same time, forces, applied to each point of it, develop in the body: i.e., mass forces and, hence, identical to the variation in the weight. This signifies, by influencing the mounting, where the forces of the attraction are located, comprising a result of the weight, we can obtain a variation in the weight, i.e., a change in the cause. Therefore, the tests conducted indicate a distinct possibility of reversing the causal relationships.

The second cycle of tests on studying the qualities of time began as a result of the observations of quite strange circumstances, interrupting a repetition of the experiments. As early as the initial experiments with the gyroscopes, it was necessary to face the fact that sometimes the tests could be managed quite easily, and sometimes they proved to be fruitless, even with a strict observance of the same conditions. These difficulties were also noted in the old experiments on the southward deflection of falling bodies. Only in those tests in which, within wide limits, it is possible to intensify the causal effect — as, e. g., during the vibrations of the mounting of the weights (scales) or of the pendulum — can we almost always attain a result. Evidently, in addition to the constant pattern  $c_2$ , in the case of time, **there also exists a variable property which can be called the density or intensity of time.** In a case of low density it is difficult for time to influence the material systems, and a requirement arises for an intensive emphasis of the causal-resultant relationship in order that a force caused by the time pattern should appear. It is possible that our psychological sensation of empty or substantive time has not only a subjective nature but also, similarly to the sensation of the flow of time, an objective physical basis.

Evidently many circumstances exist affecting the density of time in the space surrounding us. **In late autumn and in the first half of winter all of the tests could be easily managed. However, in summer these experiments became difficult to such an extent that many of them could not be completed.** Probably, in conformity with these conditions, the tests in the high altitudes can be performed much more easily than in the south; in addition to these regular variations, there often occur some changes in the conditions required for the success of the experiments: these transpired in the course of one day or even several hours. Obviously, the density of time changes within broad limits, owing to the processes occurring in nature, and our tests utilized a unique instrument to record these changes. If this be so, **it strongly suggests the possibility of having one material influence after another through time.** Such a relationship could be foreseen, since the causal-resultant phenomena occurred not only in time but also with the aid of time. Therefore, in each process of nature time can be extended or formed. These conclusions could be confirmed by a direct experiment.

Since we are studying the phenomenon of such a generality as time, it is evident that it is sufficient to take the simplest mechanical process in order to attempt to change the density of time. For example, using any motor, we can raise and lower a weight or change the tension of

a tight elastic band. We obtain a system with two poles, a source of energy and its outflow: i. e., the causal-resultant dipole. With the aid of a rigid transmission, the pole of this dipole can be separated for a fairly extensive distance. We will bring one of these poles close to a long pendulum during the vibrations of its point of suspension. It is necessary to tune the vibrations in such a way that the full effect of southward deflection would not develop, but only the tendency for the appearance of this effect. It turns out that this tendency increases appreciably and converts even to the complete effect if we bring near to the body of the pendulum or to the suspension point that pole of the dipole where the absorption of the energy is taking place. However, with the approach of the other pole (of the motor), the appearance of the effect of southern deflection in the pendulum invariably became difficult. In the case of a close juxtaposition of the poles of the dipole, their influence on the pendulum practically disappeared. Evidently in this case, a considerable compensation of their effects occurs. It turns out that **the effect of the causal pole** does not depend on the direction along which it is installed relative to the pendulum; rather its effect **depends only on the distance (spacing)**. Repeated and careful measurements demonstrated that **this effect diminishes**, not inversely proportional to the square of the distance, as in the case of force fields, but **inversely proportional to the first power of the distance**. In the raising and lowering of a 10-kg weight suspended through a unit distance, its influence was sensed at a distance of 2–3 meters from the pendulum. **Even the thick wall of the laboratory did not shield this effect**. It is necessary to comment that all of these tests, similarly to the previous ones, also were not always successful.

The results indicate that the more proximate to the system with the causal-resultant relationship, the density of time actually changes. **Near the motor there occurs a thinning (rarefaction of time), while near the energy receiver its compaction takes place. The impression is gained that time is extended by a cause and, contrariwise, it becomes more advanced in that place where the effect is located.** Therefore, in the pendulum, assistance is obtained from the receiver, while interference arises from the part on the motor. By these conditions we might also explain the easy accomplishment of these experiments in winter and in northern latitudes, while in summer and in the south it is difficult to perform the tests. The fact of the matter is that in our latitude in winter are located the effects of the dynamics of the atmosphere of the southern latitudes. This circumstance can assist the appearance of the effects of the time pattern. However,

generally and particularly **in summer the heating by solar rays creates an atmosphere loader, interfering with the effects.**

The effect of time differs basically from the effect of force fields. The effect of the causal pole on the device (pendulum) immediately creates two equal and opposite forces, applied to the body of the pendulum and the suspension point. There occurs a transmission of energy, without momentum, and, hence, also without delivery to the pole. This circumstance explains the reduction of the influences inversely proportional to the first power of the distances, since according to this law an energy decrease takes place. Moreover, this law could be foreseen, simply by proceeding from the circumstance of time as expressed by turning, and hence with it a necessity to link the plane, passing through the pole with any orientation in space. In the case of the force lines emerging from the pole, their density decreases in inverse proportion to the square of the distance; however, the density of the planes will diminish according to the law of the first power of the distance. The transmission of energy without momentum (pulse) should still have the following very important property: Such a transmission should be instantaneous: i. e., it cannot be propagating because the transmission of the pulse is associated with propagation. This circumstance follows from the most general concepts concerning time. Time in the universe is not propagated but appears immediately everywhere. On a time axis the entire universe is projected by one point. Therefore, the altered properties of a given second will appear everywhere at once, diminishing according to the law of inverse proportionality of the first power of the distance.

It seems to us that such a possibility of the instantaneous transfer of information through time should not contradict the special theory of relativity — in particular, the relativity of the concept of simultaneity. The fact is that the simultaneity of effects through time is realized in that advantageous system of coordinates with which the source of these effects is associated.

The possibility of communications through time will probably help to explain not only the features of biological relationship but also a number of puzzling phenomena of the psychics of man. Perhaps instinctive knowledge is obtained specifically in this manner. It is quite likely that in this same way are realized also the phenomena of telepathy: i. e., the transmission of thought over a distance. All these relationships are not shielded and hence have the property for the transmission of influences through time.

Further observations indicate that in the causal-resultant dipoles a complete compensation of the effect of its poles does not take place. Ob-



viously, in the process there occurs the absorption or output of certain qualities of time. Therefore, the effect of the process could be observed without a preliminary excitation of the system.

The previously applied torsion weights (balances) were modified in such a manner that, when possible, we would increase the distance between the weights suspended on the balance arm. This requirement was realized with a considerable lengthening (up to 1.5 m) of the suspension filament of one of the weights. As a result, the torsion balances came to resemble a gravitational variometer, only with the difference that in them the balance arm could be freely moved around a horizontal axis. The entire system was well grounded and shielded by a metal housing in order to avert the electrostatic effects. The masses of the weights were of the order of 5–20 grams. In the realization of any reversible process near one of the weights, we obtained a turning of the balance arm toward the meridian by a small angle  $a$  of the order of  $0.3^\circ$ , with a sensitivity of the weights (scales) corresponding to a slewing by  $9^\circ$  for the case of the effects of the forces of a time pattern of full magnitude. In this manner, the forces which were occurring proved to be quite similar to those previously investigated. They act along the axis of the Earth and yield the same series of quantized values of the slewing angle:  $\frac{1}{2}a, a, 2a, \dots$ . It turns out that the vertical components of these forces can be observed in the analytical scales, if we separate the weights in them far enough, by means of the same considerable lengthening of the suspension filament of one of the weights.

These tests indicated the basic possibility of the effect through time of an irreversible process upon a material system. At the same time, the very low value of the forces obtained testifies to a certain constructive incorrectness of the experiment, owing to which there takes place an almost complete compensation of the forces originating in the system. As a result, only a small residue of these forces acts on the system. Obviously, in our design, during the effect upon one weight, there also develops an effect upon the second weight, stopping the turning of the torsion balances. Most likely, this transmission of the effect to the second weight occurs through the suspension point. In reality, the appearance of forces of the time pattern in one of the weights signifies the transformation of the forces of the weight of this load and its reaction in the mounting point to a new time pattern, associated with the Earth's rotation. The transformation of the time pattern in the suspension point of the torsion balances can also cause the transformation of all of the forces acting here, signifying also the reaction of the second weight. However, the appearance of an additional reaction re-

quires the appearance of the additional force of the weight of the second load. Therefore, in this design, during the effect upon one load there also originates an effect upon the second load, stopping the turning of the torsion balances. The concept discussed indicates that to obtain substantial effect in the torsion balances, it is necessary to introduce an abrupt asymmetry in the suspension of the loads.

As a result of a number of tests, the following design of the asymmetrical torsion balances proved successful: one cylindrical load of considerable weight was chosen, around 300 grams. This main weight was suspended from the permanent filament made of capron, with a length of around 1.5 meters and a diameter of 0.15 mm. To this weight there was rigidly fastened, arranged horizontally, a light-weight metal plate around 10 cm in length. The free end of this plate was supported by a very thin capron filament fastened at the same point as the main filament. From this free end of the plate, we suspended on a long thin wire a weight of the order of 10 grams. For damping the system the main weight was partly lowered into a vessel containing machine oil. By a turn at the suspension point, the horizontal plate was set perpendicular to the plane of the meridian.

Let us now assume that in the system a force has developed affecting only the main weight in the plane of the meridian: i. e., perpendicularly to the plate. This force deflects the main weight by a certain angle  $a$ . The free end of the plate with a small load will also be deflected by this same angle. Therefore, upon the small load there will act a horizontal force, tending to turn the plate toward the plane of the meridian and equalizing the weight of the small load multiplied by the angle  $a$ . Since the deflection angle  $a$  equals the relative change in the weight, a force equaling the additional force of the time pattern for the weight of a small load will act on the small load. Therefore, the turning angle of the torsion balances can be computed according to the previous (13), assuming that in it  $T_0 = 0$ . The same turning, but in an opposite direction, should be obtained during the effect upon only one small load. This condition was confirmed by experiments with strong influences from close distances. However, it turned out that a heavy weight absorbed the effect better than a small weight. Therefore, weak remote forces are received (absorbed) by only one large load, which permitted us to observe the effects upon the device at very considerable distances from it, of the order of 10–20 meters. However, the optimal distance in these tests was around 5 meters.

The asymmetrical torsion balances described proved to be a successful design. The calculated angle of their turning under the effect

of additional forces of the time pattern should be of the order of  $14^\circ$ . In the case of a contactless effect over a distance, we obtained large deflections, which reached the indicated values. In these tests, as in the previous ones, we once again observed the discrete state of the stable deflections with a power of one fourth of the full effect: i. e.,  $3^\circ 5'$ .

A variety of processes caused a deflection of the weights: heating of the body; burning of an electric tube; cooling of a previously heated body; the operation of an electrical battery, closed through resistance; the dissolving of various salts in water; and even the movement of a man's head. A particularly strong effect is exerted by a nonstationary process: e. g., the blinking of an electric bulb. Owing to the processes occurring near the weights and in nature, the weights behave themselves very erratically. Their zero point often becomes displaced, shifting by the above-indicated amounts and interfering considerably with the observations. It turned out that **the balances can be shielded, to a considerable extent, from these influences by placing near them an organic substance consisting only of right-handed molecules: for example, sugar. The left-handed molecules — e. g., turpentine — evidently cause the opposite effect.**

In essence, the tests conducted demonstrate that it is possible to have the influence through time of one process upon another. In reality, the appearance of forces turning the torsion balances alters the potential energy of the balances. Therefore, in principle, there should take place a change in the physical process which is associated with them.

At a session of the International Astronomical Union in Brussels in the fall of 1966, the author presented a report concerning the physical features of the components of double stars. In binary systems a satellite constitutes an unusual star. As a result of long existence, a satellite becomes similar to a principal star in a number of physical aspects (brightness, spectral type, radius). At such great distances the possibility is excluded that the principal star will exert an influence upon its satellite in the usual manner: i. e., through force fields. Rather, the binary stars constitute an astronomical example of the effect of the processes in one body upon the processes in another, through time.

Among the many tests conducted, we should mention the observations which demonstrated the existence of yet another interesting feature in the qualities of time. It turns out that in the experiments with the vibrations of the mounting point of the balances or of the pendulum, additional forces of the time pattern which developed did not disappear immediately with the stoppage of the vibrations, but remained in the system for a considerable period. Considering that they decreased ac-

According to the exponential law  $e^{-t/t_0}$ , estimations were made of the time  $t_0$  of their relaxation, which was shown not to depend on the mass of the body but upon its density  $\rho$ . We obtained the following approximate data: for lead,  $\rho = 11 \text{ g/cm}^3$ ,  $t_0 = 14$  seconds; for aluminum,  $\rho = 2.7 \text{ g/cm}^3$ ,  $t_0 = 28$  seconds; for wood  $\rho = 0.5 \text{ g/cm}^3$ ,  $t_0 = 70$  seconds. In this manner it is possible that  $t_0$  is inversely proportional to the square root of the body's density. It is curious that the preservation of the additional forces in the system, after the cessation of the vibrations, can be observed in the balances in the simplest manner. Let us imagine balance scales in which one of the weights is suspended on rubber. Let us take this weight with one hand and, with the pressure of the other hand upon the balance arm, replace the effect of the weight taken from it. We will shake the removed weight with one hand and, with the pressure of the other hand upon the balance arm, replace the effect of the weight taken from it. We will shake the removed weight for a certain time (around a minute) on the rubber, and then we will place it back upon the scales. The scales will indicate the gradual lightening of this load, in conformity with the above-listed values for  $t_0$ . It is understandable that in this test it is necessary to take measures in order that one's hand does not heat the balance arm of the scales. In place of a hand, the end of the balance arm from which the weight is taken can be held by a mechanical clamp. Sometimes this amazingly simple test can be accomplished quite easily, but there are days when, similarly to certain other tests, it is achieved with difficulty or cannot be accomplished at all.

Based on the above-presented theoretical concepts and all of the experimental data, the following general inferences can be made:

1. The causal states, derived from three basic axioms, of the effect concerning the properties of a time pattern are confirmed by the tests. Therefore, we can consider that these axioms are substantiated by experiment. Specifically, we confirm axiom II concerning the spatial non-overlapping of causes and effects. Therefore, the force fields transmitting the influences should be regarded as a system of discrete, non-overlapping points. This finding is linked with the general philosophical principle of the possibility of cognition of the world. For the possibility of at least a marginal cognition, the combination of all material objects should be a calculated set: i.e, it should represent a discrete state, being superimposed on the continuum of space.

As concerns the actual results obtained during the experimental justification of the axiom of causality, among them the most important are

the conclusions concerning the finiteness of the time pattern, the possibility of partial reversal of the causal relationships, and the possibility of obtaining work owing to the time pattern.

2. The tests proved the existence of the effects through time of one material system upon another. This effect does not transmit a pulse (momentum), meaning it does not propagate but appears simultaneously in any material system. In this manner, in principle it proves possible to have a momentary relationship and a momentary transmission of information. Time accomplishes a relationship between all phenomena of nature and participates actively in them.
3. Time has diverse qualities, which can be studied by experiments. Time contains the entire universe of still unexplored occurrences. The physical experiments studying these phenomena should gradually lead to an understanding of what time represents. However, knowledge should show us how to penetrate into the world of time and teach us how to affect it.

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# Elastodynamics of the Spacetime Continuum

Pierre A. Millette\*

**Abstract:** We develop the Elastodynamics of the Spacetime Continuum (*STCED*) based on the analysis of the deformations of the *STC* within a general relativistic and continuum mechanical framework. We show that *STC* deformations can be decomposed into a massive dilatation and a massless wave distortion reminiscent of wave-particle duality. We show that rest-mass energy density arises from the volume dilatation of the *STC*. We derive Electromagnetism from *STCED* and provide physical explanations for the electromagnetic potential and the current density. We derive the Klein-Gordon equation and show that the quantum mechanical wavefunction describes longitudinal waves propagating in the *STC*. The equations obtained reflect a close integration of gravitational and electromagnetic interactions.

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**§1. Introduction.** The theory of General Relativity initially proposed by Einstein [1] is a theory of gravitation based on the geometry of the spacetime continuum (*STC*). The geometry of the spacetime continuum is determined by the energy-momentum present in the *STC*. This can be represented by the relation

$$\text{Energy-momentum} \longrightarrow \text{STC Geometry}$$

or, in terms of its mathematical representation,

$$T^{\mu\nu} \longrightarrow G^{\mu\nu}$$

where  $G^{\mu\nu}$  is the Einstein tensor and  $T^{\mu\nu}$  is the energy-momentum stress tensor. The spacetime continuum is thus warped by the presence of energy-momentum. This is a physical process as shown by the deflection of light by the sun, or the cosmological models resulting in a physical structure of the universe, derived from General Relativity.

Hence the theory of General Relativity leads implicitly to the proposition that the spacetime continuum must be a deformable continuum. This deformation is physical in nature. The “vacuum” that is omnipresent in Quantum Theory, is the spacetime continuum, made more evident by the microscopic scale of quantum phenomena. The physical nature of the spacetime continuum is further supported by the following evidence:

- The physical electromagnetic properties of the vacuum: characteristic impedance of the vacuum  $Z_{\text{em}} = 376.73 \Omega$ , electromagnetic permittivity of free space  $\epsilon_{\text{em}}$ , electromagnetic permeability of free space  $\mu_{\text{em}}$ .
- A straightforward explanation of the existence and constancy of the speed of light  $c$ : it is the maximum speed at which transverse deformations propagate in the *STC*.
- A physical framework for the vacuum of quantum electrodynamics with its constant creation/annihilation of (virtual) particles, corresponding to a state of constant vibration of the *STC* due to the energy-momentum continuously propagating through it.
- A physical framework to support vacuum quantum effects such as vacuum polarization, zero-point energy, the Casimir force, the Aharonov-Bohm effect.

The assignment of physical dynamic properties to the spacetime of General Relativity has been considered previously. For example, Sakharov [2] considers a “metrical elasticity” of space in which generalized forces



oppose the curving of space. Tartaglia *et al* have recently explored “strained spacetime” in the cosmological context, as an extension of the spacetime Lagrangian to obtain a generalized Einstein equation [3, 4].

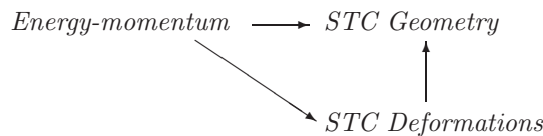
Considering the spacetime continuum to be a deformable continuum results in an alternative description of its dynamics, represented by the relation

$$\text{Energy-momentum} \longrightarrow \text{STC Deformations}$$

The energy-momentum present in the spacetime continuum, represented by the energy-momentum stress tensor, results in strains in the *STC*, hence the reference to “strained spacetime”. The spacetime continuum strains result in displacements of the elements of the spacetime continuum, hence the *STC* deformations. The spacetime continuum itself is the medium that supports those deformations. The spacetime continuum deformations result in the geometry of the *STC*.

This theory is referred to as the Elastodynamics of the Spacetime Continuum (*STCED*) (see Millette [5–8]). In this theory, we analyse the spacetime continuum within the framework of Continuum Mechanics and General Relativity. This allows for the application of continuum mechanical methods and results to the analysis of the *STC* deformations.

Hence, while General Relativity can be described as a top-down theory of the spacetime continuum, the Elastodynamics of the Spacetime Continuum can be described as a bottom-up theory of the spacetime continuum. *STCED* provides a fundamental description of the microscopic processes underlying the spacetime continuum. The relation between *STCED* and General Relativity is represented by the diagram



The combination of all deformations present in the spacetime continuum generates its geometry. *STCED* must thus be a description complementary to that of General Relativity, which is concerned with modeling the resulting geometry of the spacetime continuum rather than the deformations generating that geometry. The value of *STCED* is that it provides a microscopic description of the fundamental *STC* processes expected to reach down to the quantum level.

**§1.1. Outline of the paper.** We start by demonstrating from first principles that spacetime is strained by the presence of mass. In addition, we find that this provides a natural decomposition of the spacetime metric tensor and of spacetime tensor fields, both of which are still unresolved and are the subject of continuing investigations (see for example [9–13]).

Based on that analysis from first principles of the effect of a test mass on the background metric, we obtain a natural decomposition of the spacetime metric tensor of General Relativity into a background and a dynamical part. We find that the presence of mass results in strains in the spacetime continuum, and that those strains correspond to the dynamical part of the spacetime metric tensor. We note that these results are considered to be local effects in the particular reference frame of the observer. In addition, the applicability of the proposed metric to the Einstein field equations remains open to demonstration.

The presence of strains in the spacetime continuum as a result of the applied stresses from the energy-momentum stress tensor is an expected continuum mechanical result. The strains result in a deformation of the continuum which can be modeled as a change in the underlying geometry of the continuum. The geometry of the spacetime continuum of General Relativity resulting from the energy-momentum stress tensor can thus be seen to be a representation of the deformation of the spacetime continuum resulting from the strains generated by the energy-momentum stress tensor.

We then derive the Elastodynamics of the Spacetime Continuum by applying continuum mechanical results to strained spacetime. Based on this model, a stress-strain relation is derived for the spacetime continuum. We apply that stress-strain relation to show that rest-mass energy density arises from the volume dilatation of the spacetime continuum. Then we propose a natural decomposition of tensor fields in strained spacetime, in terms of dilatations and distortions. We show that dilatations correspond to rest-mass energy density, while distortions correspond to massless shear transverse waves. We note that this decomposition of spacetime continuum deformations into a massive dilatation and a massless transverse wave distortion is somewhat reminiscent of wave-particle duality.

From the kinematic relations and the equilibrium dynamic equation of the spacetime continuum, we derive a series of wave equations: the displacement, dilatational, rotational and strain wave equations. Hence we find that energy propagates in the spacetime continuum as wave-like deformations which can be decomposed into dilatations and distortions.

Dilatations involve an invariant change in volume of the spacetime continuum which is the source of the associated rest-mass energy density of the deformation, while distortions correspond to a change of shape of the spacetime continuum without a change in volume and are thus massless. The deformations propagate in the continuum by longitudinal and transverse wave displacements. Again, this is somewhat reminiscent of wave-particle duality, with the transverse mode corresponding to the wave aspects and the longitudinal mode corresponding to the particle aspects. A continuity equation for deformations of the spacetime continuum is derived, where the gradient of the massive volume dilatation acts as a source term. The nature of the spacetime continuum volume force and the inhomogeneous wave equations are areas of further investigation.

We then investigate the strain energy density of the spacetime continuum in the Elastodynamics of the Spacetime Continuum by applying continuum mechanical results to strained spacetime. The strain energy density is a scalar. We find that it is separated into two terms: the first one expresses the dilatation energy density (the “mass” longitudinal term) while the second one expresses the distortion energy density (the “massless” transverse term). The quadratic structure of the energy relation of Special Relativity is found to be present in the theory. In addition, we find that the kinetic energy  $pc$  is carried by the distortion part of the deformation, while the dilatation part carries only the rest-mass energy.

Since Einstein first published his Theory of General Relativity in 1915, the problem of the unification of Gravitation and Electromagnetism has been and remains the subject of continuing investigation (see for example [23–31] for recent attempts). Electromagnetism is found to come out naturally from the *STCED* theory in a straightforward manner. This theory thus provides a unified description of the spacetime deformation processes underlying general relativistic Gravitation and Electromagnetism, in terms of spacetime continuum displacements resulting from the strains generated by the energy-momentum stress tensor.

We derive Electromagnetism from the Elastodynamics of the Spacetime Continuum based on the identification of the theory’s antisymmetric rotation tensor with the electromagnetic field-strength tensor. The theory provides a physical explanation of the electromagnetic potential, which arises from transverse (shearing) displacements of the spacetime continuum, in contrast to mass which arises from longitudinal (dilatational) displacements. In addition, the theory provides a physical ex-

planation of the current density four-vector, as the 4-gradient of the volume dilatation of the spacetime continuum. The Lorentz condition is obtained directly from the theory. In addition, we obtain a generalization of Electromagnetism for the situation where a volume force is present, in the general non-macroscopic case. Maxwell's equations are found to remain unchanged, but the current density has an additional term proportional to the volume force.

The strain energy density of the electromagnetic energy-momentum stress tensor is then calculated. The dilatation energy density (the rest-mass energy density of the photon) is found to be 0 as expected. The transverse distortion energy density is found to include a longitudinal electromagnetic energy flux term, from the Poynting vector, that is massless as it is due to distortion, not dilatation, of the spacetime continuum. However, because this energy flux is along the direction of propagation (i.e. longitudinal), it gives rise to the particle aspect of the electromagnetic field, the photon.

We then investigate the volume force and its impact on the equations of the Elastodynamics of the Spacetime Continuum. First we consider a linear elastic volume force which leads to equations which are of the Klein-Gordon type. Based on the results obtained, we then consider a variation of that linear elastic volume force based on the Klein-Gordon quantum mechanical current density. We find that the quantum mechanical wavefunction describes longitudinal wave propagations in the *STC* corresponding to the volume dilatation associated with the particle property of an object. The longitudinal wave equation is then found to correspond to the Klein-Gordon equation with an interaction term of the form  $\mathbf{A} \cdot \mathbf{j}$ , further confirming that the quantum mechanical wavefunction describes longitudinal wave propagations in the *STC*. The transverse wave equation is found to be a new equation of the electromagnetic field strength  $F^{\mu\nu}$ , which includes an interaction term of the form  $\mathbf{A} \times \mathbf{j}$  corresponding to the volume density of the magnetic torque (magnetic torque density). The equations obtained reflect a close integration of gravitational and electromagnetic interactions at the microscopic level.

This paper presents a linear elastic theory of the Elastodynamics of the Spacetime Continuum based on the analysis of the deformations of the spacetime continuum. It is found to provide a microscopic description of gravitational and electromagnetic phenomena and some quantum results, based on the framework of General Relativity and Continuum Mechanics. Based on the model, the theory should in principle be able to explain the basic physical theories from which the rest of physical

theory can be built, without the introduction of inputs external to the theory. A summary of the physical phenomena derived from *STCED* in this paper are summarized in the conclusion. The direction of the next steps to further extend this theory are discussed in the concluding section of this paper.

**§1.2. A note on units and constants.** In General Relativity and in Quantum Electrodynamics, it is customary to use “geometrized units” and “natural units” respectively, where the principal constants are set equal to 1. The use of these units facilitates calculations since cumbersome constants do not need to be carried throughout derivations. In this paper, all constants are retained in the derivations, to provide insight into the nature of the equations being developed.

In addition, we use rationalized MKSA units for Electromagnetism, as the traditionally used Gaussian units are gradually being replaced by rationalized MKSA units in more recent textbooks (see for example [32]). Note that the electromagnetic permittivity of free space  $\epsilon_{em}$ , and the electromagnetic permeability of free space  $\mu_{em}$  are written with “em” subscripts as the “0” subscripts are used in *STCED* constants. This allows us to differentiate between for example  $\mu_{em}$ , the electromagnetic permeability of free space, and  $\mu_0$ , the Lamé elastic constant for the shear modulus of the spacetime continuum.

**§1.3. Glossary of physical symbols.** This analysis uses symbols across the fields of continuum mechanics, elasticity, general relativity, electromagnetism and quantum mechanics. The symbols used need to be applicable across these disciplines and be self-consistent. A glossary of the physical symbols is included below to facilitate the reading of this paper.

$\alpha$	Fine-structure constant.
$\epsilon^{\alpha\beta\mu\nu}$	Permutation symbol in four-dimensional spacetime.
$\epsilon_{em}$	Electromagnetic permittivity of free space ( <i>STC</i> ).
$\varepsilon$	Volume dilatation.
$\varepsilon^{\mu\nu}$	Strain tensor.
$\eta_{\mu\nu}$	Flat spacetime metric tensor.
$\Theta^{\mu\nu}$	Symmetric electromagnetic stress tensor.
$\kappa_0$	Bulk modulus of the <i>STC</i> .
$\lambda_0$	Lamé elastic constant of the <i>STC</i> .
$\lambda_c$	Compton wavelength of the electron.

$\mu_0$	Shear modulus Lamé elastic constant of the <i>STC</i> .
$\mu_{\text{em}}$	Electromagnetic permeability of free space ( <i>STC</i> ).
$\mu_{\text{B}}$	Bohr magneton.
$\rho$	Rest-mass density.
$\rho_0$	<i>STC</i> density.
$\varrho$	Charge density.
$\sigma^{\mu\nu}$	Stress tensor.
$\phi$	Phase of the quantum mechanical wavefunction.
$\varphi_0$	<i>STC</i> electromagnetic shearing potential constant.
$\omega^{\mu\nu}$	Rotation tensor.
$\psi$	Quantum mechanical wavefunction.
$\mathbf{A}$	Vector potential.
$A^\mu$	Four-vector potential.
$\bar{A}^\mu$	Reduced four-vector potential.
$\mathbf{B}$	Magnetic field.
$c$	Speed of light.
$e$	Electrical charge of the electron.
$e_s$	Strain scalar.
$e^{\mu\nu}$	Strain deviation tensor.
$\mathbf{E}$	Electric field.
$\hat{E}$	Total energy density.
$\mathcal{E}$	Strain energy density of the spacetime continuum.
$E^{\mu\nu\alpha\beta}$	Elastic moduli tensor of the <i>STC</i> .
$F^{\mu\nu}$	Electromagnetic field strength tensor.
$g_{\mu\nu}$	Metric tensor.
$G$	Gravitational constant.
$h$	Planck's constant.
$\hbar$	Planck's reduced constant.
$\mathbf{j}$	Current density vector.
$j^\mu$	Current density four-vector.
$\bar{j}^\mu$	Reduced current density four-vector.
$k_0$	Elastic force constant of the <i>STC</i> volume force.
$k_{\text{L}}$	<i>STC</i> longitudinal dimensionless ratio.

$k_T$	<i>STC</i> transverse dimensionless ratio.
$m$	Mass of the electron.
$\mathbf{p}$	Momentum 3-vector.
$\hat{p}$	Momentum density.
$R$	Contracted Ricci curvature tensor.
$R^{\mu\nu}$	Ricci curvature tensor.
$R^\mu{}_{\nu\alpha\beta}$	Curvature tensor.
$\mathbf{S}$	Poynting vector (electromagnetic field energy flux).
$S^\mu$	Poynting four-vector.
$t_s$	Stress scalar.
$t^{\mu\nu}$	Stress deviation tensor.
$T^{\mu\nu}$	Energy-momentum stress tensor.
$u^\mu$	Displacement four-vector.
$U_{\text{em}}$	Electromagnetic field energy density.
$x^\mu$	Position four-vector.
$X^\nu$	Volume (or body) force.
$Z_{\text{em}}$	Characteristic impedance of the vacuum ( <i>STC</i> ).

## §2. Elastodynamics of the Spacetime Continuum

### §2.1. Strained spacetime and the natural decomposition of the spacetime metric tensor.

There is no straightforward definition of local energy density of the gravitational field in General Relativity, [14, see p. 84, p. 286] and [12, 15, 16]. This arises because the spacetime metric tensor includes both the background spacetime metric and the local dynamical effects of the gravitational field. No natural way of decomposing the spacetime metric tensor into its background and dynamical parts is known.

In this section, we propose a natural decomposition of the spacetime metric tensor into a background and a dynamical part. This is derived from first principles by introducing a test mass in the spacetime continuum described by the background metric, and calculating the effect of this test mass on the metric.

Consider the diagram of Figure 1. Points  $A$  and  $B$  of the spacetime continuum, with coordinates  $x^\mu$  and  $x^\mu + dx^\mu$  respectively, are separated by the infinitesimal line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1)$$

where  $g_{\mu\nu}$  is the metric tensor describing the background state of the spacetime continuum.

We now introduce a test mass in the spacetime continuum. This results in the displacement of point  $A$  to  $\tilde{A}$ , where the displacement is written as  $u^\mu$ . Similarly, the displacement of point  $B$  to  $\tilde{B}$  is written as  $u^\mu + du^\mu$ . The infinitesimal line element between points  $\tilde{A}$  and  $\tilde{B}$  is given by  $d\tilde{s}^2$ .

By reference to Figure 1, the infinitesimal line element  $d\tilde{s}^2$  can be expressed in terms of the background metric tensor as

$$d\tilde{s}^2 = g_{\mu\nu}(dx^\mu + du^\mu)(dx^\nu + du^\nu). \quad (2)$$

Multiplying out the terms in parentheses, we get

$$d\tilde{s}^2 = g_{\mu\nu}(dx^\mu dx^\nu + dx^\mu du^\nu + du^\mu dx^\nu + du^\mu du^\nu). \quad (3)$$

Expressing the differentials  $du$  as a function of  $x$ , this equation becomes

$$d\tilde{s}^2 = g_{\mu\nu}(dx^\mu dx^\nu + dx^\mu u^\nu{}_{;\alpha} dx^\alpha + u^\mu{}_{;\alpha} dx^\alpha dx^\nu + u^\mu{}_{;\alpha} dx^\alpha u^\nu{}_{;\beta} dx^\beta), \quad (4)$$

where the semicolon (;) denotes covariant differentiation. Rearranging the dummy indices, this expression can be written as

$$d\tilde{s}^2 = (g_{\mu\nu} + g_{\mu\alpha} u^\alpha{}_{;\nu} + g_{\alpha\nu} u^\alpha{}_{;\mu} + g_{\alpha\beta} u^\alpha{}_{;\mu} u^\beta{}_{;\nu}) dx^\mu dx^\nu \quad (5)$$

and lowering indices, the equation becomes

$$d\tilde{s}^2 = (g_{\mu\nu} + u_{\mu;\nu} + u_{\nu;\mu} + u^\alpha{}_{;\mu} u_{\alpha;\nu}) dx^\mu dx^\nu. \quad (6)$$

The expression  $u_{\mu;\nu} + u_{\nu;\mu} + u^\alpha{}_{;\mu} u_{\alpha;\nu}$  is equivalent to the definition of the strain tensor  $\varepsilon^{\mu\nu}$  of Continuum Mechanics. The strain  $\varepsilon^{\mu\nu}$  is expressed in terms of the displacements  $u^\mu$  of a continuum through the kinematic relation, [17, see p. 149] and [18, see pp. 23–28]:

$$\varepsilon^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} + u^{\nu;\mu} + u^{\alpha;\mu} u_{\alpha;\nu}). \quad (7)$$

Substituting for  $\varepsilon^{\mu\nu}$  from (7) into (6), we get

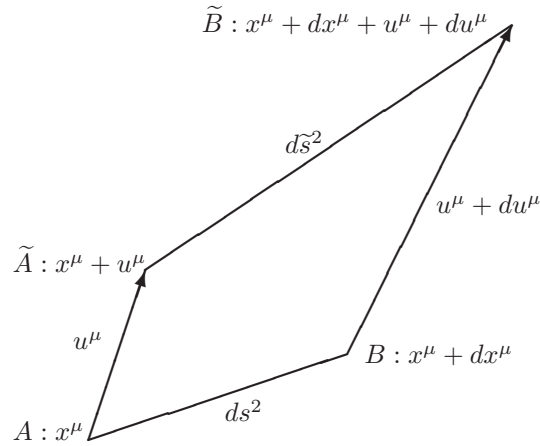
$$d\tilde{s}^2 = (g_{\mu\nu} + 2\varepsilon_{\mu\nu}) dx^\mu dx^\nu. \quad (8)$$

Setting [18, see p. 24]

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + 2\varepsilon_{\mu\nu} \quad (9)$$



Fig. 1: Effect of a test mass on the background metric tensor.



then (8) becomes

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad (10)$$

where  $\tilde{g}_{\mu\nu}$  is the metric tensor describing the spacetime continuum with the test mass.

Given that  $g_{\mu\nu}$  is the background metric tensor describing the background state of the continuum, and  $\tilde{g}_{\mu\nu}$  is the spacetime metric tensor describing the final state of the continuum with the test mass, then  $2\varepsilon_{\mu\nu}$  must represent the dynamical part of the spacetime metric tensor due to the test mass:

$$g_{\mu\nu}^{\text{dyn}} = 2\varepsilon_{\mu\nu}. \quad (11)$$

We are thus led to the conclusion that the presence of mass results in strains in the spacetime continuum. Those strains correspond to the dynamical part of the spacetime metric tensor. Hence the applied stresses from mass (i.e. the energy-momentum stress tensor) result in strains in the spacetime continuum, that is strained spacetime.

**§2.2. Model of the Elastodynamics of the Spacetime Continuum.** The spacetime continuum (*STC*) is modelled as a four-dimensional differentiable manifold endowed with a metric  $g_{\mu\nu}$ . It is a con-

tinuum that can undergo deformations and support the propagation of such deformations. A continuum that is deformed is strained.

An infinitesimal element of the unstrained continuum is characterized by a four-vector  $x^\mu$ , where  $\mu = 0, 1, 2, 3$ . The time coordinate is  $x^0 \equiv ct$ .

A *deformation* of the spacetime continuum corresponds to a state of the *STC* in which its infinitesimal elements are displaced from their unstrained position. Under deformation, the infinitesimal element  $x^\mu$  is displaced to a new position  $x^\mu + u^\mu$ , where  $u^\mu$  is the displacement of the infinitesimal element from its unstrained position  $x^\mu$ .

The spacetime continuum is approximated by a deformable linear elastic medium that obeys Hooke's law. For a general anisotropic continuum in four dimensions [18, see pp. 50–53],

$$E^{\mu\nu\alpha\beta} \varepsilon_{\alpha\beta} = T^{\mu\nu}, \quad (12)$$

where  $\varepsilon_{\alpha\beta}$  is the strain tensor,  $T^{\mu\nu}$  is the energy-momentum stress tensor, and  $E^{\mu\nu\alpha\beta}$  is the elastic moduli tensor.

The spacetime continuum is further assumed to be isotropic and homogeneous. This assumption is in agreement with the conservation laws of energy-momentum and angular momentum as expressed by Noether's theorem [21, see pp. 23–30]. For an isotropic medium, the elastic moduli tensor simplifies to [18]:

$$E^{\mu\nu\alpha\beta} = \lambda_0 (g^{\mu\nu} g^{\alpha\beta}) + \mu_0 (g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha}), \quad (13)$$

where  $\lambda_0$  and  $\mu_0$  are the Lamé elastic constants of the spacetime continuum.  $\mu_0$  is the shear modulus (the resistance of the continuum to *distortions*) and  $\lambda_0$  is expressed in terms of  $\kappa_0$ , the bulk modulus (the resistance of the continuum to *dilatations*) according to

$$\lambda_0 = \kappa_0 - \frac{1}{2} \mu_0 \quad (14)$$

in a four-dimensional continuum. A *dilatation* corresponds to a change of volume of the spacetime continuum without a change of shape while a *distortion* corresponds to a change of shape of the spacetime continuum without a change in volume.

**§2.3. Stress-strain relation of the spacetime continuum.** By substituting (13) into (12), we obtain the stress-strain relation for an isotropic and homogeneous spacetime continuum

$$2\mu_0 \varepsilon^{\mu\nu} + \lambda_0 g^{\mu\nu} \varepsilon = T^{\mu\nu}, \quad (15)$$

where

$$\varepsilon = \varepsilon^\alpha{}_\alpha \quad (16)$$

is the trace of the strain tensor obtained by contraction. The volume dilatation  $\varepsilon$  is defined as the change in volume per original volume [17, see pp. 149–152] and is an invariant of the strain tensor.

It is interesting to note that the structure of (15) is similar to that of the field equations of General Relativity, viz.

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -\varkappa T^{\mu\nu}, \quad (17)$$

where  $\varkappa = 8\pi G/c^4$  and  $G$  is the gravitational constant. This strengthens our conjecture that the geometry of the spacetime continuum can be seen to be a representation of the deformation of the spacetime continuum resulting from the strains generated by the energy-momentum stress tensor.

**§3. Rest-mass energy relation.** The introduction of strains in the spacetime continuum as a result of the energy-momentum stress tensor allows us to use by analogy results from Continuum Mechanics, in particular the stress-strain relation, to provide a better understanding of strained spacetime. As derived in (15), the stress-strain relation for an isotropic and homogeneous spacetime continuum can be written as:

$$2\mu_0 \varepsilon^{\mu\nu} + \lambda_0 g^{\mu\nu} \varepsilon = T^{\mu\nu}.$$

The contraction of (15) yields the relation

$$2(\mu_0 + 2\lambda_0)\varepsilon = T^\alpha{}_\alpha \equiv T. \quad (18)$$

The time-time component  $T^{00}$  of the energy-momentum stress tensor represents the total energy density given by [19, see pp. 37–41]

$$T^{00}(x^k) = \int d^3\mathbf{p} E_p f(x^k, \mathbf{p}), \quad (19)$$

where  $E_p = \sqrt{\rho^2 c^4 + p^2 c^2}$ ,  $\rho$  is the rest-mass energy density,  $c$  is the speed of light,  $\mathbf{p}$  is the momentum 3-vector and  $f(x^k, \mathbf{p})$  is the distribution function representing the number of particles in a small phase space volume  $d^3\mathbf{x} d^3\mathbf{p}$ . The space-space components  $T^{ij}$  of the energy-momentum stress tensor represent the stresses within the medium given by

$$T^{ij}(x^k) = c^2 \int d^3\mathbf{p} \frac{p^i p^j}{E_p} f(x^k, \mathbf{p}). \quad (20)$$

They are the components of the net force acting across a unit area of a surface, across the  $x^i$  planes in the case where  $i = j$ . In the simple case of a particle, they are given by [20, see p. 117]

$$T^{ii} = \rho v^i v^i, \tag{21}$$

where  $v^i$  are the spatial components of velocity. If the particles are subject to forces, these stresses must be included in the energy-momentum stress tensor.

The energy-momentum stress tensor thus includes the energy density, momentum density and stresses from all matter and all fields, such as for example the electromagnetic field.

Explicitly separating the time-time and the space-space components, the trace of the energy-momentum stress tensor is written as

$$T^\alpha{}_\alpha = T^0{}_0 + T^i{}_i. \tag{22}$$

Substituting from (19) and (20), using the metric  $\eta^{\mu\nu}$  of signature  $(+ - - -)$ , we obtain:

$$T^\alpha{}_\alpha(x^k) = \int d^3\mathbf{p} \left( E_p - \frac{p^2 c^2}{E_p} \right) f(x^k, \mathbf{p}) \tag{23}$$

which simplifies to

$$T^\alpha{}_\alpha(x^k) = \rho^2 c^4 \int d^3\mathbf{p} \frac{f(x^k, \mathbf{p})}{E_p}. \tag{24}$$

Using the relation [19, see p. 37]

$$\frac{1}{\overline{E}_{\text{har}}(x^k)} = \int d^3\mathbf{p} \frac{f(x^k, \mathbf{p})}{E_p} \tag{25}$$

in equation (24), we obtain the relation

$$T^\alpha{}_\alpha(x^k) = \frac{\rho^2 c^4}{\overline{E}_{\text{har}}(x^k)}, \tag{26}$$

where  $\overline{E}_{\text{har}}(x^k)$  is the Lorentz invariant harmonic mean of the energy of the particles at  $x^k$ .

In the harmonic mean of the energy of the particles  $\overline{E}_{\text{har}}$ , the momentum contribution  $\mathbf{p}$  will tend to average out and be dominated by the mass term  $\rho c^2$ , so that we can write

$$\overline{E}_{\text{har}}(x^k) \simeq \rho c^2. \tag{27}$$

Substituting for  $\overline{E}_{\text{har}}$  in (26), we obtain the relation

$$T^\alpha{}_\alpha(x^k) \simeq \rho c^2. \quad (28)$$

The total rest-mass energy density of the system is obtained by integrating over all space:

$$T^\alpha{}_\alpha = \int d^3\mathbf{x} T^\alpha{}_\alpha(x^k). \quad (29)$$

The expression for the trace derived from (22) depends on the composition of the sources of the gravitational field. Considering the energy-momentum stress tensor of the electromagnetic field, we can show that  $T^\alpha{}_\alpha = 0$  as expected for massless photons, while

$$T^{00} = \frac{\epsilon_{\text{em}}}{2} (E^2 + c^2 B^2)$$

is the total energy density, where  $\epsilon_{\text{em}}$  is the electromagnetic permittivity of free space, and  $E$  and  $B$  have their usual significance (see Page 250 for details).

Hence  $T^\alpha{}_\alpha$  corresponds to the invariant rest-mass energy density and we write

$$T^\alpha{}_\alpha = T = \rho c^2, \quad (30)$$

where  $\rho$  is the rest-mass energy density. Using (30) into (18), the relation between the invariant volume dilatation  $\varepsilon$  and the invariant rest-mass energy density becomes

$$2(\mu_0 + 2\lambda_0)\varepsilon = \rho c^2 \quad (31)$$

or, in terms of the bulk modulus  $\kappa_0$ ,

$$4\kappa_0\varepsilon = \rho c^2. \quad (32)$$

This equation demonstrates that rest-mass energy density arises from the volume dilatation of the spacetime continuum. The rest-mass energy is equivalent to the energy required to dilate the volume of the spacetime continuum, and is a measure of the energy stored in the spacetime continuum as volume dilatation.  $\kappa_0$  represents the resistance of the spacetime continuum to dilatation. The volume dilatation is an invariant, as is the rest-mass energy density.

**§4. Decomposition of tensor fields in strained spacetime.** As opposed to vector fields which can be decomposed into longitudinal (irrotational) and transverse (solenoidal) components using the Helmholtz representation theorem [17, see pp. 260–261], the decomposition

of spacetime tensor fields can be done in many ways (see for example [9–11, 13]).

The application of Continuum Mechanics to a strained spacetime continuum offers a natural decomposition of tensor fields, in terms of dilatations and distortions [18, see pp. 58–60]. A *dilatation* corresponds to a change of volume of the spacetime continuum without a change of shape while a *distortion* corresponds to a change of shape of the spacetime continuum without a change in volume. Dilatations correspond to longitudinal displacements and distortions correspond to transverse displacements [17, see p. 260].

The strain tensor  $\varepsilon^{\mu\nu}$  can thus be decomposed into a strain deviation tensor  $e^{\mu\nu}$  (the *distortion*) and a scalar  $e_s$  (the *dilatation*) according to [18, see pp. 58–60]:

$$\varepsilon^{\mu\nu} = e^{\mu\nu} + e_s g^{\mu\nu}, \quad (33)$$

where

$$e^\mu{}_\nu = \varepsilon^\mu{}_\nu - e_s \delta^\mu{}_\nu, \quad (34)$$

$$e_s = \frac{1}{4} \varepsilon^\alpha{}_\alpha = \frac{1}{4} \varepsilon. \quad (35)$$

Similarly, the energy-momentum stress tensor  $T^{\mu\nu}$  is decomposed into a stress deviation tensor  $t^{\mu\nu}$  and a scalar  $t_s$  according to

$$T^{\mu\nu} = t^{\mu\nu} + t_s g^{\mu\nu}, \quad (36)$$

where similarly

$$t^\mu{}_\nu = T^\mu{}_\nu - t_s \delta^\mu{}_\nu, \quad (37)$$

$$t_s = \frac{1}{4} T^\alpha{}_\alpha. \quad (38)$$

Using (33) to (38) into the strain-stress relation of (15) and making use of (18) and (14), we obtain separated dilatation and distortion relations respectively:

$$\text{dilatation : } t_s = 2(\mu_0 + 2\lambda_0) e_s = 4\kappa_0 e_s = \kappa_0 \varepsilon \quad (39)$$

$$\text{distortion : } t^{\mu\nu} = 2\mu_0 e^{\mu\nu}.$$

The distortion-dilatation decomposition is evident in the dependence of the dilatation relation on the bulk modulus  $\kappa_0$  and of the distortion relation on the shear modulus  $\mu_0$ . The dilatation relation of (39) corresponds to rest-mass energy, while the distortion relation is traceless and thus massless, and corresponds to shear transverse waves.

This decomposition of spacetime continuum deformations into a massive dilatation and a massless transverse wave distortion is somewhat reminiscent of wave-particle duality. This could explain why dilatation-measuring apparatus measure the massive “particle” properties of the deformation, while distortion-measuring apparatus measure the massless transverse “wave” properties of the deformation.

**§5. Kinematic relations.** The strain  $\varepsilon^{\mu\nu}$  can be expressed in terms of the displacement  $u^\mu$  through the kinematic relation [17, see pp. 149–152]:

$$\varepsilon^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} + u^{\nu;\mu} + u^{\alpha;\mu}u_{\alpha}{}^{;\nu}) \quad (40)$$

where the semicolon (;) denotes covariant differentiation. For small displacements, this expression can be linearized to give the symmetric tensor

$$\varepsilon^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} + u^{\nu;\mu}) = u^{(\mu;\nu)}. \quad (41)$$

We use the small displacement approximation in this analysis.

An antisymmetric tensor  $\omega^{\mu\nu}$  can also be defined from the displacement  $u^\mu$ . This tensor is called the rotation tensor and is defined as [17]:

$$\omega^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} - u^{\nu;\mu}) = u^{[\mu;\nu]}. \quad (42)$$

Where needed, displacements in expressions derived from (41) will be written as  $u_{\parallel}$  while displacements in expressions derived from (42) will be written as  $u_{\perp}$ . Using different symbolic subscripts for these displacements provides a reminder that symmetric displacements are along the direction of motion (longitudinal), while antisymmetric displacements are perpendicular to the direction of motion (transverse).

In general, we have [17]

$$u^{\mu;\nu} = \varepsilon^{\mu\nu} + \omega^{\mu\nu} \quad (43)$$

where the tensor  $u^{\mu;\nu}$  is a combination of symmetric and antisymmetric tensors. Lowering index  $\nu$  and contracting, we get the volume dilatation of the spacetime continuum

$$u^{\mu}{}_{;\mu} = \varepsilon^{\mu}{}_{\mu} = u_{\parallel}{}^{\mu}{}_{;\mu} = \varepsilon \quad (44)$$

where the relation

$$\omega^{\mu}{}_{\mu} = u_{\perp}{}^{\mu}{}_{;\mu} = 0 \quad (45)$$

has been used.

**§6. Dynamic equation**

**§6.1. Equilibrium condition.** Under equilibrium conditions, the dynamics of the spacetime continuum is described by the equation [18, see pp. 88–89],

$$T^{\mu\nu}{}_{;\mu} = -X^\nu, \tag{46}$$

where  $X^\nu$  is the volume (or body) force. As Wald [14, see p. 286] points out, in General Relativity the local energy density of matter as measured by a given observer is well-defined, and the relation

$$T^{\mu\nu}{}_{;\mu} = 0 \tag{47}$$

can be taken as expressing local conservation of the energy-momentum of matter. However, it does not in general lead to a global conservation law. The value  $X^\nu = 0$  is thus taken to represent the macroscopic local case, while (46) provides a more general expression.

At the microscopic level, energy is conserved within the limits of the Heisenberg Uncertainty Principle. The volume force may thus be very small, but not exactly zero. It again makes sense to retain the volume force in the equation, and use (46) in the general case, while (47) can be used at the macroscopic local level, obtained by setting the volume force  $X^\nu$  equal to zero.

**§6.2. Displacement wave equation.** Substituting for  $T^{\mu\nu}$  from (15), (46) becomes

$$2\mu_0 \varepsilon^{\mu\nu}{}_{;\mu} + \lambda_0 g^{\mu\nu} \varepsilon_{;\mu} = -X^\nu \tag{48}$$

and, using (41),

$$\mu_0 (u^{\mu;\nu}{}_\mu + u^{\nu;\mu}{}_\mu) + \lambda_0 \varepsilon^{i\nu} = -X^\nu. \tag{49}$$

Interchanging the order of differentiation in the first term and using (44) to express  $\varepsilon$  in terms of  $u$ , this equation simplifies to

$$\mu_0 u^{\nu;\mu}{}_\mu + (\mu_0 + \lambda_0) u^\mu{}_{;\mu}{}^\nu = -X^\nu, \tag{50}$$

which can also be written as

$$\mu_0 \nabla^2 u^\nu + (\mu_0 + \lambda_0) \varepsilon^{i\nu} = -X^\nu. \tag{51}$$

This is the *displacement wave equation*.

Setting  $X^\nu$  equal to zero, we obtain the macroscopic displacement wave equation

$$\nabla^2 u^\nu = -\frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{i\nu}. \tag{52}$$



**§6.3. Continuity equation.** Taking the divergence of (43), we obtain

$$u^{\mu;\nu}{}_{;\mu} = \varepsilon^{\mu\nu}{}_{;\mu} + \omega^{\mu\nu}{}_{;\mu}. \quad (53)$$

Interchanging the order of partial differentiation in the first term, and using (44) to express  $u$  in terms of  $\varepsilon$ , this equation simplifies to

$$\varepsilon^{\mu\nu}{}_{;\mu} + \omega^{\mu\nu}{}_{;\mu} = \varepsilon^{;\nu}. \quad (54)$$

Hence the divergence of the strain and rotation tensors equals the gradient of the massive volume dilatation, which acts as a source term. This is the continuity equation for deformations of the spacetime continuum.

## §7. Wave equations

**§7.1. Dilatational (longitudinal) wave equation.** Taking the divergence of (50) and interchanging the order of partial differentiation in the first term, we obtain

$$(2\mu_0 + \lambda_0)u^{\mu}{}_{;\mu}{}^{\nu}{}_{;\nu} = -X^{\nu}{}_{;\nu}. \quad (55)$$

Using (44) to express  $u$  in terms of  $\varepsilon$ , this equation simplifies to

$$(2\mu_0 + \lambda_0)\varepsilon^{;\nu}{}_{;\nu} = -X^{\nu}{}_{;\nu} \quad (56)$$

or

$$(2\mu_0 + \lambda_0)\nabla^2\varepsilon = -X^{\nu}{}_{;\nu}. \quad (57)$$

Setting  $X^{\nu}$  equal to zero, we obtain the macroscopic longitudinal wave equation

$$(2\mu_0 + \lambda_0)\nabla^2\varepsilon = 0. \quad (58)$$

The volume dilatation  $\varepsilon$  satisfies a wave equation known as the dilatational wave equation [17, see p. 260]. The solutions of the homogeneous equation are dilatational waves which are longitudinal waves, propagating along the direction of motion. Dilatations thus propagate in the spacetime continuum as longitudinal waves.

**§7.2. Rotational (transverse) wave equation.** Differentiating (50) with respect to  $x^{\alpha}$ , we obtain

$$\mu_0 u^{\nu;\mu}{}_{;\mu}{}^{\alpha} + (\mu_0 + \lambda_0)u^{\mu}{}_{;\mu}{}^{\nu\alpha} = -X^{\nu;\alpha}. \quad (59)$$

Interchanging the dummy indices  $\nu$  and  $\alpha$ , and subtracting the resulting equation from (59), we obtain the relation

$$\mu_0(u^{\nu;\mu}{}_{;\mu}{}^{\alpha} - u^{\alpha;\mu}{}_{;\mu}{}^{\nu}) = -(X^{\nu;\alpha} - X^{\alpha;\nu}). \quad (60)$$

Interchanging the order of partial differentiations and using the definition of the rotation tensor  $\omega^{\nu\alpha}$  of (42), the following wave equation is obtained:

$$\mu_0 \nabla^2 \omega^{\mu\nu} = -X^{[\mu;\nu]} \tag{61}$$

where  $X^{[\mu;\nu]}$  is the antisymmetrical component of the gradient of the volume force defined as

$$X^{[\mu;\nu]} = \frac{1}{2} (X^{\mu;\nu} - X^{\nu;\mu}). \tag{62}$$

Setting  $X^\nu$  equal to zero, we obtain the macroscopic transverse wave equation

$$\mu_0 \nabla^2 \omega^{\mu\nu} = 0. \tag{63}$$

The rotation tensor  $\omega^{\mu\nu}$  satisfies a wave equation known as the rotational wave equation [17, see p. 260]. The solutions of the homogeneous equation are rotational waves which are transverse waves, propagating perpendicular to the direction of motion. Massless waves thus propagate in the spacetime continuum as transverse waves.

**§7.3. Strain (symmetric) wave equation.** A corresponding symmetric wave equation can also be derived for the strain  $\varepsilon^{\mu\nu}$ . Starting from (59), interchanging the dummy indices  $\nu$  and  $\alpha$ , adding the resulting equation to (59), and interchanging the order of partial differentiation, the following wave equation is obtained:

$$\mu_0 \nabla^2 \varepsilon^{\mu\nu} + (\mu_0 + \lambda_0) \varepsilon^{;\mu\nu} = -X^{(\mu;\nu)} \tag{64}$$

where  $X^{(\mu;\nu)}$  is the symmetrical component of the gradient of the volume force defined as

$$X^{(\mu;\nu)} = \frac{1}{2} (X^{\mu;\nu} + X^{\nu;\mu}). \tag{65}$$

Setting  $X^\nu$  equal to zero, we obtain the macroscopic symmetric wave equation

$$\nabla^2 \varepsilon^{\mu\nu} = -\frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{;\mu\nu}. \tag{66}$$

This strain wave equation is similar to the displacement wave equation (52).

**§8. Strain energy density of the spacetime continuum.** The strain energy density of the spacetime continuum is a scalar given by [18, see p. 51]

$$\mathcal{E} = \frac{1}{2} T^{\alpha\beta} \varepsilon_{\alpha\beta}, \tag{67}$$

where  $\varepsilon_{\alpha\beta}$  is the strain tensor and  $T^{\alpha\beta}$  is the energy-momentum stress tensor. Introducing the strain and stress deviators from (33) and (36), this equation becomes

$$\mathcal{E} = \frac{1}{2} (t^{\alpha\beta} + t_s g^{\alpha\beta}) (e_{\alpha\beta} + e_s g_{\alpha\beta}). \quad (68)$$

Multiplying and using relations  $e^\alpha{}_\alpha = 0$  and  $t^\alpha{}_\alpha = 0$  from the definition of the strain and stress deviators, we obtain

$$\mathcal{E} = \frac{1}{2} (4t_s e_s + t^{\alpha\beta} e_{\alpha\beta}). \quad (69)$$

Using (39) to express the stresses in terms of the strains, this expression becomes

$$\mathcal{E} = \frac{1}{2} \kappa_0 \varepsilon^2 + \mu_0 e^{\alpha\beta} e_{\alpha\beta} \quad (70)$$

where the Lamé elastic constant of the spacetime continuum  $\mu_0$  is the shear modulus (the resistance of the continuum to *distortions*) and  $\kappa_0$  is the bulk modulus (the resistance of the continuum to *dilatations*). Alternatively, again using (39) to express the strains in terms of the stresses, this expression can be written as

$$\mathcal{E} = \frac{1}{2\kappa_0} t_s^2 + \frac{1}{4\mu_0} t^{\alpha\beta} t_{\alpha\beta}. \quad (71)$$

**§8.1 Physical interpretation of the strain energy density.** The strain energy density is separated into two terms: the first one expresses the dilatation energy density (the “mass” longitudinal term) while the second one expresses the distortion energy density (the “massless” transverse term):

$$\mathcal{E} = \mathcal{E}_{\parallel} + \mathcal{E}_{\perp}, \quad (72)$$

where

$$\mathcal{E}_{\parallel} = \frac{1}{2} \kappa_0 \varepsilon^2 \equiv \frac{1}{2\kappa_0} t^2 \quad (73)$$

and

$$\mathcal{E}_{\perp} = \mu_0 e^{\alpha\beta} e_{\alpha\beta} \equiv \frac{1}{4\mu_0} t^{\alpha\beta} t_{\alpha\beta}. \quad (74)$$

Using (32) into (73), we obtain

$$\mathcal{E}_{\parallel} = \frac{1}{32\kappa_0} (\rho c^2)^2. \quad (75)$$

The rest-mass energy density divided by the bulk modulus  $\kappa_0$ , and the transverse energy density divided by the shear modulus  $\mu_0$ , have dimen-

sions of energy density as expected.

Multiplying (71) by  $32\kappa_0$  and using (75), we obtain

$$32\kappa_0\mathcal{E} = \rho^2c^4 + 8\frac{\kappa_0}{\mu_0}t^{\alpha\beta}t_{\alpha\beta}. \tag{76}$$

Noting that  $t^{\alpha\beta}t_{\alpha\beta}$  is quadratic in structure, we see that this equation is similar to the energy relation of Special Relativity [22, see p. 51] for energy density

$$\hat{E}^2 = \rho^2c^4 + \hat{p}^2c^2, \tag{77}$$

where  $\hat{E}$  is the total energy density and  $\hat{p}$  the momentum density.

The quadratic structure of the energy relation of Special Relativity is thus found to be present in the Elastodynamics of the Spacetime Continuum. Equations (76) and (77) also imply that the kinetic energy  $pc$  is carried by the distortion part of the deformation, while the dilatation part carries only the rest mass energy.

This observation is in agreement with photons which are massless ( $\mathcal{E}_{\parallel} = 0$ ), as will be shown on Page 250, but still carry kinetic energy in the transverse electromagnetic wave distortions ( $\mathcal{E}_{\perp} = \frac{1}{4\mu_0}t^{\alpha\beta}t_{\alpha\beta}$ ).

## §9. Theory of Electromagnetism from *STCED*

**§9.1. Electromagnetic field strength.** In the Elastodynamics of the Spacetime Continuum, the antisymmetric rotation tensor  $\omega^{\mu\nu}$  is given by (42), viz.

$$\omega^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} - u^{\nu;\mu}) \tag{78}$$

where  $u^\mu$  is the displacement of an infinitesimal element of the spacetime continuum from its unstrained position  $x^\mu$ . This tensor has the same structure as the electromagnetic field tensor  $F^{\mu\nu}$  [33, see p. 550]:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{79}$$

where  $A^\mu$  is the electromagnetic potential four-vector  $(\phi, \mathbf{A})$ ,  $\phi$  is the scalar potential and  $\mathbf{A}$  the vector potential.

Identifying the rotation tensor  $\omega^{\mu\nu}$  with the electromagnetic field-strength tensor according to

$$F^{\mu\nu} = \varphi_0\omega^{\mu\nu} \tag{80}$$

leads to the relation

$$A^\mu = -\frac{1}{2}\varphi_0u_{\perp}^\mu, \tag{81}$$

where the symbolic subscript  $\perp$  of the displacement  $u^\mu$  indicates that the relation holds for a transverse displacement (orthogonal to the direction

of motion). The constant of proportionality  $\varphi_0$  will be referred to as the “*STC* electromagnetic shearing potential constant”.

Due to the difference in the definition of  $\omega^{\mu\nu}$  and  $F^{\mu\nu}$  with respect to their indices, a negative sign is introduced, and is attributed to (81). This relation provides a physical explanation of the electromagnetic potential: it arises from transverse (shearing) displacements of the spacetime continuum, in contrast to mass which arises from longitudinal (dilatational) displacements of the spacetime continuum. Sheared spacetime is manifested as electromagnetic potentials and fields.

### §9.2. Maxwell’s equations and the current density four-vector.

Taking the divergence of the rotation tensor of (78), gives

$$\omega^{\mu\nu}{}_{;\mu} = \frac{1}{2}(u^{\mu;\nu}{}_{\mu} - u^{\nu;\mu}{}_{\mu}). \quad (82)$$

Recalling (50), viz.

$$\mu_0 u^{\nu;\mu}{}_{\mu} + (\mu_0 + \lambda_0) u^{\mu}{}_{;\mu}{}^{\nu} = -X^{\nu},$$

where  $X^{\nu}$  is the volume force and  $\lambda_0$  and  $\mu_0$  are the Lamé elastic constants of the spacetime continuum, substituting for  $u^{\nu;\mu}{}_{\mu}$  from (50) into (82), interchanging the order of partial differentiation in  $u^{\mu;\nu}{}_{\mu}$  in (82), and using the relation  $u^{\mu}{}_{;\mu} = \varepsilon^{\mu}{}_{\mu} = \varepsilon$  from (44), we obtain

$$\omega^{\mu\nu}{}_{;\mu} = \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu} + \frac{1}{2\mu_0} X^{\nu}. \quad (83)$$

As seen previously on Page 239, in the macroscopic local case, the volume force  $X^{\nu}$  is set equal to zero to obtain the macroscopic relation

$$\omega^{\mu\nu}{}_{;\mu} = \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu}. \quad (84)$$

Using (80) and comparing with the covariant form of Maxwell’s equations [34, see pp. 42–43]

$$F^{\mu\nu}{}_{;\mu} = \mu_{\text{em}} j^{\nu}, \quad (85)$$

where  $j^{\nu}$  is the current density four-vector ( $c\varrho, \mathbf{j}$ ),  $\varrho$  is the charge density scalar, and  $\mathbf{j}$  is the current density vector, we obtain the relation

$$j^{\nu} = \frac{\varphi_0}{\mu_{\text{em}}} \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu}. \quad (86)$$

This relation provides a physical explanation of the current density four-vector: it arises from the 4-gradient of the volume dilatation of the

spacetime continuum. A corollary of this relation is that massless (transverse) waves cannot carry an electric charge or produce a current.

Substituting for  $j^\nu$  from (86) in the relation [35, see p. 94]

$$j^\nu j_\nu = \varrho^2 c^2, \tag{87}$$

we obtain the expression for the charge density

$$\varrho = \frac{1}{2} \frac{\varphi_0}{\mu_{\text{em}} c} \frac{2\mu_0 + \lambda_0}{2\mu_0} \sqrt{\varepsilon^{;\nu} \varepsilon_{;\nu}} \tag{88}$$

or, using the relation  $c = 1/\sqrt{\varepsilon_{\text{em}} \mu_{\text{em}}}$ ,

$$\varrho = \frac{1}{2} \varphi_0 \varepsilon_{\text{em}} c \frac{2\mu_0 + \lambda_0}{2\mu_0} \sqrt{\varepsilon^{;\nu} \varepsilon_{;\nu}}. \tag{89}$$

Up to now, our identification of the rotation tensor  $\omega^{\mu\nu}$  of the Elastodynamics of the Spacetime Continuum with the electromagnetic field-strength tensor  $F^{\mu\nu}$  has generated consistent results, with no contradictions.

**§9.3. The Lorentz condition.** The Lorentz condition can be derived directly from the theory. Taking the divergence of (81), we obtain

$$A^\mu_{;\mu} = -\frac{1}{2} \varphi_0 u_{\perp}{}^\mu_{;\mu}. \tag{90}$$

From (45), (90) simplifies to

$$A^\mu_{;\mu} = 0. \tag{91}$$

The Lorentz condition is thus obtained directly from the theory. The reason for the value of zero is that transverse displacements are massless because such displacements arise from a change of shape (distortion) of the spacetime continuum, not a change of volume (dilatation).

**§9.4. Four-vector potential.** Substituting (81) into (82) and rearranging terms, we obtain the equation

$$\nabla^2 A^\nu - A^{\mu;\nu}{}_\mu = \varphi_0 \omega^{\mu\nu}{}_{;\mu} \tag{92}$$

and, using (80) and (85), this equation becomes

$$\nabla^2 A^\nu - A^{\mu;\nu}{}_\mu = \mu_{\text{em}} j^\nu. \tag{93}$$

Interchanging the order of partial differentiation in the term  $A^{\mu;\nu}{}_\mu$  and using the Lorentz condition of (91), we obtain the well-known wave

equation for the four-vector potential [34, see pp. 42–43]

$$\nabla^2 A^\nu = \mu_{\text{em}} j^\nu. \quad (94)$$

The results we obtain are thus consistent with the macroscopic theory of Electromagnetism, with no contradictions.

**§10. Electromagnetism and the volume force  $X^\nu$ .** We now investigate the impact of the volume force  $X^\nu$  on the equations of Electromagnetism. Recalling (83), Maxwell's equation in terms of the rotation tensor is given by

$$\omega^{\mu\nu}{}_{;\mu} = \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu} + \frac{1}{2\mu_0} X^\nu. \quad (95)$$

Substituting for  $\omega^{\mu\nu}$  from (80), this equation becomes

$$F^{\mu\nu}{}_{;\mu} = \varphi_0 \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu} + \frac{\varphi_0}{2\mu_0} X^\nu. \quad (96)$$

The additional  $X^\nu$  term can be allocated in one of two ways:

- 1) either  $j^\nu$  remains unchanged as given by (86) and the expression for  $F^{\mu\nu}{}_{;\mu}$  has an additional term as developed in the first section below,
- 2) or  $F^{\mu\nu}{}_{;\mu}$  remains unchanged as given by (85) and the expression for  $j^\nu$  has an additional term as developed in the second section below.

Option 2 is shown in the following derivation to be the logically consistent approach.

**§10.1.  $j^\nu$  unchanged (contradiction).** Using (86) ( $j^\nu$  unchanged) into (96), Maxwell's equation becomes

$$F^{\mu\nu}{}_{;\mu} = \mu_{\text{em}} j^\nu + \frac{\varphi_0}{2\mu_0} X^\nu. \quad (97)$$

Using (95) into (92) and making use of the Lorentz condition, the wave equation for the four-vector potential becomes

$$\nabla^2 A^\nu - \frac{\varphi_0}{2\mu_0} X^\nu = \mu_{\text{em}} j^\nu. \quad (98)$$

In this case, the equations for  $F^{\mu\nu}{}_{;\mu}$  and  $A^\nu$  both contain an additional term proportional to  $X^\nu$ .

We show that this option is not logically consistent as follows. Using (86) into the continuity condition for the current density [34]

$$\partial_\nu j^\nu = 0 \tag{99}$$

yields the expression

$$\nabla^2 \varepsilon = 0. \tag{100}$$

This equation is valid in the macroscopic case where  $X^\nu = 0$ , but disagrees with the general case (non-zero  $X^\nu$ ) given by (57), viz.

$$(2\mu_0 + \lambda_0)\nabla^2 \varepsilon = -X^\nu{}_{;\nu}.$$

This analysis leads to a contradiction and consequently is not valid.

**§10.2.  $F^{\mu\nu}{}_{;\mu}$  unchanged (logically consistent).** Proper treatment of the general case requires that the current density four-vector be proportional to the RHS of (96) as follows ( $F^{\mu\nu}{}_{;\mu}$  unchanged):

$$\mu_{\text{em}} j^\nu = \varphi_0 \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu} + \frac{\varphi_0}{2\mu_0} X^\nu. \tag{101}$$

This yields the following general form of the current density four-vector:

$$j^\nu = \frac{1}{2} \frac{\varphi_0}{\mu_{\text{em}} \mu_0} [(2\mu_0 + \lambda_0)\varepsilon^{;\nu} + X^\nu]. \tag{102}$$

Using this expression in the continuity condition for the current density given by (99) yields (57) as required.

Using (102) into (96) yields the same covariant form of the Maxwell equations as in the macroscopic case:

$$F^{\mu\nu}{}_{;\mu} = \mu_{\text{em}} j^\nu \tag{103}$$

and the same four-vector potential equation

$$\nabla^2 A^\nu = \mu_{\text{em}} j^\nu \tag{104}$$

in the Lorentz gauge.

**§10.3. Homogeneous Maxwell equation.** The validity of this analysis can be further demonstrated from the homogeneous Maxwell equation [34]

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0. \tag{105}$$

Taking the divergence of this equation over  $\alpha$ ,

$$\partial_\alpha \partial^\alpha F^{\beta\gamma} + \partial_\alpha \partial^\beta F^{\gamma\alpha} + \partial_\alpha \partial^\gamma F^{\alpha\beta} = 0. \tag{106}$$



Interchanging the order of differentiation in the last two terms and making use of (103) and the antisymmetry of  $F^{\mu\nu}$ , we obtain

$$\nabla^2 F^{\beta\gamma} + \mu_{\text{em}} (j^{\beta;\gamma} - j^{\gamma;\beta}) = 0. \quad (107)$$

Substituting for  $j^\nu$  from (102),

$$\nabla^2 F^{\beta\gamma} = -\frac{\varphi_0}{2\mu_0} [(2\mu_0 + \lambda_0)(\varepsilon^{;\beta\gamma} - \varepsilon^{;\gamma\beta}) + (X^{\beta;\gamma} - X^{\gamma;\beta})]. \quad (108)$$

Equation (64), viz.

$$\mu_0 \nabla^2 \varepsilon^{\mu\nu} + (\mu_0 + \lambda_0) \varepsilon^{;\mu\nu} = -X^{(\mu;\nu)}$$

shows that  $\varepsilon^{;\mu\nu}$  is a symmetrical tensor. Consequently the difference term  $(\varepsilon^{;\beta\gamma} - \varepsilon^{;\gamma\beta})$  disappears and (108) becomes

$$\nabla^2 F^{\beta\gamma} = -\frac{\varphi_0}{2\mu_0} (X^{\beta;\gamma} - X^{\gamma;\beta}). \quad (109)$$

Expressing  $F^{\mu\nu}$  in terms of  $\omega^{\mu\nu}$  using (80), the resulting equation is identical to (61), viz.

$$\mu_0 \nabla^2 \omega^{\mu\nu} = -X^{[\mu;\nu]}$$

confirming the validity of this analysis of Electromagnetism including the volume force.

Equations (102) to (104) are the self-consistent electromagnetic equations derived from the Elastodynamics of the Spacetime Continuum with the volume force. In conclusion, Maxwell's equations remain unchanged. The current density four-vector is the only quantity affected by the volume force, with the addition of a second term proportional to the volume force.

It is interesting to note that the current density obtained from the quantum mechanical Klein-Gordon equation with an electromagnetic field also consists of the sum of two terms [36, see p. 35].

**§11. Electromagnetic strain energy density.** The strain energy density of the electromagnetic energy-momentum stress tensor is calculated. Starting from the symmetric electromagnetic stress tensor [34, see pp. 64–66], which has the form

$$\Theta^{\mu\nu} = \frac{1}{\mu_{\text{em}}} \left( F^\mu{}_\alpha F^{\alpha\nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \equiv \sigma^{\mu\nu}, \quad (110)$$

with  $g^{\mu\nu} = \eta^{\mu\nu}$  of signature  $(+ - - -)$ , and the field-strength tensor

components [34, see p. 43]

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad (111)$$

and

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}, \quad (112)$$

we obtain  $\sigma^{\mu\nu} \equiv \Theta^{\mu\nu}$  which is a generalization of the  $\sigma^{ij}$  Maxwell stress tensor (here  $S^j$  is the Poynting vector, see [34, p. 66], [37, p. 141])

$$\begin{aligned} \sigma^{00} &= \frac{1}{2} \left( \epsilon_{\text{em}} E^2 + \frac{1}{\mu_{\text{em}}} B^2 \right) = \frac{1}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2), \\ \sigma^{0j} &= \sigma^{j0} = \frac{1}{c \mu_{\text{em}}} (E \times B)^j = \epsilon_{\text{em}} c (E \times B)^j = \frac{1}{c} S^j, \\ \sigma^{jk} &= - \left( \epsilon_{\text{em}} E^j E^k + \frac{1}{\mu_{\text{em}}} B^j B^k \right) + \frac{1}{2} \delta^{jk} \left( \epsilon_{\text{em}} E^2 + \frac{1}{\mu_{\text{em}}} B^2 \right) = \\ &= - \epsilon_{\text{em}} \left[ (E^j E^k + c^2 B^j B^k) - \frac{1}{2} \delta^{jk} (E^2 + c^2 B^2) \right]. \end{aligned} \quad (113)$$

Hence the electromagnetic stress tensor is given by [34, see p. 66]:

$$\sigma^{\mu\nu} = \begin{pmatrix} \frac{1}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2) & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}, \quad (114)$$

where  $\sigma^{ij}$  is the Maxwell stress tensor. Using  $\sigma_{\alpha\beta} = \eta_{\alpha\mu} \eta_{\beta\nu} \sigma^{\mu\nu}$  to lower the indices of  $\sigma^{\mu\nu}$ , we obtain

$$\sigma_{\mu\nu} = \begin{pmatrix} \frac{1}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2) & -S_x/c & -S_y/c & -S_z/c \\ -S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ -S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ -S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}. \quad (115)$$

**§11.1. Calculation of the longitudinal (mass) term.** The mass term is calculated from (73) and (38):

$$\mathcal{E}_{\parallel} = \frac{1}{2\kappa_0} t_s^2 = \frac{1}{32\kappa_0} (\sigma^\alpha{}_\alpha)^2. \quad (116)$$

The term  $\sigma^\alpha{}_\alpha$  is calculated from:

$$\left. \begin{aligned} \sigma^\alpha{}_\alpha &= \eta_{\alpha\beta} \sigma^{\alpha\beta} \\ &= \eta_{\alpha 0} \sigma^{\alpha 0} + \eta_{\alpha 1} \sigma^{\alpha 1} + \eta_{\alpha 2} \sigma^{\alpha 2} + \eta_{\alpha 3} \sigma^{\alpha 3} \\ &= \eta_{00} \sigma^{00} + \eta_{11} \sigma^{11} + \eta_{22} \sigma^{22} + \eta_{33} \sigma^{33} \end{aligned} \right\}. \quad (117)$$

Substituting from (114) and the metric  $\eta^{\mu\nu}$  of signature  $(+---)$ , we obtain:

$$\sigma^\alpha{}_\alpha = \frac{1}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2) + \sigma_{xx} + \sigma_{yy} + \sigma_{zz}. \quad (118)$$

Substituting from (113), this expands to:

$$\begin{aligned} \sigma^\alpha{}_\alpha &= \frac{1}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2) + \epsilon_{\text{em}} (E_x^2 + c^2 B_x^2) + \\ &+ \epsilon_{\text{em}} (E_y^2 + c^2 B_y^2) + \epsilon_{\text{em}} (E_z^2 + c^2 B_z^2) - \\ &- \frac{3}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2) \end{aligned} \quad (119)$$

and further,

$$\begin{aligned} \sigma^\alpha{}_\alpha &= \frac{1}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2) + \epsilon_{\text{em}} (E^2 + c^2 B^2) - \\ &- \frac{3}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2). \end{aligned} \quad (120)$$

Hence

$$\sigma^\alpha{}_\alpha = 0 \quad (121)$$

and, substituting into (116),

$$\mathcal{E}_{\parallel} = 0 \quad (122)$$

as expected [34, see pp.64–66]. This derivation thus shows that the rest-mass energy density of the photon is 0.

**§11.2. Calculation of the transverse (massless) term.** The transverse term is calculated from (74), viz.

$$\mathcal{E}_{\perp} = \frac{1}{4\mu_0} t^{\alpha\beta} t_{\alpha\beta}. \quad (123)$$

Given that  $t_s = \frac{1}{4} \sigma^\alpha{}_\alpha = 0$ , then  $t^{\alpha\beta} = \sigma^{\alpha\beta}$  and the terms  $\sigma^{\alpha\beta} \sigma_{\alpha\beta}$  are calculated from the components of the electromagnetic stress tensors of (114) and (115). Substituting for the diagonal elements and making use of the symmetry of the Poynting component terms and of the Maxwell stress tensor terms from (114) and (115), this expands to:

$$\begin{aligned}
\sigma^{\alpha\beta} \sigma_{\alpha\beta} &= \frac{1}{4} \epsilon_{\text{em}}^2 (E^2 + c^2 B^2)^2 + \\
&+ \epsilon_{\text{em}}^2 \left[ (E_x E_x + c^2 B_x B_x) - \frac{1}{2} (E^2 + c^2 B^2) \right]^2 + \\
&+ \epsilon_{\text{em}}^2 \left[ (E_y E_y + c^2 B_y B_y) - \frac{1}{2} (E^2 + c^2 B^2) \right]^2 + \\
&+ \epsilon_{\text{em}}^2 \left[ (E_z E_z + c^2 B_z B_z) - \frac{1}{2} (E^2 + c^2 B^2) \right]^2 - \\
&- 2 \left( \frac{S_x}{c} \right)^2 - 2 \left( \frac{S_y}{c} \right)^2 - 2 \left( \frac{S_z}{c} \right)^2 + \\
&+ 2 (\sigma_{xy})^2 + 2 (\sigma_{yz})^2 + 2 (\sigma_{zx})^2. \tag{124}
\end{aligned}$$

The E-B terms expand to:

$$\begin{aligned}
\text{EBterms} &= \epsilon_{\text{em}}^2 \left[ \frac{1}{4} (E^2 + c^2 B^2)^2 + (E_x^2 + c^2 B_x^2)^2 - \right. \\
&- (E_x^2 + c^2 B_x^2) (E^2 + c^2 B^2) + (E_y^2 + c^2 B_y^2)^2 - \\
&- (E_y^2 + c^2 B_y^2) (E^2 + c^2 B^2) + (E_z^2 + c^2 B_z^2)^2 - \\
&- \left. (E_z^2 + c^2 B_z^2) (E^2 + c^2 B^2) + \frac{3}{4} (E^2 + c^2 B^2)^2 \right]. \tag{125}
\end{aligned}$$

Simplifying,

$$\begin{aligned}
\text{EBterms} &= \epsilon_{\text{em}}^2 \left[ (E^2 + c^2 B^2)^2 - (E_x^2 + c^2 B_x^2 + \right. \\
&+ E_y^2 + c^2 B_y^2 + E_z^2 + c^2 B_z^2) (E^2 + c^2 B^2) + \\
&+ \left. (E_x^2 + c^2 B_x^2)^2 + (E_y^2 + c^2 B_y^2)^2 + (E_z^2 + c^2 B_z^2)^2 \right], \tag{126}
\end{aligned}$$

which gives

$$\begin{aligned}
\text{EBterms} &= \epsilon_{\text{em}}^2 \left[ (E^2 + c^2 B^2)^2 - (E^2 + c^2 B^2)^2 + \right. \\
&+ \left. (E_x^2 + c^2 B_x^2)^2 + (E_y^2 + c^2 B_y^2)^2 + (E_z^2 + c^2 B_z^2)^2 \right], \tag{127}
\end{aligned}$$

and finally

$$\begin{aligned} \text{EBterms} = \epsilon_{\text{em}}^2 & \left[ (E_x^4 + E_y^4 + E_z^4) + c^4 (B_x^4 + B_y^4 + B_z^4) + \right. \\ & \left. + 2c^2 (E_x^2 B_x^2 + E_y^2 B_y^2 + E_z^2 B_z^2) \right]. \end{aligned} \quad (128)$$

Including the E-B terms in (124), substituting from (113), expanding the Poynting vector and rearranging, we obtain

$$\begin{aligned} \sigma^{\alpha\beta} \sigma_{\alpha\beta} = \epsilon_{\text{em}}^2 & \left[ (E_x^4 + E_y^4 + E_z^4) + c^4 (B_x^4 + B_y^4 + B_z^4) + \right. \\ & \left. + 2c^2 (E_x^2 B_x^2 + E_y^2 B_y^2 + E_z^2 B_z^2) \right] - \\ & - 2\epsilon_{\text{em}}^2 c^2 \left[ (E_y B_z - E_z B_y)^2 + (-E_x B_z + E_z B_x)^2 + \right. \\ & \left. + (E_x B_y - E_y B_x)^2 \right] + 2\epsilon_{\text{em}}^2 \left[ (E_x E_y + c^2 B_x B_y)^2 + \right. \\ & \left. + (E_y E_z + c^2 B_y B_z)^2 + (E_z E_x + c^2 B_z B_x)^2 \right]. \end{aligned} \quad (129)$$

Expanding the quadratic expressions,

$$\begin{aligned} \sigma^{\alpha\beta} \sigma_{\alpha\beta} = \epsilon_{\text{em}}^2 & \left[ (E_x^4 + E_y^4 + E_z^4) + c^4 (B_x^4 + B_y^4 + B_z^4) + \right. \\ & \left. + 2c^2 (E_x^2 B_x^2 + E_y^2 B_y^2 + E_z^2 B_z^2) \right] - \\ & - 2\epsilon_{\text{em}}^2 c^2 \left[ E_x^2 B_y^2 + E_y^2 B_z^2 + E_z^2 B_x^2 + B_x^2 E_y^2 + \right. \\ & \left. + B_y^2 E_z^2 + B_z^2 E_x^2 - 2(E_x E_y B_x B_y + E_y E_z B_y B_z + \right. \\ & \left. + E_z E_x B_z B_x) \right] + 2\epsilon_{\text{em}}^2 \left[ (E_x^2 E_y^2 + E_y^2 E_z^2 + E_z^2 E_x^2) + \right. \\ & \left. + 2c^2 (E_x E_y B_x B_y + E_y E_z B_y B_z + E_z E_x B_z B_x) + \right. \\ & \left. + c^4 (B_x^2 B_y^2 + B_y^2 B_z^2 + B_z^2 B_x^2) \right]. \end{aligned} \quad (130)$$

Grouping the terms in powers of  $c$  together,

$$\begin{aligned} \frac{1}{\epsilon_{\text{em}}^2} \sigma^{\alpha\beta} \sigma_{\alpha\beta} = & \left[ (E_x^4 + E_y^4 + E_z^4) + 2(E_x^2 E_y^2 + \right. \\ & \left. + E_y^2 E_z^2 + E_z^2 E_x^2) \right] + 2c^2 \left[ (E_x^2 B_x^2 + E_y^2 B_y^2 + \right. \\ & \left. + E_z^2 B_z^2) - (E_x^2 B_y^2 + E_y^2 B_z^2 + E_z^2 B_x^2 + \right. \end{aligned}$$

$$\begin{aligned}
& + B_x^2 E_y^2 + B_y^2 E_z^2 + B_z^2 E_x^2) + 4(E_x E_y B_x B_y + \\
& + E_y E_z B_y B_z + E_z E_x B_z B_x) \Big] + c^4 \left[ (B_x^4 + B_y^4 + B_z^4) + \right. \\
& \left. + 2(B_x^2 B_y^2 + B_y^2 B_z^2 + B_z^2 B_x^2) \right]. \tag{131}
\end{aligned}$$

Simplifying,

$$\begin{aligned}
\frac{1}{\epsilon_{\text{em}}^2} \sigma^{\alpha\beta} \sigma_{\alpha\beta} &= (E_x^2 + E_y^2 + E_z^2)^2 + \\
& + 2c^2 (E_x^2 + E_y^2 + E_z^2) (B_x^2 + B_y^2 + B_z^2) - \\
& - 2c^2 \left[ 2(E_x^2 B_y^2 + E_y^2 B_z^2 + E_z^2 B_x^2 + \right. \\
& + B_x^2 E_y^2 + B_y^2 E_z^2 + B_z^2 E_x^2) - 4(E_x E_y B_x B_y + \\
& \left. + E_y E_z B_y B_z + E_z E_x B_z B_x) \right] + c^4 (B_x^2 + B_y^2 + B_z^2)^2, \tag{132}
\end{aligned}$$

which is further simplified to

$$\begin{aligned}
\frac{1}{\epsilon_{\text{em}}^2} \sigma^{\alpha\beta} \sigma_{\alpha\beta} &= (E^4 + 2c^2 E^2 B^2 + c^4 B^4) - 4c^2 \left[ (E_y B_z - B_y E_z)^2 + \right. \\
& \left. + (E_z B_x - B_z E_x)^2 + (E_x B_y - B_x E_y)^2 \right]. \tag{133}
\end{aligned}$$

Making use of the definition of the Poynting vector from (113), we obtain

$$\begin{aligned}
\sigma^{\alpha\beta} \sigma_{\alpha\beta} &= \epsilon_{\text{em}}^2 (E^2 + c^2 B^2)^2 - \\
& - 4\epsilon_{\text{em}}^2 c^2 \left[ (E \times B)_x^2 + (E \times B)_y^2 + (E \times B)_z^2 \right] \tag{134}
\end{aligned}$$

and finally

$$\sigma^{\alpha\beta} \sigma_{\alpha\beta} = \epsilon_{\text{em}}^2 (E^2 + c^2 B^2)^2 - \frac{4}{c^2} (S_x^2 + S_y^2 + S_z^2). \tag{135}$$

Substituting in (123), the transverse term becomes

$$\mathcal{E}_\perp = \frac{1}{4\mu_0} \left[ \epsilon_{\text{em}}^2 (E^2 + c^2 B^2)^2 - \frac{4}{c^2} S^2 \right] \tag{136}$$

or

$$\mathcal{E}_\perp = \frac{1}{\mu_0} \left[ U_{\text{em}}^2 - \frac{1}{c^2} S^2 \right], \tag{137}$$

where  $U_{\text{em}} = \frac{1}{2} \epsilon_{\text{em}} (E^2 + c^2 B^2)$  is the electromagnetic field energy density.

**§11.3. Electromagnetic field strain energy density and the photon.**  $\mathbf{S}$  is the electromagnetic energy flux along the direction of propagation [34, p. 62]. As noted by Feynman [38, pp. 27-1–27-2], local conservation of the electromagnetic field energy can be written as

$$-\frac{\partial U_{\text{em}}}{\partial t} = \nabla \cdot \mathbf{S}, \quad (138)$$

where the term  $\mathbf{E} \cdot \mathbf{j}$  representing the work done on the matter inside the volume is 0 in the absence of charges (due to the absence of mass). By analogy with the current density four-vector  $j^\nu = (c\rho, \mathbf{j})$ , where  $\rho$  is the charge density, and  $\mathbf{j}$  is the current density vector, which obeys a similar conservation relation, we define the Poynting four-vector

$$S^\nu = (cU_{\text{em}}, \mathbf{S}), \quad (139)$$

where  $U_{\text{em}}$  is the electromagnetic field energy density, and  $\mathbf{S}$  is the Poynting vector. Furthermore, as per (138),  $S^\nu$  satisfies

$$\partial_\nu S^\nu = 0. \quad (140)$$

Using definition (139) in (137), that equation becomes

$$\mathcal{E}_\perp = \frac{1}{\mu_0 c^2} S_\nu S^\nu. \quad (141)$$

The indefiniteness of the location of the field energy referred to by Feynman [38, see p. 27-1] is thus resolved: the electromagnetic field energy resides in the distortions (transverse displacements) of the spacetime continuum.

Hence the invariant electromagnetic strain energy density is given by

$$\mathcal{E} = \frac{1}{\mu_0 c^2} S_\nu S^\nu \quad (142)$$

where we have used  $\rho = 0$  as per (121). This confirms that  $S^\nu$  as defined in (139) is a four-vector.

It is surprising that a longitudinal energy flow term is part of the transverse strain energy density i.e.  $S^2/\mu_0 c^2$  in (137). We note that this term arises from the time-space components of (114) and (115) and can be seen to correspond to the transverse displacements along the *time-space* planes which are folded along the direction of propagation in 3-space as the Poynting vector. The electromagnetic field energy density term  $U_{\text{em}}^2/\mu_0$  and the electromagnetic field energy flux term  $S^2/\mu_0 c^2$  are thus combined into the transverse strain energy density.

The negative sign arises from the signature (+ - - -) of the metric tensor  $\eta^{\mu\nu}$ .

This longitudinal electromagnetic energy flux is massless as it is due to distortion, not dilatation, of the spacetime continuum. However, because this energy flux is along the direction of propagation (i.e. longitudinal), it gives rise to the particle aspect of the electromagnetic field, the photon. As shown in [39, see pp.174–175] [40, see p.58], in the quantum theory of electromagnetic radiation, an intensity operator derived from the Poynting vector has, as expectation value, photons in the direction of propagation.

This implies that the  $(pc)^2$  term of the energy relation of Special Relativity needs to be separated into transverse and longitudinal massless terms as follows:

$$\hat{E}^2 = \underbrace{\rho^2 c^4}_{\mathcal{E}_{\parallel}} + \underbrace{\hat{p}_{\parallel}^2 c^2 + \hat{p}_{\perp}^2 c^2}_{\text{massless } \mathcal{E}_{\perp}} \tag{143}$$

where  $\hat{p}_{\parallel}$  is the massless longitudinal momentum density. (137) shows that the electromagnetic field energy density term  $U_{\text{em}}^2/\mu_0$  is reduced by the electromagnetic field energy flux term  $S^2/\mu_0 c^2$  in the transverse strain energy density, due to photons propagating in the longitudinal direction. Hence we can write [40, see p.58]

$$\int_V \frac{1}{\mu_0 c^2} S^2 dV = \sum_k n_k h \nu_k. \tag{144}$$

where  $h$  is Planck’s constant and  $n_k$  is the number of photons of frequency  $\nu_k$ . Thus the kinetic energy is carried by the distortion part of the deformation, while the dilatation part carries only the rest-mass energy, which in this case is 0.

As shown in (75), (76) and (77), the constant of proportionality to transform energy density squared ( $\hat{E}^2$ ) into strain energy density ( $\mathcal{E}$ ) is  $1/(32\kappa_0)$ :

$$\mathcal{E}_{\parallel} = \frac{1}{32\kappa_0} (\rho c^2)^2, \tag{145}$$

$$\mathcal{E} = \frac{1}{32\kappa_0} \hat{E}^2, \tag{146}$$

$$\mathcal{E}_{\perp} = \frac{1}{32\kappa_0} (\hat{p}_{\parallel}^2 c^2 + \hat{p}_{\perp}^2 c^2) = \frac{1}{4\mu_0} t^{\alpha\beta} t_{\alpha\beta}. \tag{147}$$

Substituting (137) into (147), we obtain

$$\mathcal{E}_{\perp} = \frac{1}{32\kappa_0} (\hat{p}_{\parallel}^2 c^2 + \hat{p}_{\perp}^2 c^2) = \frac{1}{\mu_0} \left( U_{\text{em}}^2 - \frac{1}{c^2} S^2 \right) \tag{148}$$



and

$$\hat{p}_{\parallel}^2 c^2 + \hat{p}_{\perp}^2 c^2 = \frac{32\kappa_0}{\mu_0} \left( U_{\text{em}}^2 - \frac{1}{c^2} S^2 \right). \quad (149)$$

This suggests that

$$\mu_0 = 32\kappa_0, \quad (150)$$

to obtain the relation

$$\hat{p}_{\parallel}^2 c^2 + \hat{p}_{\perp}^2 c^2 = U_{\text{em}}^2 - \frac{1}{c^2} S^2. \quad (151)$$

**§12. Linear elastic volume force.** The volume (or body) force  $X^\nu$  has been introduced in the equilibrium dynamic equation of the *STC* in (46) on Page 239 viz.

$$T^{\mu\nu}{}_{;\mu} = -X^\nu. \quad (152)$$

Comparison with the corresponding general relativistic expression showed that the volume force is equal to zero at the macroscopic local level. Indeed, as pointed out by Wald [14, see p. 286], in General Relativity the local energy density of matter as measured by a given observer is well-defined, and the relation

$$T^{\mu\nu}{}_{;\mu} = 0 \quad (153)$$

can be taken as expressing local conservation of the energy-momentum of matter.

It was also pointed out in that section that at the microscopic level, energy is known to be conserved only within the limits of the Heisenberg Uncertainty Principle, suggesting that the volume force may be very small, but not exactly zero. This is analogous to quantum theory where Planck's constant  $h$  must be taken into consideration at the microscopic level while at the macroscopic level, the limit  $h \rightarrow 0$  holds.

In this section, we investigate the volume force and its impact on the equations of the Elastodynamics of the Spacetime Continuum. First we consider a linear elastic volume force. Based on the results obtained, we will then consider a variation of that linear elastic volume force based on the Klein-Gordon quantum mechanical current density.

We investigate a volume force that consists of an elastic linear force in a direction opposite to the displacements. This is the well-known elastic "spring" force

$$X^\nu = k_0 u^\nu, \quad (154)$$

where  $k_0$  is the postulated elastic force constant of the spacetime continuum volume force. (154) is positive as the volume force  $X^\nu$  is defined

positive in the direction opposite to the displacement [18]. Introduction of this volume force into our previous analysis on Page 239 yields the following relations.

**§12.1. Displacement wave equation.** Substituting (154) into (51), viz.

$$\mu_0 \nabla^2 u^\nu + (\mu_0 + \lambda_0) \varepsilon^{;\nu} = -X^\nu, \quad (155)$$

the dynamic equation in terms of displacements becomes

$$\mu_0 \nabla^2 u^\nu + (\mu_0 + \lambda_0) \varepsilon^{;\nu} = -k_0 u^\nu. \quad (156)$$

This equation can be rewritten as

$$\nabla^2 u^\nu + \frac{k_0}{\mu_0} u^\nu = -\frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{;\nu}. \quad (157)$$

This displacement equation is similar to a nonhomogeneous Klein-Gordon equation for a vector field, with a source term.

**§12.2. Wave equations.** Additional wave equations as shown on Page 240 can be derived from this volume force.

**§12.2.1. Dilatational (longitudinal) wave equation.** Substituting (154) into (57), viz.

$$(2\mu_0 + \lambda_0) \nabla^2 \varepsilon = -X^\nu_{;\nu}, \quad (158)$$

the longitudinal (dilatational) wave equation becomes

$$(2\mu_0 + \lambda_0) \nabla^2 \varepsilon = -k_0 u^\nu_{;\nu}. \quad (159)$$

Using  $u^\mu_{;\mu} = \varepsilon$  from (44) and rearranging, this equation can be rewritten as

$$\nabla^2 \varepsilon + \frac{k_0}{2\mu_0 + \lambda_0} \varepsilon = 0. \quad (160)$$

This wave equation applies to the volume dilatation  $\varepsilon$ . This equation is similar to the homogeneous Klein-Gordon equation for a scalar field, a field whose quanta are spinless particles [36].

**§12.2.2. Rotational (transverse) wave equation.** Substituting (154) into (61), viz.

$$\mu_0 \nabla^2 \omega^{\mu\nu} = -X^{[\mu;\nu]}, \quad (161)$$

the transverse (rotational) wave equation becomes

$$\mu_0 \nabla^2 \omega^{\mu\nu} = -\frac{k_0}{2} (u^{\mu;\nu} - u^{\nu;\mu}). \quad (162)$$

Using the definition of  $\omega^{\mu\nu}$  from (42) and rearranging, this equation can be rewritten as

$$\nabla^2 \omega^{\mu\nu} + \frac{k_0}{\mu_0} \omega^{\mu\nu} = 0. \quad (163)$$

This antisymmetric equation is also similar to an homogeneous Klein-Gordon equation for an antisymmetrical tensor field.

**§12.2.3. Strain (symmetric) wave equation.** Substituting (154) into (64), viz.

$$\mu_0 \nabla^2 \varepsilon^{\mu\nu} + (\mu_0 + \lambda_0) \varepsilon^{;\mu\nu} = -X^{(\mu;\nu)}, \quad (164)$$

the symmetric (strain) wave equation becomes

$$\mu_0 \nabla^2 \varepsilon^{\mu\nu} + (\mu_0 + \lambda_0) \varepsilon^{;\mu\nu} = -\frac{k_0}{2} (u^{\mu;\nu} + u^{\nu;\mu}). \quad (165)$$

Using the definition of  $\varepsilon^{\mu\nu}$  from (41) and rearranging, this equation can be rewritten as

$$\nabla^2 \varepsilon^{\mu\nu} + \frac{k_0}{\mu_0} \varepsilon^{\mu\nu} = -\frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{;\mu\nu}. \quad (166)$$

This symmetric equation is also similar to a nonhomogeneous Klein-Gordon equation for a symmetrical tensor field with a source term and has the same structure as the displacement equation.

**§12.3. Electromagnetism.** We consider the impact of this volume force on the equations of electromagnetism derived previously. Substituting (154) into (95), viz.

$$\omega^{\mu\nu}{}_{;\mu} = \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu} + \frac{1}{2\mu_0} X^\nu, \quad (167)$$

Maxwell's equations in terms of the rotation tensor become

$$\omega^{\mu\nu}{}_{;\mu} = \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{;\nu} + \frac{k_0}{2\mu_0} u^\nu. \quad (168)$$

Separating  $u^\nu$  into its longitudinal (irrotational) component  $u^\nu_{\parallel}$  and its transverse (solenoidal) component  $u^\nu_{\perp}$  using the Helmholtz theorem in four dimensions [42] according to

$$u^\nu = u^\nu_{\parallel} + u^\nu_{\perp}, \quad (169)$$

substituting for  $\omega^{\mu\nu}$  from  $F^{\mu\nu} = \varphi_0 \omega^{\mu\nu}$  and for  $u_{\perp}^{\nu}$  from  $A^{\mu} = -\frac{1}{2}\varphi_0 u_{\perp}^{\mu}$ , this equation becomes

$$F^{\mu\nu}{}_{;\mu} = \varphi_0 \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{i\nu} + \frac{\varphi_0 k_0}{2\mu_0} u_{\parallel}^{\nu} - \frac{k_0}{\mu_0} A^{\nu}. \quad (170)$$

Proper treatment of this case requires that the current density four-vector be proportional to the RHS of (170) as follows:

$$\mu_{\text{em}} j^{\nu} = \frac{\varphi_0}{2\mu_0} [(2\mu_0 + \lambda_0) \varepsilon^{i\nu} + k_0 u_{\parallel}^{\nu}] - \frac{k_0}{\mu_0} A^{\nu}. \quad (171)$$

This thus yields the following microscopic form of the current density four-vector:

$$j^{\nu} = \frac{\varphi_0}{2\mu_0 \mu_{\text{em}}} [(2\mu_0 + \lambda_0) \varepsilon^{i\nu} + k_0 u_{\parallel}^{\nu}] - \frac{k_0}{\mu_0 \mu_{\text{em}}} A^{\nu}. \quad (172)$$

We thus find that the second term is proportional to  $A^{\nu}$  as is the second term of the current density obtained from the quantum mechanical Klein-Gordon equation with an electromagnetic field [36, see p. 35].

**§12.4. Discussion of linear elastic volume force results.** This section has been useful in that consideration of a simple linear elastic volume force leads to equations which are of the Klein-Gordon type. The wave equations that are obtained for the scalar  $\varepsilon$ , the four-vector  $u^{\nu}$ , and the symmetric and antisymmetric tensors  $\varepsilon^{\mu\nu}$  and  $\omega^{\mu\nu}$  respectively, are all equations that are similar to homogeneous or nonhomogeneous Klein-Gordon equations. The solutions of these equations are well understood [41, see pp. 414–433].

It should be noted that we cannot simply put

$$\frac{m^2 c^2}{\hbar^2} = \frac{k_0}{2\mu_0 + \lambda_0} \quad (173)$$

or

$$\frac{m^2 c^2}{\hbar^2} = \frac{k_0}{\mu_0} \quad (174)$$

from the Klein-Gordon equation, as the expression to use depends on the wave equation considered. This ambiguity in the equivalency of the constant  $m^2 c^2 / \hbar^2$  to *STCED* constants indicates that the postulated elastic linear volume force proposed in (154) is not quite correct, even if it is a step in the right direction. It has provided insight into the impact of the volume force on this analysis, but the volume force is not quite the simple linear elastic expression considered in (154).

In the following section, we derive a volume force from the general current density four-vector expression of (175) below. We find that the

volume force (154) and consequently the current density four-vector (172) need to be modified.

**§13. Derivation of a quantum mechanical volume force.** One identification of the volume force based on quantum mechanical considerations is possible by comparing (102), viz.

$$j^\nu = \frac{1}{2} \frac{\varphi_0}{\mu_{\text{em}} \mu_0} [(2\mu_0 + \lambda_0) \varepsilon^{i\nu} + X^\nu], \quad (175)$$

with the quantum mechanical expression of the current density four-vector  $j^\nu$  obtained from the Klein-Gordon equation for a spin-0 particle. The Klein-Gordon equation can also describe the interaction of a spin-0 particle with an electromagnetic field. The current density four-vector  $j^\nu$  in that case is written as [36, see p. 35]

$$j^\nu = \frac{i\hbar e}{2m} (\psi^* \partial^\nu \psi - \psi \partial^\nu \psi^*) - \frac{e^2}{m} A^\nu (\psi \psi^*), \quad (176)$$

where the superscript \* denotes complex conjugation.

The first term of (176) includes the following derivative-like expression:

$$i (\psi^* \partial^\nu \psi - \psi \partial^\nu \psi^*). \quad (177)$$

It is generated by multiplying the Klein-Gordon equation for  $\psi$  by  $\psi^*$  and subtracting the complex conjugate [36]. The general form of the expression can be generated by writing

$$\psi \sim \exp(i\phi), \quad (178)$$

which is a qualitative representation of the wave function. One can then see that with (178), the expression

$$\partial^\nu (\psi \psi^*) \quad (179)$$

has the qualitative structure of (177) although it is not strictly equivalent. However, given that the steps followed to generate (177) are not repeated in this derivation, strict equivalence is not expected. Replacing (177) with (179), the first term of (176) becomes

$$\frac{\hbar e}{2m} \partial^\nu (\psi \psi^*). \quad (180)$$

We see that this term is similar to the first term of (175) and setting them to be equal, we obtain

$$\frac{\varphi_0}{2\mu_{\text{em}} \mu_0} (2\mu_0 + \lambda_0) \varepsilon^{i\nu} = \frac{\hbar e}{2m} \partial^\nu (\psi \psi^*). \quad (181)$$

Similarly, the second terms of (175) and (176) are also similar and setting them to be equal, we obtain

$$\frac{\varphi_0}{2\mu_{em}\mu_0} X^\nu = -\frac{e^2}{m} A^\nu(\psi\psi^*). \quad (182)$$

The equalities (181) and (182) thus result from the comparison of (175) and (176).

The first identification that can be derived from (181) is

$$\varepsilon(x^\mu) = \psi\psi^* \quad (183)$$

to a proportionality constant which has been set to 1, given that the norm of the wavefunction itself is arbitrarily normalized to 1 as part of its probabilistic interpretation. Both are dimensionless quantities.  $\varepsilon$  is the change in volume per original volume as a function of position  $x^\mu$ , which is stated explicitly in (183), while  $\psi\psi^*$  is the probability density as a function of position, and hence is also a proportion of an overall quantity normalized to 1. There are thus many similarities between  $\varepsilon$  and  $\psi$ . This equation leads to the conclusion that the quantum mechanical wavefunction describes longitudinal wave propagations in the *STC* corresponding to the volume dilatation associated with the particle property of an object.

Using (81) viz.

$$A^\nu = -\frac{1}{2} \varphi_0 u_\perp^\nu \quad (184)$$

and (183) in (182), the quantum mechanical volume force is given by

$$X^\nu = \mu_0\mu_{em} \frac{e^2}{m} \varepsilon(x^\mu) u_\perp^\nu. \quad (185)$$

Using the definition for the dimensionless fine structure constant  $\alpha = \mu_{em}ce^2/2h$ , (185) becomes

$$X^\nu = 2\mu_0\alpha \frac{h}{mc} \varepsilon(x^\mu) u_\perp^\nu \quad (186)$$

or

$$X^\nu = 2\mu_0\alpha\lambda_c \varepsilon(x^\mu) u_\perp^\nu, \quad (187)$$

where  $\lambda_c = h/mc$  is the electron's Compton wavelength.

Thus the *STCED* elastic force constant of (154) is given by

$$k_0 = \mu_0\mu_{em} \frac{e^2}{m} = 2\mu_0\alpha \frac{h}{mc} = 2\mu_0\alpha\lambda_c. \quad (188)$$

The units are  $[N][m^{-1}]$  as expected. The volume force is proportional

to  $\varepsilon(x^\mu) u'_\perp$  as opposed to just  $u^\nu$  as in (154):

$$X^\nu = k_0 \varepsilon(x^\mu) u'_\perp. \quad (189)$$

The volume force  $X^\nu$  is proportional to the Planck constant as suspected previously. This explains why the volume force tends to zero in the macroscopic case. The volume force is also proportional to the *STC* volume dilatation  $\varepsilon(x^\mu)$  in addition to the displacements  $u'_\perp$ . This makes sense as all deformations, both distortions and dilatations, should be subject to the *STC* elastic spring force. This is similar to an elastic spring law as  $X^\nu$  is defined positive in the direction opposite to the displacement [18]. The volume force also describes the interaction with an electromagnetic field given that (176) from which it is derived includes electromagnetic interactions.

Starting from (181), and making use of (183), the *STC* electromagnetic shearing potential constant  $\varphi_0$  of (81) can be identified:

$$\varphi_0 = \frac{2\mu_0}{2\mu_0 + \lambda_0} \mu_{\text{em}} \frac{e\hbar}{2m} = \frac{2\mu_0}{2\mu_0 + \lambda_0} \mu_{\text{em}} \mu_{\text{B}} \quad (190)$$

where the Bohr magneton  $\mu_{\text{B}} = e\hbar/2m$  has been used. Using (179) and (183) in (176), we obtain

$$j^\nu = \frac{e\hbar}{2m} \varepsilon^{i\nu} - \frac{e^2}{m} A^\nu \varepsilon(x^\mu) \quad (191)$$

or

$$j^\nu = \mu_{\text{B}} \varepsilon^{i\nu} - \frac{e^2}{m} A^\nu \varepsilon(x^\mu) \quad (192)$$

with the Bohr magneton.

### §13.1. Microscopic dynamics of the STC

**§13.1.1. Dynamic equations.** Substituting (189) into (155), the dynamic equation in terms of displacements becomes

$$\mu_0 \nabla^2 u^\nu + (\mu_0 + \lambda_0) \varepsilon^{i\nu} = -k_0 \varepsilon(x^\mu) u'_\perp. \quad (193)$$

This equation can be rewritten as

$$\nabla^2 u^\nu + \frac{k_0}{\mu_0} \varepsilon(x^\mu) u'_\perp = -\frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{i\nu}. \quad (194)$$

We note that  $\varepsilon(x^\mu)$  is a scalar function of 4-position only, and plays a role similar to the potential  $V(\mathbf{r})$  in the Schrödinger equation. Indeed,  $\varepsilon(x^\mu)$  represents the mass energy structure (similar to an energy potential) impacting the solutions of this equation.

Separating  $u^\nu$  into its longitudinal (irrotational) component  $u^\nu_{\parallel}$  and its transverse (solenoidal) component  $u^\nu_{\perp}$  using the Helmholtz theorem in four dimensions [42] according to

$$u^\nu = u^\nu_{\parallel} + u^\nu_{\perp}, \tag{195}$$

we obtain the separated equations

$$\left. \begin{aligned} \nabla^2 u^\nu_{\parallel} &= -\frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{i\nu} \\ \nabla^2 u^\nu_{\perp} + \frac{k_0}{\mu_0} \varepsilon(x^\mu) u^\nu_{\perp} &= 0 \end{aligned} \right\}. \tag{196}$$

The wave equation for  $u^\nu_{\parallel}$  describes the propagation of longitudinal displacements, while the wave equation for  $u^\nu_{\perp}$  describes the propagation of transverse displacements.

**§13.1.2. Longitudinal displacements equation.** Substituting for  $\varepsilon^{i\nu}$  from (191) in the first equation of (196), we obtain

$$\nabla^2 u^\nu_{\parallel} = -\frac{2k_L}{\hbar} \left[ \frac{e^2}{m} j^\nu + eA^\nu \varepsilon(x^\mu) \right], \tag{197}$$

where the dimensionless ratio

$$k_L = \frac{\mu_0 + \lambda_0}{\mu_0} \tag{198}$$

has been introduced. Hence the source term on the RHS of this equation includes the mass resulting from the dilatation displacements, the current density four-vector, and the vector potential resulting from the distortion displacements. It provides a full description of the gravitational and electromagnetic interactions at the microscopic level.

**§13.1.3. Transverse displacements equation.** Substituting for  $u^\nu_{\perp}$  from (81) in the second equation of (196), we obtain

$$\nabla^2 A^\nu + \frac{k_0}{\mu_0} \varepsilon(x^\mu) A^\nu = 0. \tag{199}$$

Substituting for  $k_0$  from (188), this equation becomes

$$\nabla^2 A^\nu + \mu_{\text{em}} \frac{e^2}{m} \varepsilon(x^\mu) A^\nu = 0 \tag{200}$$



or

$$\nabla^2 A^\nu + 2\alpha \frac{\hbar}{mc} \varepsilon(x^\mu) A^\nu = 0 \quad (201)$$

and finally

$$\nabla^2 A^\nu + 2\alpha \lambda_c \varepsilon(x^\mu) A^\nu = 0. \quad (202)$$

This equation is similar to a Proca equation except that the coefficient of  $A^\nu$  is not the familiar  $m^2 c^2 / \hbar^2$ . Given that transverse displacements are massless, the Proca equation coefficient is not expected given its usual interpretation that it represents the mass of the particle described by the equation. This is discussed in more details in the next section.

### §13.2. Wave equations

**§13.2.1. Longitudinal wave equation.** Substituting (189) into (158), the longitudinal (dilatational) wave equation becomes

$$(2\mu_0 + \lambda_0) \nabla^2 \varepsilon = -\nabla_\nu [k_0 \varepsilon(x^\mu) u_\perp^\nu]. \quad (203)$$

Taking the divergence on the RHS, using  $u_{\perp;\nu}^\nu = 0$  from (45) and rearranging, this equation can be rewritten as

$$\nabla^2 \varepsilon = -\frac{k_0}{2\mu_0 + \lambda_0} u_\perp^\nu \varepsilon_{;\nu}. \quad (204)$$

Substituting for  $u_\perp^\nu$  from (81), for  $k_0$  from (188) and for  $\varepsilon_{;\nu}$  from (191), we obtain

$$\nabla^2 \varepsilon - 4 \frac{e^2}{\hbar^2} A^\nu A_\nu \varepsilon = 4 \frac{m}{\hbar^2} A^\nu j_\nu. \quad (205)$$

Recognizing that

$$e^2 A^\nu A_\nu = P^\nu P_\nu = -m^2 c^2, \quad (206)$$

and substituting in (205), the equation becomes

$$\nabla^2 \varepsilon + 4 \frac{m^2 c^2}{\hbar^2} \varepsilon = 4 \frac{m}{\hbar^2} A^\nu j_\nu. \quad (207)$$

This is the Klein-Gordon equation except for the factor of 4 multiplying the  $\varepsilon$  coefficient and the source term. The term on the RHS of this equation is an interaction term of the form  $\mathbf{A} \cdot \mathbf{j}$ .

As identified from (183) and confirmed by this equation, the quantum mechanical wavefunction describes longitudinal wave propagations in the *STC* corresponding to the volume dilatation associated with the particle property of an object. The RHS of the equation indicates an

interaction between the longitudinal current density  $j_\nu$  and the transverse vector potential  $A^\nu$ . This is interpreted in Electromagnetism as energy in the static magnetic induction field to establish the steady current distribution [43, see p.150]. It is also the form of the interaction term introduced in the vacuum Lagrangian for classical electrodynamics [44, see p. 428].

Although (207) with the  $m^2 c^2 / \hbar^2$  coefficient is how the Klein-Gordon equation is typically written, (205) is a more physically accurate way of writing that equation, i.e.

$$\frac{\hbar^2}{4} \nabla^2 \varepsilon - e^2 A^\nu A_\nu \varepsilon = m A^\nu j_\nu, \tag{208}$$

as the massive nature of the equation resides in its solutions  $\varepsilon(x^\mu)$ . The constant  $m$  needs to be interpreted in the same way as the constant  $e$ . The constant  $e$  in the Klein-Gordon equation is the elementary unit of electrical charge (notwithstanding the quark fractional charges), not the electrical charge of the particle represented by the equation. Similarly, the constant  $m$  in the Klein-Gordon equation needs to be interpreted as the elementary unit of mass (the electron's mass), not the mass of the particle represented by the equation. That is obtained from the solutions  $\varepsilon(x^\mu)$  of the equation.

**§13.2.2. Transverse wave equation.** Substituting (189) into (161), the transverse (rotational) wave equation becomes

$$\mu_0 \nabla^2 \omega^{\mu\nu} = -\frac{k_0}{2} [(\varepsilon u_\perp^\mu)^{;\nu} - (\varepsilon u_\perp^\nu)^{;\mu}]. \tag{209}$$

Using (42) and rearranging, this equation can be rewritten as

$$\nabla^2 \omega^{\mu\nu} + \frac{k_0}{\mu_0} \varepsilon(x^\mu) \omega^{\mu\nu} = \frac{1}{2} \frac{k_0}{\mu_0} (\varepsilon^{;\mu} u_\perp^\nu - \varepsilon^{;\nu} u_\perp^\mu). \tag{210}$$

Substituting for  $\omega^{\mu\nu}$  using  $F^{\mu\nu} = \varphi_0 \omega^{\mu\nu}$  from (80), for  $u_\perp^\nu$  from (81), for  $k_0$  from (188) and for  $\varepsilon^{;\nu}$  from (191), we obtain

$$\nabla^2 F^{\mu\nu} + \mu_{\text{em}} \frac{e^2}{m} \varepsilon(x^\mu) F^{\mu\nu} = \mu_{\text{em}} \frac{e}{\hbar} (A^\mu j^\nu - A^\nu j^\mu). \tag{211}$$

This equation can also be written as

$$\nabla^2 F^{\mu\nu} + 2\alpha \lambda_c \varepsilon(x^\mu) F^{\mu\nu} = \mu_{\text{em}} \frac{e}{\hbar} (A^\mu j^\nu - A^\nu j^\mu). \tag{212}$$

This is a new equation of the electromagnetic field strength  $F^{\mu\nu}$ . The term on the RHS of this equation is an interaction term of the form  $\mathbf{A} \times \mathbf{j}$ .

In Electromagnetism, this term is the volume density of the magnetic torque (magnetic torque density), and is interpreted as the “longitudinal tension” between two successive current elements (Helmholtz’s longitudinal tension), observed experimentally by Ampère (hairpin experiment) [45].

**§13.2.3. Strain wave equation.** Substituting (189) into (164), the strain (symmetric) wave equation becomes

$$\mu_0 \nabla^2 \varepsilon^{\mu\nu} + (\mu_0 + \lambda_0) \varepsilon^{i\mu\nu} = -\frac{k_0}{2} [(\varepsilon u_{\perp}^{\mu})^{;\nu} + (\varepsilon u_{\perp}^{\nu})^{;\mu}], \quad (213)$$

which can be rewritten as

$$\begin{aligned} \nabla^2 \varepsilon^{\mu\nu} + \frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{i\mu\nu} &= \\ &= \frac{1}{2} \frac{k_0}{\mu_0} [\varepsilon (u_{\perp}^{\mu;\nu} + u_{\perp}^{\nu;\mu}) + (\varepsilon^{i\mu} u_{\perp}^{\nu} + \varepsilon^{i\nu} u_{\perp}^{\mu})]. \end{aligned} \quad (214)$$

Substituting for  $u_{\perp}^{\nu}$  from (81), for  $k_0$  from (188) and for  $\varepsilon^{i\nu}$  from (191), we obtain

$$\begin{aligned} \nabla^2 \varepsilon^{\mu\nu} + k_{\text{L}} \varepsilon^{i\mu\nu} &= k_{\text{T}} \frac{2m}{\hbar^2} (A^{\mu} j^{\nu} + A^{\nu} j^{\mu}) + \\ &+ k_{\text{T}} \varepsilon \left[ \frac{e}{\hbar} (A^{\mu;\nu} + A^{\nu;\mu}) + \frac{2e^2}{\hbar^2} (A^{\mu} A^{\nu} + A^{\nu} A^{\mu}) \right], \end{aligned} \quad (215)$$

where the dimensionless ratio

$$k_{\text{T}} = \frac{2\mu_0 + \lambda_0}{\mu_0} \quad (216)$$

has been introduced and ratio  $k_{\text{L}}$  has been used from (198). The last term can be summed to  $2A^{\mu} A^{\nu}$ . This new equation for the symmetrical strain tensor field includes on the RHS symmetrical interaction terms between the current density four-vector and the vector potential resulting from the distortion displacements and between the vector potential and the mass resulting from the dilatation displacements.

**§13.3. Simplified wave equations.** Inspection of the wave equations derived previously shows that common factors are associated with  $A^{\nu}$  and  $j^{\nu}$  in all the equations. We thus introduce the reduced physical variables  $\bar{A}^{\nu}$  and  $\bar{j}^{\nu}$  defined according to

$$\bar{A}^{\nu} = e A^{\nu}, \quad \bar{j}^{\nu} = \frac{m}{e} j^{\nu}, \quad (217)$$

and  $\bar{A}^\nu$  and  $\bar{j}^\nu$  defined according to

$$\bar{A}^\nu = \frac{2e}{\hbar} A^\nu, \quad \bar{j}^\nu = \frac{2m}{e\hbar} j^\nu. \quad (218)$$

The various wave equations then simplify to the following.

***Longitudinal displacements equation***

$$\nabla^2 u_{\parallel}^\nu = -\frac{2k_L}{\hbar} (\bar{j}^\nu + \varepsilon \bar{A}^\nu) \quad (219)$$

$$\nabla^2 u_{\parallel}^\nu = -k_L (\bar{j}^\nu + \varepsilon \bar{A}^\nu) \quad (220)$$

***Transverse displacements equation***

$$\nabla^2 \bar{A}^\nu + 2\alpha \lambda_c \varepsilon \bar{A}^\nu = 0 \quad (221)$$

$$\nabla^2 \bar{A}^\nu + 2\alpha \lambda_c \varepsilon \bar{A}^\nu = 0 \quad (222)$$

***Longitudinal wave equation***

$$\frac{\hbar^2}{4} \nabla^2 \varepsilon - \bar{A}^\nu \bar{A}_\nu \varepsilon = \bar{A}^\nu \bar{j}_\nu \quad (223)$$

$$\nabla^2 \varepsilon - \bar{A}^\nu \bar{A}_\nu \varepsilon = \bar{A}^\nu \bar{j}_\nu \quad (224)$$

***Transverse wave equation***

$$\nabla^2 F^{\mu\nu} + 2\alpha \lambda_c \varepsilon (x^\mu) F^{\mu\nu} = \mu_{\text{em}} \frac{e}{\hbar m} (\bar{A}^\mu \bar{j}^\nu - \bar{A}^\nu \bar{j}^\mu) \quad (225)$$

$$\nabla^2 F^{\mu\nu} + 2\alpha \lambda_c \varepsilon (x^\mu) F^{\mu\nu} = \frac{1}{2} \mu_{\text{em}} \mu_B (\bar{A}^\mu \bar{j}^\nu - \bar{A}^\nu \bar{j}^\mu) \quad (226)$$

***Strain wave equation***

$$\begin{aligned} \nabla^2 \varepsilon^{\mu\nu} + k_L \varepsilon^{:\mu\nu} &= k_T \frac{2}{\hbar^2} (\bar{A}^\mu \bar{j}^\nu + \bar{A}^\nu \bar{j}^\mu) + \\ &+ k_T \varepsilon \left[ \frac{1}{\hbar} (\bar{A}^{\mu;\nu} + \bar{A}^{\nu;\mu}) + \frac{2}{\hbar^2} (\bar{A}^\mu \bar{A}^\nu + \bar{A}^\nu \bar{A}^\mu) \right] \end{aligned} \quad (227)$$

$$\begin{aligned} \nabla^2 \varepsilon^{\mu\nu} + k_L \varepsilon^{:\mu\nu} &= \frac{1}{2} k_T (\bar{A}^\mu \bar{j}^\nu + \bar{A}^\nu \bar{j}^\mu) + \\ &+ \frac{1}{2} k_T \varepsilon \left[ (\bar{A}^{\mu;\nu} + \bar{A}^{\nu;\mu}) + (\bar{A}^\mu \bar{A}^\nu + \bar{A}^\nu \bar{A}^\mu) \right] \end{aligned} \quad (228)$$

**§13.4. Microscopic theory of Electromagnetism.** We consider the impact of this volume force on the equations of electromagnetism derived previously. Substituting (186) into (95), Maxwell's equations in terms of the rotation tensor become

$$\omega^{\mu\nu}{}_{;\mu} = \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{i\nu} + \alpha \lambda_c \varepsilon(x^\mu) u_{\perp}^\nu. \quad (229)$$

Substituting for  $\omega^{\mu\nu}$  using  $F^{\mu\nu} = \varphi_0 \omega^{\mu\nu}$  from (80) and using (81) for  $u_{\perp}^\nu$ , this equation becomes

$$F^{\mu\nu}{}_{;\mu} = \varphi_0 \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{i\nu} - 2\alpha \lambda_c \varepsilon(x^\mu) A^\nu. \quad (230)$$

Proper treatment of this case requires that the current density four-vector be proportional to the RHS of (230) as follows:

$$\mu_{\text{em}} j^\nu = \varphi_0 \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{i\nu} - 2\alpha \lambda_c \varepsilon(x^\mu) A^\nu. \quad (231)$$

As seen previously, the equations of electrodynamics, in the general case, are identical to the covariant form of Maxwell's equations and are not modified by the volume force (see Page 246). This yields the following microscopic form of the current density four-vector:

$$j^\nu = \frac{\varphi_0}{\mu_{\text{em}}} \frac{2\mu_0 + \lambda_0}{2\mu_0} \varepsilon^{i\nu} - \frac{2\alpha \lambda_c}{\mu_{\text{em}}} \varepsilon(x^\mu) A^\nu. \quad (232)$$

Substituting for  $\varphi_0$  from (190) and for  $\alpha$  as in (186) into (232), we obtain

$$j^\nu = \frac{e\hbar}{2m} \varepsilon^{i\nu} - \frac{e^2}{m} A^\nu \varepsilon(x^\mu) \quad (233)$$

or

$$j^\nu = \mu_{\text{B}} \varepsilon^{i\nu} - \frac{e^2}{m} A^\nu \varepsilon(x^\mu) \quad (234)$$

using the Bohr magneton.

**§14. Discussion and Conclusion.** In this paper, we have presented the Elastodynamics of the Spacetime Continuum (*STCED*). This theory describes the deformations of the spacetime continuum by modeling and analyzing the displacements of the elements of the *STC* resulting from the spacetime continuum strains arising from the energy-momentum stress tensor, based on the application of continuum mechanical results

to the spacetime continuum. *STCED* provides a fundamental description of the microscopic processes underlying the spacetime continuum. The combination of the spacetime continuum deformations results in the geometry of the *STC*.

We have proposed a natural decomposition of the spacetime metric tensor into a background and a dynamical part based on an analysis from first principles, of the impact of introducing a test mass in the spacetime continuum. We have found that the presence of mass results in strains in the spacetime continuum. Those strains correspond to the dynamical part of the spacetime metric tensor. The applicability of the proposed metric to the Einstein field equations remains open to demonstration.

We have proposed a framework for the analysis of strained spacetime based on the Elastodynamics of the Spacetime Continuum. In this model, the emphasis is on the displacements of the spacetime continuum infinitesimal elements from their unstrained configuration as a result of the strains applied on the *STC* by the energy-momentum stress tensor, rather than on the geometry of the *STC* due to the energy-momentum stress tensor.

We postulate that this description based on the deformation of the continuum is a description complementary to that of General Relativity which is concerned with modeling the resulting geometry of the spacetime continuum. Interestingly, the structure of the resulting stress-strain relation is similar to that of the field equations of General Relativity. This strengthens our conjecture that the geometry of the spacetime continuum can be seen as a representation of the deformation of the spacetime continuum resulting from the strains generated by the energy-momentum stress tensor. The equivalency of the deformation description and of the geometrical description still remains to be demonstrated. It should be noted that these could be considered to be local effects in the particular reference frame of the observer.

We have applied the stress-strain relation of Continuum Mechanics to the spacetime continuum to show that rest-mass energy density arises from the volume dilatation of the spacetime continuum. This is a significant result as it demonstrates that mass is not independent of the spacetime continuum, but rather results from how energy-momentum propagates in the spacetime continuum. Matter does not warp spacetime, but rather matter *is* warped spacetime. The universe consists of the spacetime continuum and energy-momentum that propagates in it by deformation of its (*STC*) structure.

We have proposed a natural decomposition of tensor fields in strained

spacetime, in terms of dilatations and distortions. We have shown that dilatations correspond to rest-mass energy density, while distortions correspond to massless shear transverse waves. We have noted that this decomposition of spacetime continuum deformations into a massive dilatation and a massless transverse wave distortion is somewhat reminiscent of wave-particle duality. This could explain why dilatation-measuring apparatus measure the massive “particle” properties of the deformation, while distortion-measuring apparatus measure the massless transverse “wave” properties of the deformation.

The equilibrium dynamic equation of the spacetime continuum is described by  $T^{\mu\nu}{}_{;\mu} = -X^\nu$ . In General Relativity, the relation  $T^{\mu\nu}{}_{;\mu} = 0$  is taken as expressing local conservation of the energy-momentum of matter. The value  $X^\nu = 0$  is thus taken to represent the macroscopic local case, while in the general case, the volume force  $X^\nu$  is retained in the equation. This dynamic equation leads to a series of wave equations as derived in this paper: the displacement ( $u^\nu$ ), dilatational ( $\varepsilon$ ), rotational ( $\omega^{\mu\nu}$ ) and strain ( $\varepsilon^{\mu\nu}$ ) wave equations. The nature of the spacetime continuum volume force and the resulting inhomogeneous wave equations are areas of further investigation.

Hence energy is seen to propagate in the spacetime continuum as deformations of the *STC* that satisfy wave equations of propagation. Deformations can be decomposed into dilatations and distortions. *Dilatations* involve an invariant change in volume of the spacetime continuum which is the source of the associated rest-mass energy density of the deformation. *Distortions* correspond to a change of shape of the spacetime continuum without a change in volume and are thus massless. Dilatations correspond to longitudinal displacements and distortions correspond to transverse displacements of the spacetime continuum.

Hence, every excitation of the spacetime continuum can be decomposed into a transverse and a longitudinal mode of propagation. We have noted again that this decomposition into a dilatation with rest-mass energy density and a massless transverse wave distortion, is somewhat reminiscent of wave-particle duality, with the transverse mode corresponding to the wave aspects and the longitudinal mode corresponding to the particle aspects.

A continuity equation for deformations of the spacetime continuum has been derived; we have found that the divergence of the strain and rotation tensors equals the gradient of the massive volume dilatation, which acts as a source term.

We have analyzed the strain energy density of the spacetime con-

tinuum in *STCED*. We have found that the strain energy density is separated into two terms: the first one expresses the dilatation energy density (the “mass” longitudinal term) while the second one expresses the distortion energy density (the “massless” transverse term). We have found that the quadratic structure of the energy relation of Special Relativity is present in the strain energy density of the Elastodynamics of the Spacetime Continuum. We have also found that the kinetic energy  $pc$  is carried by the distortion part of the deformation, while the dilatation part carries only the rest mass energy.

We have derived Electromagnetism from the Elastodynamics of the Spacetime Continuum based on the identification of the theory’s anti-symmetric rotation tensor  $\omega^{\mu\nu}$  with the electromagnetic field-strength tensor  $F^{\mu\nu}$ .

The theory provides a physical explanation of the electromagnetic potential: it arises from transverse (shearing) displacements of the spacetime continuum, in contrast to mass which arises from longitudinal (dilatational) displacements of the spacetime continuum. Hence sheared spacetime is manifested as electromagnetic potentials and fields.

In addition, the theory provides a physical explanation of the current density four-vector: it arises from the 4-gradient of the volume dilatation of the spacetime continuum. A corollary of this relation is that massless (transverse) waves cannot carry an electric charge or produce a current.

The transverse mode of propagation involves no volume dilatation and is thus massless. Transverse wave propagation is associated with the distortion of the spacetime continuum. Electromagnetic waves are transverse waves propagating in the *STC* itself, at the speed of light.

The Lorentz condition is obtained directly from the theory. The reason for the value of zero is that transverse displacements are massless because such displacements arise from a change of shape (distortion) of the spacetime continuum, not a change of volume (dilatation).

In addition, we have obtained a generalization of Electromagnetism for the situation where a volume force is present, in the general non-macroscopic case. Maxwell’s equations are found to remain unchanged, but the current density has an additional term proportional to the volume force  $X^\nu$ .

The Elastodynamics of the Spacetime Continuum thus provides a unified description of the spacetime deformation processes underlying general relativistic Gravitation and Electromagnetism, in terms of spacetime continuum displacements resulting from the strains generated by the energy-momentum stress tensor.

We have calculated the strain energy density of the electromagnetic



energy-momentum stress tensor. We have found that the dilatation longitudinal (mass) term of the strain energy density and hence the rest-mass energy density of the photon is 0. We have found that the distortion transverse (massless) term of the strain energy density is a combination of the electromagnetic field energy density term  $U_{\text{em}}^2/\mu_0$  and the electromagnetic field energy flux term  $S^2/\mu_0c^2$ , calculated from the Poynting vector. This longitudinal electromagnetic energy flux is massless as it is due to distortion, not dilatation, of the spacetime continuum. However, because this energy flux is along the direction of propagation (i.e. longitudinal), it gives rise to the particle aspect of the electromagnetic field, the photon.

We have investigated the volume force and its impact on the equations of the Elastodynamics of the Spacetime Continuum. We have found that a linear elastic volume force leads to equations which are of the Klein-Gordon type. From a variation of that linear elastic volume force based on the Klein-Gordon quantum mechanical current density, we have found that the quantum mechanical wavefunction describes longitudinal wave propagations in the *STC* corresponding to the volume dilatation associated with the particle property of an object. We have derived the wave equations corresponding to the modeled volume force. The longitudinal wave equation is found to correspond to the Klein-Gordon equation with a source term corresponding to an interaction term of the form  $\mathbf{A} \cdot \mathbf{j}$ , further confirming that the quantum mechanical wavefunction describes longitudinal wave propagations in the *STC*. The transverse wave equation is found to be a new equation of the electromagnetic field strength  $F^{\mu\nu}$ , which includes an interaction term of the form  $\mathbf{A} \times \mathbf{j}$  corresponding to the volume density of the magnetic torque (magnetic torque density). The equations obtained reflect a close integration of gravitational and electromagnetic interactions at the microscopic level.

**§14.1 Future directions.** This paper has presented a linear elastic theory of the Elastodynamics of the Spacetime Continuum for the analysis of the deformations of the spacetime continuum. It provides a fundamental description of gravitational, electromagnetic and some quantum phenomena. Progress has been achieved towards the goal set initially, that the theory should in principle be able to explain the basic physical theories from which the rest of physical theory can be built, without the introduction of inputs external to the theory. Physical explanations of the following phenomena have been derived from *STCED* in this paper:

- ***Decomposition of the metric tensor.*** A decomposition of the metric tensor into its background and dynamical parts is obtained. The dynamical part corresponds to the strains generated in the spacetime continuum by the energy-momentum stress tensor.
- ***Wave-particle duality.*** Every excitation of the spacetime continuum can be separated into a transverse (distortion) and a longitudinal (dilatation) mode of propagation. This decomposition of spacetime continuum deformations into a massive dilatation (“particle”) and a massless transverse distortion (“wave”) is similar to wave-particle duality.
- ***Nature of matter.*** The longitudinal mode of propagation involves an invariant change in volume of the spacetime continuum. Rest-mass energy, and hence matter, arises from this invariant volume dilatation of the spacetime continuum.
- ***Maxwell’s equations.*** Maxwell’s equations are derived from the theory, including a generalization when a volume form  $X^\nu$  is present.
- ***Nature of Electromagnetism.*** The theory provides a physical explanation of the electromagnetic potential, which arises from transverse (shearing) displacements of the spacetime continuum, and of the current density four-vector, which is the 4-gradient of the volume dilatation of the spacetime continuum.
- ***Lorentz condition.*** The Lorentz condition is obtained directly from the theory.
- ***Electromagnetic radiation.*** The transverse mode of propagation involves no volume dilatation and is thus massless. Electromagnetic waves are transverse waves propagating in the spacetime continuum itself.
- ***Speed of light.*** Energy propagates through the spacetime continuum as deformations of the continuum. The maximum speed at which the transverse distortions can propagate through the spacetime continuum is  $c$ , the speed of light.
- ***Quadratic energy relation of Special Relativity.*** This is derived from the strain energy density which is separated into a dilatation energy density term (the “mass” longitudinal term) and a distortion energy density term (the “massless” transverse term). The kinetic energy  $pc$  is carried by the distortion part of the deformation, while the dilatation part carries only the rest mass energy.

- **Nature of photons.** The strain energy density of the electromagnetic field includes a longitudinal electromagnetic energy flux which is massless as it is due to distortion, not dilatation, of the spacetime continuum. However, because this energy flux is along the direction of propagation (i.e. longitudinal), it gives rise to the photon, the particle aspect of the electromagnetic field.
- **Nature of the wavefunction.** The quantum mechanical wavefunction describes longitudinal wave propagations in the spacetime continuum corresponding to the volume dilatation associated with the particle property of an object.
- **Klein-Gordon equation.** The longitudinal wave equation derived from a quantum mechanically derived volume force corresponds to the Klein-Gordon equation with a source term corresponding to an interaction term of the form  $\mathbf{A} \cdot \mathbf{j}$ .
- **Magnetic torque density equation.** The transverse wave equation is found to be a new equation of the electromagnetic field strength  $F^{\mu\nu}$ , which includes an interaction term of the form  $\mathbf{A} \times \mathbf{j}$  corresponding to the magnetic torque density.

A solid foundation of the *STCED* theory has been laid, from which further expansion can be achieved. The basic physical theory from which the rest of physical theory can be built is not complete. For example, the basic physical constants such as Planck's constant  $h$ , the elementary electrical charge  $e$ , the elementary mass of the electron  $m$ , should be derivable from the fundamental constants  $\kappa_0$ ,  $\mu_0$ ,  $\rho_0$  and others characterizing the spacetime continuum. They should also be physically explained by the theory.

This we believe can be achieved by using a more complete theory of the spacetime continuum and of the Elastodynamics of the Spacetime Continuum.

In this section, we suggest future directions to extend the theory of *STCED*. The following areas of exploration are being suggested as candidates worthy of further study:

- Exploration of alternative Volume forces derived from other identifications of related physical results.
- The incorporation of Torsion in the theory, based on Élie Cartan's differential forms formulation.
- Extension of the theory based on the evolution of Continuum Mechanics in the last one hundred years, including Eshelbian Mechanics [46] and the Mechanics of Generalized Continua [47].

- Extension of the theory to include Defects, such as dislocations and disinclinations. Given that the spacetime continuum behaves as a deformable medium, there is no reason not to expect dislocations and other defects to be present in the *STC*.

A more sophisticated theory of *STCED* is expected to provide additional insight into the fundamental nature of the spacetime continuum and of physical theory.

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